ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus

Mathématique

Insong Choe, George H. Hitching and Jaehyun Hong

Simplicity of Tangent bundles on the moduli spaces of symplectic and orthogonal bundles over a curve

Volume 362 (2024), p. 493-510

https://doi.org/10.5802/crmath.560

CENTRE
MERSENNE

The Comptes Rendus. Mathématique *are a member of the*
Mersenne Center for open scientific publishing
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus. Mathématique
2024, Vol. 362, p. 493-510
https://doi.org/10.5802/crmath.560

Research article / *Article de recherche*
Algebraic geometry / *Géométrie algébrique*

Simplicity of Tangent bundles on the moduli spaces of symplectic and orthogonal bundles over a curve

Simplicité des fibrés tangents des espaces de modules des fibrés symplectiques et orthogonaux sur une courbe

Insong Choe [*, a], **George H. Hitching** [b] **and Jaehyun Hong** [c]

[a] Department of Mathematics, Konkuk University, 1 Hwayang-dong, Gwangjin-Gu, Seoul 143-701, Republic of Korea

[b] Oslo Metropolitan University, Postboks 4, St. Olavs plass, 0130 Oslo, Norway

[c] Center for Complex Geometry, Institute for Basic Science (IBS), Daejeon 34126, Republic of Korea

E-mails: ischoe@konkuk.ac.kr (I. Choe), gehahi@oslomet.no (G. H. Hitching), jhhong00@ibs.re.kr (J. Hong)

Abstract. The variety of minimal rational tangents associated to Hecke curves was used by J.-M. Hwang [8] to prove the simplicity of the tangent bundle on the moduli of vector bundles over a curve. In this paper, we use the tangent maps of the symplectic and orthogonal Hecke curves to prove an analogous result for symplectic and orthogonal bundles. In particular, we show the nondegeneracy of the associated variety of minimal rational tangents, which implies the simplicity of the tangent bundle on the moduli spaces of symplectic and orthogonal bundles over a curve. We also show that for large enough genus, the tangent map is an embedding for a general symplectic or orthogonal bundle.

Résumé. La variété des tangentes des courbes minimales rationnelles associés aux courbes de Hecke, a été utilisée par J.-M. Hwang [8] pour prouver la simplicité du fibré tangent à l'espace de modules des fibrés vectoriels sur une courbe. Nous utilisons les applications tangentes des courbes de Hecke symplectiques et orthogonales pour démontrer un résultat analogue pour les fibrés symplectiques et orthogonaux. En particulier, nous prouvons que la variété des tangentes aux courbes rationnelles minimales associée est non dégénérée ; ce qui implique la simplicité des fibrés tangents aux espaces de modules des fibrés symplectiques et orthogonaux sur une courbe. Nous montrons d'ailleurs, pour genre suffisamment grand, que l'application tangente est un plongement pour un fibré symplectique ou orthogonal générique.

Keywords. symplectic bundle, orthogonal bundle, minimal rational tangents.

2020 Mathematics Subject Classification. 14D20, 53C10.

Funding. The first named author was supported by the National Research Foundation of Korea: NRF-2020R1F1A1A01068699. The third named author was supported by the Institute for Basic Science (IBS-R032-D1).

Manuscript received 4 November 2022, revised 7 August 2023, accepted 1 September 2023.

* Corresponding author

1. Introduction

Let C be a smooth projective curve of genus $g \geq 2$ over the complex numbers. Let $\mathscr{M} := \mathscr{SU}_C(n,d)$ be the moduli space of semistable vector bundles over C of rank n with fixed determinant of degree d. Note that $\mathscr{M}$ is a Fano variety of Picard number 1 and moreover smooth if n and d are coprime.

It was shown in [8, Corollary 1] that for $g \geq 4$, the tangent bundle of the smooth part $\mathscr{M}^\circ \subset \mathscr{M}$ is simple. The strategy was to exploit certain minimal rational curves called Hecke curves and the associated variety of minimal rational tangents. More precisely, it is shown that the variety of minimal rational tangents $\mathscr{C}_W$ at a generic point $W \in \mathscr{M}$ is non-degenerate in $\mathbb{P}(T_W\mathscr{M})$ and this implies the simplicity of the tangent bundle (cf. Proposition 6 below).

The goal of this paper is to prove the analogous result for the moduli spaces $\mathscr{M}S_C(n,L)$ of symplectic bundles and $\mathscr{M}O_C(n,L)$ of orthogonal bundles. The symplectic and orthogonal versions of Hecke curves were constructed in [3]. In the same paper, these curves were shown to be the minimal rational curves in the ambient varieties. Based on this, we establish the following results in this paper:

- The smoothness of the symplectic and orthogonal Hecke curves (§ 3),
- The nondegeneracy of the tangent map on the variety of minimal rational tangents of these Hecke curves (§ 4),
- The very-ampleness of the associated complete linear system (§ 5).

In particular, as a corollary of the nondegeneracy in § 4, we show that the tangent bundles of $\mathscr{M}S_C(n,L)$ and $\mathscr{M}O_C(n,L)$ are simple, under a certain genus bound (Theorem 13).

We would like to add a word of warning for the arguments that will follow. Inside the moduli space $\mathscr{SU}_C(n,d)$ of vector bundles, the locus of symplectic/orthogonal bundles form a closed subvariety, and a symplectic/orthogonal Hecke curve can be thought of as either a special kind of Hecke curve on $\mathscr{SU}_C(n,d)$ or its variation. So one might expect that the results for Hecke curves in [8, 9] or [13] directly imply the same results for the symplectic or orthogonal setting. But in most discussions of a Hecke curve on $\mathscr{SU}_C(n,d)$, one assumes that it passes through a generic point, such as a $(1,1)$-stable bundle [12, Definition 5.1]. And it is unclear if a generic point of $\mathscr{M}S_C(n,L)$ and/or $\mathscr{M}O_C(n,L)$ is $(1,1)$-stable as a vector bundle. By simple dimension comparison, it is still possible that the subvarieties $\mathscr{M}S_C(n,L)$ and/or $\mathscr{M}O_C(n,L)$ are entirely contained in the non-$(1,1)$-stable locus. For this reason, we cannot tell from the outset if the symplectic and orthogonal Hecke curves share the same properties as the Hecke curves passing through $(1,1)$-stable locus. This is why we later devise arguments based on δ-stability (cf. [3, § 4.1]) on $\mathscr{M}S_C(n,L)$ and $\mathscr{M}O_C(n,L)$.

2. Preliminary results

In this section, we gather the notations and preliminary results relevant to our discussion. Let C be a smooth projective curve of genus $g \geq 2$.

Notation 1. Given a subspace $\Lambda \subset V^*$ of a dual vector space, $\Lambda^\perp \subset V$ denotes the kernel:

$$\Lambda^\perp = \{v \in V : \lambda(v) = 0 \quad \text{for all } \lambda \in \Lambda\}.$$

Also for a subspace $U \subset V$, $U^\perp \subset V^*$ denotes the annihilator:

$$U^\perp = \left\{\phi \in V^* : \phi(u) = 0 \quad \text{for all } u \in U\right\}.$$

When V is equipped with a bilinear form $\omega\colon V \otimes V \to \mathbb{C}$, we define

$$\ker(\omega) := \left\{v_0 \in V : \omega(v_0, v) = 0 \quad \text{for all } v \in V\right\}$$

2.1. *Hecke modification*

Let W be a vector bundle over C. Choose a subspace $\Lambda \subset W_x^*$ for some $x \in C$. The *Hecke modification* W^Λ *of* W *along* Λ is given by the kernel of the composition map $W \to W_x \to W_x/\Lambda^\perp$. There is an exact sequence of sheaves:

$$0 \to W^\Lambda \xrightarrow{\phi} W \to \left(W_x/\Lambda^\perp\right) \otimes \mathscr{O}_x \to 0$$

whose restriction to the fiber at x is given by

$$0 \to \mathrm{Ker}\left(\phi_x \colon W_x^\Lambda \to W_x\right) \to W_x^\Lambda \to W_x \to W_x/\Lambda^\perp \to 0.$$

Then the locally free sheaf W^Λ corresponds to a vector bundle with $\det(W^\Lambda) = \det(W) \otimes \mathscr{O}_C(-kx)$, where k is the dimension of Λ in W_x^*.

2.2. *Hecke curves on* $\mathscr{SU}_C(n,d)$

The main reference for this subsection is [8].

Let W be a vector bundle over C. For a subspace $\theta \subset W_x^*$ of dimension one, let W^θ denote the Hecke modification of W along θ. Then we have

$$0 \to W^\theta \to W \to \left(W_x/\theta^\perp\right) \otimes \mathscr{O}_x \to 0.$$

For a subspace $\ell \subset W_x^\theta$ of dimension one, the Hecke modification V^ℓ of $V := (W^\theta)^*$ along ℓ fits into the exact sequence

$$0 \to V^\ell \xrightarrow{\beta} V \to \left(V_x/\ell^\perp\right) \otimes \mathscr{O}_x \to 0. \tag{1}$$

In particular for $\ell_0 := \mathrm{Ker}(W_x^\theta \to W_x)$, we have $(V^{\ell_0})^* \simeq W$.

For any two-dimensional subspace U with $\ell_0 \subset U \subset W_x^\theta$, the subspace $U^\perp$ has codimension two in V_x, and is contained in $\ell_0^\perp$. Hence the family

$$\left\{\left(V^\ell\right)^* : [\ell] \in \mathbb{P}(U)\right\}$$

parameterized by $\mathbb{P}(U) = \mathbb{P}(V_x/U^\perp) \cong \mathbb{P}^1$, is a deformation of W. If we choose a generic $W \in \mathscr{SU}_C(n,d)$, then this family gives a smooth rational curve through W called a *Hecke curve*. It was shown in [13] that the Hecke curves have minimal degree among the rational curves passing through a generic $W \in \mathscr{SU}_C(n,d)$.

