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Survey of Mathematical Models of Sustainability and Population Growth

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Preface

Writing this thesis was a challenge, not necessarily due to the intellectual difficulties associated with this topic but due to a series of trials that impacted my ability to dedicate myself fully to the research.

The extended thesis deadline is due to a nose operation I had in February, for which an extension was necessary due to the lengthy recovery period. During the new extension, my family and I were in the midst of selling our old house and purchasing a new one. Although the decision to move independently without a moving company was financially motivated this added considerable stress and disruption to my daily routine.

Balancing this significant life change with the demands of caring for my one-year-old child alongside working a full-time job only further restricted the time and energy I could devote to my research.

I acknowledge that these circumstances affected the quality of my thesis, and I recognize that I did not venture down certain research avenues because of this. In a perfect world, I would have liked to investigate additional variables/situations that may have altered the outcomes of my research. However, due to the constraints I faced, my focus remained on the essential components of the thesis.

This preface does not serve as an excuse but as a reflection of the reality of my situation. It is important to note that despite the many challenges I have endured this year, perseverance, along with the continued understanding and support of my lecturer, Leiv and my wife, Zoha, have allowed me to produce a meaningful piece of work.

14/08/2024 18:00 OSLO Fuwad Chaudhry

Abstract

In this paper, I present a mathematical model to analyse the Mayan civilisation's collapse by focusing on water as a critical resource. Using the mathematical model developed and used in the paper ‘Mathematical Model of Easter Island Society Collapse’ by Bologna and Flores (Bologna & Flores, 2010), I interpret the resource variable as water in my case with the Maya; it is important to state that this model is a theoretical construct and is not based on empirical knowledge.

I model Mayan society and its environment as an isolated system, with both the population and water resources treated as dynamic variables. This approach highlights the intricate relationship between the Mayans and their dependence on water, indicating how fluctuations in this essential resource could lead to significant societal impacts.

The analysis explores the interaction between the Mayan population and water availability by applying the selected mathematical equations to illustrate the dependency and predict the civilization's eventual collapse. This model offers insights into the specific case of the Maya and aligns with existing research on other extinct civilizations, providing a robust framework for understanding how human-environment interactions can drive societal collapse.

To maintain focus and clarity, the model used concentrates on water as the central variable, while other factors that contributed to the Mayan collapse, such as social, political, and economic variables, are not included in this analysis. This simplification, while necessary for the scope of this research, suggests that further studies could build on this model by integrating additional variables to offer a more comprehensive understanding of the collapse dynamics.

Table of Contents

1. Introduction 6
2. Background on the Maya 8
3. Literature Review 10
    3.1 Societal collapse: A literature review Danilo Brozovic 10
    3.2 Mathematical Model of Easter Island Society Collapse 13
    3.3 A mathematical model for the Andean Tiwanaku civilisation collapse: Climate Variations 14
    3.4 When Zombies Attack!: Mathematical Modelling of an Outbreak of Zombie Infection 15
4. Setting the scene 16
    4.1 Lotka-Volterra predator-prey models 16
    4.2 Leslie models for prey-predator 18
5. Modelling Society Collapse 19
6. Discussion 29
7. References 31

List of Figures

Figure 2.1: Map of the Mayan Civilization (Burchell, 2015)..8
Figure 2.2: Mayan Numerical System (Wikimedia Commons, 2006) 9
Figure 4.1: Lotka-Volterra Predator-Prey Relationship Model (Wikimedia Commons, 2017).16
Figure 5.1: Population Growth (dN/dt) vs N for different fixed values of R 22
Figure 5.2: Resource Availability (dR/dt) vs R for different fixed values of N 22
Figure 5.3: Survival Condition: Equilibrium Population vs Resource Consumption Impact 28

1. Introduction

Civilizations throughout history have reached impressive heights, marked by extraordinary achievements. However, many of these once-thriving societies have ultimately faced decline and collapse.

Studying the collapse of a society is a complex field that has been studied carefully to understand its root causes. Given the relevance of these studies, there is a popularity to investigate the factors contributing to societal collapse.

Some research has generally shown that the factors contributing to societal collapse are multifaceted and often interrelated. Common contributing factors that lead to societal collapse can be serious environmental degradation, resource depletion, social and political instability, warfare and economic strains. (Brozović, 2023)

In the mathematical community, there is a growing interest in studying the potential impact of societal collapse. Mathematical models have emerged as a valuable approach for analysing variables in the decline of some societies where the models vary depending on the complexities of the scenario. By creating accurate representations of real-world systems, these models can highlight critical thresholds or tipping points beyond which a society may become unsustainable. This approach not only unlocks the reasons why societies in the past have failed but also offers insight into contemporary societies facing similar challenges.

In this thesis, we will explore what has been done in the field by conducting literature reviews on relevant and important studies. We will then use a mathematical model to study the collapse of a civilization. Specifically, I will develop the model used by Bologna and Flores in their paper 'Mathematical Model of Easter Island Society Collapse' (Bologna & Flores, 2010).

By examining the studies in the literature review along with the findings from my model, I aim to identify common patterns and underlying mechanisms contributing to societal decline; in the hope of applying this framework to other civilizations, both past and present.

This research will focus on the collapse of the Mayan civilization; more specifically, I will investigate the impact that the water resource variable had on the Mayan civilization.

Brozovic suggests through his literature review that the Maya's reliance on agriculture made them particularly vulnerable to changes in climate. When faced with prolonged periods of drought, their ability to produce food was severely impacted, leading to shortages, malnutrition, and social unrest. This, in turn, weakened the political and social structures that held Maya society together, making it more susceptible to internal conflict and external pressures. (Brozović, 2023) As explained by Brozovic, this knock-on effect makes logical sense, but with a lack of empirical evidence, it is still a theory; and through mathematical modelling, we aim to gain a deeper understanding of how these factors might have combined to precipitate the collapse of the Mayan civilization.

