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Real Option Valuation

Optimal Investment in the Presence of Regime Switching & Mean Reverting Commodity Prices

Master's thesis spring 2022

Oslo Business School

Oslo Metropolitan University

Ms'c in economics and business administration

Preface

This Master thesis finalises our degree in Economics & Business Admistration at Oslo Metropolitan University. The process of writing this thesis has been challenging, but highly educational. We picked quite an unfamiliar subject and had to learn a great deal when it came to creating algorithms in MATLAB which were unknown grounds for both of us. We are very grateful for the insight and knowledge gained through writing this thesis.

We would like to express our gratitude to professor Sølve Selstø and Erik Smith-Meyer for their guidance with MATLAB and Stata.

Finally, we would like to express our sincerest gratitude to our supervisor Johann Reindl for his guidance through this thesis.

Abstract

This thesis investigates optimal investment in real options with the presence of regime switching and mean reverting commodity prices. Our aim is to provide a methodology that can address such behaviors in a commodity. Therefore, we are studying the electricity spot price as the underlying asset. In order to discover the value of the investment opportunity, we applied a combination of methods by utilizing a detailed algorithm. The algorithm accounts for early exercise values and is thus intended to value American real options. The analysis utilizes electricity price as the underlying variable on an investment in a hydropower plant. We find that electricity prices follow a mean reverting process with regime dependent parameters. Given this stochastic process, we analyze the optimal timing of an investment decision using a regime-augmented binomial tree.

We illustrated the use of the algorithm on the hypothetical investment opportunity, where it provides us with the option value tree for the two volatility states as well as the optimal early exercise boundaries within each state. Our findings show that the algorithm is an efficient and easy way to value American real options with a mean-reverting underlying variable with multiple volatility states. It gives a fair option value and supplies early exercise boundaries for the option. The algorithm can easily be applied to time series with different characteristics and behaviours as well, with minor changes.

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Chapter 1

Introduction

In today´s business world, with ever faster changes, decision makers face a range of challenges when deciding the future of their business. Managers may face decisions such as when to invest to grow their company, when to extract a resource, or when to abandon a project.

Even though these decisions are fundamentally different, they share some important features. The actions taken will be difficult if not impossible to reverse. The effect of either taking or not taking the action is uncertain. Finally, the nature and timing of the actions directly affect the cash flows generated by the entities.

The traditional analytical techniques such as the discounted cash flow(DCF) method are widely used by practitioners to analyze such projects. The problem with these techniques is that they assume that decision-makers commit to all future actions at present time and will not change this behavior in the future, despite the arrival of new information. The DCF assumes that the decision-maker do not fully exploit the flexibility available to them. This is not rational behavior and does not give a realistic reflection on decision-making. Alternative techniques should thus be applied to analyze such projects.

The approach should recognize decision-makers as rational and that they base their decisions on information that is available to them at the current date. Fortunately, there is a part of finance that covers this, the option pricing theory. It is possible to apply the financial option pricing theory to real-world projects and decisions. This approach is called real option pricing theory. The theory assumes that decision makers act to maximize market value or another objective function and use all available information in their decision making to achieve this.

In this thesis we will apply real option theory in the investment analysis of Norwegian hydropower production, a market with highly fluctuating prices and uncertainty.

1.1 Motivation and Purpose

The purpose of our thesis is to incorporate mean reversion and regime switching in a real option valuation. Option valuation often contains time series from one regime with a binomial lattice approach. Simultaneously, studies with multiple regimes predominantly uses a stochastic time series with a random walk without mean reversion, or continous time. To this end, we apply a methodology for the electricity spot price, which consists of a time series which is both mean reverting and have two regimes.

Several studies and literature that provides methodology for pricing real options uses the approach suggested by Guthrie (2009) which again builds on the framework provided by Cox, Ross & Rubenstein (1979). This framework can be effective and a plausible alternative to the traditional DCF model. The framework by Guthrie (2009) addresses the mean reversion and provides a binomial lattice approach for the spot price of copper.

Hamilton (1994) used Markov switching chain to model time series behavior which had dramatical changes. Option pricing with multiple regimes builds on the same premise, and was simplified by Aingworth, Das & Motwani (2006), who describe the approach for pricing American options with two regimes. The paper by Aingworth, Das & Motwani (2006) addresses two regimes with a Brownian motion process and lays the framework for a multinomial price tree without a mean reverting time series.

This thesis may contribute to the literature of real option pricing for two main reasons. First, the literature available on commodity time series in a real option analysis has, to the best of our knowledge, mainly binomial lattice approaches with one regime. Secondly, there are few studies that provide a multi regime model with mean reversion in the context of real option valuation without the use of continious time. Thus, we aim to contribute by combining approaches with both, regime switching and mean reversion for a real option valuation.

1.2 Research Question

To value a real option with a mean reverting and regime switching time series we offer the following research question:

How to incorporate mean reversion and regime switching in real option valuation?

1.3 Overview of Chapters

Our thesis is structured in the following manner. Chapter 2 contains an overview of the case study to which we apply our method, where we swiftly present industry details and the framework for investment decision. Chapter 3 presents the theory regarding real options and regime switching. Chapter 4 consist of methodology and combining of literature. Chapter 5 presents the technical approach to our model. Chapter 6 presents our results and findings. Chapter 7 consists of a discussion on choices and methodology. Finally, chapter 8 provides concluding remarks and suggestions for further research.

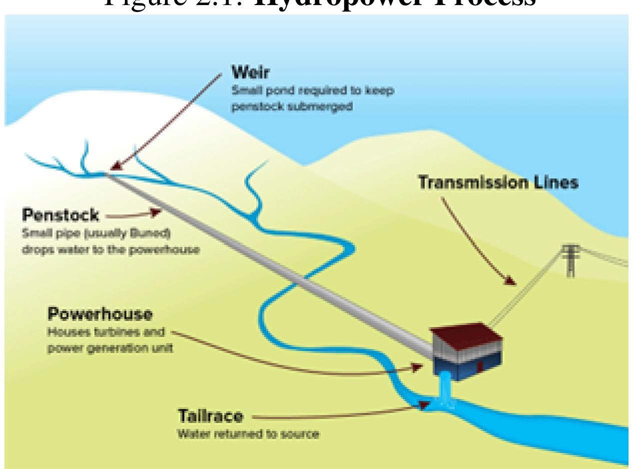
Chapter 2

Case Study

2.1 Hydropower

Hydropower is described as a source of energy using the flow of water either through stream or fall (Office of Eneregy Efficiency & Renewable Energy, 2022). Hydropower relies on a water source from a dam or a river that can provide kinetic energy. The flowing water is used to spin the turbine which again turns the shaft that is connected to a generator. Electricity from the generator flows to the grid and long-distance power lines into consumers private homes and businesses (Office of Eneregy Efficiency & Renewable Energy, 2022).

Figure 2.1: **Hydropower Process**



Process from river sourced powerplant(Hydropower, 2016)

The amount of energy generated is relied on the elevation of the fall and the volume of water that flows through the turbine. As a result, large and efficient hydropower plants demand specific locations with access to large volumes of water and elevation (Good Energy, 2022).

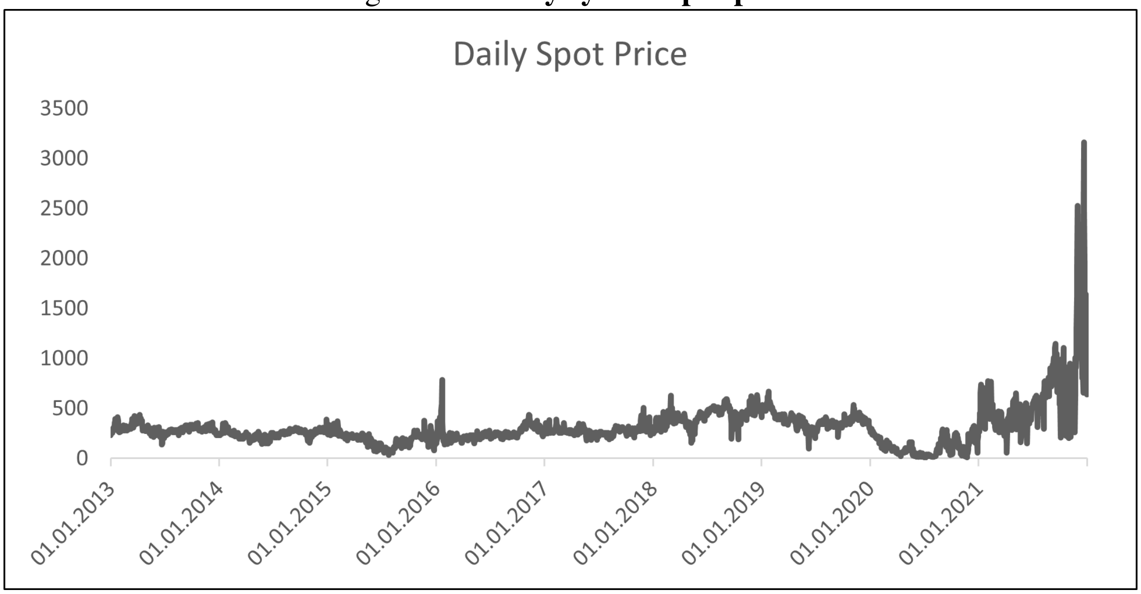
2.2 Hydropower in Norway

Large reservoirs and high capacity makes hydropower the largest source of electric energy in Norway (Energi Fakta Norge, 2021). The large reservoirs give the Norwegian power plants an advantage in flexibility. Levels of production can be adjusted accordingly to marginal cost and market swings. This flexibility is vital in competition with wind parks and solar plants. Reservoir levels are dependent on rainfall and production levels, and average levels in April of 2022 are at 22 per cent (Norwegian Water Resource Directorate, 2022b). As of March 2022, Norway's hydropower share in electricity production was 89.4 per cent (Statistisk sentralbyrå, 2022). According to The Norwegian water resource and energy directorate there are 1743 operating hydro power plants that produce 138108 GWh annually (Norwegian Water resource and Energy Directorate, 2022a). Hydropower plants in Norway has a technical-economic expansion and rearmament potential of 6-7 TWh annually, with utilization of new areas and increase efficiency on existing hydro power plants. In April of 2022 there are 82 ongoing expansions projects and 16 approved projects that has not yet started (Norwegian Water Resource & Energy Directorate, 2022c).

2.3 The Nordic and Baltic Power Market

The Nordic power market consist of multiple energy sources where hydropower is one of the largest and main sources. During the late 1990s the Nordic countries deregulated the power market and created a collected market. The Baltic countries deregulated their power market in 2013 and joined the Nordic Nord Pool market the same year (Nord Pool, 2022a).

Figure 2.2: **Daily system spot price**



Measured in NOK/kWh from January 1st 2013 to December 31st 2021 (Nord Pool, 2022b)

The price of power is set by the equilibrium of supply and demand and is affected by several different factors such as temperature, capacity, weather, and price of other energy sources. A high supply will lead to a lower price, while a low supply pulls the price upwards. The demand for electricity is influenced by the season where the colder months are characterized by high consumption and the warmer months have less consumption.

The spot price is determined through the Nordic and Baltic power market at Nord Pool Spot where spot prices and futures contracts are traded (Nord Pool, 2022a). Daily spot prices are displayed in figure (2.2).

2.4 Cloudberry Clean Energy ASA

Cloudberry is the parent company of renewable energy production. They own, develop, and operate both wind farms and hydropower plants in Sweden and Norway. Cloudberry's business model consists of three revenue generating sectors, development, production, and operation. They generated a revenue of 41 million NOK in 2021 with a production of 117 GWh. Cloudberry has 26 hydro assets and 3 wind assets producing, and 4 wind assets under development as of 31st of December 2021 (Cloudberry Clean Energy ASA, 2021, ss. 6-7). The 20th of august 2021 Cloudberry acquired Usma Kraft AS for 82.877 million NOK.

