Secant loci of scrolls over curves

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Dedicated to Professor Peter E. Newstead on the occasion of his 80th birthday conference

Abstract. Given a curve C and a linear series ℓ on C, the secant locus $V_e^{e-f}(\ell)$ parametrises effective divisors of degree e which impose at most $e-f$ conditions on ℓ. For $E \to C$ a vector bundle of rank r, we define determinantal subschemes $H_e^{e-f}(\ell) \subseteq \mathrm{Hilb}^e(\mathbb{P}E)$ and $Q_e^{e-f}(V) \subseteq \mathrm{Quot}^{0,e}(E^*)$ which generalise $V_e^{e-f}(\ell)$, giving several examples. We describe the Zariski tangent spaces of $Q_e^{e-f}(V)$, and give examples showing that smoothness of $Q_e^{e-f}(V)$ is not necessarily controlled by injectiveness of a Petri map. We generalise the Abel–Jacobi map and the notion of linear series to the context of Quot schemes.

We give some sufficient conditions for nonemptiness of generalised secant loci, and a criterion in the complete case when $f = 1$ in terms of the Segre invariant $s_1(E)$. This leads to a geometric characterisation of semistability similar to that in [**Hit19**]. Using these ideas, we also give a partial answer to a question of Lange on very ampleness of $\mathcal{O}_{\mathbb{P}E}(1)$, and show that for any curve, $Q_e^{e-1}(V)$ is either empty or of the expected dimension for sufficiently general E and V. When $Q_e^{e-1}(V)$ has and attains expected dimension zero, we use formulas of Oprea–Pandharipande and Stark to enumerate $Q_e^{e-1}(V)$.

We mention several possible avenues of further investigation.

1. Introduction

The purpose of this article is to investigate secant loci of scrolls over curves, and to begin a study of generalised secant loci on Quot schemes.

Let C be a complex projective smooth curve of genus g. For $e \geq 1$, let C_e be the eth symmetric product of C. For $n \geq 1$ and $0 \leq f < e$, we consider the locus

$$C_e^f := \{D \in C_e : h^0(C, \mathcal{O}_C(D)) \geq f+1\} = \{D \in C_e : h^0(C, K_C(-D)) \geq g-e+f\}.$$

This is a determinantal subvariety of C_e, whose tangent spaces are defined by a Petri map. The construction and properties of C_e^f are explained in detail in [**ACGH85**, Chapter IV]. The locus C_e^f is the preimage of the Brill–Noether locus

$$W_e^f := \{L \in \mathrm{Pic}^e(C) : h^0(C, L) \geq f+1\}$$

by the Abel–Jacobi map $C_e \to \mathrm{Pic}^e(C)$.

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An important characterisation of the divisors in C_e^f is given by the geometric Riemann–Roch theorem, which may be interpreted as saying that

$$C_e^f \;=\; \{D \in C_e : \dim \mathrm{Span}\,(\varphi_{K_C}(D)) \le e - f - 1\}.$$

More generally, let L be a line bundle and $\ell = (L, V)$ a linear series of dimension n. For $0 \le f \le e$, we then have a *secant locus* associated to ℓ, defined as

$$V_e^{e-f}(\ell) \;:=\; \{D \in C_e : \dim \left(V \cap H^0(C, L(-D))\right) \ge n + 1 - e + f\}. \tag{1.1}$$

Equivalently, writing φ for the natural map $C \to \mathbb{P}V^*$, we have

$$V_e^{e-f}(\ell) \;=\; \{D \in C_e : \dim \mathrm{Span}\,(\varphi(D)) \le e - f - 1\}.$$

Note that $C_e^f = V_e^{e-f}\left(K_C, H^0(C, K_C)\right)$.

The loci C_e^f are important invariants of the curve, as illustrated by the theorems of Martens and Mumford [**ACGH85**, § IV.5], and their generalisations to $V_e^{e-f}(\ell)$ proven in [**Baj15**]. Various instances and properties of C_e^f have been investigated in [**Cop95**], [**CM91**], [**GP82**] and elsewhere. The tangent spaces of C_e^f and $V_e^{e-f}(\ell)$ have been studied in [**CJ91**], [**CJ96**], [**Baj15**] and [**Baj18**].

A major topic of interest is the enumerative geometry of $V_e^{e-f}(\ell)$. This is studied for C_e^f in [**ACGH85**, Chap. VII], and that of $V_e^{e-f}(\ell)$ is investigated in [**LeB06**], [**Far08**], [**Far22**], [**Cot11**], [**Ung21a**], [**Ung21b**], [**CHZ21**], and elsewhere. In contrast to the Brill–Noether loci W_e^f, even when $V_e^{e-f}(\ell)$ has nonnegative expected dimension it is not always defined by a map with suitable ampleness properties, so the question of nonemptiness may be nontrivial. In [**Ung19**], examples are given of empty secant loci with positive expected dimension, and in [**Ung21b**] the question is addressed of when intersection-theoretic computations have enumerative meaning.

The purpose of the present work is to study two closely related generalisations of $V_e^{e-f}(\ell)$, obtained by increasing $\dim(C)$ and $\mathrm{rk}(L)$ respectively. The first is very familiar. Let S be a smooth projective variety of dimension $r \ge 1$, and $\ell = (\mathcal{L}, V)$ a linear series of dimension n on S. Let $\mathrm{Hilb}^e_{\mathrm{sm}}(S) \subseteq \mathrm{Hilb}^e(S)$ be the component containing reduced subschemes. We define

$$H_e^{e-f}(\ell) \;:=\; \{Z \in \mathrm{Hilb}^e_{\mathrm{sm}}(S) : \dim \left(V \cap H^0(S, \mathcal{L} \otimes \mathcal{I}_Z)\right) \ge n + 1 - e + f\}.$$

(The restriction to the smoothable component $\mathrm{Hilb}^e_{\mathrm{sm}}(S)$ is not necessary; see Remark 2.3.) Clearly, $H_e^{e-f}(\ell)$ reduces to $V_e^{e-f}(\ell)$ when S is the curve C. In case S is a surface, the enumerative geometry of $H_e^{e-f}(\ell)$ has been studied via integrals of Segre classes in [**EGL01**], [**Voi19**], [**MOP2019**] and elsewhere. Our focus is primarily on the case where S is a projective bundle $\pi \colon \mathbb{P}E \to C$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* M$, where techniques on vector bundles over curves can be brought to bear.

For the second generalisation of $V_e^{e-f}(\ell)$, let $E \to C$ be a vector bundle of rank r and degree d. The scheme $\mathrm{Quot}^{0,e}(E^*)$ parametrises torsion quotients of E^* of length e; equivalently, subsheaves $F^* \subset E^*$ of rank r and degree $-d-e$. Let M be a line bundle, and suppose that $V \subseteq H^0(C, E^* \otimes M)$ is a subspace of dimension $n+1$. We define

$$Q_e^{e-f}(E, M, V) \;:=\; \{[F^* \to E^*] \in \mathrm{Quot}^{0,e}(E^*) : h^0(C, F^* \otimes M) \ge n + 1 - e + f\}.$$

If E and M are clear from the context, we write simply $Q_e^{e-f}(V)$; and we abbreviate $Q_e^{e-f}\left(H^0(C, E^* \otimes M)\right)$ to Q_e^{e-f}. Note that $\mathrm{Quot}^e(L^{-1}) = C_e$ for all line bundles

L. In this case, we have

$$Q_e^{e-f}(\mathcal{O}_C, L, V) \ = \ Q_e^{e-f}(L, \mathcal{O}_C, V) \ = \ V_e^{e-f}(L^*, V).$$

Note in particular that

$$C_e^f \ = \ Q_e^{e-f}\left(K_C, \mathcal{O}_C, H^0(C, K_C)\right). \tag{1.2}$$

It is straightforward to see (§ 2.3) that $H_e^{e-f}(\ell)$ and $Q_e^{e-f}(E, M, V)$, like $V_e^{e-f}(\ell)$, are determinantal varieties with expected codimension $f(n+1-e+f)$. Moreover, $Q_e^{e-f}(H^0(C, E^*))$ is an analogue of a Brill–Noether locus for Quot schemes; see Remark 2.8. The loci $Q_e^{e-f}(V)$ are better behaved than $H_e^{e-f}(\ell)$ from many points of view. However, as the geometric intuition associated with $H_e^{e-f}(\ell)$ is illuminating, we have as far as possible developed the two notions in parallel.

Let us now give a summary of the article. After some preliminaries on scrolls and secant defect, we construct the loci $H_e^{e-f}(\ell)$ and $Q_e^{e-f}(V)$ and give some examples. In § 2.5, using an approach from [**Sta21a**] and [**Hit20**], we describe a natural surjective rational map $\alpha \colon H_e^{e-f}(\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* M, V) \dashrightarrow Q_e^{e-f}(E, M, V)$ which is birational for $f = 0$. This is a useful tool in what follows. The indeterminacy locus of α is discussed briefly in § 2.5.1. We show in § 2.6 that $Q_e^{e-f}(V)$ is not contained in $Q_e^{e-f-1}(V)$ for $n+1-e+f \geq 0$, with a partial analogue for $H_e^{e-f}(\ell)$. In § 2.7 we discuss the effect of a general projection on secant loci both of type $H_e^{e-f}(\ell)$ and $Q_e^{e-f}(V)$.

Next, suppose that $\mathrm{Quot}^{0,e}(E^*)$ contains a point $[F^* \to E^*]$ where F is stable. Then there are rational classifying maps from $\mathrm{Hilb}^e_{\mathrm{sm}}(S)$ and $\mathrm{Quot}^{0,e}(E^*)$ to the moduli space $U_C(r, d+e)$ of stable bundles of rank r and degree $d+e$. These generalise the Abel–Jacobi map $C_e \to \mathrm{Pic}^e(C)$, and map secant loci to Brill–Noether loci. In § 3, we study this situation and generalise the notion of linear series to the context of Quot schemes.

In § 4, we give some conditions for nonemptiness of the two types of generalised secant loci, extending various statements in the literature for $V_e^{e-f}(\ell)$. In § 5 the Zariski tangent spaces of $Q_e^{e-f}(E, M, V)$ are described. In the complete case, we show that smoothness is controlled by a Petri map; in general, however, the criterion for smoothness is not injectiveness of the Petri map. Indeed, in § 5.4 we show by example that when Q_e^{e-f} admits a map to a Brill–Noether locus, smoothness of the secant locus is neither necessary nor sufficient for smoothness of the Brill–Noether locus in general.

For the remainder of the article, we focus on the case $f = 1$, generalising the study of inflectional loci of linearly normal scrolls in [**Hit19**]; and using similar methods. In § 6, we construct a parameter space for Q_e^{e-1} using Quot schemes of invertible subsheaves of E. This has several applications. In Proposition 7.4, we give a criterion for nonemptiness of $\left(H_e^{e-1}\right)_{\mathrm{nd}}$ (defined in § 2.5.1) and Q_e^{e-1}, in terms of the Segre invariant $s_1(E)$. In Theorem 7.7, we offer a partial answer to a question of Lange [**Lan92**] on the relation between the Segre invariant and very ampleness of line bundles over ruled surfaces, which in fact is valid for projective bundles of any dimension over C. In § 7.4, we characterise semistable bundles in terms of secant loci. And in § 8, we show that for a general bundle E over any curve and a general $V \subseteq H^0(C, E^* \otimes M)$, the loci $Q_e^{e-f}(V)$ are either empty or of the expected dimension. The approach for the latter is similar to [**Hit19**, § 6], and relies on Kleiman's theorem on transversality of translates. We conclude by showing how results from [**EGL01**], [**OP21**] and [**Sta21b**] can be used to enumerate $Q_e^{e-1}(V)$ when it is has and attains expected dimension zero.

Topics of future inquiry. Many questions which can be asked about Brill–Noether loci on $U_C(r,d)$ or about moduli of coherent systems (see for example [**GT09**] and [**New22**]) can be formulated for $H_e^{e-f}(\ell)$ and $Q_e^{e-f}(V)$. For example, many questions of nonemptiness, components, dimension and smoothness of $H_e^{e-f}(\ell)$ and $Q_e^{e-f}(V)$ are open. The relation between smoothness of Q_e^{e-f} and the Brill–Noether locus to which it maps, and the role of semistability in the whole, would be interesting to investigate further.

The enumerative geometry of $Q_e^{e-f}(V)$ is of interest as a natural generalisation of that of $V_e^{e-f}(\ell)$, and a natural application for the recent works [**OP21**] and [**Sta21b**]. It appears a priori more tractable than that of $H_e^{e-f}(\ell)$: As a scroll contains many linear spaces, for $e \geq 3$ the loci H_e^{e-f} have components of excess dimension.

In another direction: Both $H_e^{e-f}(\ell)$ and $Q_e^{e-f}(V)$ appear naturally in the study of Mumford's notion of linear stability [**Mum77**] for scrolls in $\mathbb{P}^n$. This has relevance for questions of moduli and, as explained in [**MS12**] and [**CT-L18**], for Butler's well-known conjecture. This will be a topic of future investigation.

An object in some sense "sitting between" $H_e^{e-f}(\ell)$ and the inflectional loci studied in [**Hit19**] is the generalised Terracini locus defined in [**BC21**]. It would be interesting to see how the present techniques may be applicable to Terracini loci.

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2. Construction and first properties

In this section we define generalised secant loci. After recalling or proving some background material on linear spans, secant defect and scrolls, we construct the loci $H_e^{e-f}(\ell)$ and $Q_e^{e-f}(V)$ and give some basic results. We give a number of examples.

2.1. Linear spans and secant defect. Let S be a variety and $\mathcal{L} \to S$ a line bundle with nonempty linear series, and write $\varphi_{\mathcal{L}} \colon S \dashrightarrow |\mathcal{L}|^*$ for the standard map to the complete linear series of $\mathcal{L}$. For any closed subscheme $Z \subseteq S$, let $r_Z \colon H^0(S,\mathcal{L}) \to H^0(Z, \mathcal{L}|_Z)$ be the restriction map. Then it is easy to see that

$$\text{(2.1)} \quad \mathrm{Span}\, \varphi_{\mathcal{L}}(Z) \ = \ \mathbb{P}\mathrm{Im}\left(H^0(Z, \mathcal{L}|_Z)^* \ \to \ H^0(S,\mathcal{L})^* \right) \ = \ \mathbb{P}\mathrm{Ker}\left(H^0(S,\mathcal{L})^* \ \to \ H^0(S, \mathcal{I}_Z \otimes \mathcal{L})^* \right).$$

Now let $V \subseteq H^0(S, \mathcal{L})$ be a nonzero subspace, and let $\psi =: S \dashrightarrow \mathbb{P}V^*$ be the natural map. We have an exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(S, \mathcal{I}_Z \otimes \mathcal{L}) & \longrightarrow & H^0(S, \mathcal{L}) & \xrightarrow{r_Z} & H^0(Z, \mathcal{L}|_Z) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & V \cap H^0(S, \mathcal{I}_Z \otimes \mathcal{L}) & \longrightarrow & V & \xrightarrow{r_Z} & r_Z(V) \end{array}$$

Dualising and projectivising, we obtain a diagram of varieties

$$\begin{array}{ccccc} \operatorname{Span} \varphi_{\mathcal{L}}(Z) & \hookrightarrow & |\mathcal{L}|^* & \dashrightarrow & \mathbb{P}H^0(S, \mathcal{I}_Z \otimes \mathcal{L})^* \\ \downarrow & & \downarrow p_V & & \downarrow \\ \operatorname{Span} \psi(Z) & \hookrightarrow & \mathbb{P}V^* & \dashrightarrow & \mathbb{P}\left(V \cap H^0(S, \mathcal{I}_Z \otimes \mathcal{L})\right)^* . \end{array} \tag{2.2}$$

This description of $\operatorname{Span} \psi(Z)$ will be useful in what follows.

We recall also the notion of secant defect of a zero-dimensional scheme.

Definition 2.1. Let $Z \subset S$ be a subscheme of dimension zero. Let ℓ be a linear series on S and $\psi \colon S \dashrightarrow \mathbb{P}^n$ the associated map. The *defect* $\operatorname{def} \psi(Z)$ is the number

$$\operatorname{length} Z - \dim \operatorname{Span} \psi(Z) - 1.$$

(Here we take the empty set to have dimension -1.) We say that Z is *ℓ-nondefective* if $\operatorname{def} \psi(Z) = 0$, and *$\ell$-defective* (or simply *defective*) otherwise.

2.2. Scrolls. We briefly review scrolls over curves, which will be our primary objects of study. Let C be a complex projective smooth curve of genus g. Let $E \to C$ be a vector bundle of rank r and degree d, and $\pi \colon \mathbb{P}E \to C$ the associated projective bundle. For $M \in \operatorname{Pic}(C)$, we write $\mathcal{L}_M$ for the line bundle $\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* M$ over $\mathbb{P}E$. By the projection formula and the definition of direct image, we have a natural identification $H^0(\mathbb{P}E, \mathcal{L}_M) \xrightarrow{\sim} H^0(C, E^* \otimes M)$. Fix a subspace $V \subseteq H^0(\mathbb{P}E, \mathcal{L}_M)$ of dimension $n+1$, and consider the evaluation map

$$V \otimes \mathcal{O}_C \ \to \ E^* \otimes M. \tag{2.3}$$

Dualising, projectivising and projecting to $\mathbb{P}V^*$, we obtain a map

$$\psi \colon \mathbb{P}E \ \dashrightarrow \ \mathbb{P}V^* \times C \ \to \ \mathbb{P}V^*. \tag{2.4}$$

This is naturally identified with the standard map $\mathbb{P}E \dashrightarrow |\mathcal{L}_M|^* \dashrightarrow \mathbb{P}V^*$.