As humanity faces challenges regarding the exploitation of natural resources, climate change, and social instability, the research outcomes in this thesis become increasingly relevant. In an ideal world, the lessons we can learn from the past will help us mitigate similar outcomes in the future, and this is the hope I have for this thesis.

2. Background on the Maya

The Mayan civilization thrived in the tropical lowlands of Guatemala, Belize, Honduras, El Salvador, and parts of Mexico, particularly the Yucatán Peninsula. (See Figure 2.1) (Barrios, 2023)

Figure2.1: Map of the Mayan Civilization (Burchell, 2015)

The Mayan civilization thrived from about 250 to 900 CE. They were known for their impressive accomplishments in architecture, mathematics (Figure 2.2), astronomy, and writing. The Maya built sophisticated city-states with tall pyramids, large plazas, and detailed carvings. (Barrios, 2023)

More notably, the Mayan hieroglyphic script is one of the few fully developed writing systems of the pre-Columbian Americas. Their calendar systems, including the famous Long Count, were highly sophisticated, reflecting their deep understanding of astronomy and mathematics. (Barrios, 2023) There was speculation around 2012 when rumour swept the globe predicting the end of the world due to the Mayan calendar stopping in that year.

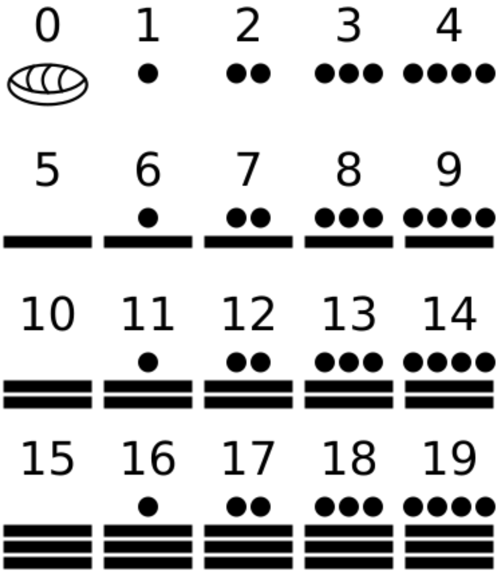

Figure 2.2: Mayan Numerical System (Wikimedia Commons, 2006)

The Mayan societal structure was complex; the empire consisted of a network of smaller, independent city-states, each with its ruler. These Mayan cities engaged in trade, cultural exchange, and warfare. (Barrios, 2023) Given their time, the Mayans were incredible farmers, which was reflected in the makeup of their economy, which was primarily agrarian. Maise was the staple crop, supplemented by beans, squash, and chilli peppers. Rare stones such as jade and obsidian propped up the economy, whilst cacao was traded across the continent. (Barrios, 2023)

Despite their achievements, the Mayan civilisation experienced a significant decline during the Terminal Classic period (approximately 800 to 1000 CE). (Barrios, 2023)
This period marks the collapse of many major Mayan city-states, particularly in the southern lowlands. The reasons behind the collapse of Mayan societies have been particularly popular, with multiple interrelated factors contributing to the downfall. (Barrios, 2023)

Environmental changes played a critical role in the collapse. A popular argument is that prolonged droughts severely affected the Maya region during the period that led up to their collapse. (Barrios, 2023)
These droughts would have devastated agriculture, leading to food shortages, malnutrition, and social unrest. The environmental change would devastate Mayan societies, making it almost impossible for them to support their populations. Since the economy was largely agrarian-based, this would, in turn, collapse the economy, leaving societies helpless. The complex trade networks would disappear, and the slow economic decline would facilitate political unrest. (Barrios, 2023)

All these factors facilitated Internal conflict and instability, which led to frequent warfare. (Barrios, 2023)

3. Literature Review

3.1 Societal collapse: A literature review Danilo Brozovic

In the paper” Societal Collapse: A Literature Review,” Brozovi organises and examines existing research on societal collapse from various scientific disciplines. The goal is to categorise the research and suggest future directions, particularly highlighting how this research can benefit the study of the future. (Brozović, 2023)

Brozovi reviews 361 articles and 73 books, identifying five main areas of discussion: past collapses, general reasons for collapse, alternatives to collapse, fictional collapses, and the impact of future climate change on societal collapse. (Brozović, 2023)

Brozovi provides an overview of essential works on societal collapse. This highlights critical contributions from various authors, offering a rich foundation for understanding the multifaceted nature of societal collapse through historical and contemporary lenses. (Brozović, 2023)

Brozovi discusses Jared Diamond’s ”Collapse: How Societies Choose to Fail or Succeed”(2005). It is important to state that Diamond examined historical societies such as the Norse in Greenland and the Maya, where similar factors were identified as contributing to the respective societal collapses. These include, but are not limited to, environmental damage, climate change, trade disruptions and warfare. A key point from the Diamond study that Brozovi discusses is that societal collapse occurs because of interrelated factors rather than a single variable. (Brozović, 2023)

Brozovi also reviews Joseph Tainter’s ”The Collapse of Complex Societies” (1988). Here, Tainter argues that as a society becomes more complex, the costs of maintaining that complexity increase, leading to the collapse of a society when the cost of maintaining the society outweighs the financial benefits the society brings in. Tainter provides an economic perspective on societal collapse, highlighting the importance of resource allocation and efficiency. (Brozović, 2023)

Peter Turchin’s ”War and Peace and War: The Life Cycles of Imperial Nations” (2006) is also highlighted in Brozovi’s review. Turchin uses mathematical models to historical data to explore the rise and fall of empires. Brozovi highlights that a key point in this review is that the cyclical nature of social cohesion and disintegration can explain the rise and fall of empires. Highlighting the key idea of ‘asabiya’ social cohesion. (Brozović, 2023)