Note that the parameter space of Hecke curves passing through $W \in \mathscr{SU}_C(n,d)$ is given by a double fibration

$$\mathbb{P}\left(T_\pi^*\right) \to \mathbb{P}\left(W^*\right) \xrightarrow{\pi} C,$$

where T_π is the vertical tangent bundle of $\pi \colon \mathbb{P}(W^*) \to C$. In the previous notation, this corresponds to the composition map

$$\left(U/\ell_0 \subset W_x^\theta/\ell_0\right) \mapsto \left(\theta \subset W_x^*\right) \mapsto x.$$

In particular for $n = 2$, this double fibration boils down to the ruled surface $\mathbb{P}(W^*)$.

2.3. *Kodaira–Spencer map*

The main references of this subsection are [11] and [12]. Consider the above family $\{V^\ell : [\ell] \in \mathbb{P}(U)\}$ as a deformation of $V^{\ell_0} \cong W^*$. Since the map $\beta_x \colon V_x^{\ell_0} \to V_x$ surjects onto $\ell_0^\perp$, we have the induced pull-back map

$$\widehat{\beta_x} \colon \mathrm{Hom}\left(\ell_0^\perp/U^\perp, V'\right) \to \mathrm{Hom}\left(V_x^{\ell_0}, V'\right)$$

for any vector space V'.

Proposition 2. *The Kodaira–Spencer map of the family* $\{V^{\ell} : [\ell] \in \mathbb{P}(U)\}$ *is given by*

$$\mathrm{Hom}\left(\ell_0^{\perp}/U^{\perp}, V_x/\ell_0^{\perp}\right) \xrightarrow{\widehat{\beta_x}} \mathrm{Hom}\left(V_x^{\ell_0}, V_x/\ell_0^{\perp}\right) \xrightarrow{\delta} H^1\left(C, \mathrm{End}\left(V^{\ell_0}\right)\right),$$

where δ *is induced from the sequence* (1).

Later we will also need to consider a slight generalization of the above family, where the Hecke modification is taken for subspaces of dimension two. For a subspace $\Theta \subset W_x^*$ of dimension 2, the Hecke modification W^{Θ} of W along Θ can be put into the following exact sequence:

$$0 \to W^{\Theta} \to W \to \left(W_x/\Theta^{\perp}\right) \otimes \mathcal{O}_x \to 0.$$

Let $\wp_0$ denote the kernel of $W_x^{\Theta} \to W_x$. The following is a straightforward generalization of Proposition 2.

Proposition 3. *Let* U *be a subspace of dimension 4 with* $\wp_0 \subset U \subset W_x^{\Theta}$*; that is,* $U^{\perp} \subset \wp_0^{\perp} \subset (W_x^{\Theta})^* =: V_x$*. Then the family* $\{V^{\wp} : [\wp] \in Gr(2, V_x/U^{\perp})\}$ *is a deformation of* $W^* = V^{\wp_0}$*, and its Kodaira–Spencer map*

$$T_{\wp_0}\left(Gr\left(2, V_x/U^{\perp}\right)\right) = \mathrm{Hom}\left(\wp_0^{\perp}/U^{\perp}, V_x/\wp_0^{\perp}\right) \to H^1\left(C, \mathrm{End}\left(V^{\wp_0}\right)\right)$$

is given by the composition

$$\mathrm{Hom}\left(\wp_0^{\perp}/U^{\perp}, V_x/\wp_0^{\perp}\right) \xrightarrow{\widehat{\beta_x}} \mathrm{Hom}\left(V_x^{\wp_0}, V_x/\wp_0^{\perp}\right) \xrightarrow{\delta} H^1\left(C, \mathrm{End}\left(V^{\wp_0}\right)\right),$$

where $\widehat{\beta_x}$ *and* δ *are induced from*

$$0 \to V^{\wp_0} \xrightarrow{\beta} V \to \left(V_x/\wp_0^{\perp}\right) \otimes \mathcal{O}_x \to 0.$$

2.4. *Symplectic Hecke curves on* $\mathcal{MS}_C(n, L)$

In this subsection, we recall the construction of symplectic Hecke curves, following [3], to which we refer the reader for the details.

For a line bundle L on C, an L-valued symplectic bundle of rank n is a vector bundle W of (even) rank n equipped with an L-valued symplectic form $\omega\colon W \otimes W \to L$. Let $\mathcal{MS}_C(n, L)$ be the moduli space of L-valued symplectic bundles of rank n. By the morphism forgetting the symplectic forms, the moduli space $\mathcal{MS}_C(n, L)$ can be thought of as a subvariety of $\mathcal{SU}_C(n, \frac{1}{2}n\ell)$, where $\ell = \deg(L)$.

The construction of symplectic Hecke curves on $\mathcal{MS}_C(n, L)$ closely follows the previous construction of Hecke curves on $\mathcal{SU}_C(n, d)$, keeping track of the deformation of symplectic forms. For a subspace $\theta \subset W_x^*$ of dimension one, let W^{θ} be the Hecke modification of W along θ, fitting into the sequence

$$0 \to W^{\theta} \to W \to \left(W_x/\theta^{\perp}\right) \otimes \mathcal{O}_x \to 0.$$

Noting that every 1-dimensional subspace θ is isotropic, we get an induced $L^*(x)$-valued skew-symmetric form on $V := (W^{\theta})^*$:

$$\omega^{\theta}\colon \left(W^{\theta}\right)^* \to W^{\theta} \otimes L^*(x).$$

Then $\ker \omega_x^{\theta}$ has codimension two in $V_x = (W^{\theta})_x^*$.

For a subspace $\ell \subset W_x^{\theta}$ of dimension one, we have the sequence

$$0 \to V^{\ell} \to V \to \left(V_x/\ell^{\perp}\right) \otimes \mathcal{O}_x \to 0.$$

Then the bundle V^{ℓ} has a skew-symmetric form induced from ω^{θ}, and it is an L^*-valued nondegenerate (symplectic) form if and only if $\ker \omega_x^{\theta} \subset \ell^{\perp}$. Now the family

$$\left\{V^{\ell} : \ker \omega_x^{\theta} \subset \ell^{\perp} \subset V_x\right\}$$

of L-valued symplectic bundles are parameterized by $\mathbb{P}(V_x/\ker\omega_x^\theta) \simeq \mathbb{P}^1$. In particular, if ℓ_0 is in the kernel of $W_x^\theta \to W_x$, then $V^{\ell_0} \cong W^*$.

Under the assumption that $g \geq 3$ and $W \in \mathscr{M}S_C(n,L)$ is a generic point, by [3, Lemma 4.5] the dual family

$$\left\{\left(V^\ell\right)^* : \ell \in \mathbb{P}\left(V_x/\ker\omega_x^\theta\right)\right\}$$

gives a rational curve on $\mathscr{M}S_C(n,L)$ passing through W, called a *symplectic Hecke curve*. Also it was shown in [3, Theorem 5.2] that these curves have minimal degree among the rational curves passing through a generic point $W \in \mathscr{M}S_C(n,L)$.

Later we need the following fact.

Proposition 4. *Assume $g \geq 3$. For a generic point $W \in \mathscr{M}S_C(n,L)$, every symplectic Hecke curve passing through W is contained in the smooth locus of $\mathscr{M}S_C(n,L)$.*

Proof. By [3, Lemma 4.5], any symplectic Hecke curve passing through a generic point W stays inside the locus of stable symplectic bundles. To see that it is contained in the smooth locus, it suffices to show that it does not touch the locus of non-regularly stable symplectic bundles which are of the form $W_1 \perp W_2$ for some stable symplectic subbundles W_1 and W_2.

This can be checked by dimension count: The locus of non-regularly stable bundles is contained in a finite union of the images of $\mathscr{M}S_C(n_1,L) \times \mathscr{M}S_C(n_2,L)$, where $n_1 + n_2 = n$. Since symplectic Hecke curves passing through a point W_0 in this subvariety are parameterized by $\mathbb{P}(W_0^*)$, it suffices to check the inequality:

$$\dim\mathbb{P}\left(W_0^*\right) + \dim\mathscr{M}S_C(n_1,L) + \dim\mathscr{M}S_C(n_2,L) \;<\; \dim\mathscr{M}S_C(n,L)$$

for any even integers n_1, n_2 with $n_1 + n_2 = n$. This boils down to $n < n_1 n_2(g-1)$, which holds for $g \geq 3$. □

2.5. *Orthogonal Hecke curves on $\mathscr{M}O_C(n,L)$*

Again, the main reference in this subsection for a construction of orthogonal Hecke curves will be [3].

For a line bundle L on C, an L-valued orthogonal bundle of rank n is a vector bundle W of rank n equipped with an L-valued orthogonal form $b\colon W \otimes W \to L$. Let $\mathscr{M}O_C(n,L)$ be the moduli space of L-valued orthogonal bundles of rank n. The moduli space $\mathscr{M}O_C(2r,L)$ has several irreducible components, due to the invariants $\det(W)$ and the 2nd Stiefel–Whitney class $w_2(W)$ (see [2, § 2]). By the morphism forgetting the orthogonal forms, each irreducible component of the moduli space $\mathscr{M}O_C(n,L)$ is sent to a subvariety of $\mathscr{SU}_C(n,\frac{1}{2}n\ell)$, where $\ell = \deg(L)$.

As in [3], we assume $n \geq 5$ throughout this paper. The reason behind this convention is that the moduli space $\mathscr{M}O_C(n,L)$ has Picard number one for $n \geq 5$, while $\mathscr{M}O_C(4,L)$ has Picard number two. Accordingly, the minimality of the orthogonal Hecke curves was discussed in [3] for $n \geq 5$. We remark that there is a standard construction of orthogonal bundles of low rank from vector bundles, described in [5].

The construction of orthogonal Hecke curves on $\mathscr{M}S_C(n,L)$ is a little bit different from that of Hecke curves on $\mathscr{SU}_C(n,d)$: the dimension of the involved subspaces are doubled.

For an isotropic subspace $\Theta \subset W_x^*$ of dimension two, let W^Θ be the Hecke modification:

$$0 \to W^\Theta \to W \to \left(W_x/\Theta^\perp\right) \otimes \mathscr{O}_x \to 0.$$

Then there is an $L^*(x)$-valued symmetric form

$$b^\Theta\colon \left(W^\Theta\right)^* \to W^\Theta \otimes L^*(x)$$

on $V := (W^\Theta)^*$ such that $\ker b_x^\Theta$ has codimension four in $V_x = (W^\Theta)_x^*$.