2.5 Usma Hydropower Plant

Usma hydropower plant is a small power plant located in Selbu, Trøndelag Norway. The power plant utilizes the flow of water from the river Usma, which provides them a water fall of 127 meters. Usma is equipped with an intake structure of 315 meters and a power station structure of 188 meters. The minimum flow of water is during the summer season 760 liters per second, and 100 liters per second during the winter (Småkraft AS, 2022). It has an annual normalized production capacity of 25.5 GWh with an installed effect of 9MW (Cloudberry Clean Energy ASA, 2021). We will assume that the annual production is as the normalized production capacity.

2.6 Investment Decision Framework

The investment decision of this thesis will consist of a one-year call option on a hypothetical hydropower plant with identical structure as the Usma plant. The investment cost of 82.877 million NOK will be the strike price of the option. The option can be exercised at any date prior to the one-year maturity date and will have a start date at 01.01.2022. Because of the complexity of the real option model, we will value the option with one-week intervals through the one-year period. The option will thus have 52 possible exercise dates.

Chapter 3

Real Options Theory

3.1 Real Options

To start describing real option theory we must draw an analogy between real options and financial options. An option gives the holder the right to take a certain action, but not the obligation to do so (Hull, 2018). The first main formula (BS) of valuating financial options was derived from economists Black and Scholes in 1973 and has opened the development of real options (Reuer & Tong, 2007). A few years later the binominal lattice approach (Cox et al., 1979) was presented to the world as an alternative to the Black & Scholes model. The binominal lattice approach gained great popularity, especially because of its convenience when valuing early exercise options. The model also opened for new opportunities when it came to real option valuation.

The start of real options was established from the influential idea that investments in real assets gives a firm the right but not the obligation to make certain decisions at the present time and in the future. A firms investment opportunities could be viewed as a financial call option. A real option uses the operative cash flows of a firm as the underlying asset, while the cost to invest is the exercise price. The time to maturity is described as the time the decision maker can change their decision before the opportunity expires (Reuer & Tong, 2007). In order to understand how real options work, it´s necessary to have introduction to option theory.

3.2 Option Theory

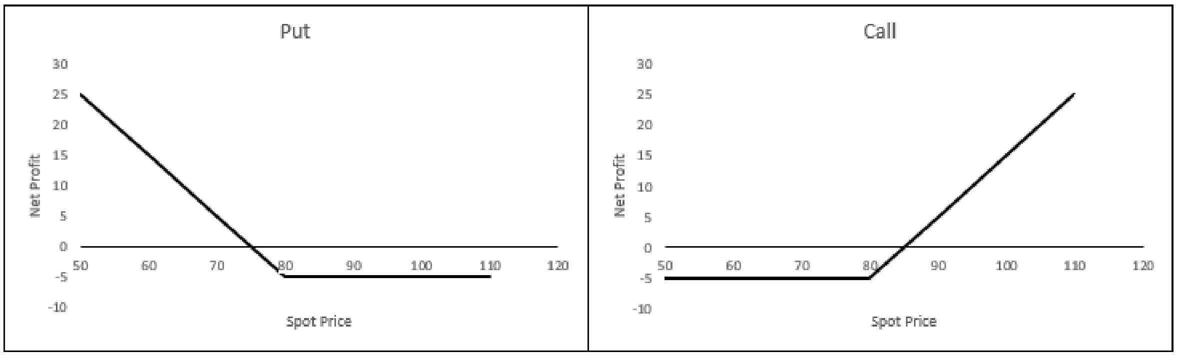
As mentioned, an option gives the holder the right to take a certain action, but not the obligation to do so (Hull, 2018). Options are therefore fundamentally different from other derivatives like futures, forwards, or swaps where the parties are obligated to commit an action. Entering the other derivatives is most often free of charge, but for an option the trader must pay a price upfront. The option price stem from the reduced downside risk of the option holder and increased downside risk of the option writer.

There are two types of options, call and put. A call option gives the holder the right to buy an asset at a certain price at a certain date. A put option gives the holder the right to sell an asset at a certain price at a certain date (MacKenzie, 2006). The price determined in the contract is called the strike or exercise price. The date specified in the option contract is called the maturity or expiration date.

Options arrive in two forms, American and European. A European option can only be exercised at expiration date. The American option on the other hand can be exercised at any time up to the expiration date (Sick, 1995).

When considering European options, a call option will always be exercised if it is "in the money", which means the spot price of the asset is larger than the exercise price at expiration (Hull, 2018). The same goes for European puts if the spot price of the asset is lower than the exercise price at expiration. The net profit will depend on the initial price of the option itself relative to the margin between the exercise price and spot price. If the spot price is lower(call) or higher(put) than the exercise price at maturity, the call/put will simply be left unexercised. The net loss will then equal the initial price of the call option.

Figure 3.1: **Option profit graph for put and call**



$Strike\ price = 80, Call/Put\text{-}price = 5$

The same dynamics are true for American options at the expiration date, but they differ from their European counterpart when it comes exercise conditions. American options can be exercised at any date before the expiration (Chen, 2011). The option can thus be exercised as soon as there are favorable price movements to lock in profit. This is rarely the optimal trading strategy because of the option´s time-value, but there are instances where early exercise is optimal. This is manly just before a dividend date for call options, and right after a dividend date for puts.

If we are to know when to exercise an American option, it is important to know how to value the option. The first step is then to understand which factors affect the option value.

3.3 Option Pricing

When valuing options there are multiple factors that affect the option value. This goes for both American and European options. The six main components affecting the price of an option are:

1. The current asset price, S_0
2. The strike price, K
3. The time to expiration, T
4. The volatility of the asset price, σ
5. The risk-free rate of interest, r
6. Expected dividend payments, Div

(Hull, 2018).

The value of a call option will increase with a rise in asset spot price S_0 as this will yield a larger margin and payoff (Hull, 2018). The opposite relationship is true for puts. A larger strike price K will result in a lower value for calls and higher for puts for the same reasons.

For an American option, a larger time to expiration T will result in a higher value for both a call and put (Barone-Adesi & Whaley, 1987). This is because of the time value of the option. Let's imagine to identical American options, with only the time variable to differ. Since an American call can be exercised at any time prior to expiration, the option with a larger maturity will have the same possibilities to be exercised as the lower maturity option, plus more potential payoff because of the additional time. The same relationship between option value and time to expiration T is true for European options as well, even though dividends can affect this relationship.

Since both call and put options have limited downside risk a rise in volatility σ will increase the value of both call and puts, American and European (Scott, 1997). A rise in volatility will increase the chance of a far in the money call/put. This results in a greater chance of large profit and increases the option value.

If the risk-free interest rate r increases and we keep the other variables the same, the value of call options will increase and put options decrease (Hull, 2018). This is mainly a result of the higher expected return from investors. This is only theoretical as in real life an increased interest rate would likely result in lower stock prices and thus the relationship would be the opposite. Dividends reduce stock prices on the ex-dividend date and thus increase the value of puts and decrease the value of call options (Hull, 2018). The increase of the variables will have the following effect on the option value:

Table 3.1: **Variable effect on option value**

Variable	European call	European put	American call	American put
S_0	**Increase**	Decrease	**Increase**	Decrease
K	Decrease	**Increase**	Decrease	**Increase**
T	-	-	**Increase**	**Increase**
σ	**Increase**	**Increase**	**Increase**	**Increase**
R	**Increase**	Decrease	**Increase**	Decrease
Div	Decrease	**Increase**	Decrease	**Increase**

(-) indicates an uncertain relationship

3.4 Brownian Motion

When valuing options there are two main techniques used. The first method is the Black-Scholes formula (Black & Scholes, 1973) and the second one is binominal lattice approach (Cox et al., 1979). Both models are based around the principles of constant volatility and drift, and that the stock price follows a geometric Brownian motion:

$$\Delta S = \mu S \Delta t + \sigma \epsilon \sqrt{\Delta t}, \qquad dz \sim N(0, dt) \tag{3.1}$$

Brownian motion is a stochastic process with drift μ, and volatility σ, and dz representing the standard wiener process (Hull, 2018). The binominal approach uses a discrete time version of geometric Brownian motion:

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

$$or \quad \epsilon \sim N(0,1) \tag{3.2}$$

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

Variable ΔS represents the change in the asset price during a short time interval Δt. Where ϵ has a normal distribution with a mean of zero and standard deviation of one (Hull, 2018). μ is the expected return of the asset. The first term on the right side of equation (3.2) represents the expected return of the asset during the time interval, and the second term on the right side represents the stochastic part of the of the return. The stochastic part of the model, in particular variable ϵ can also be approximated. We then must limit Δt to approach zero, and the equation is reduced to:

$$\Delta S = S\ exp[\sigma\epsilon\sqrt{\Delta t}] \tag{3.3}$$

Where the expected up and down factor is approximated by the stochastic part of the model:

$$\Delta Up = exp[\sigma\epsilon\sqrt{\Delta t}]$$

$$\& \tag{3.4}$$

$$\Delta Down = exp[-\sigma\epsilon\sqrt{\Delta t}]$$

This is the process that the movements in the binominal model is based on. With possible price movements after n steps equal to $n+1$. The model is versatile and can be used to value both European and American options. The binominal model is convenient when it comes to valuing potential early exercise American options. These are options where early exercise can be optimal. The Black & Scholes model do not account for early exercise for American options, which is one of the model's limitations (Whaley, 1982). The binominal model will therefore be preferred in such cases.

3.5 Risk-Neutral Valuation

When valuing options there is one fundamental assumption taken, no arbitrage opportunities. The assumption is based on the theory of highly efficient markets, where assets are fairly priced. Thus, the law on one price claims that two portfolios generating identical future cash flows must be equally priced (Guthrie, 2009). If the law of one price does not hold investors would exploit the arbitrage opportunity by short selling the expensive portfolio and buying the relatively cheap one, locking in riskless profit. The theory of efficient markets assumes that if arbitrage opportunities occur, arbitrageurs will instantly take advantage of it and the price will be corrected to the fair one.

Moving on to application of risk-neutral pricing to the binomial tree. If a state variable takes on the value of X, the next period it will either equal X_u if a up move occurs, or X_d if a down move occurs. The estimation of these values is of great interest since it influences the decision of the option holder. The cash flows corresponding to the state variable are Y_u for a up move and Y_d for a down move. Since we have the law of one price it is possible to create a synthetic portfolio of traded assets which generates a cash flow stream "close" to the one being valued and using the cost of the portfolio as the market value of the asset (Guthrie, 2009). We do this by combining a portfolio consisting of a one period risk-free bond with a certain payoff of r_f and a risky asset generating a return of either X_u or X_d with a current cost of Z. The information from Z and r_f can then be used to determine the value of the cash flows from Y_u & Y_d. The cost of the replicating cash flow portfolio is given by the formula:

$$V = \frac{\pi_u Y_u + \pi_d Y_d}{1 + r_f} \tag{3.5}$$

Where risk neutral probabilities of up $_u$ and down $_d$ moves are according to Guthrie (2009):

$$\pi_u = \frac{Z r_f - X_d}{X_u - X_d} \quad and \quad \pi_d = \frac{X_u - Z r_f}{X_u - X_d} \tag{3.6}$$

3.6 Marketed Asset Disclaimer

Traditional real option analysis uses an adequate portfolio of twin securities to value an option, and thus argues for complete markets (Barton & Lawryshyn, 2010). Copeland and Antikarov (2003) argue that markets are incomplete and because of this the best market value estimate for the project is the present value of the project itself, but with no flexibility. In other words use the traditional net present value (NPV) of the project to value the option. They argue that no portfolio will have greater correlation with the project then the cash flows generated by the project itself. We will use this assumption in our real option valuation of the hydropower investment.

3.7 Mean Reverting Price

The property of a mean reverting price is that the price will revert toward its long term mean after sudden up or down moves occur, if the moves differ from the mean. Sudden increases in the price of a commodity will cause an increase in supply, and create a reduction in the price. This is a result of the reversion to the commodity's long-term marginal production cost. The opposite will happen after a sudden price decrease. Decrease in production will lead to a fall in supply, which generates an increase in the price (Guthrie, 2009).

Regression of time series data in which the model consists of an explanatory variable and one or more lags of the explanatory variable, is called an autoregression (Gujarati & Porter, 2018). The number of lags defines the order of the regression, a model with one lag is called a first order autoregression $AR(1)$.