Brozovi references Robert J. Wenke’s ”Patterns in Prehistory: Humankind’s First Three Million Years” (1999). Wenke does not show societal collapse directly; however, it is important to note that he underscores the role of environmental factors in societal development and collapse, showing not only the resilience in societies when faced with catastrophe but the extent to which this catastrophe causes. (Brozović, 2023)

Additionally, Thomas Homer-Dixon’s ”The Upside of Down: Catastrophe, Creativity, and the Renewal of Civilization” (2006) is mentioned in Brozovic's review.
Homer-Dixon explores how complex societies can turn crises into opportunities for renewal and transformation, arguing that societal collapse can lead to creative problem-solving and innovation. This perspective suggests that the aftermath of collapse can be a period of significant change and adaptation. This is an interesting take on societal collapse. However, it is limited due to the technologies and advancements the collapsing society is restricted to. (Brozović, 2023)

The papers in Brozovic’s literature review offer different perspectives on understanding societal collapse. This not only highlights the importance of interdisciplinary research into societal collapse but also provides valuable insights into what we can learn from these situations. Depending on the situation of a country policymakers can learn from past societal collapses in order to not make the same mistakes. (Brozović, 2023)

Brozovi provides an overview of important works on societal collapse. Where several key factors and themes are identified as critical to understanding the complex phenomenon of societal collapse.

One of the primary drivers highlighted is environmental change, including climate shifts, resource depletion, and natural disasters. Brozovi states that these environmental factors initiate a cascade of adverse effects. For example, drought causes soil depletion, which in turn causes agricultural failure, which in turn causes economic downfall, which in turn causes social instability, which in turn can cause warfare, all contributing to societal collapse. Diamond’s research underscores the significant impact of environmental mismanagement and climate variations on the sustainability of societies. (Brozović, 2023)

Economic issues also emerge as a crucial aspect of societal collapse. Resource scarcity, trade disruptions, and financial crises can undermine the economic foundation of societies. Tainter’s theory is that when the costs associated with maintaining a society outweigh the financial benefits, then collapse occurs. This is an interesting take on societal collapse and can be vital in understanding the internal pressures in government that contribute to societal collapse. (Brozović, 2023)

Social and political dimensions are equally important factors contributing to societal collapse. Factors such as political corruption, social inequality, and the erosion of social cohesion can destabilise societies from within. Turchin’s concept of ”asabiya” (social cohesion) is relevant, suggesting that high social cohesion can help societies endure external pressures, whereas declining cohesion can lead to internal conflict and collapse. (Brozović, 2023)

Warfare is the simplest trigger for societal collapse. Historical examples frequently show how invasions from hostile neighbours or prolonged conflicts with other societies can accelerate societal decline. These external threats and internal vulnerabilities often create a situation where collapse becomes inevitable. (Brozović, 2023) An example of this would be when the Mongols invaded Baghdad, causing destruction and essentially exterminating a flourishing civilization.

The interplay of these factors reveals that societal collapse is rarely due to a single cause. Instead, it is typically the result of multiple interrelated elements that interact in complex ways. (Brozović, 2023)

3.2 Mathematical Model of Easter Island Society Collapse

Easter Island has long captivated scholars, raising curiosity and doubt about the construction of the moai statues. This curiosity has only increased when questions have been raised about the collapse of society.

M. Bologna and J. C. Flores dissect the dynamics of the Easter Island collapse by using sophisticated mathematical modelling techniques. This literature review attempts to integrate their research into the broader landscape of studies analysing societal collapse. (Bologna & Flores, 2010)

Bologna and Flores use adjusted Lotka-Volterra predator-prey models to highlight the resource factor, a key variable causing society's collapse. By conducting an analysis of the models, Bologna and Flores are able to deduce two equilibrium points: the first, in a scenario where the population is 0, allowing the resource variable to reach its maximum capacity, and the second, where society would collapse if the rate at which the resource is used by the population is greater than the rate at which the resource can be replenished. (Bologna & Flores, 2010)

Bologna and Flores further state that for civilisation to continue, there must be a point representing a stable balance between population size and resource level; this is done through an inequality equation. (Bologna & Flores, 2010)

This inequality equation unveils how the depletion of a vital resource can trigger catastrophic repercussions, highlighting the delicate interplay between human societies and their natural environments. It advocates for the need for sustainable resource management to mitigate a society's eventual collapse.

Using mathematical models complements traditional qualitative methodologies. This paper highlights the benefit of using mathematics as a powerful analytical tool for understanding the mechanisms of societal collapse.

It is important to note that although the results found in this paper are significant, there are limitations to using only one variable to model society collapse. Another important variable should be that Easter Island was limited in size and surrounded by water. With little technology, the Islanders could not migrate to find new resources. It is difficult to model different variables; however, to get the most accurate results, all variables must be considered together.

3.3 A mathematical model for the Andean Tiwanaku civilisation collapse: Climate variations

In this literature review, I will discuss M. Bologna and J. C. Flores' 2011 paper,” A Mathematical Model for the Andean Tiwanaku Civilization Collapse: Climate Variations.”

Bologna and Flores explore the factors contributing to the decline of the Tiwanaku civilisation in the Andean region of South America.

Flourishing from roughly 300 to 1100 CE, the Tiwanaku civilisation boasted impressive achievements in architecture, agriculture, and trade. (Flores et al., 2011)

By the 11th century, the Tiwanaku civilisation faced a downturn, which resulted from fragmentations in cultural identity and the abandonment of urban centres. Bologna and Flores investigated the influence of climatic variations on the collapse. (Flores et al., 2011)

Bologna and Flores use mathematical models to depict the relationship between climate variables and societal collapse. Using historical climatic data alongside demographic and environmental variables, Bologna and Flores are able to create a model capable of describing the tipping points that likely contributed to the societal collapse. (Flores et al., 2011)

An important finding in the study is the significant impact of climate variability on the resilience of the Tiwanaku civilisation. Through the model, we are able to see how the knock-on effect took shape. The fluctuations in rainfall and temperature could have impacted agricultural productivity and resource availability, intensifying preexisting social strains, fragmenting society and facilitating societal collapse. (Flores et al., 2011)

Through this study, Bologna and Flores highlight the necessity of adopting a holistic approach in adverse situations. The interconnectedness of human societies and their natural environments can have a ripple effect through social systems, amplifying vulnerabilities.