For an isotropic subspace $\wp \subset W_x^\Theta$ of dimension two with $\wp^\perp \subset V_x$, we have the Hecke modification

$$0 \to V^\wp \to V \to \left(V_x/\wp^\perp\right) \otimes \mathscr{O}_x \to 0.$$

Then $V^\wp$ is equipped with a symmetric form induced from b^Θ, and it is an L^*-valued nondegenerate (orthogonal) form if and only if $\ker b_x^\Theta \subset \wp^\perp$. In particular when $\wp_0$ is in the kernel of $W_x^\Theta \to W_x$, we have $V^{\wp_0} \cong W^*$.

Note that the space of two-dimensional isotropic subspaces $\wp \subset W_x^\Theta$ such that

$$\ker b_x^\Theta \subset \wp^\perp \subset V_x = \left(W^\theta\right)_x^*$$

is the isotropic Grassmannian of 2-dimensional subspaces of $V_x/\ker b_x^\Theta \cong \mathbb{C}^4$, which is a disjoint union of two projective lines. Let $\mathrm{IG}(2, V_x/\ker b_x^\Theta)$ be the line containing the point $\wp_0^\perp/\ker b_x^\Theta$. Then the family

$$\left\{V^\wp : \wp \in \mathrm{IG}\left(2, V_x/\ker b_x^\Theta\right)\right\}$$

of L^*-valued orthogonal bundles gives a deformation of W^*.

Under the assumption that $g \geq 5, n \geq 5$ and that $W \in \mathscr{MO}_C(n,L)$ is a generic point, by [3, Lemma 4.7] the dual family $\{(V^\wp)^*\}$ gives a rational curve on $\mathscr{MO}_C(n,L)$ passing through W, called an *orthogonal Hecke curve*. Also it was shown in [3, Theorem 5.3] that these curves have minimal degree among the rational curves passing through a generic point $W \in \mathscr{MO}_C(n,L)$.

Again, we show the following.

Proposition 5. *Assume $g \geq 5$ and $n \geq 5$. For a generic point $W \in \mathscr{MO}_C(n,L)$, every orthogonal Hecke curve passing through W is contained in the smooth locus of $\mathscr{MO}_C(n,L)$.*

Proof. By [3, Lemma 4.7], any orthogonal Hecke curve passing through a generic point W stays inside the locus of stable orthogonal bundles. To see that it is contained in the smooth locus, it suffices to show that it does not touch the locus of non-regularly stable orthogonal bundles which are of the form $W_1 \perp W_2$ for some stable orthogonal subbundles W_1 and W_2.

This can be checked by dimension count: The locus of non-regularly stable bundles is contained in a finite union of the images of $\mathscr{MO}_C(n_1,L) \times \mathscr{MO}_C(n_2,L)$, where $n_1 + n_2 = n$. Since orthogonal Hecke curves passing through a point $[W_0]$ in this subvariety are parameterized by $\mathrm{IG}(2, W_0^*)$, it suffices to check the inequality:

$$\dim \mathrm{IG}\left(2, W_0^*\right) + \dim \mathscr{MO}_C(n_1,L) + \dim \mathscr{MO}_C(n_2,L) < \dim \mathscr{MO}_C(n,L)$$

for any integers n_1, n_2 with $n_1 + n_2 = n$. Since $\dim IG(2, W_0^*) = 2n-6$, this boils down to $2n-6 < n_1 n_2 (g-1)$, which holds for $g, n \geq 5$. □

2.6. *Minimal rational curves*

Let M be a projective variety. Let $\mathscr{K}$ be an irreducible component of the Hilbert scheme of complete curves on M such that generic members of $\mathscr{K}$ cover an open subset of the smooth locus M° of M. For a generic point $x \in M$, denote by $\mathscr{K}_x$ the subscheme of $\mathscr{K}$ consisting of members of $\mathscr{K}$ passing through x. Assume that for a generic point $x \in M$, every member of $\mathscr{K}_x$ is an irreducible smooth rational curve contained in M° and $\mathscr{K}_x$ is an irreducible complete variety. In this case, we call $\mathscr{K}$ a *minimal rational component* of M.

A covering family of rational curves having minimal degree gives a minimal rational component. More precisely, an irreducible component $\mathscr{K}$ is a minimal rational component of M if it satisfies the following conditions:

(i) For a generic $x \in M$, every member of $\mathscr{K}_x$ is an irreducible smooth rational curve contained in M°.

(ii) The locus swept out by the curves in $\mathscr{K}$ is dense in M.

(iii) For a fixed ample line bundle ξ on M, the degree of members of $\mathscr{K}$ with respect to ξ is minimal among the curves in an irreducible family satisfying (i) and (ii).

Let $x \in M$ be a generic point. Define the tangent map $\tau_x \colon \mathscr{K}_x \dashrightarrow \mathbb{P}(T_x(M))$ by

$$\tau_x([R]) = [T_x R] \in \mathbb{P}(T_x(M)),$$

where R is a smooth rational curve in M° passing through x. The closure $\mathscr{C}_x$ of the image $\tau_x(\mathscr{K}_x) \subset \mathbb{P}(T_x(M))$ is called the *variety of minimal rational tangents* (VMRT for short) at x associated with $\mathscr{K}$.

The following is [8, Theorem 2], which connects the theory of VMRT and the simplicity of the tangent bundle.

Proposition 6. *Let M be a Fano variety which has a minimal rational component $\mathscr{K}$. If the VMRT $\mathscr{C}_x$ at a generic point $x \in M$ is non-degenerate in $\mathbb{P}T_x M$, then the tangent bundle $T(M^\circ)$ is simple.*

For the moduli space of vector bundles $\mathscr{M} = \mathscr{SU}_C(n,d)$, it is proven in [8] that for $g \geq 4$, the irreducible component $\mathscr{K}$ of the Hilbert scheme of $\mathscr{M}$ containing Hecke curves is a minimal rational component of $\mathscr{M}$. In this case, given a generic point $W \in \mathscr{M}$, $\mathscr{K}_W$ is given by

$$\mathscr{K}_W = \bigcup_{[\theta] \in \mathbb{P}(W^*)} \mathbb{P}\left(W_x^\theta / \ell_0\right) \simeq \mathbb{P}\left(T_\pi^*\right)$$

where T_π is the vertical tangent bundle of $\pi \colon \mathbb{P}(W^*) \to C$. In particular for $n = 2$, we have $\mathscr{K}_W = \mathbb{P}(W^*)$.

Moreover it is shown in [9, Theorem 3.1, Theorem 3.7] that the tangent map at a generic point $W \in \mathscr{M}$ is biregular to the image for $g \geq 5$ and birational for $g = 4$.

Finally we discuss the case of $\mathscr{M}S_C(n,L)$ and $\mathscr{M}O_C(n,L)$. By the result [3, Theorem 5.2] on the minimality of degree, we can see that there is a minimal rational component $\mathscr{K}$ of $\mathscr{M}S_C(n,L)$ containing symplectic Hecke curves such that $\mathscr{K}_W$ for a generic element W is given by

$$\mathscr{K}_W = \mathbb{P}\left(W^*\right).$$

Similarly by [3, Theorem 5.3], there is a minimal rational component $\mathscr{K}$ of $\mathscr{M}O_C(n,L)$ containing orthogonal Hecke curves such that $\mathscr{K}_{[W]}$ for a generic element $[W]$ is given by

$$\mathscr{K}_W = IG\left(2, W^*\right).$$

3. Smoothness of Hecke curves

In this section, we show the smoothness of the symplectic and orthogonal Hecke curves. To discuss deformations of symplectic and orthogonal bundles, we should first establish the structure of the tangent spaces of the moduli spaces $\mathscr{M}S_C(n,L)$ and $\mathscr{M}O_C(n,L)$. The tangent space of $\mathscr{SU}_C(n,d)$ at a stable bundle W is given by $H^1(C, \mathrm{End}_0(W))$, where $\mathrm{End}_0(W)$ is the vector bundle of traceless endomorphisms of W. By a similar argument as [1, Lemma 2.2], the tangent spaces of $\mathscr{M}S_C(n,L)$ and $\mathscr{M}O_C(n,L)$ at a regularly stable bundle W are given by $H^1(C, \mathrm{ad}\, W)$, where

$$\mathrm{ad}\, W = \begin{cases} \mathrm{Sym}^2 W \otimes L^* \text{ if } W \text{ is symplectic;} \\ \wedge^2 W \otimes L^* \text{ if } W \text{ is orthogonal.} \end{cases} \tag{2}$$

Proposition 7. *Assume $g \geq 4$ and $n \geq 4$. Then any symplectic Hecke curve passing through a generic point $W \in \mathscr{M}S_C(n,L)$ is smooth.*

Proof. From the construction, the family

$$\left\{V^\ell : \ker \omega_x^\theta \subset \ell^\perp \subset V_x\right\} \tag{3}$$

gives a deformation of V^ℓ along a subspace $U := (\ker\omega_x^\theta)^\perp \cong \mathbb{C}^2$. By Proposition 2.3, the Kodaira–Spencer map

$$T_\ell\left(\mathbb{P}\left(V_x/U^\perp\right)\right) = \mathrm{Hom}\left(\ell^\perp/U^\perp, V_x/\ell^\perp\right) \to H^1\left(C, \mathrm{End}\left(V^\ell\right)\right) \tag{4}$$

associated to this family is given by the composition

$$\mathrm{Hom}\left(\ell^\perp/U^\perp, V_x/\ell^\perp\right) \xrightarrow{\widehat{\beta_x}} \mathrm{Hom}\left(V_x^\ell, V_x/\ell^\perp\right) \xrightarrow{\delta} H^1\left(C, \mathrm{End}\left(V^\ell\right)\right), \tag{5}$$

where $\widehat{\beta_x}$ and δ are induced from

$$0 \to V^\ell \xrightarrow{\beta} V \to \left(V_x/\ell^\perp\right) \otimes \mathscr{O}_x \to 0. \tag{6}$$

Note that this can be geometrically understood as the composition map

$$T_\ell\left(\mathbb{P}\left(V_x/U^\perp\right)\right) \xrightarrow{\widehat{\beta_x}} T_{[\beta]}\mathrm{Quot}(V) \xrightarrow{\delta} H^1\left(C, \mathrm{ad}\left(V^\ell\right)\right) = T_{[V^\ell]}\mathscr{M}S_C\left(n, L^*\right),$$

where the tangent space $T_{[\beta]}\mathrm{Quot}(V)$ of the Quot scheme of V is given by

$$T_{[\beta]}\mathrm{Quot}(V) = H^0\left(C, \mathrm{Hom}\left(V^\ell, \left(V_x/\ell^\perp\right) \otimes \mathscr{O}_x\right)\right) \cong \mathrm{Hom}\left(V_x^\ell, V_x/\ell^\perp\right).$$

Hence to show that the map

$$\phi_U\colon \mathbb{P}(U) \cong \mathbb{P}^1 \to \mathscr{M}S_C(n, L)$$

which gives the symplectic Hecke curve (3) is an immersion, we need to show that the map (4) is injective.