2.3. Secant loci on Hilbert schemes. The first secant locus we will consider is a familiar and direct generalisation of the locus $C_e^f \subset C_e$. Let S be a smooth projective variety of dimension $r \geq 1$. For $e \geq 1$, we write $\operatorname{Hilb}^e_{\mathrm{sm}}(S)$ for the component of $\operatorname{Hilb}^e(S)$ containing smoothable subschemes; that is, containing an open subset of $\operatorname{Sym}^e S$.

Definition 2.2. Let S be as above, and let $\ell = (\mathcal{L}, V)$ be a linear series of dimension n. Assume that $e \geq 1$ and $f \geq 0$ and $e \geq f$. We define

$$H_e^{e-f}(\ell) \ := \ \{ Z \in \operatorname{Hilb}^e_{\mathrm{sm}}(S) : \dim \left(\operatorname{Span} \psi(Z) \right) \leq e - f - 1 \}$$

or, equivalently,

$$H_e^{e-f}(\ell) \ = \ \{Z \in \mathrm{Hilb}^e_{\mathrm{sm}}(S) : \mathrm{def}\,\psi(Z) \geq f\} = \\ \{Z \in \mathrm{Hilb}^e_{\mathrm{sm}}(S) : \dim\left(V \cap H^0(S, \mathcal{L} \otimes \mathcal{I}_Z)\right) \geq n+1-e+f\}.$$

Depending on the context, we may write $H_e^{e-f}(V)$ instead. We write H_e^{e-f} for $H_e^{e-f}\left(H^0(S,\mathcal{L})\right)$.

Remark 2.3. If $r \geq 3$ then $\mathrm{Hilb}^e(S)$ has multiple irreducible components. It is of course not necessary to restrict to the distinguished component $\mathrm{Hilb}^e_{\mathrm{sm}}(S)$; the study of secant loci on the other components of $\mathrm{Hilb}^e(S)$ is no less interesting. For the present, we restrict to $\mathrm{Hilb}^e_{\mathrm{sm}}(S)$ simply for convenience, as the other components are less well understood at the present time.

The locus $H_e^{e-f}(\ell)$ can be constructed as a determinantal variety in a standard way. Let p and q be the projections of $\mathrm{Hilb}^e_{\mathrm{sm}}(S) \times S$ to the first and second factors respectively, and let $\mathcal{Z} \subset \mathrm{Hilb}^e_{\mathrm{sm}}(S) \times S$ be the universal subscheme. Over $\mathrm{Hilb}^e_{\mathrm{sm}}(S)$ we have the diagram

$$\begin{array}{ccccc} & & V \otimes \mathcal{O}_{\mathrm{Hilb}^e_{\mathrm{sm}}(S)} & & \\ & & \downarrow & \searrow^{\varepsilon} & \\ p_*\left(\mathcal{I}_{\mathcal{Z}} \otimes q^*\mathcal{L}\right) & \longrightarrow & p_* q^* \mathcal{L} & \longrightarrow & p_*\left(\mathcal{O}_{\mathcal{Z}} \otimes q^*\mathcal{L}\right) \end{array} \quad . \tag{2.5}$$

As $\mathcal{Z}$ is flat over $\mathrm{Hilb}^e(S)$, the sheaf $p_*\left(\mathcal{O}_{\mathcal{Z}} \otimes q^*\mathcal{L}\right)$ is locally free of rank e. Thus $H_e^{e-f}(\ell)$ is the determinantal variety

$$\{Z \in \mathrm{Hilb}^e_{\mathrm{sm}}(S) : \mathrm{rk}(\varepsilon|_Z) \leq e-f\}$$

defined by the $(e-f+1)$st Fitting ideal of ε. The expected dimension of $H_e^{e-f}(\ell)$ is therefore $re - f(n+1-e+f)$.

Example 2.4. Let $L_1, \ldots, L_r$ be line bundles on C. For $1 \leq i \leq r$, let $V_i \subseteq H^0(C, L_i)$ be a subspace of dimension $n_i + 1$. Set $E := \bigoplus_{i=1}^r L_i^{-1}$ and

$$V \ := \ \bigoplus_{i=1}^r V_i \ \subseteq \ H^0(C, E^*) \ = \ H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(1)),$$

and set $\ell = (\mathcal{O}_{\mathbb{P}E}(1), V)$. Now for each i, we have a natural map

$$\iota_i \colon C \ \to \ \mathbb{P}V_i^* \ \hookrightarrow \ \mathbb{P}\left(\bigoplus_{i=1}^r V_i^*\right).$$

For each i, let e_i, f_i be such that $V_{e_i}^{e_i - f_i}(L_i, V_i)$ (defined in (1.1)) has a nonempty component of dimension s_i. Write $e := \sum e_i$ and $f := \sum f_i$. Then

$$\left\{\iota_1(D_1) \cup \cdots \cup \iota_r(D_r) : D_i \in V_{e_i}^{e_i - f_i}(L_i, V_i)\right\} \tag{2.6}$$

is a nonempty locus in $H_e^{e-f}(\ell)$ of dimension $\sum s_i$.

In § 2.5.1, Example 2.4 will motivate a conjectural dimension bound for certain secant loci.

EXAMPLE 2.5. Using the fact that $\mathbb{P}E$ is ruled, we can obtain loci in $H_e^{e-f}(\ell)$ of excess dimension. Suppose $e \geq r+1$. If $Z \in \mathrm{Hilb}^e_{\mathrm{sm}}(S)$ is reduced and has support in a fibre $\mathbb{P}E|_x$ then $\mathrm{Span}\,\psi(Z) \subseteq \mathbb{P}E|_x = \mathbb{P}^{r-1}$. Hence any such Z imposes at most r conditions on sections of $\mathcal{O}_{\mathbb{P}E}(1)$. Thus the locus

$$\bigcup_{x \in C} \{Z \text{ reduced of length } e : Z \subset \mathbb{P}E|_x\}$$

belongs to $H_e^r(V)$ and has dimension

$$\dim\left(\mathrm{Sym}^e\,\mathbb{P}^{r-1}\right) + \dim C \ = \ e(r-1)+1 \ = \ re-(e-1).$$

This exceeds the expected dimension $re-(e-r)(n+1-r)$ of $H_e^r(V)$ if $2r \leq n$.

We will return in § 2.5.1 to the study of subschemes with defect arising from points lining up inside a single fibre in this way.

REMARK 2.6. A linear series is base point free (resp., embedding) if and only if $H_1^0(\ell)$ (resp., $H_2^1(\ell)$) is empty. Most often we will consider embedded subvarieties of $\mathbb{P}^n$, but many of our results apply to arbitrary projective models.

2.4. Secant loci on Quot schemes. Let us now give another generalisation of $V_e^{e-f}(\ell)$. This time, instead of increasing $\dim(C)$, we increase $\mathrm{rk}(L)$.

DEFINITION 2.7. Let E be a vector bundle of rank r and degree d over C, and let $\pi\colon \mathbb{P}E \to C$ be the associated $\mathbb{P}^{r-1}$-bundle. Let $V \subseteq H^0(C, E^* \otimes M)$ be a subspace of dimension $n+1$. We define the generalised secant locus $Q_e^{e-f}(E, M, V)$ as

$$\left\{[F^* \to E^*] \in \mathrm{Quot}^{0,e}(E^*) : \dim(V \cap H^0(C, F^* \otimes M)) \geq n+1-e+f\right\}.$$

This can be constructed as a determinantal variety in a similar way to $H_e^{e-f}(\ell)$. We may write simply $Q_e^{e-f}(V)$ if no confusion should arise. Also, we often abbreviate $Q_e^{e-f}\left(H^0(C, E^* \otimes M)\right)$ to Q_e^{e-f}.

REMARK 2.8. As in [**HHN21**, § 3.1], given a line bundle M and any family of vector bundles $\mathcal{V} \to B \times C$, one can define the *twisted Brill–Noether locus*

$$B^k(\mathcal{V}, M) \ = \ \{b \in B : h^0(C, \mathcal{V}_b \otimes M) \geq k\} \ \subseteq \ B.$$

Let $\mathcal{F}^*$ be the universal subsheaf over $\mathrm{Quot}^{0,e}(E^*) \times C$. Then we have simply

$$Q_e^{e-f}\left(E, M, H^0(C, E^* \otimes M)\right) \ = \ Q_e^{e-f} \ = \ B^{h^0(C, E^* \otimes M)-e+f}(\mathcal{F}^*, M).$$

Other relations between secant loci and Brill–Noether loci will be studied in § 3.

2.5. The link between $H_e^{e-f}(V)$ and $Q_e^{e-f}(V)$. Following [**Hit20**] and [**Sta21a**], for $S = \mathbb{P}E$ we now describe the relation between the two generalised secant loci above. This will generalise the fact that if L is a line bundle, then

$$D \mapsto \left[L^{-1}(-D) \to L^{-1}\right] \text{ defines an isomorphism } C_e \xrightarrow{\sim} \mathrm{Quot}^{0,e}(L^{-1}).$$

Let $Z \subset \mathbb{P}E$ be a subscheme of length e. Taking direct images of $0 \to \mathcal{I}_Z(1) \to \mathcal{O}_{\mathbb{P}E}(1) \to \mathcal{O}_Z(1) \to 0$ on C, we have a sequence

$$0 \ \to \ \pi_*\mathcal{I}_Z(1) \ \to \ E^* \ \to \ \pi_*\mathcal{O}_Z(1) \ \to \ \cdots$$

As $\pi_*\mathcal{O}_Z(1)$ is a torsion sheaf on C, we see that $\pi_*\mathcal{I}_Z(1)$ is a full rank subsheaf of E^*. Following [**Hit20**], we denote this by E_Z^*. Dualising, for general $Z \in \mathrm{Hilb}^e_{\mathrm{sm}}(S)$, we have $\deg(E_Z) = \deg(E) + e$; for example if $\pi(Z)$ consists of e distinct points.

Proposition 2.9. *Let $E \to C$ be a vector bundle.*

(a) *The association $Z \mapsto [E_Z^* \to E^*]$ defines a rational map $\alpha\colon \mathrm{Hilb}^e(S) \dashrightarrow \mathrm{Quot}^{0,e}(E^*)$.*

(b) *The restriction of α to the component $\mathrm{Hilb}^e_{\mathrm{sm}}(S)$ is surjective. Moreover, α is bijective on the open set*

$$\{Z \in \mathrm{Hilb}^e_{\mathrm{sm}}(S) : \mathrm{Supp}\,\pi(Z) \text{ consists of } e \text{ distinct points}\}.$$

In particular, $\alpha|_{\mathrm{Hilb}^e_{\mathrm{sm}}(S)}$ is a birational equivalence.

Proof. (a) The existence of a suitable family of length e quotients $E^* \to \pi_* \mathcal{O}_Z(1)$ parametrised by $\mathrm{Hilb}^e(\mathbb{P}E)$ is proven[1] in [**Sta21a**, Theorem 3 (i)] for $e = 2$ when $\mathbb{P}E$ is a projective bundle over a surface, and the same argument works for $e \geq 1$ when the base is a curve. In this case, over the open subset

$$\{Z \in \mathrm{Hilb}^e(S) : E^* \to \pi_* \mathcal{O}_Z(1) \text{ is surjective}\}$$

we obtain a family of elementary transformations $E_Z^* \to E^*$ of degree $\deg(E^*) - e$. This locus is nonempty, as it contains all Z projecting to e distinct points of C, so we obtain the desired rational map α.

(b) This follows from the proof of [**Hit20**, Theorem 2.6]. □

Remark 2.10. If $e \geq 2$ and $r \geq 2$, then α has fibres of positive dimension. For example, suppose $e = 2$ and let μ_1 and μ_2 be points of a fibre $\mathbb{P}E|_x$ spanning a $\mathbb{P}^1$. Then for any μ_1', μ_2' spanning the same $\mathbb{P}^1$, we have $E_{\{\mu_1', \mu_2'\}} = E_{\{\mu_1, \mu_2\}}$.

2.5.1. *Relatively defective subschemes.* In contrast to the situation studied in [**Sta21a**], the map α is not a morphism when $e \geq 3$. The indeterminacy locus is the set of $Z \in \mathrm{Hilb}^e(S)$ such that $E^* \to \pi_*(\mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}E}(1))$ is not surjective; equivalently, $\deg(E_Z^*) > \deg E^* - e$. For example, if Z consists of three collinear points in a fibre of $\mathbb{P}E$, then $\deg(E_Z^*) = \deg E^* - 2$. This motivates the following, which is [**Hit20**, Definition 3.4]:

Definition 2.11. Let $Z \subset \mathbb{P}E$ be a subscheme of dimension zero. Then Z is said to be *π-nondefective* if $\deg(E_Z^*) = \deg(E^*) - \mathrm{length}(Z)$, and *$\pi$-nondefective* otherwise.

Now we can formulate an important statement and a conjecture. We denote by $\mathrm{Hilb}^e_{\mathrm{sm}}(S)_{\mathrm{nd}}$ the locus of π-nondefective subschemes in $\mathrm{Hilb}^e_{\mathrm{sm}}(S)$, and we write

$$H_e^{e-f}(V)_{\mathrm{nd}} \ := \ H_e^{e-f}(V) \cap \mathrm{Hilb}^e_{\mathrm{sm}}(S)_{\mathrm{nd}}.$$

Proposition 2.12. *Let V be an $n+1$-dimensional subspace of*

$$H^0(C, E^* \otimes M) \ \cong \ H^0(S, \mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* M).$$

Suppose $n + 1 \geq e - f$. Let $Z \in \mathrm{Hilb}^e_{\mathrm{sm}}(S)$ be π-nondefective. Then $Z \in H_e^{e-f}(V)$ if and only if $[E_Z^* \to E^*]$ belongs to $Q_e^{e-f}(V)$.

Proof. By construction of E_Z, we have

$$H^0(C, E_Z^* \otimes M) \cap V \ = \ H^0(S, \mathcal{I}_Z(1) \otimes \pi^* M) \cap V. \tag{2.7}$$

The statement follows. □

[1] It was stated, unfortunately without an adequate justification, in [**Hit20**, Theorem 2.6 (a)].

In view of the proposition, we will switch freely between $H_e^{e-f}(V)_{\mathrm{nd}}$ and $Q_e^{e-f}(V)$, depending on what is more convenient or illuminating. We refer to [**Hit20**, § 3.1] for discussion and some equivalent definitions of π-nondefectivity. Example 2.5 illustrates that π-defective subschemes can give rise to artificially large components of $H_e^{e-f}(\ell)$, which do not reflect the behaviour of $Q_e^{e-f}(V)$. For the most part, we will exclude π-defective subschemes from our consideration.

Next, we turn briefly to the question of dimension bounds for $H_e^{e-f}(V)$ and $Q_e^{e-f}(V)$. Returning to Example 2.4: If for each i the pair (C, L_i) is one of the four special types described in [**Baj15**, Theorem 4.6], and if $e_i \le h^0(C, L_i) - 2$, then we may take s_i to be the maximal value $e_i - f_i - 1$ proven in loc. cit., and then the locus (2.6) above has dimension $e - f - r$. Note that in this situation we have

$$e \ = \ \sum_{i=1}^{r} e_i \ \le \ \sum_{i=1}^{r} \left(h^0(C, L_i) - 2 \right) \ = \ h^0(C, E) - 2r.$$

Based on this information, we make the following naive conjecture.

Conjecture 2.13. *Suppose C is nonhyperelliptic of genus $g \ge 9$. Let $E \to C$ be a bundle of rank r and slope at most $g - 2$ such that $\mathcal{O}_{\mathbb{P}E}(1)$ is very ample. For $e \le h^0(C, E) - 2r$, we have*

$$\dim Q_e^{e-f} \ \le \ \dim \left(H_e^{e-f} \right)_{\mathrm{nd}} \ \le \ e - f - r.$$

We also remark the following fact, which generalises [**Hit20**, Proposition 4.4 and Remark 4.5 (a)] in the complete case.

Lemma 2.14. *Let $[F^* \to E^*]$ be a point of $\mathrm{Quot}^{0,e}(E^*)$. Let $Z \subset \mathbb{P}E$ be any subscheme such that $F = E_Z$. Then*

$$\mathrm{Span}\, \psi(Z) \ = \ \mathbb{P}\mathrm{Ker} \left(V^* \ \to \ (V \cap H^0(C, E_Z^* \otimes M)^*) \ = \ \mathbb{P} \left(V \cap H^0(C, E_Z^* \otimes M) \right)^{\perp} \right. .$$

In particular, $\mathrm{Span}\, \psi(Z)$ depends only on F^*, and in fact does not require that Z be π-nondefective or even zero-dimensional.