3.4 When Zombies Attack!: Mathematical Modelling of an Outbreak of Zombie Infection

Although this paper does not directly relate to societal collapse, I thought reviewing a paper that uses the SIR model to describe an outbreak would be important. The reason for doing this is that the same argument made in this study using an infection can be used in the case of a societal collapse with a disease outbreak in the civilization.

The 2009 paper ”When Zombies Attack!: Mathematical Modelling of an Outbreak of Zombie Infection” by Robert J. Smith models a hypothetical zombie outbreak, offering insights into disaster preparedness. (Munz et al., n.d.)

Smith uses the classical SIR model as a template for his model, adapting this concept to suit the features of a zombie outbreak. The model includes variables such as zombie bite transmission rates, human mortality rates, and the effectiveness of zombie eradication efforts, allowing for exploring various scenarios and mitigation strategies. (Munz et al., n.d.)

Smith assigns a part of the population as susceptible, infected, or recovered (or deceased). Using differential equations, Smith is able to calculate the rate of change for each population over time. Susceptible individuals would become infected when in contact with a Zombie, and infected individuals would either recover or die, with recovered individuals remaining immune to reinfection. (Munz et al., n.d.)

Smith can determine thresholds and containment measures by varying the population size, geographical distribution and intervention strategies. (Munz et al., n.d.)

One of the paper's key findings is the identification of critical thresholds, known as the ”zombie outbreak equilibrium,” which determines whether the human population can successfully contain or repel the zombie invasion. (Munz et al., n.d.)

Smith is able to compare the effectiveness of different containment strategies, from quarantine measures to military interventions. (Munz et al., n.d.)

This study's important aspect is that it attempts to model a fictional scenario that provides emergency response and disaster recovery solutions. While the zombie outbreak scenario may seem far-fetched, Smith argues that the principles of mathematical modelling can be applied to a range of plausible disaster scenarios, including infectious disease outbreaks, natural disasters, and terrorist attacks (Munz et al., n.d.).

By examining the societal response to a zombie apocalypse, Smith encourages readers to reconsider their assumptions about risk assessment, resource allocation, and public policy in the face of existential threats. (Munz et al., n.d.)

4. Setting the scene

4.1 Lotka-Volterra predator-prey models

Predator-prey interactions are used to describe the relationship between a predator and a prey over time. These models are crucial in ecological studies and can be effectively described by the Lotka-Volterra equations. These equations capture how predator x(t) and prey y(t) populations change over time: (Shim & Fishwick, 2008)

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \delta \mathrm{xy} - \gamma \mathrm{y}$$

In these equations, x represents the predator population, y represents the prey population, and α, β, δ, γ are parameters referring to birth rates, death rates, and interactions between predator and prey populations.

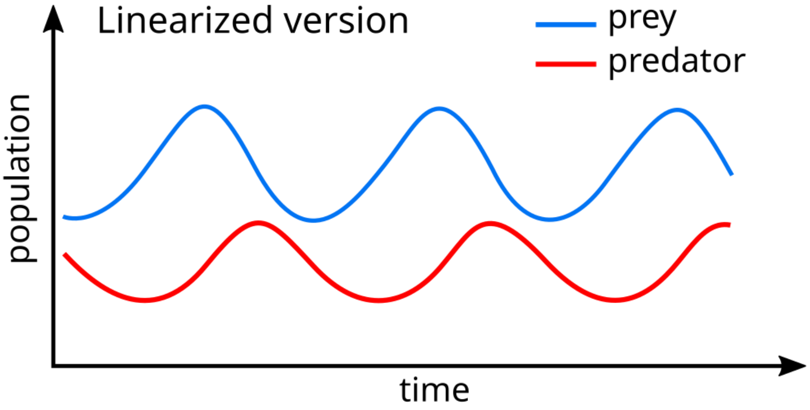

Figure 4.1: Lotka-Volterra Predator-Prey Relationship Model (Wikimedia Commons, 2017)

As shown in Figure 4.1, when prey populations (such as rabbits) are abundant, the predator population (like foxes) increases due to an excessive food supply. This relationship is captured by the term βxy in the predator equation, indicating that predators consume prey at a rate proportional to both predator and prey densities. As predator numbers rise, It becomes more difficult to hunt for the prey as the prey are limited in number. The predators hunt excessively, causing the prey population to decline (δxy term in the prey equation). With fewer prey available, predator numbers eventually decline due to reduced food supply γy term), allowing the prey population to recover. (Shim & Fishwick, 2008)

This cyclic pattern of predator-prey interactions demonstrates natural ecosystems' dynamic balance between predator and prey populations.

It is important to note that Stable equilibrium points occur where $\frac{dx}{dt} = 0 \; and \, \frac{dy}{dt} = 0$, indicating a point of balance where predator and prey populations stabilise. These equilibrium points show how closely linked predator and prey dynamics are, and although it is not always the case, this highlights the importance of balance in maintaining stable populations. (Shim & Fishwick, 2008)

Using the Lotka-Volterra model to simulate different scenarios, ecologists can predict and mitigate potential disruptions to predator-prey relationships, ensuring the resilience and sustainability of natural systems. (Shim & Fishwick, 2008)

4.2 Leslie models for prey-predator

The Leslie model continues the basic principles of the Lotka-Volterra equations by incorporating age-structured population dynamics. This is particularly useful for species with varying developmental stages (Rozikov & Shoyimardonov, 2020). This model is advantageous for studying populations like mammals, where discrete time steps align well with life stages. While the Leslie matrix model is a powerful tool in ecology, especially for examining predator-prey dynamics with detailed life history information, it is not directly relevant to the current focus of this thesis.