We assume that the spot price of electricity follows a first order autoregressive process, which is given by:

$$p_{j+1} - p_j = a_0 + a_1 p_j + u_{j+1}, \quad u_{j+1} \sim N(0, \phi^2) \tag{3.7}$$

Where p_j represents the log price of the j^{th} observation and a_0, a_1 and ϕ are constants. Since the process is mean reverting, it is given that a_1 is negative. (Guthrie, 2009)

The behavior of an economical or financial commodity in a time series has in some instances breaks. These changes in behavior can come from numerous reasons such as policy changes, financial crises, or weather changes (Durlauf Blume, 2009). Separating periods and characterizing them with different levels of volatility can be done by using a Markov switching model. Utilizing the principals for characterizing regimes, proposed by Durlauf & Blume (2009) gives us:

$$(p_{j+1} - p_j) = a_{0st} + a_{1st} p_j + u_{j+1}, \quad u_{j+1} \sim N(0, \phi^2) \tag{3.8}$$

Where st represent the regime the time series can be characterized by. Its transition probabilities are given by:

$$Pr(s_t = j | s_{t-1} = i, s_{t-2} = k, ..., y_{t-1}, y_{t-2}...) = Pr(s_t = j | s_{t-1} = i) = p_{ij} \tag{3.9}$$

Which is given from the switching Markov chain (Durlauf & Blume, 2009).

What is known as the Ornstein-Uhlenbeck process is an alternative for geometric Brownian motion. The process is meant to model mean reverting to properly describe what is modeled (Andersen, Davis, Kreiß & Mikosch, 2009).

The process can be used to generalize the $AR(1)$ processes so that arbitrary frequency is found in the observed price using rate of mean reversion a, long-term level b, and the volatility σ (Guthrie, 2009). From time t, the difference in the price over the next change in time Δt has a normal distribution with volatility of $\frac{\sigma^2(1-e^{-2a\Delta t})}{2a}$ and a mean $(-e^{-a\Delta t})(b-p_t)$ thus:

$$p_{t+\Delta t} - p_t \sim N\left((1-e^{-a\Delta t})(b-p_t), \frac{\sigma^2(1-e^{-2a\Delta t})}{2a}\right) \tag{3.10}$$

(Guthrie, 2009)

Chapter 4

Real Option Valuation Method

This chapter contains a description of the valuation method we will use. It consists of the recombination of two binomial lattices to one quadrinomial tree.

4.1 Autoregression with Regime Switching

The spot price of electricity is mean reverting and follows a first order autoregression which can be represented by equation (3.8). We assume the spot price has breaks and periods that can be characterized by two different regimes st, which gives us the following autoregression:

$$(p_{j+1} - p_j) = a_{0s_t} + a_{1s_t} p_j + u_{j+1}, \quad u_{j+1} \sim N(0, \phi_{s_t}^2)$$

Where the changes in the sample average is defined by the random variable st and there is an assumption that $s_t = H$ for the period with high volatility and $s_t = L$ for the periods with low volatility, and for period that have the same characteristics as either $s_t = H$ or $s_t = L$. This will give us two $AR(1)$ processes, one for $s_t = H$ and one for $s_t = L$, defining two different regimes, one for high volatility H, and one for low volatility L. The two different $AR(1)$ processes are defined by:

$$(p_{j+1} - p_j) = a_{0H} + a_{1H} p_j + u_{j+1}, \quad u_{j+1} \sim N(0, \phi_H^2)$$

$$(p_{j+1} - p_j) = a_{0L} + a_{1L}p_j + u_{j+1}, \quad u_{j+1} \sim N(0, \phi_L^2)$$

To obtain the probability law governing $(p_{j+1} - p_j)$ we need the parameters from the $AR(1)$ process and the transition probabilities pHH, pHL, pLH and pLL given by the regime switching model.

4.2 From Data to Normalized Estimated of the Parameters

Under the AR(1) processes, changes in p are normally distributed with a mean of $a_0 + a_{1pj}$ and variance ϕ^2. The parameters of a_0, a_1 and ϕ are thus related to the Ornstein-Uhlenbeck parameters by the following equations (Guthrie, 2009):

$$a_0(1 - e^{-a\Delta t})b, \qquad a_1 = -(1 - e^{-a\Delta t}), \qquad \phi^2 = \frac{\sigma^2}{2a}(1 - e^{-2a\Delta t}), \tag{4.1}$$

If the autoregression offers us estimates $\hat{a}_0$, $\hat{a}_1$ and $\hat{\phi}$ from our daily price data, reasonable estimates of parameters a, b and σ are the values $\hat{a}$, $\hat{b}$ and $\hat{\sigma}$.

$$\hat{a}_0(1 - e^{-\hat{a}\Delta t})\hat{b}, \qquad \hat{a}_1 = -(1 - e^{-\hat{a}\Delta t}), \qquad \hat{\phi}^2 = \frac{\hat{\sigma}^2}{2\hat{a}}(1 - e^{-2\hat{a}\Delta t}), \tag{4.2}$$

We get the normalized parameter estimates by solving the equations for $\hat{a}$, $\hat{b}$ and $\hat{\sigma}$.

$$\hat{a} = \frac{-log(1 + \hat{a}_1)}{\Delta t_d}, \qquad \hat{b} = \frac{-\hat{a}_0}{\hat{a}_1}, \qquad \hat{\sigma} = \hat{\phi}\left(\frac{2log(1 + \hat{a}_1)}{\hat{a}_1(2 + \hat{a}_1)\Delta t_d}\right)^{\frac{1}{2}} \tag{4.3}$$

Once the estimates $\hat{a}_0$, $\hat{b}_0$ and $\hat{\phi}$ from the auto regression is obtained we substitute the parameter estimated into the equations for $\hat{a}$, $\hat{b}$ and $\hat{\sigma}$. in order to obtain the normalized parameter estimates (Guthrie, 2009).

Since we are using a two-state regime model and therefore two sets of parameters of a_0, a_1 and ϕ from the $AR(1)$ model, this process must be done two times separately for each of the states. Thus, we get a different set of normalized parameter estimates $\hat{a}$, $\hat{b}$ and $\hat{\sigma}$ for each state.

4.3 From Normalized Estimates to the Price Tree

The normalized estimates are utilized to create the binominal tree for the prices. Each period in the tree represents Δt_d years, and the tree for the log price starts at $x(0,0) = logP_0$. Each following period, the log price either increases or decreases by the stochastic factor $\hat{\sigma}\sqrt{\Delta t_d}$, depending on whether an up or down move occur (Guthrie, 2009). At node (i,n) we can calculate the amount of up and down moves with up moves being the sum of $n-i$ and down moves being the sum of i. The log price will thus equal:

$$\underbrace{logP_0}_{Starting\ value} + \underbrace{(n-i)(\hat{\sigma}\sqrt{\Delta t_d})}_{effect\ of\ up\ move} + \underbrace{i(-\hat{\sigma}t_d)}_{effect\ of\ down\ move} \tag{4.4}$$

Which simplifies to:

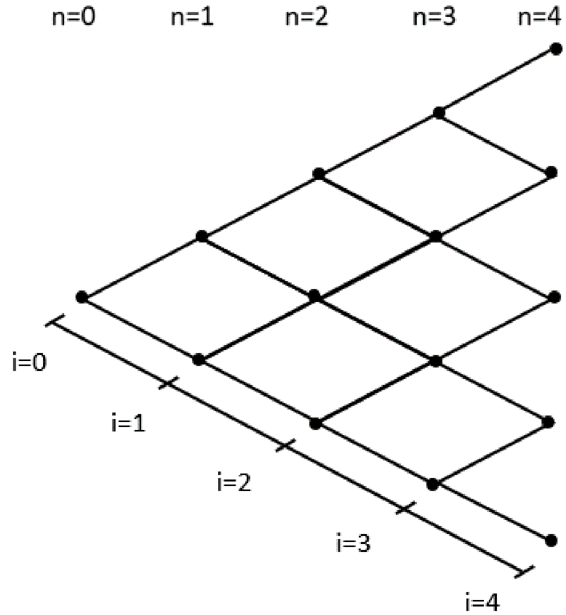
$$X(i,n) = logp_0 + (n-2i)\hat{\sigma}\sqrt{\Delta t_d} \tag{4.5}$$

By taking the exponential of both sides of the equation we can find the price at node (i,n) with:

$$X(i,n) = e^{x(i,n)} = P_0 e^{(n-2i)\hat{\sigma}\sqrt{\Delta t_d}} \tag{4.6}$$

This closed-form expression for the price enables us to calculate the expected price at any node of the binominal tree without having to iterate from x(0,0) to the respective node:

Figure 4.1: **Binomial tree**



We can use the closed-form expression to calculate the size of up and down movements. We then consider a up movement from node (i, n) to $(i, n+1)$, where the price will be:

$$X(i, n+1) = P_0 e^{((n+1)-2i)\hat{\sigma}\sqrt{\Delta t_d}} = e^{\hat{\sigma}\sqrt{\Delta t_d}} X(i, n) \tag{4.7}$$

The size of an up move at this node equals:

$$U = \frac{X(i, n+1)}{X(i, n)} = e^{\hat{\sigma}\sqrt{\Delta t_d}} \tag{4.8}$$

The same principle is true for a down move, where the price will be:

$$X(i, n+1) = P_0 e^{((n+1)-2(i+1))\hat{\sigma}\sqrt{\Delta t_d}} = e^{-\hat{\sigma}\sqrt{\Delta t_d}} X(i, n) \tag{4.9}$$

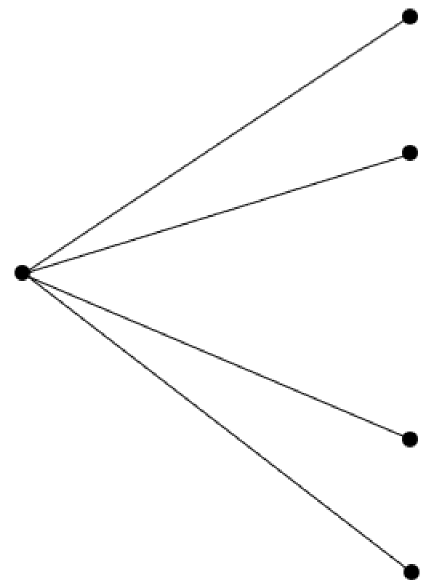
The size of a down move at this node equals:

$$D = \frac{X(i, n+1)}{X(i, n)} = e^{-\hat{\sigma}\sqrt{\Delta t_d}} \tag{4.10}$$

This illustrates that the up and down movements are constant throughout the tree. Since we have a two-state regime model its necessary to calculate the up and down movements for both the regimes. The calculation of the movements is the same, but the size of the up and down movements will differ between the states. The reason for this is the different volatility between the states. The high volatility state $\hat{\sigma}^H$ will yield larger up and down movements for the price, than the low volatility state $\hat{\sigma}^L$.

The up and down moves from the Ornstein-Ulenbeck process for each regimes D^H, D^L, U^H, and U^L are used to calculate the possible moves. From each node the price can move four different ways.

Figure 4.2: **Possible movements from each node**



Each node can move up and down in the current regime, or up and down with a regime switching as displayed in figure (4.2). The four possibilities at each node makes the tree grow at a large scale for each added time step.

4.4 Probabilities of Up and Down Moves

The next step is to calculate the probabilities of up and down moves for the electricity spot price at each node in the tree. We need to determine the probabilities so that the expected value of the change in log price over the next period is equal to the value implied by our normalized parameter estimates $\hat{a}$, $\hat{b}$ and $\hat{\sigma}$ (Guthrie, 2009).

If the up-move probability at node (i,n) equals:

$$\theta_u(i,n) = \frac{1}{2} + \frac{(1 - e^{-\hat{a}\Delta t_d})(\hat{b} - x(i,n))}{2\hat{\sigma}\sqrt{\Delta t_d}} \tag{4.11}$$

The expected change in the log price will be:

$$(1 - e^{-\hat{a}\Delta t_d})(\hat{b} - x(i, n)) \tag{4.12}$$

This matches the expected value of the Onrstein-Uhlenbeck process. The mean-reversion in the electricity spot price is illustrated in the up-move probability formula as a rise in the log price $x(i, n)$ will result in a lower up-move probability. If the spot price $x(i, n)$ rises above its long-run level $\hat{b}$ a down-move will be more likely than an up-move. If the spot price $x(i, n)$ falls under its long-run level $\hat{b}$ an up-move will be more likely than a down-move. The log-price will thus always be drawn towards its long-run level by the mean reverting probability structure (Guthrie, 2009).