Proof. The statement is a consequence of (2.2) and (2.7). □

2.6. Distinctness of the strata. For fixed e, the determinantal varieties $H_e^{e-f}(\ell)$ and $Q_e^{e-f}(V)$ define stratifications of $\mathrm{Hilb}^e_{\mathrm{sm}}(S)$ and $\mathrm{Quot}^{0,e}(E^*)$ respectively as f varies. Generalising [**Baj15**, Lemma 4.1], [**AS15**, Lemma 2.2] and [**ACGH85**, Lemma IV.1.7], we will now show that $Q_e^{e-f}(V)$ is strictly bigger than $Q_e^{e-f-1}(V)$ whenever $n + 1 - e + f \ge 0$, and similarly for $H_e^{e-f}(\ell)$ under a certain condition. The result for higher dimensional varieties rests on the following statement for curves, which is a straightforward generalisation of [**AS15**, Lemma 2.2]; as the proof is essentially identical, we omit it.

Lemma 2.15. *Let T be a smooth projective curve and $\ell = (\mathcal{L}, V)$ a very ample linear series on T of dimension $n+1$. Let e and f be integers satisfying $0 \le f \le e$ and $n + 1 - e + f \ge 0$. Then no component of $V_e^{e-f}(\ell)$ is fully contained in $V_e^{e-f-1}(\ell)$.*

Proposition 2.16. *Let S be a smooth projective variety, and $\ell = (\mathcal{L}, V)$ a very ample linear series of dimension $n + 1$. Let e and f be integers with $e \ge 1$ and $f \ge 0$ and $e \ge f$, and satisfying $n + 1 - e + f \ge 0$.*

(a) *Let $\mathcal{X}$ be a component of $H_e^{e-f}(\ell)$. Suppose $Z_0 \in \mathcal{X}$ is such that the subscheme $Z_0 \subset S$ is curvilinear. Then $\mathcal{X}$ is not fully contained in $H_e^{e-f-1}(\ell)$.*

(b) *Let $E \to C$ be a vector bundle, and $\mathcal{L}_M = \mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* M$. Let $V \subseteq H^0(C, E^* \otimes M)$ be a subspace of dimension $n+1$. Then no component of $Q_e^{e-f}(V)$ is fully contained in $Q_e^{e-f-1}(V)$.*

Proof. (a) As curvilinearity is an open condition, clearly we may assume that Z_0 belongs to no component of $H_e^{e-f}(\ell)$ other than $\mathcal{X}$. By curvilinearity, $T_\nu Z_0$ has dimension one for all $\nu \in \mathrm{Supp}(Z_0)$. Hence, by considering the closure of a suitable affine curve contained in some affine open subset of S, we may assume that Z_0 belongs to $\mathrm{Hilb}^e(T) \cong T_e$ for a curve $T \subseteq S$ whose image in $\mathbb{P}V^*$ is nondegenerate. Then we obtain an identification of V with a subspace of $H^0(T, \mathcal{L}|_T)$. Setting $\ell_T := (\mathcal{L}|_T, V)$, we can consider the usual secant locus $V_e^{e-f}(\ell_T)$ as defined in (1.1). For all $f \geq 0$, there are natural inclusions $V_e^{e-f}(\ell_T) \hookrightarrow H_e^{e-f}(\ell)$. As Z_0 belongs only to the component $\mathcal{X}$ of $H_e^{e-f}(\ell)$, if $\mathcal{V}$ is any component of $V_e^{e-f}(\ell_T)$ containing Z_0 then we have $\mathcal{V} \subseteq \mathcal{X}$.

Now suppose that $Z_0 \in H_e^{e-f-1}(\ell)$. Then $Z_0 \in V_e^{e-f-1}(\ell_T) \cap \mathcal{V}$ also. Since $e \leq n+1+f$, by Lemma 2.15 we may deform Z_0 inside $\mathcal{V}$ to a point $Z_1 \in \mathcal{V} \setminus V_e^{e-f-1}(\ell)$. Then Z_1 has defect precisely f in $\mathbb{P}V^*$. But as $\mathcal{V} \subseteq \mathcal{X}$, we see that $\mathcal{X}$ contains the point $Z_1 \in H_e^{e-f}(\ell) \setminus H_e^{e-f-1}(\ell)$. This proves (a).

(b) Let $\mathcal{X}$ be a component of $Q_e^{e-f}(V)$. By Lemma 2.9 (a) and Proposition 2.12, we see that $\alpha^{-1}(\mathcal{X})$ is a union $\bigcup_i \widetilde{\mathcal{X}}_i$ of nonempty open subsets of components of $H_e^{e-f}(V)_{\mathrm{nd}}$. By the proof of [**Hit20**, Theorem 2.6 (a)], at least one $\widetilde{\mathcal{X}}_{i_0}$ contains a point Z_0 corresponding to a curvilinear subscheme. By part (a), therefore, $\widetilde{\mathcal{X}}_{i_0}$ is not contained in $H_e^{e-f-1}(V)$. Again by Proposition 2.12, the image of any point of $\widetilde{\mathcal{X}}_{i_0} \setminus H_e^{e-f-1}(V)$ is a point of $\mathcal{X} \setminus Q_e^{e-f-1}(V)$, and we obtain (b). □

2.7. Secant loci under general projections. We will now show that the secant loci $H_e^{e-f}(V)$ and $Q_e^{e-f}(V)$ behave predictably under projection from a general centre. The main ingredient is the theorem of Kleiman on transversality of a general translate, which will be used several times in what follows:

Theorem 2.17 ([**Kle74**, Theorem 2 (i)]). *Let G be a connected algebraic group and Z an irreducible variety admitting a transitive action of G. Suppose that $a\colon X \to Z$ and $b\colon Y \to Z$ are maps of irreducible varieties. For any $\gamma \in G$, we write $\gamma \cdot Y$ for Y considered as a Z-variety via the map $y \mapsto \gamma \cdot b(y)$. Then there is a dense open subset $U \subseteq G$ such that $(\gamma \cdot Y) \times_Z X$ is either empty or equidimensional of dimension $\dim X + \dim Y - \dim Z$ for all $\gamma \in U$.*

Our results will follow from this theorem and linear algebra. To avoid repetition, **during this subsection, we fix integers n, m, e, and f such that**

$$0 \leq f \leq e \leq m+1 \leq n+1.$$

Let H be a variety and $\beta\colon \mathcal{E} \to V^* \otimes \mathcal{O}_H$ a map of vector bundles over H, where $\mathrm{rk}\, \mathcal{E} = e$ and $V^* = \mathbb{C}^{n+1}$. We consider the degeneracy loci

$$D_{e-f}(\beta) = \{h \in H : \mathrm{rk}\,(\beta|_h) \leq e-f\} \quad \text{and} \quad D^\circ_{e-f}(\beta) = \{h \in H : \mathrm{rk}\,(\beta|_h) = e-f\}.$$

For each f' with $0 \leq f' \leq f$, we write

$$d_m(f') := \dim H - f'(m+1-e+f).$$

Lastly, if $W \subseteq V$ is a subspace, we write p_W for the projection $V^* \to W^*$, and set $W^\perp := \mathrm{Ker}(p_W)$.

PROPOSITION 2.18. *Let H and $\beta\colon \mathcal{E} \to V^* \otimes \mathcal{O}_H$ be as above. Assume that for each $f' \in \{0, \ldots, f\}$ satisfying $f - f' \leq n - m$ we have*

$$\dim D_{e-f'}(\beta) \ \leq\ d_m(f'). \tag{2.8}$$

Then for a general subspace $W \subseteq V$ of dimension $m+1$, the degeneracy locus $D_{e-f}(p_W \circ \beta)$ is empty or of the expected dimension $\dim H - f(m+1-e+f)$ when this is nonnegative.

PROOF. Firstly, suppose $f' = f$. Clearly $D_{e-f}(\beta) \subseteq D_{e-f}(p_W \circ \beta)$. In this case, $d_m(f)$ is exactly the expected dimension of $D_{e-f}(p_W \circ \beta)$. Thus, by (2.8), any component of $D_{e-f}(p_W \circ \beta)$ of excess dimension must intersect $D^\circ_{e-f'}(\beta)$ for some $f' < f$.

Suppose that $h \in D^\circ_{e-f'}(\beta)$ for some $f' \in \{0, \ldots, f-1\}$. Then $\mathrm{rk}(p_W \circ \beta|_h) \leq e - f$ if and only if

$$\dim\left(W^\perp \cap \mathrm{Im}\,\beta|_h\right) \ \geq\ f - f'. \tag{2.9}$$

If $f - f' > n - m = \dim W^\perp$, then condition (2.9) is not satisfied for any $h \in D^\circ_{e-f'}(\beta)$, so $D_{e-f}(p_W \circ \beta) \cap D^\circ_{e-f'}(\beta)$ is empty for all W of dimension $m+1$.

On the other hand, suppose that $1 \leq f - f' \leq n - m$. Let $a\colon D^\circ_{e-f'}(\beta) \to \mathrm{Gr}(e-f', V^*)$ be the map $h \mapsto \mathrm{Im}\,\beta|_h$. Consider the Schubert cycle

$$\Sigma^{f-f'}_{W^\perp} \ :=\ \{\Lambda \in \mathrm{Gr}(e-f', V^*) : \dim(\Lambda \cap W^\perp) \geq f - f'\}.$$

In the notation of [**GH78**, p. 194–196], we have $\Sigma^{f-f'}_{W^\perp} = \overline{W_{a_1,\ldots,a_{e-f'}}}$ where

$$a_1 \ =\ \cdots \ =\ a_{f-f'} \ =\ m+1-e+f \quad \text{and} \quad a_{f-f'+1} \ =\ \cdots \ =\ a_{e-f'} \ =\ 0.$$

By loc. cit.,

$$\dim \Sigma^{f-f'}_{W^\perp} \ =\ \dim \mathrm{Gr}(e-f', V^*) - (f-f')(m+1-e+f). \tag{2.10}$$

For each component Y of $D^\circ_{e-f'}(\beta)$, we consider the fibre product diagram

$$\begin{array}{ccc} Y_W & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle a|_Y} \\ \Sigma^{f-f'}_{W^\perp} & \hookrightarrow & \mathrm{Gr}(e-f', V^*). \end{array}$$

If $h \in Y$, then by (2.9) we see that $h \in D_{e-f}(p_W \circ \beta)$ if and only if $h \in Y_W$.

Now $\mathrm{GL}(V^*)$ acts transitively on $\mathrm{Gr}(\ell, V^*)$ for all ℓ. For $\gamma \in \mathrm{GL}(V^*)$, unwinding definitions, we see that $\gamma \cdot \Sigma^{f-f'}_{W^\perp} = \Sigma^{f-f'}_{\gamma \cdot W^\perp}$. Hence by Theorem 2.17 and by (2.10), for a general choice of W the locus Y_W is empty or equidimensional of dimension

$$\dim Y - (f-f')(m+1-e+f). \tag{2.11}$$

As by hypothesis (2.8) we have $\dim Y \leq d_m(f')$, the number (2.11) is bounded above by the expected dimension of $D_{e-f}(p_W \circ \beta)$, as desired. (In fact, as the latter is determinantal, Y_W is nonempty only if $\dim Y = d_m(f')$.) □

Remark 2.19. An easy computation shows that for $f' \geq 0$, the condition $f - f' \leq n - m$ in the Proposition 2.18 is equivalent to

$$d_m(f') \ \geq \ \dim H - f'(n + 1 - e - f'),$$

the expected dimension of $D_{e-f'}(\beta)$.

We will apply the foregoing to the study of secant loci. The following generalises [**Hit19**, Theorem A.1]. It may be viewed as an elementary contribution to the line of inquiry in for example [**GP13**] and [**Ran15**]. For a projective model $\psi \colon S \dashrightarrow \mathbb{P}V^*$ and $W \subseteq V$, let ψ_W be the composition $S \dashrightarrow \mathbb{P}V^* \dashrightarrow \mathbb{P}W^*$.

Corollary 2.20. *Let S be a projective variety, and $\psi \colon S \dashrightarrow \mathbb{P}^n = \mathbb{P}V^*$ a projective model. Let $\mathcal{Z} \subset H \times S$ be a family of length e subschemes of S parametrised by a variety H. For each subspace $W \subseteq V$ and for each f', we define*

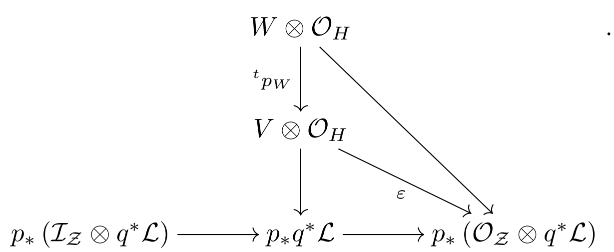
$$H_e^{e-f'}(H, W) \ := \ \{h \in H : \operatorname{def} \psi_W(\mathcal{Z}_h) \geq f'\}.$$

Assume that for each $f' \in \{0, \ldots, f\}$ such that $f - f' \leq n - m$ we have

$$\dim H_e^{e-f'}(H, V) \ \leq \ d_m(f').$$

Then for a general subspace $W \subseteq V$ of dimension $m+1$, the secant locus $H_e^{e-f}(H, W)$ is empty or of the expected dimension $\dim H - f(m + 1 - e + f)$.

Proof. For any $W \subseteq V$, the locus $H_e^{e-f'}(H, W)$ can be constructed as a determinantal variety exactly as in § 2.3. Let $\mathcal{L}$ be the line bundle inducing the map to $\mathbb{P}^n$. Let p and q be the projections of $H \times S$ to the first and second factors respectively. We have a diagram of sheaves over $H \times S$ as follows:



As $\mathcal{Z}$ is flat over H, the sheaf $p_* (\mathcal{O}_{\mathcal{Z}} \otimes q^* \mathcal{L})$ is locally free of rank e. Now for $h \in H$, by (2.1) and (2.2) we have $\operatorname{Span} \psi_W(\mathcal{Z}_h) = \mathbb{P}\mathrm{Im}\, (p_W \circ {}^t\varepsilon|_h)$. Thus $H_e^{e-f'}(H, W)$ is the determinantal variety

$$\{Z \in H : \operatorname{rk}(p_W \circ {}^t\varepsilon|_Z) \leq e - f'\} \ = \ D_{e-f'}(p_W \circ {}^t\varepsilon).$$

Now by assumption, $D_{e-f'}({}^t\varepsilon)$ satisfies the hypothesis of Proposition 2.18 for $0 \leq f' \leq f$. Therefore, for W general, $H_e^{e-f}(H, W) = D_{e-f}(p_W \circ {}^t\varepsilon)$ is empty or of the expected dimension by Proposition 2.18 when this is nonnegative, and empty otherwise. □

We will now give a corresponding statement for secant loci of type $Q_e^{e-f}(V)$. This will be used in § 8.

Corollary 2.21. *Let C be a curve and $E \to C$ a vector bundle. Let $V \subseteq H^0(C, E^*)$ be a subspace of dimension $n + 1$. Let $\mathcal{F}^\vee \to p_C^* E^*$ be a family of* *elementary transformations of degree* $\deg E^* - e$ *parametrised by a variety* Q*. For each* f' *and for each subspace* $W \subseteq V$ *of dimension* $m+1$*, set*

$$Q_e^{e-f'}(Q, W) \;:=\; \left\{ q \in Q : \dim \left(H^0(C, \mathcal{F}_q^*) \cap W \right) \geq m + 1 - e + f' \right\}.$$

Assume that for each $f' \in \{0, \dots, f\}$ *such that* $f - f' \leq n - m$ *we have*

$$\dim Q_e^{e-f'}(Q, V) \;\leq\; d_m(f').$$

Then for a general subspace $W \subseteq V$ *of dimension* $m+1$*, the locus* $Q_e^{e-f}(Q, W)$ *is empty or of the expected dimension* $\dim Q - f(m+1-e+f)$ *when this is nonnegative, and empty otherwise.*

Proof. This is similar to the proof of the previous corollary. □

This completes our overview of basic properties of scrolls and their secant loci. In the sections that follow, we will apply these to various questions on $H_e^{e-f}(\ell)$ and $Q_e^{e-f}(V)$.

3. Secant loci and Brill–Noether loci

We now further discuss the connection between secant loci and Brill–Noether loci, and generalise the notions of Abel–Jacobi map and linear series to the context of Quot schemes. This expands [**Hit20**, Remark 4.7].

It is well known that for $e \geq f \geq 0$ there is a diagram

$$\begin{array}{ccccc} C_e & \xrightarrow{\sim} & \mathrm{Quot}^{0,e}(\mathcal{O}_C) & \longrightarrow & \mathrm{Pic}^e(C) \\ \uparrow & & \uparrow & & \uparrow \\ C_e^f & \xrightarrow{\sim} & Q_e^{e-f}\left(\mathcal{O}_C, K_C, H^0(C, K_C)\right) & \longrightarrow & B_{1,e}^{1+f} \end{array} \tag{3.1}$$

The rows are given by $D \mapsto [\mathcal{O}_C(-D) \to \mathcal{O}_C] \mapsto \mathcal{O}_C(D)$. This composition is an Abel–Jacobi map, with fibre over L given by the linear series $|L|$.