In the Leslie model, prey and predator populations are represented as vectors, with matrix equations governing their transitions across time steps (Rozikov & Shoyimardonov, 2020). The model captures the complexity of life stages and interactions between different age classes, offering insights into population trends and ecological responses to environmental changes (Rozikov & Shoyimardonov, 2020).

However, this thesis does not take from the Leslie model as it focuses on a different methodological approach to understanding predator-prey dynamics. Instead, the focus is on using the mathematical model developed and used in the paper ‘Mathematical Model of Easter Island Society Collapse’ by Bologna and Flores (Bologna & Flores, 2010), interpreted using the resource variable water.

5. Modelling Society Collapse

During this study on the collapse of the Mayan civilisation, I will adapt the mathematical framework used in "Easter Island: Mathematical Model of Easter Island Society Collapse" by M. Bologna and J. C. Flores in the attempt to determine whether there is a similar case with the Mayan civilization.

The Easter Island model provided a structured approach to understanding societal collapse through resource depletion and environmental dynamics. I will focus on modelling the impact of resource depletion on the Mayan society, using equations that track how population growth and resource consumption interact over time. I will use the Mayan population as the population variable and water availability as the resource variable.

This will help simulate how the over-extraction of water resources affected the sustainability of Mayan city-states.

This approach will allow for a comprehensive understanding of the interplay between human activities and environmental conditions in the context of the Mayan collapse. By comparing the results to the Easter Island case study, I will assess whether similar dynamics can explain the rise and fall of the Mayan civilisation, offering insights into the broader patterns of societal resilience and vulnerability in the face of ecological challenges.

Definitions of Variables

R_i : Quantity of resource i.

Represents the available amount of a resource such as palm trees, oil, or food.

N_i : Population size of species i.

Indicates the number of inhabitants or distinct subgroups within the population (e.g., different species or tribes).

k :Index for types of resources.

Used to distinguish different kinds of resources in the model (e.g., R_1, R_2 etc)

m : Index for types of populations.

Used to distinguish different population groups or species in the model (e.g., N_1, N_2, etc.).

r_i : Growth rate of population i.

Describes the rate at which population i increases over time.

r'_i : Renewability rate of resource i.

Describes how quickly resource i can replenish itself over time.

$N_{ci}(R_1, \ldots, R_k)$: Carrying capacity with respect to resource availability.

Indicates the maximum population i that the environment can sustain, depending on the availability of various resources

χ_{ij} : Interaction effect between populations i and$j$.

Measures the impact of population j on population i where $\chi_{ij} \neq \chi_{ji}$ as prey is a resource for the predator.

α_{ij} : Effect of population j on resource i.

Measures the rate at which population j consumes or impacts resource i.

R_{ci} : Maximum capacity of resource i.

Represents the upper limit of how much resource i can exist in the environment.

Model Equations:

Population Dynamics Equation: (Bologna & Flores, 2010)

$$\frac{dN_i}{dt} = r_i \ N_i \left[1 - \frac{N_i}{N_{ci}(R_1, \ldots., R_K)}\right] - \sum\nolimits_{\mathrm{j=1,j\neq i}}^{m} Xij^{N_j N_i}$$

This describes how the population N_{i} grows and interacts with other populations and the environment.

Resource Dynamics Equation: (Bologna & Flores, 2010)

$$\frac{dR_i}{dt} = r'_i \ R_i \left[1 - \frac{R}{R_{ci}}\right] - \sum\nolimits_{\mathrm{j=1}}^{m} \alpha ij^{N_j R_i}$$

This describes how the resource R_{i} regenerates and is consumed by different populations.

When modelling the collapse of the Mayan civilisation with a single species (the Mayans) and one primary resource (water), we can replace variables in order to simplify our equations. The Population Dynamics Equation for the Mayans (N) and the Resource Dynamics Equation for water (R) are then formulated as follows (Bologna & Flores, 2010)

$$\frac{dN}{dt} = rN \left[1 - \frac{N}{N_c(R)}\right]$$

And

$$\frac{dR}{dt} = r'R \left[1 - \frac{R}{R_c}\right] - \alpha NR$$

Here, N represents the Mayan population, and R stands for water availability.

The growth rate of the population is denoted by r, and the renewability rate of water by r' .

The carrying capacity $Nc(R)$ describes the maximum population the water resource can sustain, adjusting as the resource level R changes.

The term α captures the rate at which the Mayan population consumes water.

Rc is the maximum capacity or upper limit of water that the environment can sustain.