A complication of the model is that $\theta_u(i, n)$ can become negative or over 1 for sufficiently large $x(i, n)$ values. We solve this by resetting $\theta_u(i, n)$ to 1 if $\theta_u(i, n) \geq 1$ and $\theta_u(i, n)$ to 0 if $\theta_u(i, n) \leq 0$. The probability of an up or down move at node (i,n) thus equals:

$$\theta_u(i, n) = \begin{cases} 0 & \text{if,} \quad \frac{1}{2} + \frac{(1-e^{-\hat{a}\Delta t_d})(\hat{b}-logX(i,n))}{2\hat{\sigma}\sqrt{\Delta t_d}} \leq 0, \\ \frac{1}{2} + \frac{(1-e^{-\hat{a}\Delta t_d})(\hat{b}-logX(i,n))}{2\hat{\sigma}\sqrt{\Delta t_d}} & \text{if,} 0 < \frac{1}{2} + \frac{(1-e^{-\hat{a}\Delta t_d})(\hat{b}-logX(i,n))}{2\hat{\sigma}\sqrt{\Delta t_d}} < 1, \\ 1 & \text{if,} \quad \frac{1}{2} + \frac{(1-e^{-\hat{a}\Delta t_d})(\hat{b}-logX(i,n))}{2\hat{\sigma}\sqrt{\Delta t_d}} \geq 1, \end{cases} \tag{4.13}$$

Therefore, at certain nodes we can be sure that the next move for the spot price will be down. This is the nodes where $\theta_u(i, n) = 0$:

$$\theta_u(i, n) = \frac{1}{2} + \frac{(1 - e^{-\hat{a}\Delta t_d})(\hat{b} - logX(i, n))}{2\hat{\sigma}\sqrt{\Delta t_d}} \leq 0 \tag{4.14}$$

We can calculate at which log price a down move will be certain:

$$logX(i,n) \geq \hat{b} + \frac{\hat{\sigma}\sqrt{\Delta t_d}}{1 - e^{-\hat{a}\Delta t_d}} \tag{4.15}$$

The reverse is true for nodes where $\theta_u(i,n) = 1$:

$$\theta_u(i,n) = \frac{1}{2} + \frac{(1 - e^{-\hat{a}\Delta t_d})(\hat{b} - logX(i,n))}{2\hat{\sigma}\sqrt{\Delta t_d}} \geq 1 \tag{4.16}$$

At these nodes we can be sure that the next move is a up move. And the log price where this happens is:

$$logX(i,n) \leq \hat{b} + \frac{\hat{\sigma}\sqrt{\Delta t_d}}{1 - e^{-\hat{a}\Delta t_d}} \tag{4.17}$$

When we take the exponentials of the right side of the expression, we get the price in which the up or down moves are certain. The probabilities are regime dependent and must be estimated for each of the volatility states $(\hat{\sigma}^H, \hat{\sigma}^L)$. When valuing an option using a single state regime it is enough to use a binomial tree. When working with two states it is required to use a tree recombined from two different binomial trees, one for each state. The recombined tree has four possible moves for each node and is therefore called a quadrinomial tree.

Constructing the tree will entail the transition probabilities between the two regimes, expressed as a matrix:

$$P = \begin{bmatrix} pHH & pHL \\ pLH & pLL \end{bmatrix} \tag{4.18}$$

Where pLL represents the probability of staying in regime H, pLH represents the probability of switching state from H to L, pHH is the probability of moving from state L to H, and pHH is the probability of staying in state L.

With the Ornstein-Uhlenbeck parameters for probabilities of up θ_u and down θ_d moves we can calculate the probabilities of each node in the recombined tree. Where the transition probability is used with the probability of up and down moves from each state. Giving us four different possibilities for each node in both regimes:

Table 4.1: **Probabilities in both regimes**

High volatility regime	Low volatility regime
$\theta_u^H(i,n) * pHH$	$\theta_u^L(i,n) * pLL$
$\theta_u^H(i,n) * pHL$	$\theta_u^L(i,n) * pLH$
$\theta_d^H(i,n) * pHH$	$\theta_d^L(i,n) * pLL$
$\theta_d^H(i,n) * pHL$	$\theta_d^L(i,n) * pLH$

The probability changes either to an up or down move in the given regime or goes up and down with a regime switch.

4.5 Risk-Neutral Probabilities

The risk neutral probabilities for an up move are:

$$\pi_u(i,n) = \frac{Zr_f - X_d}{X_u - X_d} \tag{4.19}$$

Where Z represents the price of the asset after either and up move X_u or a down move X_d. When calculating the risk-neutral probabilities we must be careful so that the risk neutral probabilities are correct for each node. The risk-neutral probability of an up move is calculated such that:

$$\pi_u(i,n) = \frac{Z(i,n)r_f - X(i+1,n+1)}{X(i,n+1) - X(i+1,n+1)}, \tag{4.20}$$

The expressions X_d and X_u are replaced by $X(i+1,n+1)$ and $X(i,n+1)$. The risk-neutral probability of a down move is:

$$\pi_d(i,n) = 1 - \pi_u(i,n) \tag{4.21}$$

One complication occurs when calculating the risk-neutral probabilities, the probability can reach either zero or one at some nodes, which results in no risk in the movements because of mean reversion. Therefore, the the risk-neutral probability of an up move is given by:

$$\pi_u(i,n) = \begin{cases} 0 & \text{if,} \quad \theta_u(i,n) = 0, \\ \frac{Z(i,n)r_f - X(i+1,n+1)}{X(i,n+1) - X(i+1,n+1)} & \text{if,} 0 < \theta_u(i,n) < 1, \\ 1 & \text{if,} \quad \theta_u(i,n) = 1, \end{cases} \tag{4.22}$$

Using this formula simplifies the process and allows us to use the standard valuation approach (Guthrie, 2009).

4.6 Using the CAPM

To estimate the risk neutral probabilities, we will use the CAPM with the same data used to calibrate the tree, thus:

$$K = E[\tilde{R}_x] - (E[\tilde{R}_m] - r_f)\beta_x \tag{4.23}$$

Where $\tilde{R}_m$ is the return of the market portfolio, $\beta_x = \frac{Cov[\tilde{R}_x, \tilde{R}_m]}{Var[\tilde{R}_m]}$ is the electricity spot price beta, and $\tilde{R}_x = \frac{\tilde{X}}{X}$. The probability of a risk neutral up move at node (i, n) is:

$$\pi_u(i,n) = \theta_u(i,n) - \frac{(E[\tilde{R}_m] - r_f)\beta_x}{U - D} \tag{4.24}$$

To calculate this, we only need market risk premium and the price beta since we already have the other estimates.

As there are numerous market risk premium estimates available, and the estimation can be a controversial issue, we will, as Guthrie (2009) suggest apply an available estimate for the Norwegian market.

For the price beta β_x we will use the residuals from the Markov-Switching model. The residuals for each regime will be regressed as the dependent variable, on the returns of a market portfolio proxy $r_{m,j}$. Which gives us:

$$\hat{u}_j = y_0 + y_1 r_{m,j} + v_j, \tag{4.25}$$

Using a broad stock index as the market portfolio proxy is popular, but it is important that the returns of the proxy is continuously compounded to keep it consistent with the composition of the residuals (Guthrie, 2009).

After obtaining the β_x and market risk premium, we adjust the time increment of the market risk premium so that we use the same time increment as our tree is. Which gives us $(E[\tilde{R}_m] - r_f) * \Delta t$. To make the beta applicable such that the time increment for the log price of the residuals is the same as for the market proxy return, we will use:

$$\beta_x = \hat{y}_1 \sqrt{\frac{\Delta t_2}{\Delta t_1} * \frac{1 - e^{-2\hat{a}\Delta t_1}}{1 - e^{-2\hat{a}\Delta t_2}}} \tag{4.26}$$

Where t_1 and t_2 represent the time increment for the return of market proxy portfolio and the time increment for our tree (Guthrie, 2009).

Next, we find the adjustment factor for the probability of an up move:

$$\frac{(E[\tilde{R}_m] - r_f)\beta_x}{U - D} \tag{4.27}$$

4.7 Constructing the Tree

To price the real options, we first must create and value the binominal tree for the underlying electricity price $V(i, n)$. We will assume the that the electricity spot price follows a mean reverting process with two volatility regimes, and that historic price fluctuations reflect the future volatility of the price.

Thus, we have to fit an $AR(1)$ model with two regimes to the historic price data and follow the steps from section (4.2) to obtain the normalized parameter estimates $\hat{a}$, $\hat{b}$ and $\hat{\sigma}$. The next task is to follow the steps from section (4.4) to obtain the sizes of U and D movements for both states.

We start with the present value of the spot price $V(0,0)$ and compute the tree by multiplying it with the U and D movements for both states $\hat{\sigma}^H$ and $\hat{\sigma}^L$. This results in four possible price movements at $n = 1$: $V(0,0)U^H$, $V(0,0)U^L$, $V(0,0)D^H$, $V(0,0)D^L$. At any following state $V(i,n)$ there will be four possible price movements: $V(i,n)U^H$, $V(i,n)U^L$, $V(i,n)D^H$, $V(i,n)D^L$. We therefore construct the quadrinomial tree by calculating the four possible price movements for each node.

4.8 Finding the Option value

After constructing the quadrinomial price tree for the underlying electricity spot price, the next step is to value the option $C(i,n)$. The option in question is an American option. To value the option, we must work backward through the tree, and value the option at each node $n(i,n)$ based on one period ahead option values $(i, n+1)$. The option value $C(i,n)$ at each node is thus derived from the four possible (U^H, U^L, D^H, D^L) next period option values, the probabilities of the U and D movements within each state, and the probabilities of a regime shift between the two states $(\hat{\sigma}^H, \hat{\sigma}^L)$:

The first task will be to derive the option value for the range of possible prices $V(i,n)$ at the last date before expiration. Consider that the number of time periods n in our tree represents the number of possible exercise dates $T = \{t_1 < t_2 < ... < t_{(n-1)}\}$, as well as investment costs associated with the exercise date $I = \{I_1, I_2, ..., I_{(n-1)}\}$. In our case the investment cost is identical for all possible exercise dates, and we can simply refer to it as I. The last possible exercise date t_n represents an investment cost of $I_n = I$. The option value C(i,n) at the last possible exercise date N, will be the greatest of zero and the options intrinsic value:

$$C(i,N) = max\{V(i,t_n) - I, 0\} \tag{4.28}$$

When we then iterate backward through the tree from the last exercise date option, we use the following expression for the value of the underlying $V(i,n)$ at any node (i,n):

$$V_{st}(i,n) = e^{-r\Delta t}\{|\pi_u^H(i,n)C_u^H + \pi_d^H(i,n)C_d^H|p_{st}^H + |\pi_u^L(i,n)C_u^L + \pi_d^L(i,n)C_d^L|p_{st}^L\} \tag{4.29}$$

Where:

$$C^H(i,n+1) = C_u^H,$$
$$C^H(i+1,n+1) = C_d^H,$$
$$C^L(i,n+1) = C_u^L,$$
$$C^L(i+1,n+1) = C_d^L,$$

In the expression we have the node specific risk neutral-probabilities up and down moves within each regime $[\pi_u^H(i,n), \pi_d^H(i,n), \pi_u^L(i,n), \pi_d^L(i,n)]$ and the probabilities of a regime shift from equation (4.18) dependent on the present regime:

Table 4.2: **Node specific risk neutral-probabilities**

High volatility regime	Low volatility regime
$\pi_u^H(i,n) * pHH$	$\pi_u^L(i,n) * pLL$
$\pi_u^H(i,n) * pHL$	$\pi_u^L(i,n) * pLH$
$\pi_d^H(i,n) * pHH$	$\pi_d^L(i,n) * pLL$
$\pi_d^H(i,n) * pHL$	$\pi_d^L(i,n) * pLH$

The option value $C(i,n)$ is the greater of $V(i,n) - I$ and zero:

$$\begin{gathered} C_{st}(i,n) = max\{e^{-r\Delta t}\{|\pi_u^H(i,n)C_u^H + \pi_d^H(i,n)C_d^H|p_{st}^H + \\ |\pi_u^L(i,n)C_u^L + \pi_d^L(i,n)C_d^L|p_{st}^L\} - I, 0\} \end{gathered} \tag{4.30}$$

This will be the risk-neutral pricing formula we use to value the real options. We use this formula to iterate backwards through the tree, starting with the option value $C(i,n)$ at the last possible exercise date. The backward recursion must be done two times, once for the high $\hat{\sigma}^H$, and once for the low $\hat{\sigma}^L$ state as the initial regime.