Now let $U_C(r, d+e)$ be the moduli space of stable bundles of rank r and degree $d+e$ over C. For $k \geq 0$, we consider the higher rank Brill–Noether locus

$$B_{r,d+e}^k \;=\; \{ F \in U_C(r, d+e) : h^0(F) \geq k \}.$$

Let E be a vector bundle of rank r and degree d, not necessarily semistable. Suppose that $0 \leq f \leq e$ and $Q_e^{e-f}(E, K_C, H^0(E^* \otimes K_C))$ contains a point $[F^* \to E^*]$ such that F is a stable bundle. Now by Serre duality, there is an exact sequence of vector spaces

$$0 \;\to\; H^0(C, E) \;\to\; H^0(C, F) \;\to\; H^0(C, F/E) \;\to\; H^0(C, K_C \otimes E^*)^* \;\to\; H^0(C, K_C \otimes F^*)^* \;\to\; 0,$$

from which it follows that

$$h^0(C, F) = h^0(C, E) + f \text{ if and only if } h^0(C, F^* \otimes K_C) = h^0(C, E^* \otimes K_C) - e + f. \tag{3.2}$$

Thus, generalising (3.1), we have a Cartesian diagram
(3.3)

$$\begin{array}{ccccc} \mathrm{Hilb}^e_{\mathrm{sm}}(S)_{\mathrm{nd}} & \xrightarrow{\alpha} & \mathrm{Quot}^{0,e}(E^*) & \overset{a}{\dashrightarrow} & U_C(r, d+e) \\ \uparrow & & \uparrow & & \uparrow \\ H^{e-f}_e(\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* K_C)_{\mathrm{nd}} & \xrightarrow{\alpha} & Q^{e-f}_e\left(E, K_C, H^0(C, E^* \otimes K_C)\right) & \overset{a}{\dashrightarrow} & B^{h+f}_{r,d+e} \end{array},$$

where α is as defined in Proposition 2.9, and $h := h^0(C, E)$, and $a([F^* \to E^*]) = F$. The maps a and $a \circ \alpha$ are generalisations of the Abel–Jacobi map.

Now for any F of rank r, we write $H^0(C, \mathrm{Hom}(F^*, E^*))_{\mathrm{inj}}$ for the open subset of generically injective maps. Notice that there is a natural action of $\mathrm{Aut}\,(F^*)$ on $H^0(C, \mathrm{Hom}(F^*, E^*))$ that fixes $H^0(C, \mathrm{Hom}(F^*, E^*))_{\mathrm{inj}}$.

Proposition 3.1. *Let $E \to C$ be a bundle of rank r and degree d.*

(a) *For a fixed bundle F of rank r and degree $d + e$, the locus*

$$\{[F_1^* \xrightarrow{j} E^*] \in \mathrm{Quot}^{0,e}(E^*) : F_1 \cong F \text{ as vector bundles}\} =: Q_E(F)$$

is in bijection with $H^0(C, \mathrm{Hom}(F^*, E^*))_{\mathrm{inj}}/\mathrm{Aut}\,(F^*)$.

(b) *Suppose that $\mathrm{Quot}^{0,e}(E^*)$ contains a point $[F^* \to E^*]$ such that F is a stable vector bundle. Let a be as defined in (3.3). Then*

$$a^{-1}(F) = Q_E(F) \cong \mathbb{P}H^0(C, \mathrm{Hom}(F^*, E^*))_{\mathrm{inj}}.$$

Proof. (a) The association $j \mapsto [F^* \xrightarrow{j} E^*]$ defines a map

$$H^0(C, \mathrm{Hom}(F^*, E^*))_{\mathrm{inj}} \to Q_E(F)$$

which is clearly surjective. By definition of the Quot scheme and the universal property of kernels, two maps j' and j define the same element of $\mathrm{Quot}^{0,e}(E^*)$ if and only if $j' = j \circ \gamma$ for some $\gamma \in \mathrm{Aut}\,(F^*)$. This proves (a). Part (b) follows from (a) since $\mathrm{Aut}\,(F^*) = \mathbb{C}^*$ if F is stable. □

Remark 3.2. If L is a line bundle of degree e, by Proposition 3.1 we obtain

$$a^{-1}(L) \cong Q_{\mathcal{O}_C}(L) \cong \mathbb{P}H^0(C, \mathrm{Hom}(L^{-1}, \mathcal{O}_C))_{\mathrm{inj}} \cong |L|.$$

Thus it makes sense to call $Q_E(F)$ a "generalised linear series of F with respect to E". If $\Omega \subseteq H^0(C, \mathrm{Hom}(F^*, E^*))$ is an $\mathrm{Aut}\,(F^*)$-invariant subspace, then the locus $\Omega_{\mathrm{inj}}/\mathrm{Aut}\,(F^*)$ generalises the notion of possibly incomplete linear subseries.

Remark 3.3. The Riemann–Kempf singularity theorem [**ACGH85**, Chap. VI.2], gives a description of the tangent cones to the rank one Brill–Noether loci W^f_e. For $k \geq r$ and $W \in B^k_{r,d}$ a generated vector bundle, there is a generalisation of the Riemann–Kempf singularity theorem given as follows. For $\ell \geq r$, let $U^\ell \subseteq \mathrm{Gr}(\ell, H^0(C, W))$ be the open subset of subspaces which generically generate W. Assuming this is nonempty, by [**Hit20**, Theorem 5.6] the projectivised tangent cone to $B^k_{r,d}$ at W is given by

$$\mathbb{P}\mathcal{T}_W B^k_{r,d} = \overline{\bigcup_{\Lambda \in U^k} \left(\bigcap_{\Pi \in (U^r \cap \mathrm{Gr}(r, \Lambda))} \mathrm{Span}\,\psi\,(Z_\Pi \times_C \mathbb{P}W) \right)}. \quad (3.4)$$

Here $\psi\,(Z_\Pi \times_C \mathbb{P}W)$ is a certain generalisation of the canonical image of the divisor associated to a section of a line bundle, defined in [**Hit20**, § 5.2]. The point of interest for us here is that $U^r \cap \mathrm{Gr}(r, \Lambda)$ appears in (3.4) in place of the linear subseries g_d^{k-1} in the original Riemann–Kempf singularity theorem [**ACGH85**, p. 241]. Continuing from the previous remark, we will interpret $U^r \cap \mathrm{Gr}(r, \Lambda)$ as a generalised linear series.

Firstly, the map $(s_1, \ldots, s_r) \mapsto \mathrm{Span}\{s_1, \ldots, s_r\}$ defines a morphism

$$H^0(C, \mathrm{Hom}(\mathcal{O}_C{}^{\oplus r}, W))_{\mathrm{inj}} \ \to \ \mathrm{Gr}(r, H^0(C, W)),$$

whose image is exactly U^r, and which descends to an isomorphism

$$H^0(C, \mathrm{Hom}(\mathcal{O}_C{}^{\oplus r}, W))_{\mathrm{inj}}/\mathrm{GL}_r \ \xrightarrow{\sim}\ U^r. \tag{3.5}$$

By Proposition 3.1 (a), this is exactly the generalised linear series $Q_{W^*}(\mathcal{O}_C{}^{\oplus r})$. Moreover, for $\Lambda \in U^k$ the isomorphism (3.5) restricts to an isomorphism

$$\Lambda_{\mathrm{inj}}^{\oplus r}/\mathrm{GL}_r \ \xrightarrow{\sim}\ U^r \cap \mathrm{Gr}(r, \Lambda).$$

Thus $U^r \cap \mathrm{Gr}(r, \Lambda)$ does indeed play the role of a linear series in the generalised Riemann–Kempf theorem.

4. Some nonemptiness results

In this section we give various sufficient or necessary conditions for nonemptiness of $H_e^{e-f}(V)_{\mathrm{nd}}$ and $Q_e^{e-f}(V)$, for some values of e and f. We begin with an easy "theoretical bound" valid for any smooth nondegenerate variety S of dimension at least 2 (it is trivial for curves).

LEMMA 4.1. *Let S be a smooth variety of dimension r and $\ell = (\mathcal{L}, V)$ a very ample linear series of dimension n on S. Set $e_0 := n - r + 2$. Then for any $e > e_0$ and $f \leq e - e_0$, the secant locus $H_e^{e-f}(\ell)$ is nonempty.*

PROOF. By Theorem 2.17, for a general $\Pi = \mathbb{P}^{n-r+1} \subset \mathbb{P}V^*$ the intersection $\Pi \cap \psi(S)$ is of dimension 1. Any reduced subscheme $Z \subset (\Pi \cap \psi(S))$ of length $e > e_0$ has linear span contained in Π, so has defect at least $e - e_0$. Thus Z defines a point of $H_e^{e-f}(\ell)$ for any $f \leq e - e_0$. □

REMARK 4.2. If $e = n$, then $e - e_0 = r - 2$. Thus for $r \geq 3$, Lemma 4.1 implies that $H_n^{n-f}(\ell)$ is nonempty for $1 \leq f \leq r - 2$. This generalises the nonemptiness statement of [**AS15**, Lemma 2.1] for $r \geq 3$.

Let us now focus on secant loci of scrolls over curves. Firstly, we have a technical statement.

LEMMA 4.3. *Let $\pi \colon \mathbb{P}E \to C$ be a projective bundle, and $\iota \colon C \to \mathbb{P}E$ a section. Let $x \in C$ be a point.*

(a) *There exists a local parameter z on C at x and a frame $\phi_1, \ldots, \phi_r$ for E^* near x such that for $k \geq 1$ the one-point subscheme $\iota(kx) \subset \mathbb{P}E$ is defined by the ideal*

$$\left(\pi^* z^k, \frac{\phi_2}{\phi_1}, \ldots, \frac{\phi_r}{\phi_1} \right). \tag{4.1}$$

(b) *Write $Z := \iota(kx)$ and set $E_Z^* = \pi_* \mathcal{I}_Z(1)$. Then $\{\pi^* z^k \cdot \phi_1, \phi_2, \ldots, \phi_r\}$ is a frame for E_Z^* near x.*

Proof. Let $U := \operatorname{Spec} A$ be an affine neighbourhood of x in C upon which E is trivial. Let $\nu_1, \ldots, \nu_r$ be a frame such that $\iota \colon C \to \mathbb{P}E$ corresponds to the subbundle $\mathcal{O}_C \cdot \nu_1$. Let $\phi_1, \ldots, \phi_r$ be the dual frame of $E^*|_U$, and let z be a local parameter on C at x. We consider the affine variety

$$\Omega \ := \ \operatorname{Spec} A \left[\frac{\phi_2}{\phi_1}, \ldots, \frac{\phi_r}{\phi_1} \right] \ \cong \ U \times \mathbb{A}^{r-1} \ \subset \ \mathbb{P}E.$$

In these coordinates, the embedding

$$kx \ = \ \operatorname{Spec} \left(A/\mathfrak{m}_x^k \right) \ \hookrightarrow \ U \ \xrightarrow{\iota} \ \Omega$$

corresponds to the composed map of rings

$$A \left[\frac{\phi_2}{\phi_1}, \ldots, \frac{\phi_r}{\phi_1} \right] \ \xrightarrow{\sigma^*} \ A \ \to \ A/\mathfrak{m}_x^k.$$

where $\sigma^* z = z$ and $\sigma^*(\phi_i/\phi_1) = 0$ for $2 \le i \le r$. The last composed map has kernel exactly (4.1), and we obtain (a).

For the rest: Using (a), we see that $\mathcal{I}_{\iota(kx)}(1)$ is generated on $\pi^{-1}(U)$ by the elements $\pi^* z^k \cdot \phi_1, \phi_2, \ldots, \phi_r$. Statement (b) follows. □

Corollary 4.4. *Let $\mathbb{P}E$ and ι be as above. For any effective divisor D of degree e on C, the subscheme $\iota(D) \subset \mathbb{P}E$ defines a point of* $\mathrm{Hilb}^e(\mathbb{P}E)$ *which is π-nondefective and smoothable.*

Proof. Firstly, as $\iota(D)$ is the image of a curvilinear scheme by an embedding, $\iota(D)$ is smoothable. For the rest: As π-nondefectivity is local on C, it suffices to consider the case $D = ex$. In this case, by Lemma 4.3 (b) we see that

$$E_Z^*|_U \ = \ \pi_* \mathcal{I}_Z(1)|_U \ \cong \ \mathcal{O}_U(-ex) \oplus \mathcal{O}_U^{\oplus(r-1)}.$$

Therefore, E^*/E_Z^* has length e; whence $\iota(D)$ is π-nondefective by Definition 2.11. □

Suppose now that $S = \mathbb{P}E$ is a scroll of dimension $r \ge 2$ over C. We will use existing results on secant loci of curves to prove nonemptiness of secant loci of S whose expected dimension is large.

Proposition 4.5. *Let E be a bundle of rank $r \ge 2$ and degree d, and write $S := \mathbb{P}E$. Suppose that $V \subseteq H^0(S, \mathcal{L}_M)$ is a linear subspace of dimension $n+1$. Suppose that*

$$0 \ < \ f \ < \ e \quad \textit{and} \quad re - f(n+1-e+f) \ \ge \ (r-1)e. \tag{4.2}$$

Then $H_e^{e-f}(V)_{\mathrm{nd}}$ and $Q_e^{e-f}(V)$ are nonempty.

Proof. Write $d' := \deg(E \otimes M^{-1}) = d - r \cdot \deg(M)$. By [**PR03**, Proposition 6.1], for all $k \gg 0$, there exists a short exact sequence $0 \to L \to E \otimes M^{-1} \to F \to 0$ where F is a stable vector bundle of rank $r-1$ and degree k. We choose

$$k \ \ge \ \max\{0, 2g + d'\}. \tag{4.3}$$

Since $k \ge 0$, we have $h^0(C, F^*) = 0$ by stability. Taking global sections of $0 \to F^* \to E^* \otimes M \to L^{-1} \to 0$ and dualising, we obtain

$$\cdots \ \to \ H^0(L^{-1})^* \ \to \ H^0(E^* \otimes M)^* \ \to \ 0. \tag{4.4}$$

Projectivising and identifying C with $\mathbb{P}L$, we obtain a commutative diagram

$$\begin{array}{ccc} |L^{-1}|^* & \dashrightarrow\!\!\!\!\rightarrow & \mathbb{P}V^* \\ \uparrow{\scriptstyle \varphi_{L^{-1}}} & & \uparrow \\ C & \stackrel{\iota}{\hookrightarrow} & \mathbb{P}E. \end{array}$$

Now $\deg L^{-1} = k - d'$, which by (4.3) is at least $2g$. Hence $h^1(C, L^{-1}) = 0$ and $\varphi_{L^{-1}}$ is an embedding. As C is nondegenerate in $|L^{-1}|^*$, we obtain an identification of $\mathbb{P}V$ with a subseries $g^n_{k-d'}$ of $|L^{-1}|$.

We recall next from [**ACGH85**, p. 355] that the *relative deficiency* of this $g^n_{k-d'}$ is defined to be $\deg L^{-1} - g - n$. As $h^1(C, L^{-1}) = 0$, we have

$$\deg L^{-1} - g - n \ = \ \chi(C, L^{-1}) - (n+1) \ = \ h^0(C, L^{-1}) - \dim V,$$

which is nonnegative since we have identified $\mathbb{P}V$ with a subseries of $|L^{-1}|$ in (4.4). Furthermore, by (4.2) we have $e - f(n+1-e+f) \geq 0$. Hence by loc. cit. the usual secant locus $V^{e-f}_e(g^n_{d+k}) \subseteq C_e$ is nonempty. Thus there exists a divisor D on C of degree e such that $\mathrm{Span}\,\psi(\iota(D))$ has dimension $e - f - 1$. Then $\iota(D)$ is a length e subscheme of $\mathbb{P}E$ spanning at most a $\mathbb{P}^{e-f-1}$ in $|\mathcal{L}_M|^*$; and $\iota(D)$ is π-nondefective and smoothable by Corollary 4.4. Hence $H^{e-f}_e(V)_{\mathrm{nd}}$ is nonempty, and so therefore is $Q^{e-f}_e(V)$ by Proposition 2.12. □

Remark 4.6. If $f = 1$, then (4.2) reduces to $e \geq 2$ and $2e - 1 \geq n + 1$. If $e = n$, then this is satisfied for $n \geq 2$. Thus for $r \geq 2$, the statement of Proposition 4.5 again generalises the nonemptiness part of [**AS15**, Lemma 2.1].

We conclude this section with a necessary condition for nonemptiness.

Proposition 4.7. *Let E be a bundle of rank r and degree d. Set $\widehat{h} := h^0(C, K_C \otimes E)$. If Q^{e-f}_e is nonempty, then the twisted Brill–Noether locus*

$$B^{\widehat{h}+f}_{1,e}(K_C \otimes E) \ = \ \{L \in \mathrm{Pic}^e(C) : h^0(C, K_C L \otimes E) \geq f\}$$

contains an effective line bundle.