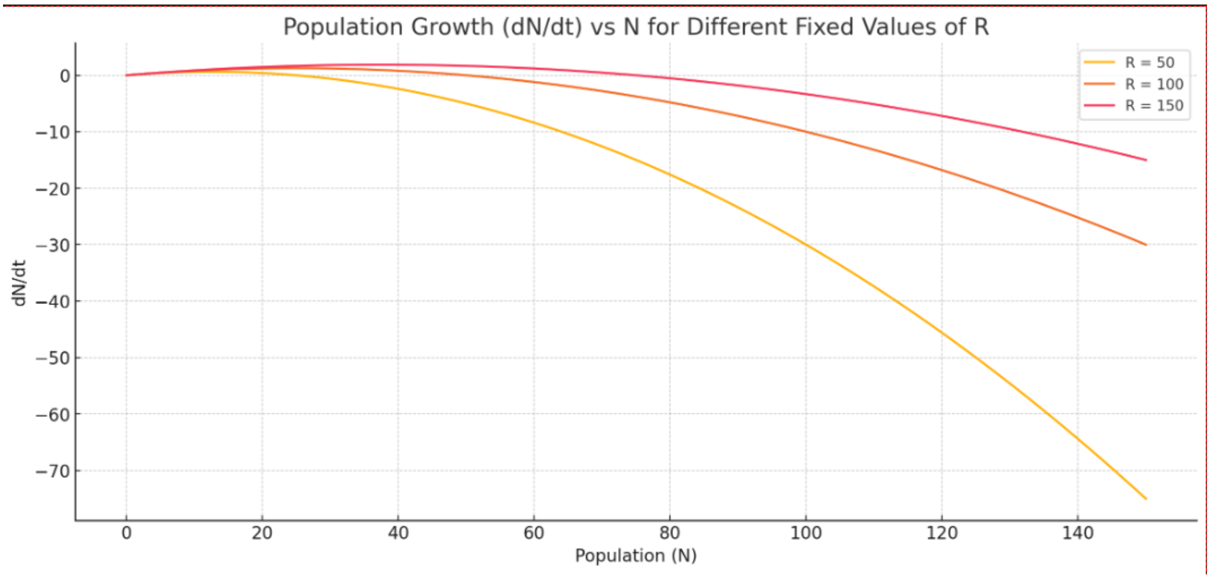

Figure 5.1: Population Growth (dN/dt) vs N for different fixed values of R

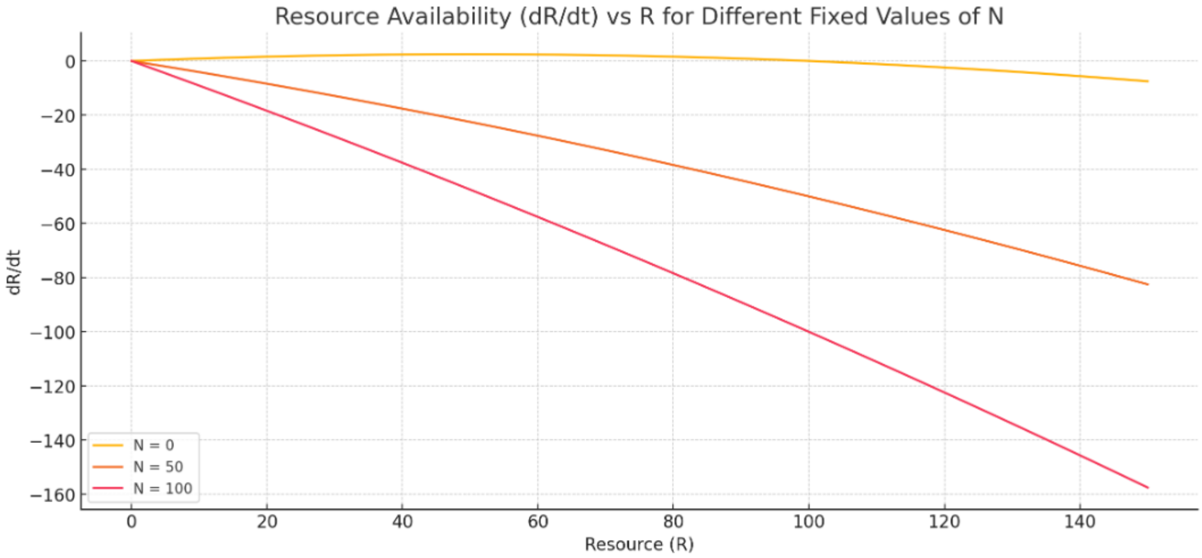

Figure 5.2: Resource Availability (dR/dt) vs R for different fixed values of N

Figure 5.1:

The population growth graph shows different curves for dN/dt as N changes, with each curve corresponding to a fixed value of R.

When R is larger, Nc(R) is also larger, allowing the population to grow before reaching a point where growth slows and eventually stops.

As N approaches Nc(R), dN/dt decreases, indicating that the population is nearing its carrying capacity and thus slowing down growth.

Figure 5.2:

Figure 5.2 shows different curves for dR/dt as R changes, with each curve corresponding to a fixed value of N.

When the population N is small, the resource level R tends to stabilize near its carrying capacity Rc.

As N increases, the term −αNR becomes more significant, representing increased resource consumption by the population. This causes resources to deplete more rapidly, as seen by the more negative values of dR/dt when N is higher.

Both figures 5.1 and 5.2 show the interdependence between population size and resource availability. As resources become scarce, population growth slows. Conversely, as the population grows, resources deplete faster.

If we recall the equations,

$$\frac{dN}{dt} = rN\left[1 - \frac{N}{N_c(R)}\right]$$

And

$$\frac{dR}{dt} = r'R\left[1 - \frac{R}{R_c}\right] - \alpha NR$$

The first term in each describes natural growth: the population grows according to a logistic model that considers the carrying capacity based on water availability, while water renews naturally up to its capacity Rc.

The second term in the resource equation accounts for the depletion of water due to its consumption by the Mayan population.

This model captures the dynamic interplay between population growth and resource availability, reflecting how increasing population pressures on water resources can lead to depletion and, ultimately, societal collapse if the resource is overused and cannot replenish quickly enough.

As the water remains plentiful, the Mayan population can expand, but as the population grows, it consumes more water, leading to resource depletion. This depletion causes the carrying capacity ($Nc(R)$), or the maximum sustainable population size, to decline as water becomes scarce.

This scarcity subsequently slows or even reverses population growth. The system ideally reaches a dynamic balance where water replenishes at a rate that sustains the population. However, if the population grows too rapidly or water is consumed faster than it can be replenished, the balance is disrupted, leading to resource depletion and potential population collapse.