In the underlying price tree at any time n there will be matching values for some of the nods within the possible up and down movements combined with regime shifts. The tree therefore grows at polynomial complexity rather than exponential. This will result in a less complex tree. If we are to analyze the complexity of the model, we must find a counting system for the number of nodes at any time step n.

Porposition 1

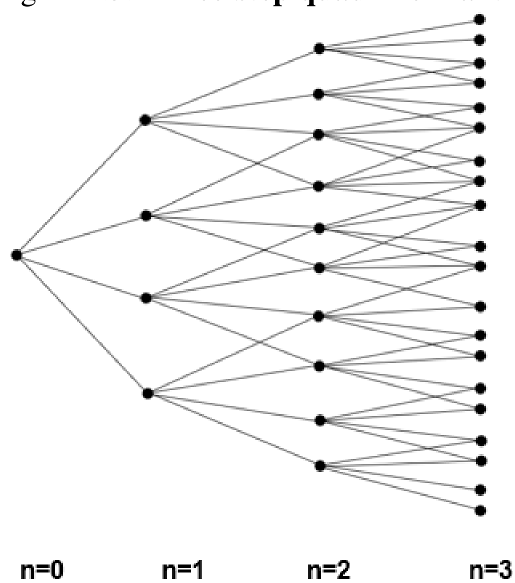
The number of distinct possible nodes at time step n is $\binom{n+2P-1}{2P-1}$where P is the number of regimes(Aingworth et al., 2006).

When determining the total number of possible nodes after n steps, we can look at the underlying electricity spot price S:

$$\begin{gathered} \text{S}(\text{u}_1)^{U_1}(d_1)^{D_1}(u_2)^{U_2}(d_2)^{D_2}...(u_P)^{U_P}(d_P)^{D_P}, \\ \sum_{i=1}^{P} U_i + D_i = n, \end{gathered} \tag{4.31}$$

Were $u_1, d_1, u_2, d_2 ... u_p, d_p$ are the up and down moves for regime P, and U_1, D_1, U_2, D_2... U_P, D_P are the number of times the moves occur(Aingworth et al., 2006).

Figure 4.3: **Three-step quadrinomial tree**



The number of distinct prices is in a one-to-one correlation with $\binom{n+2P-1}{2P-1}$ the number of different ways to select $2P-1$ distinct values in the range $1, 2, \ldots, n+2P-1$ (Aingworth et al., 2006).

The n unselected numbers represent the up and down moves, with the number of elements before the first one chosen representing the level of up moves in the regime one, between the first and second represents the level of down moves in regime one, between the second and third representing the level of up moves in regime two and so on, depending on the number of regimes P. In a two regime model the number of elements after the third chosen will represent the level of down moves in the second regime and will be the final one. Every recombinant path has one representation, and each set of choices represents a valid path.

Porposition 2

The total number of nodes in the tree after n steps is $\binom{n+2P}{2P}$ where P is the number of regimes (Aingworth et al., 2006).

$\sum_{j=0}^{n}\binom{j+2P-1}{2P-1}=\binom{2P-1}{2P-1}+\binom{2P}{2P-1}+\sum_{j=2}^{n}\binom{j+2P-1}{2P-1}$,

we can expand the two first terms on the right side of the equation by using the definition of combination:

$$\sum_{j=0}^{n}\binom{j+2P-1}{2P-1}=\frac{(2P-1)!}{(2P-1)!}+\frac{(2P)!}{(2P-1)!}+\sum_{j=2}^{n}\binom{j+2P-1}{2P-1}$$

$$\sum_{j=0}^{n}\binom{j+2P-1}{2P-1}=\frac{(2P)!}{(2P)!}+\frac{2P(2P)!}{(2P)!}+\sum_{j=2}^{n}\binom{j+2P-1}{2P-1}$$

$$\sum_{j=0}^{n}\binom{j+2P-1}{2P-1}=\frac{(2P)!(1+2P)}{(2P)!}+\sum_{j=2}^{n}\binom{j+2P-1}{2P-1}$$

$$\sum_{j=0}^{n}\binom{j+2P-1}{2P-1}=\frac{(2P+1)!}{(2P)!}+\sum_{j=2}^{n}\binom{j+2P-1}{2P-1}$$

$$\sum_{j=0}^{n}\binom{j+2P-1}{2P-1}=\frac{(2P+1)!}{(2P)!}+\frac{2P(2P+1)!}{(2P)!(2!)}+\sum_{j=3}^{n}\binom{j+2P-1}{2P-1}$$

$$\sum_{j=0}^{n}\binom{j+2P-1}{2P-1}=\frac{2(2p+1)!+2P(2P+1)!}{(2P)!(2!)}+\sum_{j=3}^{n}\binom{j+2P-1}{2P-1}$$

$$\sum_{j=0}^{n}\binom{j+2P-1}{2P-1}=\frac{2(2P+1)!+2P(2P+1)!}{(2P)!(2!)}+\sum_{j=3}^{n}\binom{j+2P-1}{2P-1}$$

$$\sum_{j=0}^{n}\binom{j+2P-1}{2P-1}=\frac{(2P+2)!}{(2P)!(2!)}+\sum_{j=3}^{n}\binom{j+2P-1}{2P-1}$$

$$=\frac{(2P+n)!}{(2P)!(n!)}$$

The proposition shows that the quadrinomial tree has a polynomial growth:

$$\binom{n+2P}{2P}=\frac{(2P+n)!}{(2P)!(n!)}$$

$$=\frac{(2P+n)(2P+n-1)(2P+n-2)..(n+1)(n)(n-1)...1}{(2P)!n!}$$

$$=\frac{(2P+n)(2P+n-1)(2P+n-2)..(n+1)n!}{(2P)!n!}$$

$= \frac{(2P+n)(2P+n-1)(2P+n-2)..(n+1)}{(2P)!}$

$= \frac{\Pi_{i=0}^{2P-1}(2P+n-1)}{(2P)!}$

The expression is a polynomial of degree 2P.

Chapter 5

Algorithm

The algorithm we implement to value the real option starts at the maturity date and loops backward one step at the time until it reaches the present date. We start by defining the input variables and estimating the necessary variables discussed in chapter 4. We then define zero matrices for the necessary variables and zero vectors for the early exercise values. We do this to pre allocate the memory and speed up the process time of the code.

The algorithm starts by using the formula in proposition 1 under section (4.8) to calculate the number of nodes at maturity. The next step is to estimate the maturity values for the electricity spot price S and the option values C^H, C^L. Then we start the backward recursion loop filling in all the time steps back to $n = 0$, where we also have node loop covering all the nodes at a certain time n. We calculate the mean-reverting probabilities for each node based on S section (4.4). Each node evaluates the option continuation value equation (4.30) relative to the intrinsic value in case of exercise. The option value will continue as the most valuable of the two. The code therefore accounts for if its most valuable to exercise or hold the option at any node. The code saves the values in which its more valuable to exercise the option at each state in vector E^H and E^L. When all nodes at a certain time level is iterated the early exercise values are applied to the cell where early exercise is optimal. Thus, we end up with the option value with early exercise boundaries based on the simulated price path. See Appendix (A.1) for full MATLAB code.

MATLAB Algorithm Main Steps:

Input

$S0$-current electricity spot price,

K-Strike price,

PM-Regime transition matrix,

$Nsteps$-n time steps,

$delt1$-1/252,

$delt2$=1/52,

NPV-NVP underlying Investment,

MRP-Market risk premium,

$BetaRNP$-Beta values for S in each state(H, L),

R-Discount factor,

(a_0, a_1 and σ)-AR(1) parameters for each state(H, L).

Output(Remark: (H, L)=High low volatility state)

$ValueEl1$ - The value of the electricity price option at time 0 in H.

$ValueEl2$ - The value of the electricity price option at time 0 in L.

$ValueInvest1$ - The value of the investment option at time 0 in H.

$ValueInvest2$ - The value of the investment option at time 0 in L.

Steps

1 Estimate $\hat{a}$, $\hat{b}$ and $\hat{\sigma}$

2 Estimate u and d

3 Estimate Risk-Neutral beta correction constant

4 Estimate Discount factor

5 Define Complexity of the tree NLeaves= $\binom{Nsteps+2P-1}{2P-1}$

Define Matrices with size (NLeaves, Nsteps) for: S, CH and CL

Define Early exercise vectors with size (Nsteps) for: $E1, E2\ldots$

Compute Intrinsic value (CH, CL) spot price S for nodes at maturity

Backward recursion:

for $n = 1$ to $Nsteps$

    Define Nodes at current time step Nnodes: =$\binom{n+2P-1}{2P-1}$

    Define Nodes at next time step nLeaves: = $\binom{n-1+2P-1}{2P-1}$

    for $h = 1$ to $Nnodes$

        Compute Expected asset value S Intrinsic value $V(i, t_n)$ at nodes

        Compute Probabilities puH puL and boundaries

        Identify All 4 corresponding nodes

        Compute Node payoffs: UD probabilities*regime transition*node value

        Compute Payoff node early exercise

        Compute Investment option values tree: $C(i, n)^{Investment}(H, L)$

    end

end

Compute Value of electricity price option at time 0: $C(i, 0)^{Electricity}(H, L)$

Compute Value of investment option at time 0: $C(i, 0)^{Investment}(H, L)$

The algorithm is tested by applying two identical regimes with arbitrary transition probabilities. By using an example from Guthrie (2009), we could utilize the given parameters and use the findings from the book to check if our algorithm provides the correct results. The algorithm gave the same results as in the book which implies that it works correctly when valuing options on mean-reverting variables with multiple volatility states as well.

Chapter 6

Results and Findings

The modeling and valuation process of the real option from chapter 2.5 can be reduced to seven main steps as discussed in chapter 4:

1. Estimate the $AR(1)$ processes with Markov switching model(MSM) in STATA.
2. Find the normalized estimates of the parameters from the MSM with Ornstein-Uhlenbeck parameters from equation (4.3) in MATLAB.
3. Calculating the up and down movements within each regime from equations (4.8) and (4.10), and construct the quadrinomial tree in MATLAB.
4. Estimating the probabilities of up and down moves at each node within each regime from equation (4.13), and the probabilities of a regime shift from equation (4.18) in MATLAB.
5. Estimating the market price of risk to compute the risk neutral probabilities from equation (4.22) in MATLAB.
6. Compute a quadrinomial tree for the electricity price and the investment as discussed in section (4.7) to (4.8).
7. Estimate the intrinsic value of the option with equation (4.28) and use backward recursion to find the present value of the real option from equation (4.30) in MATLAB.

Most of the steps are conducted by using the Matlab code mentioned in chapter 5 which is found in section (A.1). We will in this part present the most important aspects of the analysis itself as well as the results from the analysis.

6.1 Collection of Data

Our thesis utilizes time series data of spot prices from the Nordic-Baltic electricity market. The data is collected from Nord Pool's websites (Nord Pool, 2022b) and Investing.com's database (Investing.com, 2022). The data has a daily frequency over the time period 01.01.2013 to 31.12.2021.

Determining the mean reverting process

To determine mean reversion in our time series we performed multiple tests for stationarity. Both the augmented dickey fuller test, and the Phillips-Perron test gives us a p-value of 0.00 from section (A.5). In this case we must reject the null hypothesis which says that the spot price has a unit root and is non-stationary. Rejecting the null-hypothesis means that our time-series is stationary. Stationarity in a timeseries tells us that there is mean reversion in the spot price of electricity.