Proof. Suppose $[F^* \to E^*]$ belongs to Q^{e-f}_e. Using (3.2), we obtain

$$h^0(C, K_C \otimes F) \ \geq \ h^0(K_C \otimes E) + f.$$

Now $F \subseteq E(D)$ for some $D \in C_e$. Then $\mathcal{O}_C(D)$ belongs to $B^{\widehat{h}+f}_{1,e}(K_C \otimes E)$. □

This observation will be further developed in § 6 and thereafter.

5. Tangent spaces to secant loci

In this section, we will describe the Zariski tangent spaces of $Q^{e-f}_e(V)$, generalising [**ACGH85**, Lemma IV.1.5] and [**Cop95**, Theorem 0.3]. In preparation, we give a construction of the first-order infinitesimal deformations of an element of any Quot scheme, which will be convenient for calculations.

5.1. Tangent spaces to Quot schemes. Let $(X, \mathcal{O}_X(1))$ be a polarised variety and $\mathcal{V}$ a coherent sheaf over X. The scheme $\mathrm{Quot}^{\Phi}(\mathcal{V})$ parametrises equivalence classes of coherent quotients $\mathcal{V} \xrightarrow{q} \mathcal{Q}$ where $\mathcal{Q}$ has Hilbert polynomial Φ. The first-order infinitesimal deformations of a given $[\mathcal{V} \xrightarrow{q} \mathcal{Q}]$ are parametrised by the Zariski tangent space $T_q\mathrm{Quot}^{\Phi}(\mathcal{V})$. This is canonically isomorphic to $H^0(X, \mathrm{Hom}(\mathcal{W}, \mathcal{Q}))$, where $\mathcal{W} = \mathrm{Ker}(q)$; see for example [**HL10**, Chapter 2]. These deformations naturally induce deformations of the sheaf $\mathcal{W}$, which we will now construct. (A similar approach to deformations is given in [**Eis95**, Exercise 6.12 c].)

For each $u \in H^0(X, \mathrm{Hom}(\mathcal{W}, \mathcal{Q}))$, let $\mathbb{W}_u$ be the sheaf over $\mathrm{Spec}\,\mathbb{C}[\varepsilon] \times X$ given over each open set $U \subseteq X$ by

$$\mathbb{W}_u(U) \ = \ \{\varepsilon t + s \in H^0(U, \varepsilon \cdot \mathcal{V} \oplus \mathcal{W}) : q(t) = u(s)\} \tag{5.1}$$

(Here we use the fact that $\mathrm{Spec}\ \mathbb{C}[\varepsilon] \times X$ has the same ambient topological space as X.) The sheaf $\mathbb{W}_u$ is naturally contained in $\varepsilon \cdot \mathcal{V} \oplus \mathcal{V}$, and $\mathbb{W}_u = \mathbb{W}$ is easily checked to fit into an exact diagram of the form

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varepsilon \cdot \mathcal{W} & \longrightarrow & \mathbb{W} & \longrightarrow & \mathcal{W} & \longrightarrow & 0 \\
 & & \downarrow j & & \downarrow & & \downarrow j & & \\
0 & \longrightarrow & \varepsilon \cdot \mathcal{V} & \longrightarrow & \varepsilon \cdot \mathcal{V} \oplus \mathcal{V} & \longrightarrow & \mathcal{V} & \longrightarrow & 0 \\
 & & \downarrow q & & \downarrow q' & & \downarrow q & & \\
0 & \longrightarrow & \varepsilon \cdot \mathcal{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array} \tag{5.2}$$

Thus $\mathbb{W}_u$ defines a first-order infinitesimal deformation of $\mathcal{V} \xrightarrow{q} \mathcal{Q}$.

LEMMA 5.1. *The assignment*

$$u \ \mapsto \ \left[\varepsilon \cdot \mathcal{V} \oplus \mathcal{V} \to \frac{\varepsilon \cdot \mathcal{V} \oplus \mathcal{V}}{\mathbb{W}_u}\right]$$

defines a bijection between $H^0(C, \mathrm{Hom}(\mathcal{W}, \mathcal{Q}))$ *and the set of isomorphism classes of first-order infinitesimal deformations of* $[\mathcal{V} \xrightarrow{q} \mathcal{Q}]$.

PROOF. We indicate only the main steps. Recall firstly that the *graph* of a map of $\mathcal{O}_X$-modules $u \colon \mathcal{W} \to \mathcal{Q}$ is the $\mathcal{O}_X$-submodule $\Gamma_u \subseteq \mathcal{Q} \oplus \mathcal{W}$ given over open $U \subseteq X$ by

$$\Gamma_u(U) \ = \ \{(u(s), s) : s \in \mathcal{W}(U)\}.$$

By construction, $\mathbb{W}_u$ is exactly the inverse image of $\Gamma_{\varepsilon u}$ by the projection $\varepsilon \cdot \mathcal{V} \oplus \mathcal{W} \to \varepsilon \cdot \mathcal{Q} \oplus \mathcal{W}$.

Now by for example [**HL10**, § 2.2], each first-order infinitesimal deformation of $\mathcal{V} \xrightarrow{q} \mathcal{Q}$ corresponds to an exact diagram of the form (5.2). As mentioned above, $\mathbb{W} = \mathbb{W}_u$ fits into such a diagram. Conversely, given a diagram of the form (5.2), write $\widetilde{\mathbb{W}} := (q \oplus \mathrm{Id}_{\mathcal{V}})\,(\mathbb{W})$. Then

(i) $\widetilde{\mathbb{W}} \cap (\varepsilon \cdot \mathcal{Q} \oplus 0) = 0$ since $\mathbb{W} \cap (\varepsilon \cdot \mathcal{V} \oplus 0) = \varepsilon \cdot \mathcal{W}$; and
(ii) the image of $\widetilde{\mathbb{W}}$ in $\mathcal{V}$ is $\mathcal{W}$ since $\mathbb{W} \to \mathcal{W}$ is surjective.

Linear algebra arguments then show that $\widetilde{\mathbb{W}} = \Gamma_{\varepsilon u}$ for a uniquely determined $u \colon \mathcal{W} \to \mathcal{Q}$, and that that $\mathbb{W} = \mathbb{W}_u$. □

5.2. Zariski tangent spaces of $Q_e^{e-f}(V)$. We now consider a smooth curve C and a vector bundle $E \to C$ of rank r and degree d. Let M be a line bundle and $V \subseteq H^0(C, E^* \otimes M)$ a subspace of dimension $n+1$. We will study the tangent spaces to $Q_e^{e-f}(E, M, V) =: Q_e^{e-f}(V)$.

For the remainder of the paper, for any sheaf $\mathcal{W}$ over C we abbreviate $H^i(C, \mathcal{W})$ and $h^i(C, \mathcal{W})$ to $H^i(\mathcal{W})$ and $h^i(\mathcal{W})$ respectively.

Following the treatment in [**ACGH85**, Chap. IV] of the tangent spaces of C_e^f, which is exactly $Q_e^{e-f}(K_C, \mathcal{O}_C, H^0(K_C))$, we fix the following. Consider an elementary transformation $0 \to E \to F \to \mathcal{T} \to 0$ where τ has length e. Taking $\mathrm{Hom}(-, M)$, we obtain a sequence

$$0 \ \to \ F^* \otimes M \ \to \ E^* \otimes M \ \xrightarrow{q} \ \tau \ \to \ 0 \tag{5.3}$$

where $\tau := Ext^1_{\mathcal{O}_C}(\mathcal{T}, M)$ is noncanonically isomorphic to $\mathcal{T}$. By the previous section, and since M is invertible,

$$T_{F^*}\mathrm{Quot}^{0,e}(E^*) \ \cong \ H^0(C, \mathrm{Hom}(F^*, \tau \otimes M^{-1})) \ \cong \ H^0(C, \mathrm{Hom}(F^* \otimes M, \tau)).$$

Also, recall that there is a natural map

$$c \colon H^0(\mathrm{Hom}(F^* \otimes M, \tau)) \ \to \ \mathrm{Hom}\left(H^0(F^* \otimes M), H^0(\tau)\right).$$

Proposition 5.2. *Let E, M and V be as above. Suppose that $[F^* \to E^*]$ is a point of $Q_e^{e-f}(V) \setminus Q_e^{e-f-1}(V)$. Then*

$$T_{F^*}Q_e^{e-f}(V) \ = \ \{u \in H^0(\mathrm{Hom}(F^* \otimes M, \tau)) : c(u)(V \cap H^0(F^* \otimes M)) \subseteq q(V)\}.$$

Moreover, this is precisely the kernel of the map

$$H^0(\mathrm{Hom}(F^* \otimes M, \tau)) \ \to \ \mathrm{Hom}\left(V \cap H^0(F^* \otimes M), \frac{H^0(\tau)}{q(V)}\right)$$

given by composing c, restriction to $V \cap H^0(F^* \otimes M)$ and quotient by $q(V)$.

Proof. A tangent vector $u \in H^0(\mathrm{Hom}(F^* \otimes M, \tau))$ belongs to $T_{F^*}Q_e^{e-f}(V)$ if and only if every $s \in V \cap H^0(F^* \otimes M)$ lifts to the deformation $\mathbb{W}_u$ given as in (5.1) by

$$\mathbb{W}_u(U) \ = \ \{\varepsilon t + s \in \varepsilon \cdot (E^* \otimes M)(U) \oplus (F^* \otimes M)(U) : q(t) = u(s)\}$$

over each open subset $U \subseteq C$. This is equivalent to saying that for each element $s \in V \cap H^0(F^* \otimes M)$, there exists $t \in V$ such that $q(t) = c(u)(s)$; in other words, that $c(u)\left(V \cap H^0(\widehat{F})\right) \subseteq q(V)$. This proves the first statement; and the second is then clear. □

5.3. The complete case. During this subsection, we fix $M = K_C$ and assume that $V = H^0(E^* \otimes K_C)$. In this case, we can say more about $T_{F^*}Q_e^{e-f}$.

Firstly, we recall some familiar facts. To ease notation, for any locally free sheaf W over C we will denote $K_C \otimes W^*$ by $\widehat{W}$. Fix an elementary transformation $0 \to E \to F \to \mathcal{T} \to 0$. This canonically determines a sequence $0 \to \widehat{F} \to \widehat{E} \xrightarrow{q} \tau \to 0$ and an element of $\mathrm{Quot}^{0,e}(E^*)$ as above. Let $\partial \colon H^0(\tau) \to H^1(\widehat{F})$ denote the associated coboundary map. Then we obtain a commutative diagram with exact rows

(5.4)

$$\begin{array}{ccccccc} H^0(\mathrm{Hom}(\widehat{F},\tau)) & \xrightarrow{\delta} & H^1(\mathrm{End}\,\widehat{F}) & \longrightarrow & H^1(\mathrm{Hom}(\widehat{F},\widehat{E})) & \longrightarrow & 0 \\ \downarrow c & & \downarrow \cup & & \downarrow & & \\ \mathrm{Hom}\left(H^0(\widehat{F}),H^0(\tau)\right) & \xrightarrow{\partial_*} & \mathrm{Hom}\left(H^0(\widehat{F}),H^1(\widehat{F})\right) & \longrightarrow & \mathrm{Hom}\left(H^0(\widehat{F}),H^1(\widehat{E})\right) & \longrightarrow & 0 \end{array}$$

where $\cup$ is the cup product map. We recall the well known fact that via Serre duality, $\cup$ is dual to the Petri map

$$\mu\colon H^0(\widehat{F})\otimes H^0(F) \ \to \ H^0(\widehat{F}\otimes F) \ = \ H^0(K_C\otimes \mathrm{End}\,F).$$

PROPOSITION 5.3. *Let* $0 \to E \to F \to \mathcal{T} \to 0$ *be as above. Suppose that* $[F^* \to E^*]$ *belongs to* $Q_e^{e-f} \setminus Q_e^{e-f-1}$.

(a) *We have* $T_{F^*}Q_e^{e-f} = \mathrm{Ker}(\cup\circ\delta) \ = \ \mathrm{Im}\,({}^t\delta\circ\mu)^{\perp}$.

(b) *The locus* Q_e^{e-f} *is smooth and of the expected dimension at* $[F^*\to E^*]$ *if and only if* $\mu^{-1}\left(H^0(\widehat{F}\otimes E)\right) = H^0(\widehat{F})\otimes H^0(E)$.

PROOF. (a) Since $V = H^0(\widehat{E})$, we have $\mathrm{Im}(c(u)) \subseteq q(V)$ if and only if $\partial\circ c(u)$ is zero in $\mathrm{Hom}\left(H^0(\widehat{F}),H^1(\widehat{F})\right)$. Hence by Proposition 5.2 and commutativity of (5.4), it follows that

$$T_{F^*}Q_e^{e-f} \ = \ \mathrm{Ker}(\partial_*\circ c) \ = \ \mathrm{Ker}(\cup\circ\delta).$$

As $\mathrm{Ker}(\varphi) = \mathrm{Im}({}^t\varphi)^{\perp}$ for any vector space map φ, and since ${}^t\cup = \mu$, this in turn coincides with $\mathrm{Im}\,({}^t\delta\circ\mu)^{\perp}$.

(b) By part (a), the codimension of $T_{F^*}Q_e^{e-f}$ in $T_{F*}\mathrm{Quot}^{0,e}(E^*)$ is exactly $\dim\mathrm{Im}({}^t\delta\circ\mu)$. Let us compute the latter. Dualising (5.4), we obtain

(5.5)

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\widehat{F})\otimes H^0(E) & \longrightarrow & H^0(\widehat{F})\otimes H^0(F) & \longrightarrow & H^0(\widehat{F})\otimes H^0(\tau)^* \\ & & \downarrow & & \downarrow \mu & & \downarrow \\ 0 & \longrightarrow & H^0(\widehat{F}\otimes E) & \longrightarrow & H^0(\widehat{F}\otimes F) & \xrightarrow{{}^t\delta} & H^0(\mathrm{Hom}(\widehat{F},\widehat{\tau}))^*. \end{array}$$

Thus $\mathrm{Ker}({}^t\delta\circ\mu) = \mu^{-1}\left(H^0(\widehat{F}\otimes E)\right)$, and so $\mathrm{Im}({}^t\delta\circ\mu)$ has dimension

$$h^0(\widehat{F})\cdot h^0(F) - \dim\mu^{-1}\left(H^0(\widehat{F}\otimes E)\right) \ =$$

$$h^0(\widehat{F})\cdot\left(h^0(F)-h^0(E)\right) - \dim\frac{\mu^{-1}\left(H^0(\widehat{F}\otimes E)\right)}{H^0(\widehat{F})\otimes H^0(E)}.$$

By hypothesis, $h^0(\widehat{F}) = h^0(\widehat{E}) - e + f$. Hence by (3.2) we have $h^0(F) - h^0(E) = f$. Thus $\mathrm{Im}({}^t\delta\circ\mu)$ has dimension

$$\left(h^0(\widehat{E}) - e + f\right)\cdot f - \dim\frac{\mu^{-1}\left(H^0(\widehat{F}\otimes E)\right)}{H^0(\widehat{F})\otimes H^0(E)}.$$

As the expected codimension of Q_e^{e-f} in $\mathrm{Quot}^{0,e}(E^*)$ is $(h^0(\widehat{E}) - e + f) \cdot f$ (for in this case $n+1 = h^0(\widehat{E})$), we see that $T_{F^*} Q_e^{e-f}$ has the expected codimension if and only if $\mu^{-1}\left(H^0(\widehat{F} \otimes E)\right) = H^0(\widehat{F}) \otimes H^0(E)$. □

5.4. Some examples. Continuing to use the notation of the previous subsection, let us return to the study of the generalised Abel–Jacobi map

$$a \colon \mathrm{Quot}^{0,e}(E^*) \dashrightarrow U_C(r, d+e)$$

given by $[F^* \to E^*] \mapsto F$. As noted in § 3, we have

$$a^{-1}\left(B^{h^0(E)+f}_{r,d+e}\right) \ = \ Q_e^{e-f}(E, K_C, H^0(E^* \otimes K_C)).$$

It is well known that $B^{h^0(E)+f}_{r,d+e} \backslash B^{h^0(E)+f+1}_{r,d+e}$ is smooth and of the expected dimension at a stable bundle F if and only if μ is injective. In the special case $E = \mathcal{O}_C$ and $F = \mathcal{O}_C(D)$ with $D \in C_e^f \setminus C_e^{f+1}$, by [**ACGH85**, Lemma IV.1.6], injectiveness of μ is also equivalent to smoothness of C_e^f at D. One can check that in this situation, the condition in Proposition 5.3 (b) is equivalent to injectiveness of μ, so there is no contradiction. However, we will now give examples showing that in general, smoothness of $B^{h^0(E)+f}_{r,d+e}$ at F is neither necessary nor sufficient for smoothness of $Q_e^{e-f}(E, K_C, H^0(E^* \otimes K_C))$ at F^*.