Since $\widehat{\beta_x}$ is injective by definition of Quot schemes, we need to check that δ is injective. The map δ fits into the long exact sequence associated to (6) tensored by $(V^\ell)^*$:

$$0 \to H^0\left(C, \mathrm{End}\left(V^\ell\right)\right) \to H^0\left(C, \left(V^\ell\right)^* \otimes V\right) \to \mathrm{Hom}\left(V^\ell, \left(V_x/\ell^\perp\right)\right) \xrightarrow{\delta} H^1\left(C, \mathrm{End}\left(V^\ell\right)\right)$$

Hence δ is injective everywhere if we know:

- $\dim H^0(C, \mathrm{End}(V^\ell)) = 1$ for all ℓ and
- $\dim H^0(C, (V^\ell)^* \otimes V) = 1$ for all ℓ.

The first condition holds if V^ℓ is regularly stable. By [3, Lemma 4.2], the second condition holds if every point V is a generic point and $g \geq 3$.

Now it remains to show that the map ϕ_U is injective. It was shown in [3, Lemma 4.5] that ϕ_U is generically injective if $g \geq 3$ and $n \geq 4$. Its proof can be slightly modified to show the injectiveness (under a stronger bound on g and n). The point of the proof was to choose $W = (V^{\ell_0})^*$ as a "1-stable" symplectic bundle (see [3, § 4.1]). By the same argument, if we choose W to be 2-stable, then every bundle V^ℓ can be shown to be 1-stable, and hence $V^{\ell_1} \cong V^{\ell_2}$ implies $\ell_1 = \ell_2$. By [3, Lemma 4.1] a generic point of $\mathscr{M}S_C(n, L)$ is 2-stable for $g \geq 4, n \geq 4$ and we are done. □

Proposition 8. *Assume $g \geq 5$ and $n \geq 5$. Then any orthogonal Hecke curve passing through a generic point $W \in \mathscr{M}O_C(n, L)$ is smooth.*

Proof. From the construction of orthogonal Hecke curves, the family

$$\left\{V^\wp : \mathrm{Ker}\,\omega_x^\Theta \subset \wp^\perp \subset V_x\right\} \tag{7}$$

gives a deformation of $V^\wp$ along a subspace $U := (\mathrm{Ker}\,\omega_x^\Theta)^\perp \cong \mathbb{C}^4$. By Proposition 3, the Kodaira–Spencer map

$$T_\wp Gr\left(2, V_x/U^\perp\right) = \mathrm{Hom}\left(\wp^\perp/U^\perp, V_x/\wp^\perp\right) \to H^1\left(C, \mathrm{End}\left(V^\wp\right)\right) \tag{8}$$

associated to this family is given by the composition

$$\mathrm{Hom}\left(\wp^\perp/U^\perp, V_x/\wp^\perp\right) \xrightarrow{\widehat{\beta_x}} \mathrm{Hom}\left(V_x^\wp, V_x/\wp^\perp\right) \xrightarrow{\delta} H^1\left(C, \mathrm{End}\left(V^\wp\right)\right), \tag{9}$$

where $\widehat{\beta_x}$ and δ are induced from

$$0 \to V^{\wp} \xrightarrow{\beta} V \to \left(V_x/\wp^{\perp}\right) \otimes \mathcal{O}_x \to 0. \tag{10}$$

Note that this can be geometrically understood as the composition map

$$T_\ell Gr(2, V_x/U^{\perp}) \xrightarrow{\widehat{\beta_x}} T_{[\beta]}\mathrm{Quot}(V) \xrightarrow{\delta} H^1\left(C, \mathrm{ad}\left(V^{\wp}\right)\right) = T_{[V^{\wp}]}\mathcal{MO}_C\left(n, L^*\right),$$

where the tangent space $T_{[\beta]}\mathrm{Quot}(V)$ of the Quot scheme of V is given by

$$T_{[\beta]}\mathrm{Quot}(V) = H^0\left(C, \mathrm{Hom}\left(V^{\wp}, \left(V_x/\wp^{\perp}\right) \otimes \mathcal{O}_x\right)\right) \cong \mathrm{Hom}\left(V_x^{\wp}, V_x/\wp^{\perp}\right).$$

Hence to show the immersedness of the orthogonal Hecke curve (7), it suffices to show that the map (8) is injective.

Since $\widehat{\beta_x}$ is injective by definition of Quot schemes, we need to check that δ is injective. The map δ fits into the long exact sequence associated to (10) tensored by $(V^{\wp})^*$:

$$0 \to H^0\left(C, \mathrm{End}\left(V^{\wp}\right)\right) \to H^0\left(C, \left(V^{\wp}\right)^* \otimes V\right) \to \mathrm{Hom}\left(V^{\wp}, \left(V_x/\wp^{\perp}\right)\right) \xrightarrow{\delta} H^1\left(C, \mathrm{End}\left(V^{\wp}\right)\right)$$

Hence δ is injective everywhere if we know:

- $\dim H^0(C, \mathrm{End}(V^{\wp})) = 1$ for all $\wp$ and
- $\dim H^0(C, (V^{\wp})^* \otimes V) = 1$ for all $\wp$.

The first condition holds if $V^{\wp}$ is regularly stable. The second condition holds if V is general and $g \geq 3$ by [3, Lemma 4.2].

Now to show the injectiveness, as in the symplectic case, it suffices to choose W to be 3-stable in order that every bundle $V^{\wp}$ is 2-stable. By [3, Lemma 4.1] a generic point of $\mathcal{MO}_C(n, L)$ is 3-stable for $g \geq 5, n \geq 5$ and we are done. □

4. Nondegeneracy of the tangent map

In this section, we discuss the tangent map of the variety of minimal rational tangents for the moduli spaces $\mathcal{MS}_C(2r, L)$ and $\mathcal{MO}_C(n, L)$. In particular, we study the complete linear system which defines the tangent map. This confirms that the image of the tangent map is nondegenerate, and as a consequence we get the simpleness of the tangent bundle of the moduli space. Basically we follow the computations in [7] of the Kodaira–Spencer map of the Hecke curves on the moduli space $\mathcal{SU}_C(2, d)$.

4.1. *Symplectic bundles*

Proposition 9. *Assume $g \geq 4$ and $n \geq 4$ as in Proposition 7. Let $\mathcal{K}$ be the minimal rational component consisting of symplectic Hecke curves on $\mathcal{MS}_C(2r, L)$. Then for a generic $W \in \mathcal{MS}_C(2r, L)$, the tangent map*

$$\tau_W : \mathcal{K}_W = \mathbb{P}W^* \to \mathbb{P}T_W\mathcal{MS}_C(n, L) = \mathbb{P}H^1\left(C, \mathrm{Sym}^2 W^* \otimes L\right)$$

is the composition $\Phi_W \circ \iota$, where Φ_W is given by the complete linear system $|\mathcal{O}(1) \otimes \pi^* K_C|$. Also, $\iota(\theta) = \theta \otimes (\omega(\theta, \cdot))$ for $\theta \in W_x^*$, and the image of $\iota(\theta)$ in $\mathbb{P}(H^1(C, \mathrm{Sym}^2 W^* \otimes L))$ is given by the linear functional $H^0(C, \mathrm{Sym}^2 W \otimes L^* \otimes K_C) \to (K_C)_x$ taking the trace of endomorphisms of W_x.

$$\begin{tikzcd}
\mathcal{K}_W = \mathbb{P}W^* \arrow[r] \arrow[dr, "\iota"'] & \mathbb{P}(\mathrm{End}_0(W)) \arrow[r, dashed, "\Psi_W"'] & \mathbb{P}H^1(C, \mathrm{End}_0 W) \\
 & \mathbb{P}\left(\mathrm{Sym}^2 W^* \otimes L\right) \arrow[u, hook] \arrow[r, "\Phi_W"'] & \mathbb{P}H^1\left(C, \mathrm{Sym}^2 W^* \otimes L\right) \arrow[u, hook]
\end{tikzcd}$$

Remark 10. The upper arrow Ψ_W is the natural mapping of the ruled variety $\mathbb{P}(\mathrm{End}_0(W))$, which is not necessarily defined everywhere at this stage, but it will turn out in §5 to be a morphism and furthermore an embedding under certain genus assumption. On the other hand, the lower arrow Φ_W is a morphism by Proposition 7.