Regression and Ornstein-Uhlenbeck parameters

The first parameters we obtain comes from running a regression on equation (3.8) which gives us two $AR(1)$ processes. The results from the regressions are for the high volatility regime (A.7):

$$p_{j+1} - p_j = \underset{(0.00)}{0.2617441} - \underset{(0.00)}{0.0490557} p_j, \sigma = 0.301582$$

and for the low volatility state (A.7):

$$p_{j+1} - p_j = \underset{(0.00)}{0.1186061} - \underset{(0.00)}{0.0208533} p_j, \sigma = 0.06522$$

The results show us that the a_1 is negative in both regimes, this is a necessary condition to generalize the $AR(1)$ parameters with the Ornstein-Uhlenbeck process. The values displayed in the parenthesis represents the P-values from the regression. A P-value lower than 0.05 can be allowed to reject the null hypothesis which says that the coefficient equates zero and has no influence. From our regression we get significant P-values where we can reject the null hypothesis, meaning the coefficient are affecting our model.

Table 6.1: **Regression statistic**

Parameter	Highvolatility	Low volatility
Sample period	1/1/13-31/12/21	1/1/13-31/12/21
Standard error of $AR(1)$	0.301582	0.06522
Observations	3268	3268

From the log spot price from Nord Pool(Nord Pool, 2022b)

Table 6.2: **Ornstein-Uhlenbeck parameters**

Parameter	Highvolatility	Low volatility
$\hat{a}$, mean reversion	0.0502998	0.0210738
$\hat{b}$, long-term mean	5.3356511	5.6876418
$\hat{\sigma}$, standard deviation	0.301582	0.06522

The standard errors displayed in table (6.1) is an estimate of the standard error which is the standard deviation of the residuals in the regressions. The MATLAB codes for building the price tree is attached in the appendix(A.1). Other than the Ornstein-Uhlenbeck parameters displayed in table (6.2), we need multiple more parameters. The different variables needed follows below.

Current asset value S_0

For the price tree the current asset value should be the day of the valuation. But since our regression on historic spot prices ends at 31.12.2021, we will use the spot price from 01.01.2022 collected from Nord Pool's spot prices for the Nordic-Baltic market. The price was at 01.01.2022 761.64 NOK/MWh.

Risk-free rate, r_f

The risk-free interest rate we have chosen for the investment is the rate of 12-month treasury bonds from Norges Bank. We are using the rate from 03.01.22 which was 1.018% (Norges Bank, 2022).

State beta, β_x

Determining the beta for the investment can be computed in different manners. We have chosen to estimate the state state beta by regressing the regime dependent residuals on the Oslo Børs Benchmark Index(OSEBX) daily returns. We thus get two sets of Betas, one for each state(β^H, β^L). The high state beta β^H have a value of 1.048, and the low state beta β^L have a value of 1.02.

Market risk premium, MRP

The market risk premium is collected from PwC's annual report for the Norwegian market. We will use the 2021 market risk premium which was according to PwC 5.00% (PwC, 2022).

Time to maturity, T

The time to maturity for our thesis is 12 months. In this relation we will build our tree of 52 weekly steps.

Transition matrix, P

The transition matrix used to calculate the mean reverting probabilities in our quadrinomial tree was obtained from running a markov switching model (A.7). The transition matrix used in our model is:

$$P = \begin{bmatrix} 0.912385 & 0.087615 \\ 0.9611189 & 0.0388811 \end{bmatrix} \tag{6.1}$$

6.2 The risk-neutral probabilities of an up and down move

To address the mean reversion in the spot price we need to apply risk neutral probability for each node from equation (4.24). The probabilities will be in accordance with mean reversion from equation (4.13) and are estimated in MATLAB (A.1). To obtain the risk-neutral adjustment factor we need to adjust the daily Beta values (β^H, β^L) to weekly Beta estimates in accordance with equation (4.27). From equation (4.26) we get the adjustment factor of 0.0025 for each node in the low volatility state, and 0.000626 for the high volatility state. The up move probability at each node are adjusted for the state dependent adjustment factor to obtain the risk neutral probabilities in each state.

6.3 Quadrinomial Price Tree for the Underlying Investment

When we are to compute the tree for the underlying there are two important assumptions we make. The first assumption is that the net present value of the underlying investment is reflected from process that drives the value, the current electricity production. The present value of production is price dependent, and thus we assume that the NPV of the investment at any time n is reflected from the present electricity price S at that time n.

Since the value of the underlying investment is derived from the electricity price, the second assumption is that the value of the underlying investment will follow the same process as the electricity price.

Since we have these assumptions, we can compute the option tree for the electricity spot price S with the strike price K and use the tree as a proxy for the underlying investment. S_0 will then represent the NPV_0 (S_0=NPV_0), K will represent the investment cost I (K=I), and the S will represent the changes in the NPV relative to moves in the electricity spot price S (ΔS=ΔNPV). The relative sizes of up and down moves within both regimes (U^H, U^L, D^H, D^L) will thus be the same for the NPV and S (A.8) figure (A.6).

6.4 Real Option Value

When finding the option value at $n = 0$ we start with computing the intrinsic values of the of the option at the last possible exercise date as described in chapter (4.8). We still use the underlying electricity price S as a proxy for the NPV (S=NPV), and the strike price K as a proxy for the investment cost I (K=I). The reason for this is that the mean reverting property of the probabilities is price dependent. And the price dependency is estimated from parameters on historical electricity prices. Any other variable than the electricity price S would thus not be compatible with the probabilities. Since we have the assumptions discussed in chapter (6.3) this does not cause any major problems when finding the option value of the investment opportunity.

We use the intrinsic values for S and iterate backwards through the tree to find the option value at each possible node $C(i, n)$ as explained in chapter (4.8). When we have iterated all the option values, we can transform them from option values on the electricity price to the option values for the investment. The electricity option values $C(i, n)^{Electricity}$ relative to the present spot price S_0 will be identical to relation between the investment option values $C(i, n)^{Investment}$ and the NPV_0, due to the assumptions in chapter (6.3):

$$\frac{C(i,n)^{Electricity}}{S_0} = \frac{C(i,n)^{Investment}}{NPV_0} \tag{6.2}$$

The option value on the investment at any node must therefore be:

$$C(i,n)^{Investment} = \frac{C(i,n)^{Electricity}}{S_0} * NPV_0 \tag{6.3}$$

With the relation between the strike price K and investment price I being:

$$\begin{aligned} \tfrac{K}{S_0} &= \tfrac{I}{NPV_0} \\ K &= \tfrac{I}{NPV_0} * S_0 \\ I &= \tfrac{K}{S_0} * NPV_0 \end{aligned} \tag{6.4}$$

We can thus calculate the option values for the investment $C(i,n)^{Investment}$ at any node with equation (6.3).

In our case we have a "at a money call option" where $S0 = K = 761.64$ NOK and n=52. We get the following option trees for the electricity price in each of the two volatility states $(\hat{\sigma}^H, \hat{\sigma}^L)$, with the algorithm in (A.1).

Table 6.3: **Option value electricity price**

0	1	2	3	..52	0	1	2	3	..52
434.20	742.75	2209.82	5107.57		**263.99**	742.75	2209.82	5107.57	
	503.39	977.65	2673.80			270.40	977.65	2673.80	
	330.75	567.66	1249.23			232.33	287.55	1249.23	
	414.27	429.20	623.4			253.38	255.89	415.39	
		360.39	742.75				233.59	742.75	
		281.46	498.04				211.39	262.46	
		668.71	381.95				539.57	235.30	
		429.20	324.12				255.89	223.86	
		320.54	297.35				224.12	210.76	
		397.42	251.68				243.39	195.07	
			1808.51					1808.51	
			742.75					742.75	
			498.04					262.46	
			407.05					245.07	
			324.12					223.86	
			273.77					204.01	
			606.41					363.83	
			407.05					245.07	
			313.80					216.50	
			375.17					233.29	

$C(i,n)^{Electricity}$: $\hat{\sigma}^H(left), \hat{\sigma}^L(right)$: for the $n(0 \rightarrow 3)$. Figure (A.7)

We utilize equation (6.3) at each node to get the option tree for the investment opportunity with $NPV = I = 82.88$ MNOK and $n = 52$:

$$C(i,n)^{Investment} = \frac{C(i,n)^{Electricity}}{S_0} * NPV_0$$

Table 6.4: **Option value investment**

0	1	2	3	..52	0	1	2	3	..52
47.25	80.82	240.45	555.77		**28.73**	80.82	240.45	555.77	
	54.22	106.38	290.94			29.42	106.38	290.94	
	35.35	61.19	135.93			25.28	31.29	135.93	
	44.35	46.21	67.23			27.57	27.84	45.20	
		38.56	80.82				25.42	80.82	
		29.98	53.67				23.00	28.56	
		72.19	40.89				58.71	26.60	
		46.21	34.68				27.84	24.36	
		34.15	31.69				24.39	22.93	
		42.53	26.75				26.48	21.23	
			196.79					196.79	
			80.82					80.82	
			53.67					28.56	
			43.61					26.67	
			34.68					24.36	
			29.09					22.20	
			65.39					39.59	
			43.61					26.67	
			33.34					23.56	
			40.16					25.39	

$C(i,n)^{Investment}$: $\hat{\sigma}^H(left), \hat{\sigma}^L(right)$: for the $n(0 \rightarrow 3)$. Figure (A.8)

As we can see in table (6.3) the present value $(n = 0)$ of a 1 year American "at the money" call option on the electricity price is valued at 434.20 NOK in the high volatility state and 263.99 NOK in the low volatility state.

As for the hydropower investment, table (6.4) illustrates that the present value of the real option is 47.25 million NOK at $n = 0$ in a high volatility state, and a value of 28.73 million NOK if in a low volatility state. Remembering table (3.1) from chapter (3) volatility has positive correlation with the option value and it is expected that the value of investment opportunity is larger in the high volatility state.

The nodes in table (6.5) marked in bold are the nodes where the intrinsic value is larger than the continuation value of the option. This is instances where its optimal to exercise the option rather than holding it. These are nodes where the underlying electricity price is higher than the expected electricity prices between this date and maturity. Since the price is mean reverting, this will be the case at certain relatively high prices. In the low volatility regime, there is more optimal early exercise dates since the probability of large upside fluctuations will be lower, and thus early exercise will be optimal at lower prices than in the high volatility regime. The remaining nodes are instances where the continuation value of the option is larger than the intrinsic value, and the optimal strategy is to hold the option.

Table 6.5: **Early exercise boundaries for investment**

1	2	3	..52	1	2	3	..52
80.82	**240.46**	**555.77**		**80.82**	**240.46**	**555.77**	
54.22	**106.38**	**290.95**		29.42	**106.38**	**290.95**	
35.36	61.19	**135.93**		25.28	31.29	**135.93**	
44.35	46.21	67.23		27.57	27.84	**45.20**	
	38.56	**80.82**			25.42	**80.82**	
	29.98	53.67			23.00	28.56	
	72.19	40.89			**58.71**	25.60	
	46.21	34.68			27.84	24.36	
	34.15	31.69			24.391	22.93	
	42.53	26.75			26.48	21.23	
		196.79				**196.79**	
		80.82				**80.82**	
		53.67				28.56	
		43.61				26.67	
		34.68				24.36	
		29.09				22.20	
		65.39				**39.59**	
		43.61				26.67	
		33.34				23.56	
		40.16				25.39	

$C(i,n)^{Investment}$: $\hat{\sigma}^H(left), \hat{\sigma}^L(right)$: for the $n(1 \to 3)$. Figure (A.9)

6.5 Sensitivity Analysis

To analyze the key driver of the option valuation we will perform a sensitivity analysis on the volatility and the strike price. As we have two regimes, we will perform a change in volatility in both the high and low volatility regime. The strike price will remain the same for both regimes.

First, we will present an analysis of changing the strike price with 10% both upwards and downwards. Table (6.5) display the effects of changing the strike price on the real option value.