Applying these equations to the Mayan civilization allows us to simulate the effects of water availability fluctuations on the population over time. Heavy exploitation of water resources without adequate replenishment would lead to a decrease in the carrying capacity ($Nc(R)$), stressing the population and potentially causing a decline.

Incorporating environmental factors like droughts into the model can further demonstrate how sudden or extended reductions in water availability could intensify these effects and contribute to the civilisation's collapse.

In the original equations, we introduced a concept called $N_c(R)$, which represents how the population size N depends on the availability of resources R. This concept helps us understand how populations grow or decline based on their access to resources.

The function $N_c(R)$ needs to follow certain rules: when resources R are plentiful (approaching infinity), $N_c(R)$ also increases without limit, suggesting the potential for the population to grow indefinitely.

Conversely, when resources are scarce (approaching zero), $N_c(R)$ also approaches zero, indicating that the population cannot be sustained without adequate resources. If resources remain constant, $N_c(R)$ simplifies to a constant value, similar to how populations stabilise under normal conditions.

To keep things simple, we chose $N_c(R)=\beta R$, where β is a positive number. This choice means that the maximum population size that can be supported by the available resources R is directly proportional to R.

Previously, the term αNR represented how the population N interacts with the resource R, with α indicating how strong this interaction is.

To expand the model, we could include fishing as an additional resource. In that case, $N_c(R)$ might be adjusted to $N_c(R)=\beta R+R_f$, where R_f represents the carrying capacity of fishing resources. However, for our study, we simplified our model by ignoring fishing resources due to the historical limitations such as the absence of boats during the Mayan Civilization.

Hence our equations can be rewritten as: (Bologna & Flores, 2010)

$$\frac{dN}{dt} = rN\left[1 - \frac{N}{\beta \mathrm{R}}\right]$$

where r is the intrinsic growth rate of the population, and β is a parameter that relates the carrying capacity N_c to the available water R.

What this equation now means is that the population grows exponentially when water is abundant but slows as the water becomes scarce, limiting the population's potential size. The water resource dynamics are represented by: (Bologna & Flores, 2010)

$$\frac{dR}{dt} = r'R\left[1 - \frac{R}{R_c}\right] - \alpha \mathrm{ER}$$

where r' is the renewal rate of the water resource, R_c is the maximum sustainable level of water, and αE is the water consumption rate by the population.

What this equation now means is that the natural replenishment of water is balanced against its consumption by the population, modelling how increasing population leads to greater water depletion.

Finding equilibrium points in the context of modelling the collapse of the Mayan civilisation is crucial in analysing the importance of the variables.

These equilibrium points provide insights into the sustainability of human-environment interactions. Equilibrium points represent stable conditions where the population N and the water resource R reach a balance over time.

For instance, (N_e, R_e) denotes a stable equilibrium where the Mayan population and the water resources are in a sustainable relationship. This means that under these conditions, the available water resources could support the population without depleting them beyond their capacity to regenerate.

By solving the equations, we found two key equilibrium points:

Trivial Equilibrium (No Population):

$N_0 = 0$, $R_0 = R_c$.

This point represents a scenario in which the population is zero, allowing the water level to reach its maximum capacity R_c.

It is an unstable saddle point, implying that any small population increase will disturb this balance.

Non-trivial Equilibrium (Population-Water Balance): (Bologna & Flores, 2010)

$$N_e = \frac{\beta R_c}{1 + \alpha E \beta R_c}$$

$$R_e = \frac{R_c}{1 + \alpha E \beta R_c}$$

This point represents a stable balance where the population size N_e and water level R_e are in equilibrium.

The parameter αE, which measures the effectiveness of the population's water consumption relative to its renewal, plays a crucial role in determining the population and resource levels at equilibrium.

The relevance of these equilibrium points allows us to understand how the Mayan society could maintain or fail to maintain balance with their water resources.

The stable equilibrium indicates a sustainable state where water replenishment can match or exceed consumption.

However, if water consumption surpasses the replenishment rate, the system moves away from equilibrium, leading to resource depletion and potential population collapse.

It is important to understand how the population N and the water resources R interact. Although it is possible to find a stable equilibrium point for both population and water mathematically, biological realities must be considered: a minimum population size N_{min} is necessary for the society's survival.

This minimum ensures sufficient genetic diversity, social structures, and the ability to maintain a functioning society. If the population drops below this critical number, the civilization risks collapse, even if water resources remain stable.

Equilibrium Condition

In a simplified form, the equilibrium condition for water resources without human interference can be expressed by an equation like: $R_e = R$

However, for the population N, the equilibrium point considering water R is: $N = \beta R$

Here, β is a parameter indicating how the maximum sustainable population size is proportional to available water.

Survival Condition

To ensure the Mayan civilization survives, the population at equilibrium N_e must be at least N_{min}, the minimum viable number.

To find the upper limit for the parameter αE (which describes how population consumption impacts water resources), we impose the condition that the equilibrium population N_e is no less than N_{min} : (Bologna & Flores, 2010)

$$\alpha E \geq \frac{1}{N_{min}} - \frac{1}{\beta R_c}$$

In simpler terms, this inequality ensures that the population doesn't collapse due to insufficient water.

If we assume that N_{min} is much smaller than βR_c, this can be approximated to: (Bologna & Flores, 2010)

$$\alpha E N_{min} \geq 1$$

or equivalently:

$$\alpha N_{min} \geq r'$$

This condition means that civilization will collapse if the population's use of water exceeds the rate at which water can be replenished.

Numerical Example

In practical terms, if we start with an initial population N(0) less than the carrying capacity given by $N_c = \beta R_c$, the population may initially grow, potentially exceeding this carrying capacity depending on initial conditions.

This growth, however, is not sustainable. For example, if the population exceeds Br_c, it might peak at values much higher than the sustainable level but will eventually decline rapidly, falling to or below N_{mi} and collapsing.