Example 5.4. Let D be an effective divisor of degree $e \leq g-1$, and $E := \mathcal{O}_C(-D)$. Set $F = E(D) = \mathcal{O}_C$, so that $\widehat{F} = K_C$. Here $\deg(E) = -e$, and $f = 1$, and

$$F \ \in \ Q_e^{e-1}(E, K_C, H^0(E^* \otimes K_C)) \ = \ Q_e^{e-1}(\mathcal{O}_C(-D), K_C, H^0(K_C(D))).$$

Trivially, $B^1_{1,d+e} = B^1_{1,0} = \{\mathcal{O}_C\}$ is smooth and of the expected dimension at F.

Now $H^0(\widehat{F}) \otimes H^0(F) = H^0(K_C) \otimes H^0(\mathcal{O}_C)$. Clearly μ is an isomorphism, and

$$\mu^{-1}\left(H^0(\widehat{F} \otimes E)\right) \ = \ \mu^{-1}\left(H^0(K_C(-D))\right) \ = \ H^0(\mathcal{O}_C) \otimes H^0(K_C(-D)),$$

which is nonzero since $\deg(D) \leq g-1$. On the other hand, $H^0(\mathcal{O}_C(-D)) \otimes H^0(K_C)$ is zero. By Proposition 5.3 (b), the secant locus Q_e^{e-1} is not smooth at F. (One can check that the expected dimension of Q_e^{e-1} is $e - g < 0$.)

The next example shows that Q_e^{e-f} can be smooth at F^* even though $B^k_{r,d}$ is not smooth at F. The idea is to produce an E and F such that $\mathrm{Ker}\,\mu$ is nonzero but contained in $H^0(\widehat{F}) \otimes H^0(E)$ so that the condition $\mu^{-1} H^0(\widehat{F} \otimes E) = H^0(\widehat{F}) \otimes H^0(E)$ can still obtain. Perhaps somewhat artificially, we will start with the bundle F and choose an appropriate E. The bundle F will have rank two and canonical determinant; the Brill–Noether loci of such bundles have been studied in [**TiB04**], [**TiB08**] and elsewhere, and the Petri map is known to have nonzero kernel in many cases.

Example 5.5. Suppose $g \geq 6$, so that $B^2_{1,g-2}$ is nonempty. Let L_1 be a line bundle of degree $g-2$ with $h^0(L_1) = 2$. Let x_1 and x_2 be general points of C, and set $L_2 := K_C L_1^{-1}(-x_1 - x_2)$. Then $h^0(L_2) = 1$. Let

$$0 \ \to \ L_1 \oplus L_2 \ \to \ F \ \to \ \mathcal{O}_{x_1} \oplus \mathcal{O}_{x_2} \ \to \ 0 \tag{5.6}$$

be an elementary transformation such that neither L_1 nor L_2 is a quotient of F. By [**Mer99**, Théorème A.5], we may assume that F is stable.

We now claim that $H^0(F) = H^0(L_1) \oplus H^0(L_2)$. It will suffice to show that the composed map

$$H^0(\mathcal{O}_{x_1} \oplus \mathcal{O}_{x_2}) \ \to \ H^1(L_1) \oplus H^1(L_2) \ \to \ H^1(L_1) \ \cong \ H^0(K_C L_1^{-1})^* \tag{5.7}$$

is injective. It is not hard to see that the image of this composed map is exactly the cone over the secant spanned by the points x_1 and x_2 on the plane curve $\varphi_{K_C L_1^{-1}}(C)$. As already used, by generality of x_1 and x_2 we have $h^0(K_C L_1^{-1}(-x_1 - x_2))) = 1$, so this cone has dimension 2. Hence (5.7) is an isomorphism, and in particular injective as desired.

Thus F defines a point of $B^3_{2,2g-2}$. Moreover, since $\det(F) \cong K_C$, we have $F \cong K_C \otimes F^*$, and the Petri map can be identified with the natural map

$$H^0(F) \otimes H^0(F) \ \to \ H^0(F \otimes F).$$

Abusing notation, we also denote this map by μ. Let s_1, s_2 be a basis for $H^0(L_1)$. Then $s_1 \wedge s_2$ is a nonzero element of $\mathrm{Ker}(\mu)$, so $B^3_{2,2g-2}$ is *not* smooth at F.

We will now construct a bundle E such that $[F^* \to E^*]$ is a smooth point of a certain $Q^{e-f}_e(E^*, K_C, H^0(E^* \otimes K_C))$. Let E be an elementary transformation $0 \to E \to F \to \mathcal{O}_D \to 0$ where D is a general effective divisor of degree $e \geq 2$, and

$$\mathrm{Im}\,(E|_p \to F|_p) \ = \ L_1|_p \text{ for each } p \in D. \tag{5.8}$$

As $L_1 \to F \to \mathcal{O}_D$ is zero, $L_1 \subset E$. (Note that E is therefore not stable, as $\deg(L_1) = g - 2 \geq \frac{2g-2-e}{2}$, but this does not affect the argument.) In particular $H^0(E)$ contains $H^0(L_1)$. Furthermore, let s_3 be a generator of $H^0(L_2)$. By generality of D, we may assume that $s_3(p) \notin L_1|_p$ for all $p \in D$. Thus s_3 is not a section of E, whence $h^0(E) = h^0(L_1) = 2$. By Riemann–Roch, $h^0(E^* \otimes K_C) = e + 2$. As

$$h^0(F^* \otimes K_C) \ = \ h^0(F) \ = \ 3 \ = \ h^0(E^* \otimes K_C) - e + 1,$$

we have $[F^* \to E^*] \in Q^{e-1}_e \left(E, K_C, H^0(C, E^* \otimes K_C)\right)$. We will show that it is a smooth point.

By Proposition 5.3 (b), noting that $F \cong F^* \otimes K_C = \widehat{F}$, we must show that

$$\mu^{-1}\left(H^0(F \otimes E)\right) \ = \ H^0(F) \otimes H^0(E). \tag{5.9}$$

Let $s := \sum_{i,j=1}^3 \alpha_{ij} s_i \otimes s_j$ be an element of $H^0(F) \otimes H^0(F)$, and assume that $(\mu(s))(p)$ belongs to the image of $F \otimes E|_p$ for all $p \in C$. This is a nonempty condition only at $p \in D$, since $E|_p = F|_p$ for $p \notin D$. For $p \in D$, using (5.6), there is a canonical splitting of $(F \otimes F)|_p$ as

$$(L_1|_p \otimes L_1|_p) \oplus (L_2|_p \otimes L_1|_p) \oplus (L_1|_p \otimes L_2|_p) \oplus (L_2|_p \otimes L_2|_p)\,.$$

We write out $(\mu(s))\,(p)$ in terms of this splitting:

(5.10)
$$\begin{aligned}(\alpha_{11} s_1(p) \otimes s_1(p) + \alpha_{21} s_2(p) \otimes s_1(p) + \alpha_{12} s_1(p) \otimes s_2(p) + \alpha_{22} s_2(p) \otimes s_2(p),\\ \alpha_{31} s_3(p) \otimes s_1(p) + \alpha_{32} s_3(p) \otimes s_2(p), \alpha_{13} s_1(p) \otimes s_3(p) + \alpha_{23} s_2(p) \otimes s_3(p),\\ \alpha_{33} s_3(p) \otimes s_3(p))\,.\end{aligned}$$

Now in view of (5.8), by hypothesis $(\mu(s))\,(p)$ belongs to

$$\mathrm{Im}\,(E|_p \otimes F|_p \ \to \ F|_p \otimes F|_p) \ = \ (L_1|_p \oplus L_2|_p) \otimes L_1|_p$$

for $p \in D$. Thus the third and fourth components of (5.10) must be zero for all $p \in D$. Firstly, since $s_3(p)$ can be assumed to be nonzero for all $p \in D$ by generality of D, we obtain $\alpha_{33} = 0$. Furthermore, considering the third component of (5.10), we obtain

$$(\alpha_{13} s_1(p) + \alpha_{23} s_2(p)) \otimes s_3(p) \ = \ 0 \text{ for all } p \in D.$$

Using the nonvanishing of $s_3(p)$ above, we obtain $\alpha_{13} s_1(p) + \alpha_{23} s_2(p) = 0$ for all $p \in D$. Again using generality of D, we may assume that no two points of D are identified under the map $C \to |L_1|^* = \mathbb{P}^1$. Thus this condition places e independent linear conditions on $(\alpha_{13}, \alpha_{23})$. Since $e \geq 2$, the only solution is $\alpha_{13} = \alpha_{23} = 0$. It follows that $s \in H^0(F) \otimes H^0(E)$. Thus (5.9) is satisfied and Q_e^{e-1} is smooth at $[F^* \to E^*]$.

Remark 5.6. In both of these examples, the singularity arises from a determinantal variety which does not have the expected dimension[2]. Moreover, the bundle E in Example 5.5 is not semistable. It would be interesting to have examples of other kinds, and to investigate more thoroughly the relationships between smoothness of $B^k_{r,d}$, smoothness of Q_e^{e-f} and semistability of E.

6. Parameterisation of Q_e^{e-1}

As before, let E be a vector bundle of rank r and degree d over C. In [**Hit19**, § 4.3], a parameterisation is given for the inflectional loci of a map $\mathbb{P}E \dashrightarrow |\mathcal{L}_M|^*$. We will now construct a similar parameterisation of $Q_e^{e-1}(E, M, H^0(E^* \otimes M))$. In the sections that follow, we will give some applications similar to those in [**Hit19**, § 5–6] for this parameterisation.

For $M \in \mathrm{Pic}^0(C)$, consider the Quot scheme

$$\mathrm{Quot}^{r-1, d+2r(g-1)+e}(K_C M^{-1} \otimes E)$$

parametrising equivalence classes of quotients $[K_C M^{-1} \otimes E \to \mathcal{Q}]$ where $\mathcal{Q}$ is coherent of rank $r-1$ and degree $\deg(K_C M^{-1} \otimes E) + e$; equivalently, invertible subsheaves of $K_C M^{-1} \otimes E$ of degree $-e$. As we wish to focus on subsheaves, and to ease notation, we denote this Quot scheme by $Q_{1,-e}(K_C M^{-1} \otimes E)$.

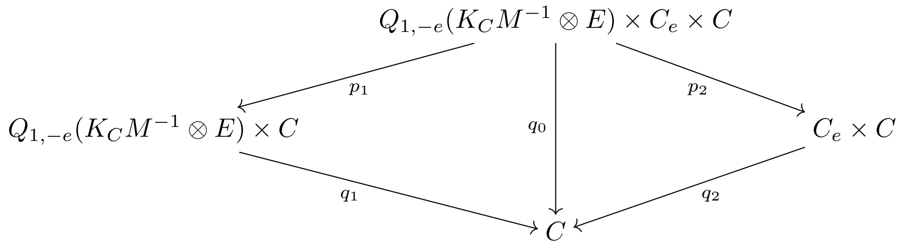
Let $a \colon C_e \to \mathrm{Pic}^{-e}(C)$ be the Abel–Jacobi-type map $D \mapsto \mathcal{O}_C(-D)$, and let $b \colon Q_{1,-e}(K_C M^{-1} \otimes E) \to \mathrm{Pic}^{-e}(C)$ be the forgetful map $\left[L \xrightarrow{\sigma} K_C M^{-1} \otimes E\right] \mapsto L$. Consider the fibre product

$$\begin{array}{ccc} S^e_M & \longrightarrow & Q_{1,-e}(K_C M^{-1} \otimes E) \\ \big\downarrow & & \big\downarrow b \\ C_e & \xrightarrow{\ a\ } & \mathrm{Pic}^{-e}(C). \end{array} \tag{6.1}$$

Set-theoretically, S^e_M parametrises pairs (σ, D) where D is an effective divisor of degree e on C and $\sigma \colon \mathcal{O}_C(-D) \to K_C M^{-1} \otimes E$ is a sheaf injection. We will now construct a family of elementary transformations of E^* parametrised by S^e_M.

[2]Note however that the fixed determinant Brill–Noether locus B^3_{2,K_C} may well have the expected dimension as a *symmetric* determinantal variety. See for example [**TiB08**].

Consider the following diagram where all maps are projections.



Recall also the exact sequence $0 \to \mathcal{O}_{C_e \times C}(-\mathscr{D}) \to \mathcal{O}_{C_e \times C} \xrightarrow{\rho} \mathcal{O}_{\mathscr{D}} \to 0$ where $\mathscr{D}$ is the universal divisor.

Let $\Sigma \colon \mathcal{L} \to q_1^*(K_C M^{-1} \otimes E)$ be the universal subsheaf over $Q_{1,-e}(K_C M^{-1} \otimes E) \times C$. A key point is that $\mathcal{L}$ is locally free, and hence $p_1^* \mathcal{L}$ is too (this is the reason we work with $Q_{1,-e}(K_C M^{-1} \otimes E)$ instead of $\mathrm{Quot}^{1,e}({K_C}^{-1} M \otimes E^*)$ directly). Thus $p_1^* \Sigma$ can be considered as a map of vector bundles over $Q_{1,-e}(K_C M^{-1} \otimes E) \times C_e \times C$ (with a degeneracy locus). Hence we can take its transpose, twist by $q_0^*(K_C M^{-1})$ and form the composed map

$$q_0^* E^* \ \to \ q_0^*(K_C M^{-1}) \otimes p_1^* \mathcal{L}^\vee \ \to \ q_0^*(K_C M^{-1}) \otimes p_1^* \mathcal{L}^\vee \otimes p_2^* \mathcal{O}_{\mathscr{D}}. \tag{6.2}$$

Now C_e is also a Quot scheme, parametrising degree e torsion quotients of any line bundle over C. Thus, by the universal property of Quot schemes, the restriction of this map to any fibre

$$\left\{ [L \xrightarrow{\sigma} K_C M^{-1} \otimes E], D \right\} \times C$$

is identified with the composed map

$$E^* \ \xrightarrow{\widehat{\sigma}} \ K_C M^{-1} L^{-1} \ \to \ K_C M^{-1} L^{-1}|_D$$

where $\widehat{\sigma} = {}^t\sigma \otimes \mathrm{Id}_{K_C M^{-1}}$. The kernel of this is an elementary transformation of E^*. As the restriction $K_C M^{-1} L^{-1} \to K_C M^{-1} L^{-1}|_D$ is always surjective, the elementary transformation has degree $\deg E^* - e$ if and only if σ is a vector bundle injection at each point of D. Pulling back to S_M^e as defined in (6.1), we obtain a family of elements of $\mathrm{Quot}^{0,e}(E^*)$ parametrised by the open subset

$$\left\{ \left([L \xrightarrow{\sigma} K_C M^{-1} \otimes E], D \right) : L \cong \mathcal{O}_C(-D) \text{ and } \sigma \text{ is a vector bundle injection along } D \right\}$$

of S_M^e. By the universal property of $\mathrm{Quot}^{0,e}(E^*)$, we obtain a map $\gamma_M^e \colon S_M^e \dashrightarrow \mathrm{Quot}^{0,e}(E^*)$ defined precisely on the above open set, given by sending

$$(\sigma, D) \ \mapsto \ \mathrm{Ker} \left(E^* \ \xrightarrow{\widehat{\sigma}} \ K_C M^{-1}(D) \ \to \ K_C M^{-1}(D)|_D \right). \tag{6.3}$$

Now we characterise the locus $Q_e^{e-1}(E, M, H^0(E^* \otimes M)) = Q_e^{e-1}$ using the spaces S_M^e. The description is analogous to that of the osculating spaces to the scroll $\mathbb{P}E$ given in [**Hit19**, Proposition 4.1].

Proposition 6.1. *Let E and M be as above, and γ_M^e as defined in (6.3).*

(a) *We have* $\mathrm{Im}(\gamma_M^e) \subseteq Q_e^{e-1}$.

(b) *Suppose that* $[F^* \to E^*]$ *belongs to* Q_e^{e-1}. *Then for some* $e' \in \{1, \ldots, e\}$ *there exists* $[G^* \to E^*]$ *belonging to* $\mathrm{Im}\left(\gamma_M^{e'}\right) \subseteq \mathrm{Quot}^{e'}(E^*)$ *such that* $[F^* \to G^*]^* \in \mathrm{Quot}^{e-e'}(G^*)$.

Proof. (a) Suppose that $[F^* \to E^*]$ is of the form $\gamma_M^e(\sigma, D)$. By definition of γ_M^e, we have a exact diagram

$$\begin{array}{ccccc} F^* & \longrightarrow & E^* & \longrightarrow & E^*/F^* \\ \downarrow \widehat{\sigma}' & & \downarrow \widehat{\sigma} & & \downarrow \wr \\ K_C M^{-1} & \longrightarrow & K_C M^{-1}(D) & \longrightarrow & K_C M^{-1}(D)|_D \end{array}$$

Thus $\widehat{\sigma}'$ defines an element of $H^0(K_C M^{-1} \otimes F)$. Now by hypothesis, $\widehat{\sigma}$ is a vector bundle surjection along D. It follows that $\widehat{\sigma}'$ does not factorise via E^*. Hence $h^0(K_C M^{-1} \otimes F) \geq h^0(K_C M^{-1} \otimes E) + 1$. Using (3.2), we obtain $h^0(F^* \otimes M) \geq h^0(E^* \otimes M) - e + 1$. Hence $[F^* \to E^*]$ belongs to Q_e^{e-1}.