Proof. Recall that $\ker\omega^\theta$ is a subspace of V_x of codimension two, and after we put $U^\perp = \ker\omega^\theta_x \subset V_x$, the family $\{V^\ell : [\ell] \in \mathbb{P}(V_x/U^\perp)\}$ is a deformation of $W^* = V^{\ell_0}$. Applying Proposition 2 to the kernel of ω^θ, we get that the Kodaira-Spencer map

$$T_{\ell_0}\left(\mathbb{P}\left(V_x/U^\perp\right)\right) = \mathrm{Hom}\left(\ell_0^\perp/U^\perp, V_x/\ell_0^\perp\right) \to H^1\left(C, \mathrm{End}\left(V^{\ell_0}\right)\right)$$

for the family $\{V^\ell : [\ell] \in \mathbb{P}(V_x/U^\perp)\}$, is given by the composition

$$\mathrm{Hom}\left(\ell_0^\perp/U^\perp, V_x/\ell_0^\perp\right) \xrightarrow{\widehat{\beta_x}} \mathrm{Hom}\left(V_x^{\ell_0}, V_x/\ell_0^\perp\right) \xrightarrow{\delta} H^1\left(C, \mathrm{End}\left(V^{\ell_0}\right)\right),$$

where $\widehat{\beta_x}$ and δ are induced from

$$0 \to V^{\ell_0} \xrightarrow{\beta} V \to \left(V_x/\ell_0^\perp\right) \otimes \mathscr{O}_x \to 0.$$

Furthermore, as in the case of the moduli space of vector bundles discussed in [7], the element $\delta(\widehat{\beta}_x(v))$ in $H^1(C, \mathrm{End}(V^{\ell_0}))$ for $v \in \mathrm{Hom}(\ell_0^\perp/U^\perp, V_x/\ell_0^\perp)$, is represented by the cocycle $\left\{\frac{e_2^* \otimes e_1}{z} \text{ on } \mathscr{U}_0 \cap \mathscr{U}_j\right\}$. Here, $\{\mathscr{U}_0, \mathscr{U}_1, \ldots, \mathscr{U}_N\}$ is a coordinate covering of C such that

- all the involved vector bundles are trivial on $\mathscr{U}_0$ and $\mathscr{U}_j$ for $1 \le j \le N$,
- $x \in \mathscr{U}_0$ and $x \notin \mathscr{U}_j$ for $1 \le j \le N$ so that on each $\mathscr{U}_j$, we identify V^{ℓ_0} and V via $V^{\ell_0} \xrightarrow{\beta} V$,
- z is a coordinate on $\mathscr{U}_0$ centered at x,
- $\{e_1, e_2, \ldots, e_n\}$ and $\{f_1, f_2, \ldots, f_n\}$ are frames of $V^{\ell_0}|_{\mathscr{U}_0}$ and $V|_{\mathscr{U}_0}$, respectively such that $e_{1,x} \in \theta \subset V_x^{\ell_0} = W_x^*$, $f_{2,x} \in \ell_0^\perp/U^\perp \subset V_x/U^\perp = (W_x^\theta)^*/U^\perp$ and
- β sends $e_1, e_2, \ldots, e_n$ to $zf_1, f_2, \ldots, f_n$.

Thus $\delta(\widehat{\beta}_x(v))$ corresponds to the image of $\iota(\theta) = \theta \otimes \omega(\theta, \cdot)$ via the duality $H^1(C, \mathrm{End}_0(W)) \simeq H^0(C, K_C \otimes \mathrm{End}_0(W))^*$ induced by the residue pairing. □

4.2. *Orthogonal bundles*

Proposition 11. *Assume $g \ge 5$ and $n \ge 5$ as in Proposition 8. Let $\mathscr{K}$ be the minimal rational component consisting of orthogonal Hecke curves on $\mathscr{M}O_C(n, L)$. Then for a generic $W \in \mathscr{M}O_C(n, L)$, the tangent map τ_W*

$$\tau_W : \mathscr{K}_W = IG\left(2, W^*\right) \to \mathbb{P}T_W \mathscr{M}O_C(n, L) = \mathbb{P}H^1\left(C, \wedge^2 W^* \otimes L\right)$$

is the composition $\Phi_W \circ \iota$, where Φ_W is given by the complete linear system $|\mathscr{O}(1) \otimes \pi^* K_C|$. Also, $\iota(v_1 \wedge v_2) = b(v_1, \cdot) \otimes v_2 - b(v_2, \cdot) \otimes v_1$ for $[v_1 \wedge v_2] \in IG(2, W_x^*)$ and the image of $\iota(v_1 \wedge v_2)$ in $\mathbb{P}(H^1(C, \wedge^2 W^* \otimes L))$ is given by the linear functional $H^0(C, \wedge^2 W^* \otimes L^* \otimes K_C) \to (K_C)_x$ taking the trace of endomorphisms of W_x.

$$\begin{tikzcd}
\mathscr{K}_W = IG(2, W^*) \arrow[r] \arrow[dr, "\iota"'] & \mathbb{P}(\mathrm{End}_0(W)) \arrow[r, dashed, "\Psi_W"'] & \mathbb{P}H^1(C, \mathrm{End}_0 W) \\
& \mathbb{P}\left(\wedge^2 W^* \otimes L\right) \arrow[u, hook] \arrow[r, "\Phi_W"'] & \mathbb{P}H^1\left(C, \wedge^2 W^* \otimes L\right) \arrow[u, hook]
\end{tikzcd}$$

Remark 12. As before, the upper arrow Ψ_W is not necessarily defined everywhere at this stage, but it will turn out in §5 to be a morphism and furthermore an embedding under certain genus assumption. On the other hand, the lower arrow Φ_W is a morphism by Proposition 8.

Proof. Recall that the kernel of b^Θ is a subspace of codimension 4, and if we put $U^\perp = \mathrm{Ker}\, b_x^\Theta \subset V_x = (W_x^\Theta)^*$, then the family $\{V^\wp : [\wp] \in IG(2, V_x/U^\perp)\}$ is a deformation of $W^* = V^{\wp_0}$. As in the proof of Proposition 8, we apply Proposition 3 to the kernel of b^Θ and we get that the Kodaira-Spencer map

$$T_{\wp_0}\left(Gr\left(2, V_x/U^\perp\right)\right) = \mathrm{Hom}\left(\wp_0^\perp/U^\perp, V_x/\wp_0^\perp\right) \to H^1\left(C, \mathrm{End}\left(V^{\wp_0}\right)\right)$$

for the family $\{V^\wp : [\wp] \in Gr(2, V_x/U^\perp)\}$ is given by the composition

$$\mathrm{Hom}\left(\wp_0^\perp/U^\perp, V_x/\wp_0^\perp\right) \xrightarrow{\widehat{\beta_x}} \mathrm{Hom}\left(V_x^{\wp_0}, V_x/\wp_0^\perp\right) \xrightarrow{\delta} H^1\left(C, \mathrm{End}\left(V^{\wp_0}\right)\right),$$

where $\widehat{\beta_x}$ and δ are induced from

$$0 \to V^{\wp_0} \xrightarrow{\beta} V \to \left(V_x/\wp_0^\perp\right) \otimes \mathcal{O}_x \to 0.$$

Choose a coordinate covering $\{\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_N\}$ of C such that

- all the involved vector bundles are trivial on $\mathcal{U}_0$ and $\mathcal{U}_j$ for $1 \le j \le N$,
- $x \in \mathcal{U}_0$ and $x \notin \mathcal{U}_j$ for $1 \le j \le N$ so that on each $\mathcal{U}_j$, we identify $V^{\wp_0} = W^*$ and $V = (W^\Theta)^*$ by $V^{\wp_0} \xrightarrow{\beta} V$,
- z is a coordinate on $\mathcal{U}_0$ centered at x,
- $\{e_1, e_2, \ldots, e_n\}$ and $\{f_1, f_2, \ldots, f_n\}$ are frames of $V^{\wp_0}|_{\mathcal{U}_0}$ and $V|_{\mathcal{U}_0}$ respectively such that $e_{1,x}, e_{2,x} \in \Theta \subset V_x^{\wp_0} = W_x^*$, $f_{3,x}, f_{4,x} \in \wp_0^\perp/U^\perp \subset V_x/U^\perp = (W_x^\Theta)^*/U^\perp$ and
- β sends e_1, e_2, e_3, e_4 to zf_1, zf_2, f_3, f_4.

Then the tangent space $T_{\wp_0}(IG(2, V_x/U^\perp))$ is generated by

$$v = f_{3,x}^* \otimes f_{1,x} - f_{4,x}^* \otimes f_{2,x} \in \mathrm{Hom}\left(\wp_0^\perp/U^\perp, V_x/\wp_0^\perp\right).$$

Furthermore, $v \circ \beta$ maps $e_{1,x}$, $e_{2,x}$, $e_{3,x}$, $e_{4,x}$ to 0, 0, $f_{1,x}$, $f_{2,x}$ up to a constant multiple. Thus $\widehat{\beta_x}(v)$ can be extended to $\widetilde{v}$ with $\widetilde{v}(e_1) = 0$ and $\widetilde{v}(e_2) = 0$ and $\widetilde{v}(e_3) = f_1$ and $\widetilde{v}(e_4) = f_2$. Then $\delta(\widehat{\beta_x}(v))$ is defined by the cocycle

$$\widehat{v}_j = \left(\beta|_{\mathcal{U}_0 \cap \mathcal{U}_j}\right)^{-1} \circ \widetilde{v} \in H^0\left(\mathcal{U}_0 \cap \mathcal{U}_j, \mathrm{End}\, V^{\ell_0}\right),$$

where $\beta|_{\mathcal{U}_0 \cap \mathcal{U}_j} : V^{\ell_0}|_{\mathcal{U}_0 \cap \mathcal{U}_j} \to V|_{\mathcal{U}_0 \cap \mathcal{U}_j}$ is the isomorphism. Therefore, $\delta(\widehat{\beta}_x(v))$ is represented by the cocycle $\left\{\frac{e_3^* \otimes e_1 - e_4^* \otimes e_2}{z} \text{ on } \mathcal{U}_0 \cap \mathcal{U}_j\right\}$, where $\{e_1^*, e_2^*, \ldots, e_n^*\}$ is the dual frame.

Thus $\delta(\widehat{\beta}_x(v))$ corresponds to the image of $\iota(v_1 \wedge v_2) = b(v_1, \cdot) \otimes v_2 - b(v_2, \cdot) \otimes v_1$ via the duality $H^1(C, \mathrm{End}_0(W)) \simeq H^0(C, K_C \otimes \mathrm{End}_0(W))^*$ induced by the residue pairing. □

4.3. *Simplicity of the tangent bundles*

The simplicity of the tangent bundle of $\mathcal{S}U_C(n,d)$ has been proven in [8], based on the non-degeneracy of VMRT as stated in Proposition 6. Now we show the parallel results for symplectic and orthogonal bundes. In the following, we assume that the rank is at least 5 in the orthogonal case to guarantee the minimality of the orthogonal Hecke curves (see § 2.5).