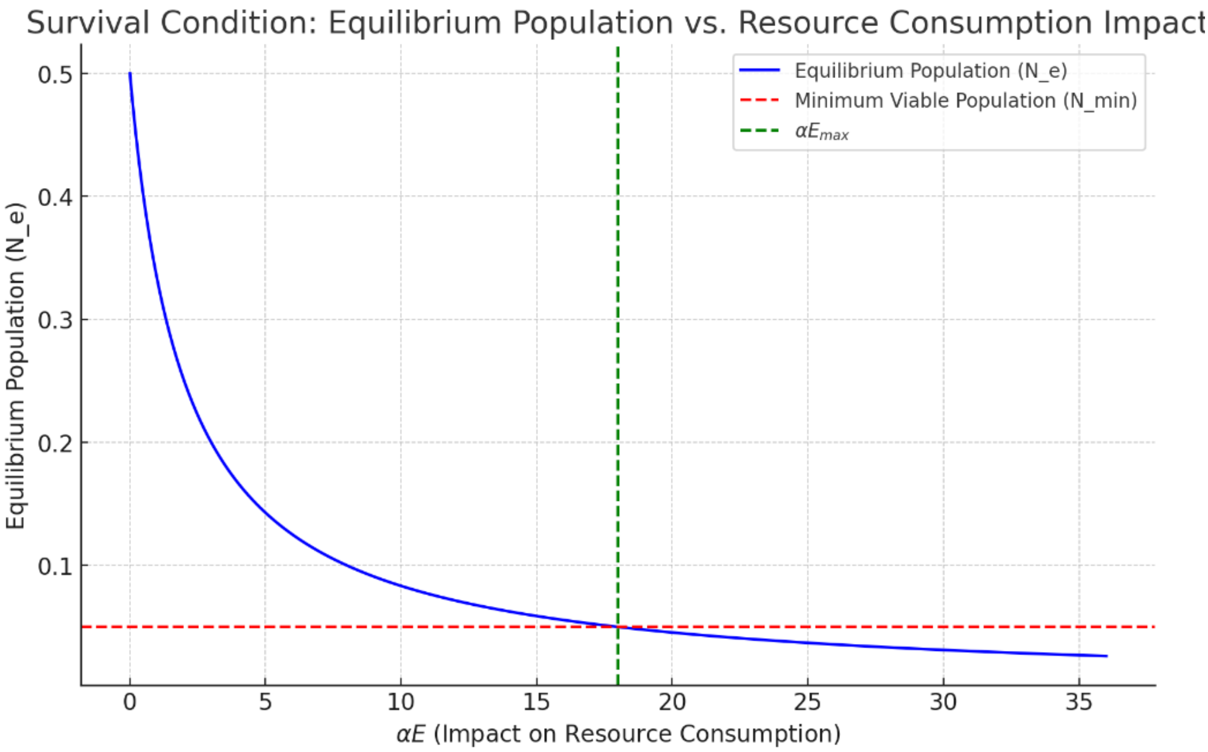

Figure 5.3: Survival Condition: Equilibrium population vs Resource Consumption Impact

Blue Curve (Equilibrium Population Ne):

- The curve shows how the equilibrium population decreases as the parameter αE increases.
- This is consistent with the idea that higher values of αE (greater impact of population consumption on water resources) would reduce the sustainable population.

Red Dashed Line (Minimum Viable Population Nmin):

- This line represents the minimum population needed for survival.
- The point where the blue curve crosses this line indicates the critical value of αE beyond which the population falls below the viable level.

Green Dashed Line (αEmax):

- This line represents the upper limit for αE. Beyond this point, the equilibrium population Ne drops below Nmin, leading to collapse.

6. Discussion

By modelling the interaction between the Mayan population and water as a critical resource, we have highlighted how resource depletion and environmental factors could have influenced the society's stability. Although this is a bold link to make the models in this paper, along with the multiple literature reviews, allow us to make this link confidently.

This simplified yet robust approach reveals that as the Mayan population grew, their increasing water demand could have led to resource scarcity. This scarcity, facilitated by environmental changes such as prolonged droughts, might have resulted in significant stress on the population, ultimately contributing to societal decline. (Bologna & Flores, 2010)

The equilibrium points identified in our model illustrate crucial thresholds where the population and water availability balance each other. These points help us understand the conditions under which the Mayan society could sustain itself or face collapse.

For example, an equilibrium point where the population and water resources are balanced indicates a stable state where both can coexist without leading to resource exhaustion. However, if the water availability drops below a critical threshold, the model predicts a shift away from this balance, leading to a decline in population and potential collapse. (Bologna & Flores, 2010)

Understanding the importance of resource management protects the population from the vulnerabilities that civilizations experience during hardship.

One significant insight from this study is the role of water resource management in the resilience or collapse of the Mayan civilization. The findings suggest that the Mayans' reliance on water for their agricultural and societal needs made them particularly sensitive to changes in water availability. Future research could focus on more detailed historical data to refine the model, incorporating factors such as variations in rainfall, changes in water storage technologies, and the impact of agricultural practices on water usage. In a perfect world, it would be ideal to have all these variables to ensure the utmost accuracy of our study, but the Mayan civilisation collapsed over 1000 years ago. Finding accurate data is difficult for these additional variables. We are limited to studying the basic variables.

Moving forward, our methodology could be expanded to include other critical resources such as food or timber and how their availability interacted with water resources to influence Mayan society. However, it is difficult to include this variable in our model as we do not have a way of interpreting this into empirical data.

The limitation of this study is the number of variables I have used. Returning to the literature review, it is essential to include an interdisciplinary view when analysing a distant civilisation. This is why I found it relevant to go into the depth I did for the literature review to add a scope for the research on the other variables.

In the context of this study, I am restricted to mathematical modelling, but it would be interesting to include other variables, such as archaeological data, alongside the existing model to see whether this would support my analysis or raise some questions.

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