(b) Suppose that $h^0(F^* \otimes M) \geq h^0(E^* \otimes M) - e + 1$. Again using (3.2), we see that there exists $\sigma \in H^0(C, K_C M^{-1} \otimes F)$ whose image in $H^0(C, K_C M^{-1} \otimes (F/E))$ is nonzero.

Now we view sections of F as rational sections of E with at worst poles limited by F/E. (This generalises the definition of $\mathcal{O}_C(D)$ for an effective divisor D.) Let $x_1, \ldots, x_s \in \mathrm{Supp}(F/E)$ be the points where $(\sigma \mod K_C M^{-1} \otimes E)$ is supported; as $\sigma \notin H^0(K_C M^{-1} \otimes E)$, there is at least one such point. For $1 \leq j \leq s$, we may choose a suitable neighbourhood U_j of x_j and a frame $\nu_1^{(j)}, \ldots, \nu_r^{(j)}$ for $K_C M^{-1} \otimes E|_{U_j}$ and a local parameter z_j on C at x_j, such that the image of $\sigma|_{U_j}$ is $\mathcal{O}_{U_j} \cdot \left(z_j^{-e_j} \cdot \nu_1^{(j)}\right)$. Let G be the elementary transformation of E such that $K_C M^{-1} \otimes G$ has frame

$$z_j^{-e_j} \cdot \nu_1^{(j)},\ \nu_2^{(j)},\ \ldots,\ \nu_r^{(j)} \tag{6.4}$$

over each U_j, and is equal to $K_C M^{-1} \otimes E$ otherwise. Now $z_j^{-e_j} \cdot \nu_1^{(j)} \subseteq \mathrm{Im}(\sigma)$ belongs to F, and the remaining frame elements $\nu_2^{(j)}, \ldots, \nu_r^{(j)}$ belong to $E \subset F$. Therefore, we have inclusions $E \subset G \subseteq F$. Writing $e' := \sum_{i=1}^s e_j$, we have

$$\deg E + e' \ = \ \deg G \ \leq \ \deg F \ = \ \deg E + e.$$

In particular $e' \leq e$, and

$$[F^* \to G^*] \text{ is an element of } \mathrm{Quot}^{e-e'}(G^*). \tag{6.5}$$

Let D' be the effective divisor $\sum_{i=1}^s e_j x_j \in C_{e'}$. From the construction of G it follows that

$$\sigma \ \in \ H^0(K_C M^{-1} \otimes G) \ \subseteq \ H^0(K_C M^{-1}(D') \otimes E)$$

and moreover, that the image of $\sigma \colon \mathcal{O}_C \to K_C M^{-1} \otimes G$ is nonzero at each of the x_j. Twisting by $\mathcal{O}_C(-D')$, we see that σ defines a sheaf injection $\sigma' \colon \mathcal{O}_C(-D') \to K_C M^{-1} \otimes E$ which is a vector bundle injection along $\mathrm{Supp}\, D'$.

Now we claim that

$$[G^* \to E^*] \ = \ \gamma_M^{e'}\left([\sigma' \colon \mathcal{O}_C(-D') \to K_C M^{-1} \otimes E], D'\right).$$

To see this: For each U_j, let $\phi_1^{(j)}, \ldots, \phi_r^{(j)}$ be the frame for $E^*|_{U_j}$ dual to $\nu_1^{(j)}, \ldots, \nu_1^{(j)}$. Then by construction of G, we see that $G^*|_{U_j} \subset E^*|_{U_j}$ has the frame

$$z_j^{e_j} \cdot \phi_1^{(j)},\ \phi_2^{(j)},\ \ldots,\ \phi_r^{(j)} \tag{6.6}$$

dual to (6.4); and G^* coincides with E^* on $C \setminus D'$. On the other hand, the map

$$E^* \xrightarrow{{}^t\sigma' \otimes \mathrm{Id}_{K_C M^{-1}}} K_C M^{-1}(D') \ \to\ K_C M^{-1}(D')|_{D'}$$

is given in our chosen coordinates over U_j by

$$\sum_{i=1}^r f_i \phi_i^{(j)} \ \mapsto\ z_j^{-e_j} \cdot f_1 \ \mapsto\ z_j^{-e_j} \cdot f_1 \mod \mathcal{O}_{U_j}.$$

Comparing with (6.6), we see that this map has kernel exactly G^*, proving the claim. Statement (b) now follows from the claim and (6.5). $\square$

REMARK 6.2. Suppose $[L \xrightarrow{\sigma} K_C \otimes E]$ is an element of $Q_{1,-e}(K_C M^{-1} \otimes E)$ where $|L^{-1}|$ has dimension $k \geq 1$. For simplicity, suppose $|L^{-1}|$ is base point free. Then $S^e_{\mathcal{O}_C}$ contains the locus

$$\{(\sigma, D) : D \in |L^{-1}|\} \ \cong\ \mathbb{P}^k.$$

By Proposition 6.1 (a), we obtain an element of Q_e^{e-1} for each $D \in |L^{-1}|^*$ along which σ is a bundle injection. By Propositions 2.9 and 2.12 the projective model $\mathbb{P}E \dashrightarrow |\mathcal{L}_M|^*$ has a family of 1-defective e-secants parametrised by an open subset of $|L^{-1}|$.

REMARK 6.3. One may ask whether there is a relation between S^e_M and H_e^{e-1} akin to that described in Proposition 6.1 for Q_e^{e-1}. There does exist a map $\beta \colon S^e_M \dashrightarrow \mathrm{Hilb}^e_{\mathrm{sm}}(S)_{\mathrm{nd}}$ by $(\sigma, D) \mapsto \mathbb{P}\sigma(D)$, globalising the construction of Lemma 4.3. This is again defined exactly when σ is a vector bundle injection at D, and in fact one can show that $\alpha \circ \beta = \gamma$, where $\alpha \colon \mathrm{Hilb}^e(S) \dashrightarrow \mathrm{Quot}^{0,e}(E^*)$ is the map studied in § 2.5. An analogue of Proposition 6.1 (a) then follows from Proposition 2.12. However, Proposition 6.1 (b) is harder to generalise, as one cannot always choose a subscheme playing the role of the sheaf G above; essentially due to issues of identifiability (defined for example in [**BC21**, Definition 6.1]) arising from the existence of linear spaces contained in $\mathbb{P}E$. Thus we content ourselves for now with the study of Q_e^{e-1}.

In the sections that follow, we use Proposition 6.1 to characterise semistability, to study questions of very ampleness of $\mathcal{O}_{\mathbb{P}E}(1)$, and to show that the secant loci are well behaved when certain parameters are chosen generally.

7. The Segre invariant s_1 and secant loci

In this section, we use the parameter spaces S^e_M to characterise nonempty secant loci for $f = 1$ in terms of the Segre invariant s_1. Firstly, we review the notion of Segre invariants.

7.1. Segre invariants.

DEFINITION 7.1. Let $E \to C$ be a vector bundle of rank r and degree d. For $1 \leq t \leq r-1$, the Segre invariant $s_t(E)$ is defined by

$$s_t(E) \ := \ \min\{td - r \cdot \deg(F) : F \subset E \text{ a subbundle of rank } t\}.$$

Segre invariants are "degrees of stability": E is stable if and only if $s_t(E) > 0$ for $1 \leq t \leq r-1$. It is clear that $s_1(E) = s_1(E \otimes L)$ for any line bundle L. Moreover, $s_t(E)$ is lower semicontinuous as E varies in families. Taking direct sums of line bundles, one can produce bundles E with arbitrarily low $s_t(E)$. However, Hirschowitz [**Hir86**, Théorème 4.4] (see also [**CH10**]) proved the following sharp upper bound on $s_t(E)$.

THEOREM 7.2 (Hirschowitz's bound). *Let $E \to C$ be a vector bundle of rank r and degree d. Then for $1 \leq t \leq r-1$, we have $s_t(E) \leq t(r-t)(g-1)+\delta$, where δ is the unique integer such that $0 \leq \delta \leq r-1$ and $t(r-t)(g-1)+\delta \equiv td \mod r$. Moreover, a general stable E attains this upper bound.*

In particular, this implies that Quot schemes of subsheaves of E are always nonempty for low enough degree. Let us make this precise for line subbundles. The following is well known, but we include a proof for convenience.

LEMMA 7.3. *Let C be any curve. Let $E \to C$ be a vector bundle which is general in moduli. Then $Q_{1,-e}(K_C M^{-1} \otimes E)$ is nonempty and smooth of dimension $re+d+(r+1)(g-1)$ whenever this is nonnegative, and empty otherwise. Moreover, when $Q_{1,-e}(K_C M^{-1} \otimes E)$ is nonempty, a general element is a vector bundle injection.*

PROOF. One computes easily that $re+d+(r+1)(g-1) \geq 0$ if and only if

$$\deg(K_C \otimes E) - r \cdot (-e) \ \geq \ (r-1)(g-1)+\delta,$$

with δ as in Theorem 7.2, using the fact that left side is congruent to d modulo r. In this case, by the Hirschowitz bound, $K_C \otimes E$ has a line subbundle of degree at least $-e$, so $Q_{1,-e}(K_C M^{-1} \otimes E)$ is nonempty. Smoothness and dimension of $Q_{1,-e}(K_C M^{-1} \otimes E)$ for general E then follow from [**LN03**, Lemma 3.3] and the discussion before it. To see that a general element is saturated, we note that the locus of nonsaturated subsheaves in $Q_{1,-e}(K_C M^{-1} \otimes E)$ has dimension at most

$$\max\{\dim Q_{1,-e_1}(K_C \otimes E) + \dim C_{e-e_1} : e_1 < e\},$$

which one checks is strictly less than $re+d+(r+1)(g-1)$.

On the other hand, if if $re+d+(r+1)(g-1) < 0$ then $\deg(K_C \otimes E) + re < (r-1)(g-1)$. For such e, there is an invertible subsheaf of degree $-e$ in $K_C \otimes E$ if and only if $s_1(E)$ is smaller than the generic value, so E is not general. □

7.2. Nonemptiness criterion for $f = 1$. The following is a generalisation of [**Hit19**, Theorem 5.2], with a very similar proof.

THEOREM 7.4. *Let $E \to C$ be a vector bundle of rank r and degree d. For $e \geq 1$, the following are equivalent.*

1. *$s_1(E) > d + r(2g-2+e)$.*
2. *For all $M \in \mathrm{Pic}^0(C)$ and all $Z \in \mathrm{Hilb}^e_{\mathrm{sm}}(S)_{\mathrm{nd}}$, the span of $\phi_{\mathcal{L}_M}(Z)$ is of dimension $e-1$ in $|\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* M|^*$.*
3. *For all $M \in \mathrm{Pic}^0(C)$ and all $[F^* \subset E^*] \in \mathrm{Quot}^{0,e}(E^*)$, we have*

$$h^0(F^* \otimes M) \ = \ h^0(E^* \otimes M) - e.$$

PROOF. The equivalence of (2) and (3) follows from Proposition 2.12 and surjectivity of $\alpha\colon \mathrm{Hilb}^e_{\mathrm{sm}}(S) \to \mathrm{Quot}^{0,e}(E^*)$. Let us show the equivalence of (1) and (3).

Assume (1). As $s_1(K_C M^{-1} \otimes E) = s_1(E) > d + r(2g-2+e)$, if $L \subset K_C M^{-1} \otimes E$ is an invertible subsheaf, then $\deg(L) < -e$. In particular, $Q_{1,-e'}(K_C M^{-1} \otimes E)$ and hence $S^{e'}_M$ are empty for all $M \in \mathrm{Pic}^0(C)$ and $e' \le e$. By Proposition 6.1 (b), also Q^{e-1}_e is empty.

Conversely, suppose that $s_1(E) = s_1(K_C \otimes E) \le d + r(2g-2+e)$. Then there exists $L \in \mathrm{Pic}^{-e}(C)$ and a sheaf injection $\sigma'\colon L \to K_C \otimes E$ of degree $-e$. Let $D \in C_e$ be any divisor along which $L \to K_C \otimes E$ is a vector bundle injection. Set $M := L(D) \in \mathrm{Pic}^0(C)$ and

$$\sigma \ := \ \sigma' \otimes \mathrm{Id}_{L^{-1}(-D)} \colon \mathcal{O}_C(-D) \ \to \ K_C M^{-1} \otimes E.$$

Then (σ, D) defines an element of S^e_M at which γ^e_M is defined. Hence Q^{e-1}_e is nonempty by Proposition 6.1 (a). □

REMARK 7.5. As in [**Hit19**, Remark 5.6], we note that Theorem 7.4 does not hold for incomplete linear series. For any $Z \in \mathrm{Hilb}^e_{\mathrm{sm}}(S)$, if we project from a point of $\mathrm{Span}\,\psi(Z)$ then, whatever the value of $s_1(E)$, the image of $\mathbb{P}E$ has a defective e-secant space. (However, as noted in § 2.7, under certain conditions the H^{e-1}_e and Q^{e-1}_e do behave as expected under projection from a general centre.)

We mention now some special cases, generalising those discussed in [**Hit19**, Corollary 5.4]. We omit the proofs, as they are practically identical to those in [**Hit19**]. Recall that the *slope* of a bundle $W \to C$ is the ratio $\mu(W) := \deg(W)/\mathrm{rk}(W)$.

COROLLARY 7.6. *Let E be a bundle of rank r and degree d, where $d \le r(1-2g)$.*

(a) *Suppose that $s_1(E) > 0$ (this is the case for example if E is stable). Then for $e \le \mu(E^*) - (2g-2)$, for all $M \in \mathrm{Pic}^0(C)$ the loci $\left(H^{e-1}_e\right)_{\mathrm{nd}}$ and Q^{e-1}_e are empty.*

(b) *If $e \ge \mu(E^*) - \frac{1}{r}(r+1)(g-1)$, then for some $M \in \mathrm{Pic}^0(C)$, the loci $\left(H^{e-1}_e\right)_{\mathrm{nd}}$ and Q^{e-1}_e are nonempty.*

(c) *If $s_1(E)$ is the generic value $(r-1)(g-1)+\delta$, then the converse to (b) also holds.*

7.3. A question of Lange. If $r = 2$, then $\mathbb{P}E$ is a ruled surface over C. For rational curves and elliptic curves, Lange [**Lan92**, Lecture 2 (a)] (see also [**Har77**, Exercise V.2.12]) gives criteria for very ampleness of $\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* M$ in terms of $\deg(M)$ and $s_1(E)$. He then poses the problem of finding similar criteria for ruled surfaces when $g \ge 2$. We now offer a result in this direction, which in fact holds for projective bundles of any dimension $r \ge 2$ over C.

THEOREM 7.7. *Let $E \to C$ be a bundle of rank r and degree d. Then the line bundle $\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* L$ is very ample for all $L \in \mathrm{Pic}^\ell(C)$ if and only if*

$$\ell \ > \ \mu(E) - \frac{s_1(E)}{r} + 2g. \tag{7.1}$$

PROOF. A map $S \to \mathbb{P}^n$ is an embedding if and only if all 2-secants are nondefective. Furthermore, when S is a projective bundle $\mathbb{P}E$ over C, it follows from the argument of [**Sta21a**, Theorem 3 (i)] that all 2-secants are π-nondefective (cf.

Definition 2.11). Thus it suffices to show that (7.1) is equivalent to emptiness of the secant locus $H_2^1(\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* L)_{\mathrm{nd}}$ for all $L \in \mathrm{Pic}^\ell(C)$.

Fix now a line bundle L_0 of degree ℓ. For any $M \in \mathrm{Pic}^0(C)$, we have

$$H_2^1(H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* L_0 M)) \ \cong\ H_2^1\left(H^0(\mathbb{P}(E \otimes L_0^{-1}, \mathcal{O}_{\mathbb{P}(E \otimes L_0^{-1})}(1) \otimes \pi^* M))\right).$$

By Theorem 7.4, the right-hand side is empty for all $M \in \mathrm{Pic}^0(C)$ if and only if

$$s_1(E \otimes L_0^{-1}) \ > \ d - r\ell + 2rg.$$

As $s_1(E \otimes L_0^{-1}) = s_1(E)$, this is equivalent to $\ell > \mu(E) - \frac{s_1(E)}{r} + 2g$, as desired. □

Theorem 7.7 is a statement concerning $E \otimes L$ for all line bundles L of a given degree. We make one more observation showing that very ampleness may also depend on $L \in \mathrm{Pic}^\ell(C)$. Fix such an L, and write $h := h^0(K_C L^{-1} \otimes E)$. We consider the twisted Brill–Noether locus

$$B_{1,2}^{h+1}(K_C L^{-1} \otimes E) \ = \ \{N \in \mathrm{Pic}^2(C) : h^0(K_C L^{-1} N \otimes E) \geq h+1\}.$$

Also on $\mathrm{Pic}^2(C)$ we have the standard Brill–Noether locus

$$B_{1,2}^1 \ = \ \{\mathcal{O}_C(x+y) : x, y \in C\} \ \subseteq \ \mathrm{Pic}^2(C)$$

parametrising effective line bundles of degree two over C.