Table 6.6: **Sensitivity anlysis of strike price,** K

Strike price, k	685.48 $(10\% decrease)$	**761.64**	837.80 $(10\% increase)$
*Real option value in H	51.42 $(8.82\% increase)$	**47.25**	43.69 $(7.53\% decrease)$
*Real option value in L	30.58 $(6.44\% increase)$	**28.73**	26.50 $(7.76\% decrease)$

(*) indicates variable in MNOK

Our sensitivity analysis on the strike price, K tells us that a decrease in the strike price by 10% causes an increase in both the value if in the high volatility state and the low volatility state by approximately the same percentage. However, an increase in the strike price by 10% provides a smaller decrease in both regimes. The value if in a high volatility state decreases by 8.82% while the decrease if in the low volatility state is only 6.44%.

The sensitivity analysis will keep the volatility in one of the regimes constant while changing the volatility in the other regime. Table (6.7) and (6.8) displays the change in real option value when preforming a change in volatility.

Table 6.7: **Sensitivity analysis of volatility** H

Volatility in H	28.15% $(6.6\% decrease)$	**30.15%**	32.15% $(6.6\% increase)$
Volatility L	**6.52%** $(Constant)$	**6.52%**	**6.52%** $(Constant)$
*Real option value in H	39.96 $(15.4\% derease)$	**47.25**	57.36 $(21.4\% increase)$
*Real option value in L	22.87 $(20.4\% decrease)$	**28.73**	34.77 $(21.02\% increase)$

Table 6.8: **Sensitivity analysis of volatility** L

Volatility in L	4.52% $(30.21\% decrease)$	**6.525%**	8.52% $(30.21\% increase)$
Volatility H	**30.15%** $(Constant)$	**30.15%**	**30.15%** $(Constant)$
*Real option value in H	46.48 $(1.63\% derease)$	**47.25**	47.91 $(1.40\% increase)$
*Real option value in L	27.46 $(4.42\% decrease)$	**28.73**	30.60 $(6.51\% increase)$

(*) indicates variable in MNOK

The results from the sensitivity analysis tells us that a change in the high volatility state yields a larger change in the real option valuation. The results tell us that the largest change in the real option value comes from an increase in the volatility in H by two per centage points which equates a change of 6.6% in volatility gives us an increase of 21.4% in the real option if in the high volatility state, and an increase of 21.02% if in the low volatility regime. The results are aligned with real option theory where a higher volatility drives the value and creates a larger upside. This was discussed in chapter 3 and displayed in table (3.1). To conclude, the key driver which affects the real option value the most is the volatility in the high volatility regime.

Chapter 7

Discussion

Our thesis contains a real option valuation of an investment opportunity, but the essence of the thesis is to model mean reversion and dramatical breaks in the behavior of a time series. We have combined methodologies and presented an approach which can be an alternative to traditional methods. The methodology can be applied to processes and time series that share behavior similarities with the electricity spot price in the Nordic-Baltic power market. In this chapter we will first, discuss the results and our findings. Secondly, we will discuss the methodology, and its benefits and limitations as well as providing a description to critical choices.

7.1 Methodology

As presented in chapter one, our thesis aims to combine methodology such that we can design an approach that can address specific processes and behaviors. In our research, we have to the best our ability, structured a method that can on a surface level capture the behavior of the spot price of electricity. We are aware of the limits as well as the benefit the model provides.

7.2 Modeling the Time Series

Let us now analyze the mean reversion process. The model does what our intension was from the start. It is able to integrate the mean reversion of the electricity price by using a standard Ornstein-Uhlenbeck process. With this process, the probability and price have a development which mimics the realistic properties of the commodity. This is contrary to widespread methods. The Ornstein-Uhlenbeck process is in our case used instead of a geometric Brownian Motion which is is the continuous time equivalent of a random walk with drift for log prices. The simplicity of only using geometric Brownian motion demands just the calculation of volatility and the term for drift, while our method of choice, demands the calculation of both the long-term mean and the level of mean reversion as described in chapter 3, which complicates the method to some degree. Further, the mean reverting process does not address every part of the electricity price, there are unfortunately some drawbacks and areas to improve. Commodities and time series may in some instances be affected by seasonality. This means that a season can be characterized with specific behavior. For instance, remembering back to chapter 2, the spot price of electricity is higher during the winter season and lower during the summer. Our methodology does not address and model this aspect of the time series. This is one drawback in our thesis which can be improved by further research. There are multiple ways to take seasonality into account. It can be done by applying dummy-variables to the AR-process, using a sine cosine function in the regression, or utilizing seasonality filters to the time series. Our reasoning for not applying sine cosine function was its complexity in the technical aspect. We also found that utilizing smoothing filters did virtually nothing to the seasonality.

Further, the model carries the benefit of capturing what is described as extraordinary changes in behavior which can occur from multiple different reasons such as policy changes, weather changes or geo-political shifts. By dividing the time series into two different regimes with a Markov switching model, our model captures these changes. With few adjustments it is also possible to apply more than two regimes. This benefit makes it more appealing than a regular binomial lattice approach which does not hold the same properties. Next, we wish to address the choice of using two regimes. Our research consists of one high volatility regime and one low volatility regime. Our logic for using only two regimes comes from two main reasons. First, it would limit the possible steps, as more than two regimes would increase the number of possible movements and the run time for the code would substantially longer. Secondly, our statistical program did not manage to apply a regression with switching variance over more than two states, which indicates that the time series only contained two apparent regimes. However, there is evidence of jumps and spikes in our time series. This can be observed in the graph displayed in figure (2.2) These jumps are not accounted for in the model.

On a broader level the methodology holds the benefits of its applicability to other commodities other than the price of electricity. It is easy to change certain aspects of the model such that you can address different process and behaviors. To end, the theory and intuition easy to understand and digestible.

7.3 Obtaining the Risk Adjustment

As there are no concrete answer to what the correct required rate of return or beta for an investment is, we wish to discuss the method for obtaining the risk neutral probabilities and discount factor. Guthrie (2009) suggest using the relationship between spot and futures prices to estimate risk neutral probabilities since CAPM does not capture risk reward in a sufficient way. Our thesis does not apply this method as there is not enough data on futures prices in the Nordic-Baltic market. On this note, we decided to apply the CAPM even if it assumes that historical data is a reliable indicator of the future and that the CAPM correctly compensates investors for risk. When calculating the CAPM we used the residuals of the markov switching model, and ran a regression with the return on the Oslo benchmark Index.

7.4 Marketed Asset Disclaimer

The markedet asset disclaimer is used to solve the problem that occurs from incomplete markets. The disclaimer makes an assumption that market value estimate for the project is the present value of the project itself, but with no flexibility, as first described in chapter 3. When we valuate options in our thesis we use this assumption, but it comes with some problems. The valuation is prone to inaccuracies as there are no way to test the assumption in a market. For this reason, it is important to emphasize that the value of the underlying is not in its entirety solved. As the production of Usma is electricity, we have made the assumption that the volatility of the investment is equal to the commodity. This assumption creates openings for errors in the valuations. An alternative to fix this issue can be to individualize the operational income and spendings and assume fixed costs of operation per unit produced and continuous production. In our case this is difficult as the cost of production in a river sourced hydropower plant is significantly low and does not vary much. At the same time both income and costs are not deterministic.

7.5 Technicality

To apply our method, we utilized both STATA and MATLAB. Both the programs and our application of the programs have their benefits and difficulties. Using STATA to perform the $AR(1)$ regression is intuitive and straight forward. It was easy to transform data and manage variables. However, there are limitations to what the program can do. We were not able to build our own models, and we could only use the models which are included in the program. We recognize that we failed to include the weekend effect in our $AR(1)$ regression in STATA. The spot price has differences over the weekend, but the price for the weekend is set before the Friday. This thesis ran an autoregression with a daily frequency without weekend breaks. Addressing this effect and integrate it in the technical aspect would improve the thesis.

As the mathematical and technical aspect of model increased in complexity we had to switch to MATLAB where there is more room to design and build your own models. We experienced more possibilities and more accessibility to information and tutorials online. Coding in MATLAB did have its drawbacks. We were not as experienced with the program and had to spend time learning the basics and the fundamentals. For this reason, we believe we have a MATLAB-code which can be simplified and made more efficient by a seasoned programmer. Running the code with 52 steps required approximately 85 minutes to finish on a computer with 8 GB ram and a Ryzen AMD 5 processor. A better computer may perform the code faster. Even though there are areas to improve, the code has the benefit of being easy to adapt. Applying the code to a different methodology can be done with only a few adjustments.

As mentioned in chapter 5 we have tested our MATLAB code by applying two exact regimes with arbitrary transition probabilities. By using an example from Guthrie (2009), we could utilize the given parameters and use the findings from the book to check if our algorithm provides the correct results. The algorithm gave the same results as in the book which implies that it works correctly when valuing options on mean-reverting variables with multiple volatility states as well. Unfortunately, we cannot generalize to multiple regimes. There might still be a problem that did not show up when having two identical regimes.

Chapter 8

Concluding Remarks

The presence of regime switching commodity prices in real option valuation is to the best of our knowledge, limited in real option theory. It mostly consists of either binomial lattice approaches or regime switching but, in continuous time. To this end, we believe that our thesis contributes to the limited literature. We believe the contribution is combining methodologies, such that mean reversion and regime switching is disclosed in the algorithm.

We can conclude this thesis with the fact that our algorithm is an efficient and easy way to value American real options with a mean-reverting underlying variable with multiple volatility states. It gives a fair option price and supplies early exercise boundaries. We have illustrated how one could apply the algorithm in a real-life investment in a hydropower plant. Under the marketed asset disclaimer, we assume the NPV of the underlying investment is reflected from the present value of production and thus the present electricity price, and that the underlying investment will follow the same process as the electricity price in table (6.3). Under these assumptions we value the 1-year real option with 52 possible exercise dates. We estimate the present value of the option as displayed in table (6.4) with early exercise boundaries presented in table (6.5) for both volatility states.

The algorithm can easily be transformed to value an option on an underlying variable that follows the geometric Brownian-motion as well. One would merely have to change the probabilities from mean reverting to static, and the use Brownian motion or another stochastic model to estimate the movement factors. We can thus conclude that our model is quite versatile as well.

8.1 Further Research

The core of our thesis is to model the volatility of the electricity prices. We believe our research provides a simplistic and solid model. However, there are areas that can be extended for further research. First, we believe that if the seasonal behavior of electricity is implemented in the model, it would prove better accuracy of the result. Next, the reality of spikes and jumps in the time series can argue for the necessity of a process that captures akin behavior. Additionally, the exploration of including more regimes can provide an interesting extension, either through elimination or addition. Hence, we propose that any further research consider the possibilities of such additions. Although it is not an essential part of the thesis, an interesting inclusion would be to apply the relationship between futures and spot prices as the estimate for the risk neutral probabilities as Guthrie (2009) suggests, if possible.

In addition, there are some technical aspects to consider expanding. First, the data treatment to address the weekend effect would be beneficial to the proficiency of the model. Lastly, we believe there are possibilities to streamline the algorithm, such that the code runs faster and can be applied to a larger number of steps.