Theorem 13. *Let $\mathcal{M}^\circ$ be the smooth locus of the moduli space $\mathcal{M}S_C(2r, L)$ ($\mathcal{M}O_C(n, L)$, respectively) of L-valued symplectic (orthogonal, respectively) bundles on C of genus g. Assume that $g \ge 4$ for symplectic case, and $g \ge 5, n \ge 5$ for orthogonal case. Then the tangent bundle $T(\mathcal{M}^\circ)$ is simple.*

Proof. Symplectic bundles of rank $n = 2$ are nothing but the rank two vector bundles and so $\mathcal{M}S_C(2, L) = \mathcal{S}U_C(2, L)$. In this case, the simplicity of the tangent bundle has been proven in [8, Corollary 1] under the assumption $g \ge 4$. (Moreover, the stability of the tangent bundle of $\mathcal{S}U_C(2,1)$ was shown in [7, Theorem 1] for $g \ge 2$.)

Now we assume $g \geq 4, n \geq 4$ for symplectic case (and $g \geq 5, n \geq 5$ for orthogonal case). By Proposition 9 and Proposition 11, the tangent maps of symplectic and orthogonal Hecke curves are given by the corresponding complete linear systems. Hence for each case, the VMRT at a generic point is non-degenerate, and the wanted result follows from Proposition 6. □

Remark 14. In the above theorem, we made the assumptions on the genus and rank to guarantee the smoothness of the symplectic and orthogonal Hecke curves, which is necessary to get a tangent morphism τ_W. As can be seen from the proofs of Propositions 7 and 8, technically we need these genus assumptions to guarantee that a generic point of $\mathscr{M}S_C(n,L)$ (resp. $\mathscr{M}O_C(n,L)$) is 2-stable (resp. 3-stable). This corresponds to the assumption $g \geq 4$ in [8, Proposition 1 and Corollary 1] to guarantee the $(1,1)$-stability of a generic vector bundle. Still, it is an interesting question if the simplicity of the tangent bundle still holds in the low genus cases and for orthogonal bundles of rank three and four.

5. Biregularity of the tangent map

Let W be an L-valued symplectic or orthogonal bundle over C. Throughout this section, we will assume that an orthogonal bundle of even rank, say $2r$, admits an isotropic subbundle of rank r. As shown in [2, Lemma 2.5], this is equivalent to that $\det(W) = L^r$. Accordingly, $\mathscr{M}O_C(2r,L)$ denotes one of the moduli components which parameterizes L-valued orthogonal bundles of rank $2r$ with determinant L^r. On the other hand, every orthogonal bundle of odd rank, say $2r+1$, admits an isotropic subbundle of rank r by [2, Lemma 2.7]. Hence $\mathscr{M}O_C(2r+1,L)$ denotes any moduli component which parameterizes L-valued orthogonal bundles of rank $2r+1$.

As in (2), we write

$$\operatorname{ad} W = \begin{cases} \operatorname{Sym}^2 W \otimes L^* & \text{if } W \text{ is symplectic;} \\ \wedge^2 W \otimes L^* & \text{if } W \text{ is orthogonal.} \end{cases}$$

In either case, $\operatorname{ad} W$ is a self-dual subbundle of $\operatorname{End}_0 W$.

Let $\pi \colon \mathbb{P}(\operatorname{ad} W) \to C$ be the associated projective bundle of $\operatorname{ad} W$. We consider the natural map

$$\Phi_W \colon \mathbb{P}(\operatorname{ad} W) \dashrightarrow \mathbb{P}H^0(C, K_C \otimes \operatorname{ad} W)^*,$$

given by the complete linear system $|\mathscr{O}_{\mathbb{P}(\operatorname{ad} W)}(1) \otimes \pi^* K_C|$. Our goal will be to prove that Φ_W is an embedding. In fact, we show a stronger statement:

Theorem 15. *Let $\ell := \deg(L) \in \{0,1\}$. The map*

$$\Psi_W \colon \mathbb{P}(\operatorname{End}_0 W) \dashrightarrow \mathbb{P}H^0(C, K_C \otimes \operatorname{End}_0 W)^*$$

given by the complete linear system $|\mathscr{O}_{\mathbb{P}(\operatorname{End}_0 W)}(1) \otimes \pi^* K_C|$ is an embedding for:

(1) *a generic $W \in \mathscr{M}S_C(2r,L)$ if $r \geq 2$ and $g \geq 5 + 2\ell$,*
(2) *a generic $W \in \mathscr{M}O_C(2r,L)$ if either ($r = 3$, $g \geq 8$) or ($r \geq 4$, $g \geq 7$),*
(3) *a generic $W \in \mathscr{M}O_C(2r+1,L)$ if either ($r = 2$, $g \geq 14$) or ($r \geq 3$, $g \geq 9$).*

Corollary 16. *Let $\mathscr{M}$ be one of the moduli spaces in the above theorem. Let*

$$\tau_W \colon \mathscr{K}_x \to \mathbb{P}(T_W \mathscr{M})$$

be the tangent map of the minimal rational component $\mathscr{K}$ associated to the symplectic or orthogonal Hecke curves. Then τ_W is an embedding of the corresponding VMRT under the same assumption as in Theorem 15.

Proof. This follows from the fact that Ψ_W, hence Φ_W, is an embedding, together with the pictures of Proposition 9 and Proposition 11. □

The remaining parts of this section are devoted to proving Theorem 15. Following [9, Proof of Theorem 3.1], we shall use the fact that Ψ_W is an embedding if and only if

$$h^0(\mathscr{O}_C(D) \otimes \mathrm{End}_0 W) = 0 \text{ for all } D \in C^{(2)}, \tag{11}$$

where $C^{(2)}$ parameterizes the effective divisors of degree two.

We first show the following:

Lemma 17. *Let $\mathscr{E} \to B \times C$ be a family of vector bundles over C. Then the subset*

$$\left\{b \in B : h^0(\mathscr{O}_C(D) \otimes \mathscr{E}_b) = 0 \textit{ for all } D \in C^{(2)}\right\}$$

is open in B, where $\mathscr{E}_b = \mathscr{E}|_{\{b\}\times C}$.

Proof. The complement of the locus in question is

$$\left\{b \in B : h^0(\mathscr{O}_C(D) \otimes \mathscr{E}_b) \geq 1 \text{ for some } D \in C^{(2)}\right\}.$$

This is the image in B of the closed set

$$\left\{(b, D) \in B \times C^{(2)} : h^0(\mathscr{O}_C(D) \otimes \mathscr{E}_b) \geq 1\right\}$$

by the projection from $B \times C^{(2)}$. As $C^{(2)}$ is projective and in particular complete, this projection is closed. The statement follows. □

Applying Lemma 17 to a suitable étale cover of each of the moduli spaces in question, we see that it suffices to exhibit a single bundle V with property (11). We now assemble some vanishing results which we shall use to this end.

Lemma 18. *Let E_1 and E_2 be vector bundles such that for all $D \in C^{(2)}$ we have*

(i) $h^0(\mathscr{O}_C(D) \otimes \mathrm{End}_0 E_1) = 0$
(ii) $h^0(\mathscr{O}_C(D) \otimes \mathrm{End}_0 E_2) = 0$
(iii) $h^0(\mathrm{Hom}(E_1, E_2(D))) = 0$
(iv) $h^0(\mathrm{Hom}(E_2, E_1(D))) = 0$

Let $\Pi \subseteq H^1(\mathrm{Hom}(E_2, E_1))$ be a subspace of dimension at least $2 \cdot \mathrm{rk}(E_1) \cdot \mathrm{rk}(E_2) + 3$. Then if $0 \to E_1 \to W \to E_2 \to 0$ is an extension whose class δ is a general element of Π, we have $h^0(\mathscr{O}_C(D) \otimes \mathrm{End}_0 W) = 0$ for all $D \in C^{(2)}$.

Proof. Let W be an extension whose class is a general element of Π. Let D be an element of $C^{(2)}$, and suppose that $\alpha\colon W \to W(D)$ is a nonzero map. We shall show that under the hypotheses above, $\alpha = \mathrm{Id}_W \otimes \mathbf{s}$ for some $\mathbf{s} \in H^0(\mathscr{O}_C(D))$. This will suffice to prove the statement.

By (iii), the restriction $\alpha|_{E_1}$ factorizes via $E_1(D)$. Therefore, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & E_1 \arrow[r] \arrow[d] & W \arrow[r, "q"] \arrow[d, "\alpha"] & E_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_1(D) \arrow[r] & W(D) \arrow[r, "q'"] & E_2(D) \arrow[r] & 0.
\end{tikzcd}$$

By (i), then, $\alpha|_{E_1} = \mathrm{Id}_{E_1} \otimes \mathbf{s}$ for some $\mathbf{s} \in H^0(\mathscr{O}_C(D))$. (If C is nonhyperelliptic then $(\mathbf{s}) = D$.) Therefore, $\alpha - \mathrm{Id}_W \otimes \mathbf{s}$ is a map $W \to W(D)$ vanishing on E_1. Hence

$$\alpha - \mathrm{Id}_W \otimes \mathbf{s} = \beta \circ q \tag{12}$$

for some $\beta \in H^0(\mathrm{Hom}(E_2, W(D)))$.