Proposition 7.8. *Let E be a bundle of rank r and degree d. For $L \in \mathrm{Pic}^\ell(C)$, the line bundle $\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* L$ is very ample if and only if*

$$B_{1,2}^{h+1}(K_C L^{-1} \otimes E) \cap B_{1,2}^1 \ = \ \emptyset.$$

Proof. By [**Hit20**, Proposition 4.1] (substituting $K_C L^{-1} \otimes E$ for "V"), the line bundle $\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* L$ is very ample on $\mathbb{P}E$ if and only if

$$h^0(C, K_C L^{-1} \otimes E(x+y)) \ = \ h^0(C, K_C L^{-1} \otimes E) \text{ for all } x+y \in C_2. \tag{7.2}$$

This is equivalent to saying that $h^0(C, K_C L^{-1} \otimes E(x+y)) = h$ for all $x + y \in C_2$; in other words, that no line bundle of the form $\mathcal{O}_C(x+y)$ belongs to the locus $B_{1,2}^{h+1}(K_C L^{-1} \otimes E)$. □

7.4. A criterion for semistability. Here we use Theorem 7.4 to give a criterion for semistability. This is entirely analogous to [**Hit19**, Theorem 5.7]; as the proof is practically identical, we only sketch it. For a given $E \to C$, for $t \geq 1$ we write $\pi_t \colon \mathbb{P}(\wedge^t E) \to C$ for the projection.

Theorem 7.9. *Let E be a vector bundle of rank r with $\mu(E) < 1 - 2g$. Then the following are equivalent.*

(1) *The bundle E is semistable.*
(2) *For $1 \leq t \leq r-1$, for all $M \in \mathrm{Pic}^0(C)$, and for $1 \leq e < t \cdot \mu(E^*) - (2g-2)$, the secant locus*
$$H_e^{e-1}\left(H^0(\mathbb{P}(\wedge^t E), \mathcal{O}_{\mathbb{P}(\wedge^t E)}(1) \otimes \pi_t^* M\right)_{\mathrm{nd}}$$
is empty.
(3) *For $1 \leq t \leq r-1$, and for M and e as in (2), the secant locus*
$$Q_e^{e-1}\left(\wedge^t E, M, H^0(C, \wedge^t E \otimes \pi_t^* M)\right)$$
is empty; that is, $h^0(C, \mathcal{F}_n) = h^0(C, M \otimes \wedge^n E^*) - e$ for all elementary transformations $0 \to \mathcal{F}_n \to \wedge^n E^* \to T \to 0$ where T is torsion of length e.

Proof. The equivalence of (2) and (3) is proven as in Theorem 7.4. If E is semistable then, since $\mathbb{C}$ has characteristic zero, also $\wedge^t E$ is semistable for $1 \leq t \leq r-1$. In particular, $s_1(\wedge^t E) \geq 0$. Theorem 7.4 then implies (3).

Conversely, assume (3). Then for each $t \in \{1, \ldots, r-1\}$, by Theorem 7.4 we have

$$s_1(\wedge^t E) \ > \ \deg(\wedge^t E) + \mathrm{rk}(\wedge^t E) \cdot (2g - 2 - e_t)$$

where $e_t \ = \ \max\{e : 1 \leq e < t \cdot \mu(E^*) - (2g-1)\}$. By definition of s_1, the left and right sides of the last inequality are congruent modulo $\mathrm{rk}(\wedge^t E)$. Therefore

$$s_1(\wedge^t E) \ \geq \ \deg(\wedge^t E) + \mathrm{rk}(\wedge^t E) \cdot (2g - 2 + e_t + 1).$$

By definition of e_t, this becomes $s_1(\wedge^t E) \geq 0$. As in the proof of [**Hit19**, Theorem 5.7], we conclude that $\mu(F) \leq \mu(E)$ for all rank t subbundles $F \subset E$. □

8. Secant loci of general scrolls for $f = 1$

In this final section we study questions of expected dimension, nonemptiness and enumeration for the secant loci $Q_e^{e-1}(E, M, V) =: Q_e^{e-1}(V)$ when the parameters E, M and V are chosen generally. The statements are valid for any curve C. We obtain also the emptiness of certain $H_e^{e-1}(\mathcal{L}_M, V)_{\mathrm{nd}}$; however, in the lack of a complete description of the fibres of $\alpha\colon \mathrm{Hilb}^e(\mathbb{P}E) \dashrightarrow \mathrm{Quot}^{0,e}(E^*)$, we do not prove further dimension bounds on $H_e^{e-1}(\mathcal{L}_M, V)$ at present.

8.1. Expected dimension of $Q_e^{e-1}(V)$. Here we show that if the parameters are chosen generally, $\left(H_e^{e-1}(V)\right)_{\mathrm{nd}}$ and $Q_e^{e-1}(V)$ are empty if the expected dimension is zero, and $Q_e^{e-1}(V)$, if nonempty, is of the expected dimension otherwise. This generalises [**Hit19**, Theorem 6.4], and uses a similar method. We begin by treating the complete case. Fix $r \geq 2$ and $d < r(1-g) - 1$, so that $n := -d - r(g-1) - 1 \geq 0$. Recall that for $f = 1$, the expected dimension of H_e^{e-1} and Q_e^{e-1} is $re - (n + 1 - e + 1) = (r+1)e - n - 2$.

Theorem 8.1. *Let $E \to C$ be a general vector bundle of rank r and degree d. Then there is a nonempty open subset $U \in \mathrm{Pic}^0(C)$ such that for $M \in U$, the following hold.*

(a) *$h^0(C, E^* \otimes M) = n + 1$, so $|\mathcal{L}_M|$ has dimension n.*
(b) *If $(r+1)e - n - 2 < 0$, then Q_e^{e-1} and $\left(H_e^{e-1}\right)_{\mathrm{nd}}$ are empty.*
(c) *If $(r+1)e - n - 2 \geq 0$, then Q_e^{e-1} is empty or of the expected dimension.*

Proof. Part (a) is exactly [**Hit19**, Theorem 6.4 (a)]. For (b): We recall the construction of S_M^e as a fibre product in (6.1), and the forgetful map

$$b\colon Q_{1,-e}(K_C M^{-1} \otimes E) \ \to \ \mathrm{Pic}^{-e}(C).$$

Unwinding definitions, we see for any $M \in \mathrm{Pic}^0(C)$ that

$$M^{-1} \cdot b\left(Q_{1,-e}(K_C \otimes E)\right) \ = \ b\left(Q_{1,-e}(K_C M^{-1} \otimes E)\right),$$

where the action on the left hand side is that of $\mathrm{Pic}^0(C)$ on itself by translation. Thus, by Theorem 2.17, for general $M \in \mathrm{Pic}^0(C)$ the space S_M^e is empty or of dimension

$$\dim Q_{1,-e}(K_C M^{-1} \otimes E) + \dim C_e - \dim \mathrm{Pic}^{-e}(C)$$

when this is nonnegative. By Lemma 7.3 and the assumption of generality of E, this number is

$$re + d + (r+1)(g-1) + e - g \; = \; (r+1)e - n - 2,$$

the last equality by the definition of n above. This is exactly the expected dimension of Q_e^{e-1} and H_e^{e-1}.

Now by Proposition 6.1 (b), the locus Q_e^{e-1} is nonempty only if $S_M^{e'}$ is nonempty for some $e' \leq e$. The above computation shows that $\dim S_M^{e'} < \dim S_M^e$ for $e' < e$. Therefore, if $(r+1)e - n - 2 < 0$ then Q_e^{e-1} is empty. As $\alpha \left(H_e^{e-1}\right)_{\mathrm{nd}} = Q_e^{e-1}$ by Proposition 2.12, also $\left(H_e^{e-1}\right)_{\mathrm{nd}}$ is empty. This proves (b).

For the rest: If $[F^* \to E^*]$ is a point of Q_e^{e-1} then, by Proposition 6.1 (b), we have $F^* \subseteq G^*$, where $[G^* \to E^*]$ belongs to the image of

$$\gamma_M^{e'} \colon S_M^{e'} \dashrightarrow \mathrm{Quot}^{0,e'}(E^*),$$

for some $e' \leq e$. Thus $\dim Q_e^{e-1}$ is at most

$$\dim S_M^{e'} + \dim \mathrm{Quot}^{e-e'}(G^*) \; = \; (r+1)e' - n - 2 + r(e-e') \; = \; (r+1)e - n - 2 - (e-e').$$

As $e \geq e'$, part (c) follows. (Note that we have equality for $e' = e$; compare with Proposition 2.16 (b).) □

Using Corollary 2.21, we can extend Theorem 8.1 to the case of incomplete linear series.

THEOREM 8.2. *Let E and M be general in the sense of Theorem 8.1. Let $W \subseteq H^0(C, E^* \otimes M)$ be a general subspace of dimension $m+1 < n+1$.*

(a) *If $(r+1)e - m - 2 < 0$, then $Q_e^{e-1}(W)$ is empty.*
(b) *If $(r+1)e - m - 2 < 0$, then $\left(H_e^{e-1}(W)\right)_{\mathrm{nd}}$ is empty.*
(c) *If $(r+1)e - m - 2 \geq 0$, then $Q_e^{e-1}(W)$ is empty or of the expected dimension.*

PROOF. For (a) and (c), we will apply Corollary 2.21. As $f = 1$, we have $0 \leq f' \leq 1$, so $f - f' \leq 1 \leq n - m$. Thus, by Remark 2.19, to satisfy the hypothesis of Corollary 2.21 it will suffice to show that $Q_e^{e-1}(H^0(E^* \otimes M))$ is empty or has the expected dimension (trivially, Q_e^e is always of the expected dimension). This follows from Theorem 8.1. Then (b) follows as before from Propositions 2.9 (b) and 2.12. □

8.2. Nonemptiness of $Q_e^{e-1}(V)$. Theorems 8.1 and 8.2 show that $Q_e^{e-1}(V)$, if nonempty, is of the expected dimension for a general choice of the parameters. We now describe two situations in which $Q_e^{e-1}(V)$ is in fact nonempty.

PROPOSITION 8.3. *Let $E \to C$ be a bundle which is general in moduli. Suppose $e \geq \max\{g, -d/r - (g-1)(r+1)/r\}$. Then $Q_e^{e-1}(V)$ is nonempty for generic $M \in \mathrm{Pic}^0(C)$.*

PROOF. As $e \geq -d/r - (g-1)(r+1)/r$, we have $re + d + (r+1)(g-1) \geq 0$. Since E is general, by Lemma 7.3 the scheme $Q_{1,-e}(K_C M^{-1} \otimes E)$ is nonempty and a general element is of the form $[L \xrightarrow{\sigma} K_C M^{-1} \otimes E]$ where σ is a vector bundle injection. But since also $e \geq g$, for any L so occurring we have $L = \mathcal{O}_C(-D)$ for some $D \in C_e$. Therefore, (σ, D) is a point of S_M^e where γ_M^e is defined. Hence Q_e^{e-1} is nonempty by Proposition 6.1 (a). □

Proposition 8.4. *Let E be any bundle, and suppose that $Q_{1,-e}(K_C \otimes E)$ is nonempty for some $e \geq 1$. Then $Q_e^{e-1}(E, M, H^0(E^* \otimes M))$ is nonempty for some $M \in \mathrm{Pic}^0(C)$.*

Proof. Suppose $[L \xrightarrow{\sigma} K_C \otimes E]$ is a point of $Q_{1,-e}(K_C \otimes E)$. Now σ may not be a bundle injection at all points, but we can choose a sufficiently general $D \in C_e$ along which σ is a bundle injection. Then $L^{-1}(-D)$ has degree zero, and

$$\left(\left[\mathcal{O}_C(-D) \xrightarrow{\sigma \otimes L^{-1}(-D)} K_C L^{-1}(-D)) \otimes E \right], D \right)$$

is a point of $S^e_{L^{-1}(-D)}$ at which $\gamma^e_{L^{-1}(-D)}$ is defined. Then

$$Q_e^{e-1}(E, L^{-1}(-D), H^0(C, E^* \otimes L^{-1}(-D)))$$

is nonempty by Proposition 6.1 (a). □

Remark 8.5. Proposition 8.4 applies whenever $Q_{1,-e}(K_C \otimes E)$ has nonnegative expected dimension. If E is generic, then this follows from Lemma 7.3. For arbitrary E, one can find possibly nonsaturated elements of $Q_{1,-e}(K_C M^{-1} \otimes E)$ by taking subsheaves of elements of $Q_{1,h}(K_C \otimes E)$ for $h > -e$.

8.3. Enumeration of $Q_e^{e-1}(V)$. By the previous two subsections, one knows that $Q_e^{e-1}(E, M, V) = Q_e^{e-1}(V)$, if nonempty, is of the expected dimension for any C and for a general choice of (E, M, V); and furthermore that $Q_e^{e-1}(V)$ is indeed nonempty under certain conditions. We now use results of [**EGL01**], [**OP21**] and [**Sta21b**] to enumerate $Q_e^{e-1}(V)$ when it has and attains expected dimension zero.

Following the notation of [**OP21**] and [**Sta21b**], we denote by $M^{[e]}$ the vector bundle over $\mathrm{Quot}^{0,e}(E^*)$ with fibre $H^0(C, (E^*/F^*) \otimes M)$ at the point $[F^* \to E^*]$. For $V \subseteq H^0(E^* \otimes M)$, let $\varepsilon \colon V \otimes \mathcal{O}_{\mathrm{Quot}^{0,e}(E^*)} \to M^{[e]}$ be the evaluation map. Then $Q_e^{e-1}(V)$ is the determinantal locus $\{F^* \in \mathrm{Quot}^{0,e}(E^*) : \mathrm{rk}\, \varepsilon|_{F^*} \leq e-1\}$. When $re = n + 2 - e$, so that $Q_e^{e-1}(V)$ has expected dimension zero, using the Porteous formula we obtain

$$[Q_e^{e-1}(V)] \;=\; s_{n+2-e}\left(M^{[e]}\right) \;=\; \int_{[\mathrm{Quot}^{0,e}(E^*)]} s(M^{[e]}). \tag{8.1}$$

The main goal of this subsection will be to sketch the proof of the following.

Theorem 8.6. *Let C be a smooth curve, $E \to C$ a vector bundle of rank r and $M \to C$ a line bundle. Then there is an equality of formal power series*

$$\sum_{e \geq 0} \int_{[\mathrm{Quot}^{0,e}(E^*)]} s(M^{[e]})\, q^e \;=\; A_1(q)^{\deg(E^* \otimes M)} \cdot B(q)^{1-g} \tag{8.2}$$

where

$$q = (-1)^r t(1+t)^r, \quad A_1(q) = 1 + t \quad \text{and} \quad B(q) = \frac{(1+t)^{r+1}}{1 + t(r+1)}.$$

Sketch of proof. All the ingredients for proving this theorem are present in [**EGL01**], [**OP21**] and [**Sta21b**], so we give only an outline. The formula (8.2) is proven in [**OP21**, Theorem 6 and Theorem 8] when E is the trivial bundle of rank $r \geq 1$. The generalisation of the corresponding formula [**OP21**, Corollary 16] for surfaces to arbitrary E is [**Sta21b**, Proposition 4.3], and we use the same strategy.

To show the existence of the power series A_1 and B, we follow essentially word for word the proof of [**EGL01**, Theorem 4.2], replacing $\mathcal{K}_r$ with the set

$\mathcal{T}_r := \{(C, E, M) : C$ a smooth curve, $E \to C$ a bundle of rank r and $M \in \mathrm{Pic}(C)\}$,

and $H_{\Psi,\Phi}(S, x)$ with the map $\mathcal{T}_r \to \mathbb{Q}[[q]]$ given by

$$(C, E, M) \mapsto \sum_{e \geq 0} q^e \int_{[\mathrm{Quot}_C^{0,e}(E^*)]} s(M^{[e]});$$

and γ with the map $(C, E, M) \mapsto (\deg(E^* \otimes M), \chi(\mathcal{O}_C))$. The role of [**EGL01**, Theorem 4.1] is played in our situation by the following statement which emerges in the proof of [**Sta21b**, Theorem 3.3]:

Proposition 8.7. *For each $e \geq 0$, there exists a polynomial $u_e(x, y)$ which is universal in the sense that for any $(C, E, M) \in \mathcal{T}_r$ we have*

$$\int_{[\mathrm{Quot}^{0,e}(E^*)]} s(M^{[e]}) = u_e\left(\deg(E^* \otimes M), \chi(\mathcal{O}_C)\right).$$

Once the existence of A_1 and B is established, they can be computed by comparing with [**OP21**, Theorem 6 and Theorem 8] for suitable triples (C, E, M) with E trivial and C rational or elliptic. □

The following enumeration result is immediate from Theorem 8.6 and (8.1).

Corollary 8.8. *Suppose E, M and $V \subseteq H^0(E^* \otimes M)$ are such that $Q_e^{e-1}(V)$ has and attains expected dimension zero. Then the number of points of $Q_e^{e-1}(V)$, counted with multiplicity, is the coefficient of q^e in the expression (8.2).*

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