If $\beta \neq 0$ then by (iv) we see that $q' \circ \beta$ is a nonzero map $E_2 \to E_2(D)$. By (ii), we have

$$q' \circ \beta = \mathrm{Id}_{E_2} \otimes \mathbf{s}_{D'}$$

for some $D' \in |D|$ and $\mathbf{s}_{D'} \in H^0(\mathcal{O}_C(D))$ satisfying $(\mathbf{s}_{D'}) = D'$. (If C is nonhyperelliptic then $D' = D$.) In particular, the map $\mathrm{Id}_{E_2} \otimes \mathbf{s}_{D'} \colon E_2 \to E_2(D')$ lifts to $W(D)$. This means that

$$\delta(W) \in \ker\left(\mathrm{Id}_{E_2} \otimes \mathbf{s}_{D'} \colon H^1\left(\mathrm{Hom}(E_2(D), E_1(D))\right) \to H^1\left(\mathrm{Hom}(E_2(D-D'), E_1(D))\right)\right).$$

In view of the exact sequence

$$0 \to E_2^* \otimes E_1 \to E_2^* \otimes E_1(D') \to E_2^* \otimes E_1(D')|_{D'} \to 0,$$

this kernel has dimension at most $2 \cdot \mathrm{rk}(E_2) \cdot \mathrm{rk}(E_1)$. The union

$$\bigcup_{D' \in C^{(2)}} \ker\left(\left(\mathrm{Id}_{E_2} \otimes \mathbf{s}_{D'}\right)^*\right)$$

is therefore of dimension at most $2 \cdot \mathrm{rk}(E_2) \cdot \mathrm{rk}(E_1) + 2 < \dim \Pi$. But since $\delta(W)$ was assumed to be general in Π, we may assume that the map $\mathrm{Id}_{E_2} \otimes \mathbf{s}_{D'}$ does not lift to W. Therefore, we must have $\beta = 0$.

By (12), we obtain $\alpha = \mathrm{Id}_W \otimes \mathbf{s}$, as desired. □

Lemma 19. *Suppose t and s are integers with $s \geq 0$ and $t + 2s < g$. Let N be a generic line bundle in $\mathrm{Pic}^t(C)$. Then $h^0(N(D)) = 0$ for all $D \in C^{(s)}$, where $C^{(s)}$ parameterizes the effective divisors of degree s.*

Proof. If $t + s < 0$ then this is clear. Otherwise: If $h^0(C, N(D)) \neq 0$ for some $D \in C^{(s)}$, then N is of the form $\mathcal{O}_C(D_1 - D)$ for some $D_1 \in C^{(t+s)}$ and $D \in C^{(s)}$. Hence the locus of such N is of dimension at most $t + 2s$ in $\mathrm{Pic}^t(C)$. (We take $C^{(0)} = \{\mathcal{O}_C\}$.) By hypothesis, this locus is not dense in $\mathrm{Pic}^t(C)$. The statement follows. □

Lemma 20. *Let r, s, t be integers with $r \geq 2$, $s \geq 0$ and $t + rs < (r-1)(g-1)$. Let G be a generic stable bundle of rank r and degree t. Then $h^0(G(D)) = 0$ for all $D \in C^{(s)}$.*

Proof. A nonzero section of $G(D)$ gives a sheaf injection $\mathcal{O}_C(-D) \to G$. This implies that the first Segre invariant of G is bounded by

$$s_1(F) \leq \deg(G) - \mathrm{rk}(G) \cdot \deg(\mathcal{O}_C(-D)) = t + rs.$$

But as G is generic, by [10, Satz 2.2] we have

$$s_1(G) \geq (r-1)(g-1).$$

Hence we have an inequality $(r-1)(g-1) \leq t + rs$, which contradicts the assumption. Thus $h^0(C, G(D)) = 0$ for all D. □

Lemma 21. *Let E be a generic stable bundle over C of rank $r \geq 1$ and degree e.*

(1) *If $e \geq 0, s \geq 1$ and $2e + 2s < g$, then $h^0(C, E \otimes E(D)) = 0$ for all $D \in C^{(s)}$.*
(2) *If $e < 0, s \geq 1$ and $2s < g$, then $h^0(C, E \otimes E(D)) = 0$ for all $D \in C^{(s)}$.*

Proof. By Lemma 17, the locus of points E which satisfy the desired vanishing property is open. Thus it suffices to exhibit a single E with the desired property. (Note that even if such an E is unstable, we can continuously deform it to get a generic stable one with the same vanishing property.)

We first assume $e \geq 0$ and proceed by induction on r. For $r = 1$, this holds by Lemma 19. Suppose now that $r \geq 2$. By induction hypothesis, there is a stable vector bundle F of rank $r - 1$ and degree e with the desired vanishing property, which can be assumed to be generic in the moduli. Fix a generic line bundle $L_0 \in \mathrm{Pic}^0(C)$ and put $E = L_0 \oplus F$. Then $E \otimes E(D)$ has four direct summands, and it suffices to show all of them have no sections.

By induction we may assume that $h^0(C, F \otimes F(D)) = 0$ for all $D \in C^{(2)}$. Also we note that $h^0(C, (L_0)^2(D)) = 0$ for all $D \in C^{(2)}$ by Lemma 19. Finally we show that $h^0(C, F \otimes L_0(D)) = 0$ for all D. If F is a line bundle, the condition for the vanishing is $e + 2s < g$ by Lemma 19, and this holds by assumption. If $\mathrm{rk}(F) = r-1 \geq 2$, then the condition for the vanishing is $e+(r-1)s < (r-2)(g-1)$ by Lemma 20. Also this holds by assumption, noting that $s \geq 1$ and $r = \mathrm{rk}(F)+1 \geq 3$.

It follows that for a generic extension E given above, we have $h^0(C, E \otimes E(D)) = 0$ for all $D \in C^{(s)}$.

To finish, we need to find a bundle of degree $e < 0$ with the vanishing property. By the part (1), we may choose a stable bundle E of rank r and degree 0 satisfying $h^0(E \otimes E(D)) = 0$ for all $D \in C^{(s)}$. Let $\widetilde{E}$ be obtained by an elementary transformation $0 \to \widetilde{E} \to E \to \tau \to 0$, where $\deg(\tau) = e$. Then $\widetilde{E} \otimes \widetilde{E}(D)$ is a subsheaf of $E \otimes E(D)$, and by the vanishing result for E, we obtain $h^0(C, \widetilde{E} \otimes \widetilde{E}(D)) = 0$. This completes the proof. □

Also we record the following cohomology vanishing result [9, Proposition 3.2], which will be used later:

Proposition 22. *For a general stable vector bundle F of arbitrary rank and degree, $H^0(C, (\mathrm{End}_0 F)(D)) = 0$ for any effective divisor D of degree d whenever $g \geq \frac{3}{2}d + 2$.*

Now we apply these results to show the desired vanishing property for a generic symplectic and orthogonal bundle of even rank. Consider an extension

$$0 \to E \to W \to E^* \otimes L \to 0. \qquad (*)$$

for $E \in \mathscr{SU}_C(r, e)$ which is a generic stable bundle. Recall that, by [6, Criterion 2.1], we get a symplectic bundle W if we choose $(*)$ in $H^1(\mathrm{Sym}^2 E \otimes L^*)$, and an orthogonal bundle if we choose $(*)$ in $H^1(\wedge^2 E \otimes L^*)$. Also in both cases, $E \to W$ is an isotropic subbundle. (From now on whenever we discuss orthogonal bundles of even rank, we consider those bundles with an isotropic subbundle of the half rank only.)

Once we find a symplectic/orthogonal bundle W in this extension which has the desired vanishing property (11), by deformation this will show that the vanishing property holds for a general stable symplectic/orthogonal bundle with the same topological invariants. The only topological invariants of a symplectic bundle are rank and degree, while we need to additionally consider the 2nd Stiefel–Whitney class for an orthogonal bundle. Note that we may assume $L = \mathscr{O}_C$ when $\deg(L)$ is even and $L = \mathscr{O}_C(x)$ for some $x \in C$ when $\deg(L)$ is odd.

If $L = \mathscr{O}_C$ so that $\deg(W) = 0$, the moduli space $\mathscr{M}O_C(2r, \mathscr{O}_C)$ has two components classified by the 2nd Stiefel–Whitney class $w_2(W)$ such that the degree of any rank r isotropic subbundle of W has the same parity as $w_2(W)$. On the other hand, if $L = \mathscr{O}_C(x)$ so that $\deg(W) = r$, then by [2, § 2] every orthogonal bundle $W \in \mathscr{M}O_C(2r, \mathscr{O}_C(x))$ has rank r isotropic subbundles both of even degree and of odd degree.

Hence it suffices to show that a bundle W obtained by a generic extension $(*)$ has the vanishing property (11) in each of the following cases:

- For symplectic bundles: $e = \deg E = 0$ and $L = \mathscr{O}_C$ or $\mathscr{O}_C(x)$.
- For orthogonal bundles: ($L = \mathscr{O}_C$ and $e = -1, 0$) or ($L = \mathscr{O}_C(x)$ and $e = 0$).

Proposition 23. *Let $\ell = \deg(L) \in \{0, 1\}$.*

(1) *For $r \geq 2$, suppose $g \geq 5 + 2\ell$. If $(*)$ is a symplectic extension defined by a generic class in $H^1(\mathrm{Sym}^2 E \otimes L^*)$ where $e := \deg E = 0$, then $h^0(\mathscr{O}_C(D) \otimes \mathrm{End}_0 W) = 0$ for all $D \in C^{(2)}$.*

(2) *Suppose either ($r = 3$, $g \geq 8$) or ($r \geq 4$, $g \geq 7$). If $(*)$ is an orthogonal extension defined by a generic class in $H^1(\wedge^2 E \otimes L^*)$ where $e := \deg E \in \{-1, 0\}$, then $h^0(\mathscr{O}_C(D) \otimes \mathrm{End}_0 W) = 0$ for all $D \in C^{(2)}$.*

Proof. For the vanishing, let us apply Lemma 18 with $E_1 = E^*$ and $E_2 = E \otimes L$. As E is general in moduli and $g \geq 5$, conditions (i) and (ii) follow from Proposition 22.

Conditions (iii) and (iv) read $h^0(E \otimes E \otimes L(D)) = 0$ and $h^0(E^* \otimes E^* \otimes L^*(D)) = 0$ for any $D \in C^{(2)}$, respectively. These vanishing conditions are checked by Lemma 21 under the assumption $g \geq$

$5+2\ell$ in each case for $L=\mathscr{O}_C$ or $L=\mathscr{O}_C(x)$. To get a conclusion from Lemma 18, it will suffice to check that $h^1(\mathrm{Sym}^2 E\otimes L^*)$ and $h^1(\wedge^2 E\otimes L^*)$ are bigger than $2r^2+2$, respectively.

In the symplectic case, by Riemann–Roch the desired inequality would follow from

$$2r^2+3 \ \le\ \frac{1}{2}r(r+1)\left(g-1+\deg(L)\right).$$

This reads

$$\frac{g-1+\ell}{2} \ge \frac{2r^2+3}{r(r+1)},$$

which holds for $r\ge 2$ and $g\ge 5-\ell$.

In the orthogonal case, the desired inequality would follow from

$$2r^2+3 \ \le\ -(r-1)e+\frac{1}{2}r(r-1)(g-1+\deg(L)).$$

This reads

$$\frac{g-1+\ell}{2} \ge \frac{2r^2+3+(r-1)e}{r(r-1)},$$

which holds if either $(r=3,\ g\ge 8)$ or $(r\ge 4,\ g\ge 7)$. □

Now we discuss the case of orthogonal bundles of odd rank. In this case, we require some more vanishing results. To get a better genus bound, we assume here $e=\deg E\in\{-2,-1\}$ instead of $\{-1,0\}$.

Lemma 24. *Suppose $g\ge 8$. Suppose $r\ge 2$, and let E be a generic stable bundle of rank r and degree e, where $e\in\{-2,-1\}$. Let $0\to E\to F\to\mathscr{O}_C\to 0$ be a generic extension. Then $h^0(\mathscr{O}_C(D)\otimes\mathrm{End}_0 F)=0$ for all $D\in C^{(2)}$.*

Proof. Let us apply Lemma 18 with $E_1=E$ and $E_2=\mathscr{O}_C$. Condition (i) follows as above from [9, Proposition 3.2] since E is general and $g\ge 5$. Condition (ii) is trivial. Conditions (iii) and (iv) follow from Lemma 20 under the assumption $g\ge 8$.

It remains to check that $\dim H^1(\mathrm{Hom}(\mathscr{O}_C,E)) > 2r+3$. As E is stable of negative degree, $h^0(E)=0$. By Riemann–Roch, $h^1(E)=-e+r(g-1)\ge r(g-1)+1$. Using the inequalities $r\ge 2$ and $g\ge 5$, one checks that this exceeds $2r+3$. The statement now follows from Lemma 18. □

Lemma 25. *Let E and F be as in Lemma 24, and suppose $g\ge 8$. Then for all $D\in C^{(2)}$ we have $h^0(E^*\otimes F^*(D))=0$ and $h^0(F\otimes E(D))=0$.*

Proof. By construction of F, for each $D\in C^{(2)}$ we have exact sequences

$$H^0(E\otimes E(D)) \ \to\ H^0(F\otimes E(D)) \ \to\ H^0(E(D)) \ \to\ \cdots$$

and

$$H^0\left(E^*(D)\right) \ \to\ H^0\left(E^*\otimes F^*(D)\right) \ \to\ H^0\left(E^*\otimes E^*(D)\right) \ \to\ \cdots$$

By Lemma 21, we have $h^0(E\otimes E(D))=0=h^0(E^*\otimes E^*(D))$ for all $D\in C^{(2)}$. The vanishing $h^0(E^*(D))=0=h^0(E(D))$ is a consequence of Lemma 20. The statement follows. □

By [2, Lemma 2.4], for any L-valued orthogonal bundle W of odd rank, there is a line bundle N such that $W\otimes N$ is an $\mathscr{O}_C$-valued orthogonal bundle of trivial determinant. So we may work only for the moduli component of $\mathscr{O}_C$-valued orthogonal bundles of trivial determinant. As in the even rank case, the moduli space $\mathscr{M}O_C(2r+1,\mathscr{O}_C)$ has two components classified by the 2^{nd} Stiefel–Whitney class $w_2(W)$. By [4, Theorem 3.1], the degree of any rank n isotropic subbundle of W has the same parity as $w_2(W)$. To construct such orthogonal bundles of rank $2r+1$ as extensions, we use some results from [4, § 3]. Let $0\to E\xrightarrow{j} F\to\mathscr{O}_C\to 0$ be an extension as above, and let Π_j be the subspace of $H^1(F\otimes E)$ as defined in [4, § 3], which contains $H^1(C,\wedge^2 E)$ as a subspace of codimension 1. By [4, Lemma 3.2], an extension $0 \to E \to W \to F^* \to 0$ defined by a class contained in $\Pi_j \setminus H^1(C, \wedge^2 E)$ is an orthogonal bundle.

Proposition 26. *Suppose either ($r = 2$, $g \geq 14$) or ($r \geq 3$, $g \geq 9$). Let* $0 \to E \to W \to F^* \to 0$ *be a stable orthogonal bundle of rank* $2r+1$ *as above, whose extension class is general in* Π_j. *Then* $h^0(\mathscr{O}_C(D) \otimes \mathrm{End}_0 W) = 0$ *for all* $D \in C^{(2)}$.

Proof. Again, we use Lemma 18; this time with $E_1 = E$ of rank r, degree e and $E_2 = F^*$ of rank $r+1$, degree $-e$ where $e \in \{-2, -1\}$. Condition (i) follows from Proposition 22 as before. Condition (ii) follows from Lemma 24. Conditions (iii) and (iv) follow from Lemma 25.

Lastly, we must show that $\dim(\Pi_j) \geq 2 \cdot \mathrm{rk}(E) \cdot \mathrm{rk}(F^*) + 3 = 2r(r+1) + 3$. By Riemann–Roch,

$$\dim(\Pi_j) \ = \ h^1\left(\wedge^2 E\right) + 1 \ \geq \ -(r-1)e + \frac{r(r-1)}{2}(g-1) + 1.$$

Hence it suffices to have

$$-(r-1)e + \frac{r(r-1)}{2}(g-1) + 1 \ \geq \ 2r(r+1) + 3.$$

Simplifying this by using $e \leq -1$, we get the wanted inequality if

$$\frac{r(r-1)}{2}(g-1) \geq 2r^2 + r + 3. \tag{13}$$

This holds for $(r = 2, g \geq 14)$ or $(r \geq 3, g \geq 9)$. □

Proof of Theorem 15. By Lemma 17, it suffices to exhibit a single element W_0 of each of the moduli spaces satisfying the vanishing property (11). This follows from Propositions 23 and 26. □

Remark 27.

(1) The genus bound in Theorem 15 is not optimal, and can be improved simply by computing inequalities above more accurately. For instance, the above inequality (13) for orthogonal bundles reads $g \geq \frac{2(2r^2+r+3)}{r(r-1)} + 1$, which becomes $g \geq 6$ in the limit $r \to \infty$. But there seems to be some obstruction to apply our method to the curves of lower genus. This can be compared with [9, Theorem 3.1, Theorem 3.7] which shows in the vector bundle case that the tangent map of the VMRT at a generic point of $\mathscr{S}U_C(n,d)$ is a morphism for $g \geq 5$ and birational for $g = 4$.

(2) We did not consider orthogonal bundles of rank ≤ 4 according to the convention in § 2.5. But all the arguments in § 5 are valid for arbitrary rank, hence we can apply the same argument to get a very ampleness result for orthogonal bundles of rank ≤ 4. For example, we could state Proposition 23 (2) for $r = 2$, in which case the genus assumption would be $g \geq 12$. Hence the map Ψ_W is an embedding for a generic $W \in \mathscr{M}O_C(4, L)$ if $g \geq 12$.

(3) Theorem 15 shows that the VMRT of $\mathscr{M}S_C(n,L)$ and $\mathscr{M}O_C(n,L)$ at a generic point W is biregular to $\mathbb{P}(W^*)$ and $IG(2,W)$ respectively, under the assumption on the genus. As remarked in [3, Remark 6.2], this improves the involved genus bound in [3, § 6].

Acknowledgements

The authors would like thank the anonymous referees whose suggestions were very helpful to clarify the presentation.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1] A. Bajravani and G. H. Hitching, "Brill–Noether loci on moduli spaces of symplectic bundles over curves", *Collect. Math.* **72** (2021), no. 2, pp. 443–469.

[2] D. Cheong, I. Choe and G. H. Hitching, "Isotropic Quot schemes of orthogonal bundles over a curve", *Int. J. Math.* **32** (2021), no. 8, article no. 2150047.

[3] I. Choe, K. Chung and S. Lee, "Minimal rational curves on the moduli spaces of symplectic and orthogonal bundles", *J. Lond. Math. Soc.* **105** (2022), no. 1, pp. 543–564.

[4] I. Choe and G. H. Hitching, "Maximal isotropic subbundles of orthogonal bundles of odd rank over a curve", *Int. J. Math.* **26** (2015), no. 13, article no. 1550106.

[5] I. Choe and G. H. Hitching, "Low rank orthogonal bundles and quadric fibrations", *J. Korean Math. Soc.* **60** (2023), no. 6, pp. 1137–1169.

[6] G. H. Hitching, "Subbundles of symplectic and orthogonal vector bundles over curves", *Math. Nachr.* **280** (2007), no. 13-14, pp. 1510–1517.

[7] J.-M. Hwang, "Tangent vectors to Hecke curves on the moduli space of rank 2 bundles over an algebraic curve", *Duke Math. J.* **101** (2000), no. 1, pp. 179–187.

[8] J.-M. Hwang, "Hecke curves on the moduli space of vector bundles over an algebraic curve", in *Algebraic geometry in East Asia (Kyoto, 2001)*, World Scientific, 2002, pp. 155–164.

[9] J.-M. Hwang and S. Ramanan, "Hecke curves and Hitchin discriminant", *Ann. Sci. Éc. Norm. Supér.* **37** (2004), no. 5, pp. 801–817.

[10] H. Lange, "Zur Klassifikation von Regelmannigfaltigkeiten", *Math. Ann.* **262** (1983), no. 4, pp. 447–459.

[11] M. S. Narasimhan and S. Ramanan, "Deformations of the moduli space of vector bundles over an algebraic curve", *Ann. Math.* **101** (1975), pp. 391–417.

[12] M. S. Narasimhan and S. Ramanan, "Geometry of Hecke cycles. I", in *C. P. Ramanujam – A tribute*, Springer, 1978, pp. 291–345.

[13] X. Sun, "Minimal rational curves on moduli spaces of stable bundles", *Math. Ann.* **331** (2005), no. 4, pp. 925–937.