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Appendix A

Appendix

A.1 MATLAB Code

```
% Input in the model:
    % S0=start price
    % K=Strike Price
    % PM=Transition probability matrix for the regimes
    % R=Discount rate
    % Sigma=Volatility in the two different regimes [H, L]
    % opttype=0 for call option
    % Nsteps=number of timesteps
    % S0=start price
    % delt1=time increment for Ornstein-Uhlenbeck parameters
    % delt2=time increment for Nsteps
    % BetaRNP=Beta for risk-neutural-probabilities
    % MRP=Market risk premium
    % NPV=Net present value investment

% Output in the model:
```

```
% ValueInvestH=Investment Option Value in the high
% volatility state.
% ValueInvestL=Investment Option Value in the low
% volatility state.

% Input
PM=[0.9122651, 0.0877349; 0.0390129, 0.9609871];
sigma=[0.301582, 0.06522];
opttype=0;
S0=761.64;
Nsteps=52;
R=0.01018;
K=761.64;
delt1=(1/252);
delt2=(1/52);
BetaRNPH=1.048;
BetaRNPL=1.020;
MRP=0.05;
NPV=82.877;

% Input from AR(1) Markov switching model:
a0H=0.2617441;
a1H=-0.0490557;
```

```
sdh=0.301582;
a0L=0.1186061;
a1L=-0.0208533;
sdl=0.06522;

% Normalized yearly estimates of the parameters Ornstein-
% Uhlenbeck:
alfaH=(-log(1+a1H))/delt1;
betaH=-(a0H/a1H);
sdH=sdh*((2*log(1+a1H))/(a1H*(2+a1H)*delt1))^(1/2);
alfaL= (-log(1+a1L))/delt1;
betaL= -(a0L/a1L);
sdL= sdl*((2*log(1+a1L))/(a1L*(2+a1L)*delt1))^(1/2);s

% Estimating the size of up and down moves:
uH=exp(sdH*sqrt(delt2));
uL=exp(sdL*sqrt(delt2));
dH=exp(-sdH*sqrt(delt2));
dL=exp(-sdL*sqrt(delt2));
u=[ uH, uL];
d=[ dH, dL];

% Estimating the risk-neutural Beta
```

```
BetaRNH=BetaRNP*sqrt((delt1/delt2)*((1-exp(-2*alfaH*delt2))/
(1-exp(-2*alfaH*delt1))));
BetaRNL=BetaRNP*sqrt((delt1/delt2)*((1-exp(-2*alfaL*delt2))/
(1-exp(-2*alfaL*delt1))));

% Calculating the up move risk-neutural correction
% constant:
RNpH=(MRP*delt2*BetaRNH)/(uH-dH);
RNpL=(MRP*delt2*BetaRNL)/(uL-dL);

% Calculate the discount factor DF
DF= exp((R)*(delt2));

% Number of regimes
Nsigma = length(sigma);

% Calculate the complexity of the tree
NLeaves= nchoosek(Nsteps-1+(2*Nsigma), (2*Nsigma)-1);

% Adding continuation value and price in the tree with early
% exercise vectors
CH=zeros(NLeaves, Nsteps+1);
CL=zeros(NLeaves, Nsteps+1);
```

```
S=zeros(NLeaves, Nsteps+1);
B=ones(NLeaves, 1);
EH=zeros(Nsteps+1, 1);
EL=zeros(Nsteps+1, 1);

% Adding the rest of the matrices with one step less
% then Nleaves
NLeaves1=nchoosek(Nsteps-2+(2*Nsigma), (2*Nsigma)-1);

puH=zeros(NLeaves1,Nsteps);
puL=zeros(NLeaves1,Nsteps);
pdH=zeros(NLeaves1,Nsteps);
pdL=zeros(NLeaves1,Nsteps);
phi1=zeros(NLeaves1,Nsteps);
FH=zeros(NLeaves1,Nsteps);
FL=zeros(NLeaves1,Nsteps);
psiH=zeros(NLeaves1,Nsteps);
psiL=zeros(NLeaves1,Nsteps);
CVH=zeros(NLeaves1,Nsteps);
CVL=zeros(NLeaves1,Nsteps);

% adding the movements vector:
ud= [u d];
```

```

% Obtaining the powers of the movements value for the leaves
[Nmatrix] =multinomial_powers_recursive(Nsteps, Nsigma*2);

% Compute the price and the option values in the leave
% nodes at the maturity
for h=1:NLeaves
          for j=1: 4
          B(h) = ud(j)^(Nmatrix(h,5-j))*B(h);
          end
 S(h,1)= S0 * B(h);
end
CH(:,1)=opttype*max(K-S(:,1) ,0)+(1- opttype)*max(S(:,1)-K,0);
CL(:,1)=CH(:,1);

% the last value of the early exercise boundary is always
% equal to the strike
EH(Nsteps+1)=K;
EL(Nsteps+1)=K;

% Variables for holding the early exercise values at each
% level
eH=zeros(NLeaves+1,1);
```

```
eL=zeros(NLeaves+1,1);

%Starting the backward recursion:
for n=1:Nsteps

  %Number of nodes at current time step:
  Nnodes= nchoosek(Nsteps-1-n+(2*Nsigma), (2*Nsigma)-1);

  %Number of nodes at time step after the current:
  Nleaves=nchoosek(Nsteps-n+(2*Nsigma), (2*Nsigma)-1);

  %Power matrix that corresponding to the time steps above:
  [Nmatrix]=multinomial_powers_recursive(Nsteps+1-n,
   Nsigma*2);
  [Nmatri]=multinomial_powers_recursive(Nsteps-n,
   Nsigma*2);

  B=ones(Nnodes, 1);
  for h=1:Nnodes
    % B = movement factors accumulation at current node:
    for j=1:4
      B(h) =ud((j))^(Nmatri(h, 5-j))*B(h);
    end
```

```
    % S=expected value of the asset at current node:
    S(h, n+1)=S0* B(h);

    % phi= intrinsic value of the option at current node.
    phi1(h, n)=opttype*(K-S(h, n+1))+(1-opttype)*
     (S(h, n+1)-K);

    % Calculate risk neutrual up move probabilities for each
    % node for high state(puH) and low state(puH).
    puH(h,n)=0.5+(((1-exp(-alfaH*delt2))*(betaH-log
     (S(h,n+1))))/(2*sdH*sqrt(delt2)))-RNpH;
    puL(h,n)=0.5+(((1-exp(-alfaL*delt2))*(betaL-log
     (S(h,n+1))))/(2*sdL*sqrt(delt2)))-RNpL;

    % Adding upper and lower boundaries for the probabilities.
    puH(puH<=0)=0;
    puH(puH>=1)=1;

    puL(puL<=0)=0;
    puL(puL>=1)=1;

    % Calculate down move probabilities for each node for
```

```
    % high state(puH) and low state(puH)
    pdH(h, n)=(1-puH(h, n));
    pdL(h, n)=(1-puL(h, n));

    % In order to identify the transition direction to the
    % previous step, subtract powers of the current node
    % from the powers of all the nodes in the next time step,
    % then identify the indices of the only 4 positively
    % valued rows
    Trans=bsxfun(@(A,B) A-B, Nmatrix, Nmatri(h,:));
    [Row, ~]=find(Trans<0);
    Trans(Row,:)=0;
    [cc, ~]=find(Trans==1);

    cc=flip(cc);

    %Calculating the node continuation value in both states by
    %matching the probabilities appropriately with the values:
    FH(h,n)=(puH(h,n)*PM(1,1)*CH(cc(1),n)) + (puL(h,n)*PM(1,2)
    *CL(cc(2),n))+ (pdH(h,n)*PM(1,1)*CH(cc(3),n))+ (pdL(h,n)*
    PM(1,2)*CL(cc(4),n));
    FL(h,n)=(puH(h,n)*PM(2,1)*CH(cc(1),n)) + (puL(h,n)*PM(2,2)
    *CL(cc(2),n))+ (pdH(h,n)*PM(2,1)*CH(cc(3),n))+ (pdL(h,n)*
```

```
    PM(2,2)*CL(cc(4),n));

    %psi=Continuation value at the nodes:
    psiH(h, n)=exp(-R*delt2)*FH(h,n);
    psiL(h, n)=exp(-R*delt2)*FL(h,n);

    %Checking for early exercise:
    CH(h,n+1)=max(phi1(h,n),psiH(h,n));
    CL(h,n+1)=max(phi1(h,n),psiL(h,n));

    %save early exercise value:
    if phi1(h,n)>psiH(h,n)
       eH(h)=CH(h,n+1);
    end
    if phi1(h,n) > psiL(h,n)
       eL(h)=CL(h,n+1);
    end

    %Calculating the investment option value(millions NOK)
    %for each node, in both states.
    CVH(h,n)=(CH(h,n+1)/(S0))*NPV;
    CVL(h,n)=(CL(h,n+1)/(S0))*NPV;

```

```
end

%Filtering the minimum of the early exercise values:
eH(eH==0)=NaN;
eL(eL==0)=NaN;
[eeH, IH]=min(eH);
[eeL, IL]=min(eL);
%Taking the spot prices that corresponds to the minimum
%early exercise values:
EH(Nsteps+1-n)=S(IH, n+1);
EL(Nsteps+1-n)=S(IL, n+1);

%Filling the emty values of the vectors E with zeros
if isempty(eeH) || isnan(eeH)
  EH(Nsteps+1-n)=0;
end
if isempty(eeL) || isnan(eeL)
  EL(Nsteps+1-n)=0;
end

% Reset the vectors e for the next round:
eH=zeros(NLeaves+1,1);
eL=zeros(NLeaves+1,1);
```

```
end

%The value of the electricity price option at time 0:
ValueELH=CH(1, Nsteps+1);
ValueELL=CL(1, Nsteps+1);

%The value of the option on the investment at time 0:
ValueInvestH=(ValueELH/S0)*NPV;
ValueInvestL=(ValueELL/S0)*NPV;
```

A.2 MATLAB Code function for powers

Our algorithm uses the following algorithm to find the movement factors for the matrices at each time level. The algorithm is collected and directly quoted from: (Isaac, 2014)

```
function [Nmatrix] = multinomial_powers_recursive(pow,ndim)
% computes the multinomial expansion of
% (x_0 + x_1 + x_2 + ... + x_ndim)^pow
% Nmatrix is a matrix of powers
% This is equivalent to finding all multi-indices with norm=1
% Need another thing to calculate the coefficients, but that
% is easy recursive on dimension!
if ndim==1,
  Nmatrix = pow;
else
  % recurse
  Nmatrix = [];
  for pow_on_x1 = 0:pow,
    % say we fix the power in the first dimension to be
    % "pow_on_x1" (0,1,2,...) then the possible terms are
    % all terms for [(pow-pow_on_x1),ndim-1]
    [newsubterms] = multinomial_powers_recursive
    (pow-pow_on_x1,ndim-1);
```

```
    % stick on the power for the x1 part and add to Nmatrix
    Nmatrix = [Nmatrix; [pow_on_x1*ones
    (size(newsubterms,1),1) , newsubterms] ];
  end
end
```

A.3 STATA Commands

```
import excel, sheet("elspot-prices-2013-daily-nok") firstrow
rename Date date
rename Price price
tsset date, daily
gen ln_price=ln(price)
reg d.ln_price l1.ln_price, robust
arima d.ln_price l1.ln_price
mswitch ar d.ln_price, switch(l1.ln_price) ar(0) varswitch
predict residH residL, residuals
rename Close price_market
gen ln_price_market = ln(price_market)
gen returnm = d1.(ln_price_market)
reg residH returnm, robust
reg residL returnm, robust
```

A.4 STATA $AR(1)$

Figure A.1: **Auto regression**

```
. arima dln_price l1.ln_price, robust

(setting optimization to BHHH)
Iteration 0:   log pseudolikelihood =  1048.7765
Iteration 1:   log pseudolikelihood =  1048.7765

ARIMA regression

Sample: 1/3/2013 thru 12/31/2021                Number of obs     =       3285
                                                Wald chi2(1)      =      17.85
Log pseudolikelihood = 1048.776                 Prob > chi2       =     0.0000

------------------------------------------------------------------------------
             |              Semirobust
   dln_price | Coefficient  std. err.      z    P>|z|     [95% conf. interval]
-------------+----------------------------------------------------------------
dln_price    |
    ln_price |
         L1. |   -.038559   .0091268    -4.22   0.000    -.0564472   -.0206708
             |
       _cons |   .2151596    .052143     4.13   0.000     .1129612     .317358
-------------+----------------------------------------------------------------
      /sigma |   .1758365    .005703    30.83   0.000     .1646587    .1870142
------------------------------------------------------------------------------
Note: The test of the variance against zero is one sided, and the two-sided
      confidence interval is truncated at zero.
```

A.5 STATA ADF and PP test

Figure A.2: **ADF test**

```
. dfuller ln_price

Dickey-Fuller test for unit root          Number of obs  = 3,285
Variable: ln_price                        Number of lags =     0

H0: Random walk without drift, d = 0

                                       Dickey-Fuller
                   Test        -------- critical value --------
              statistic           1%           5%          10%
---------------------------------------------------------------
 Z(t)            -8.002       -3.430       -2.860       -2.570
---------------------------------------------------------------
MacKinnon approximate p-value for Z(t) = 0.0000.
```

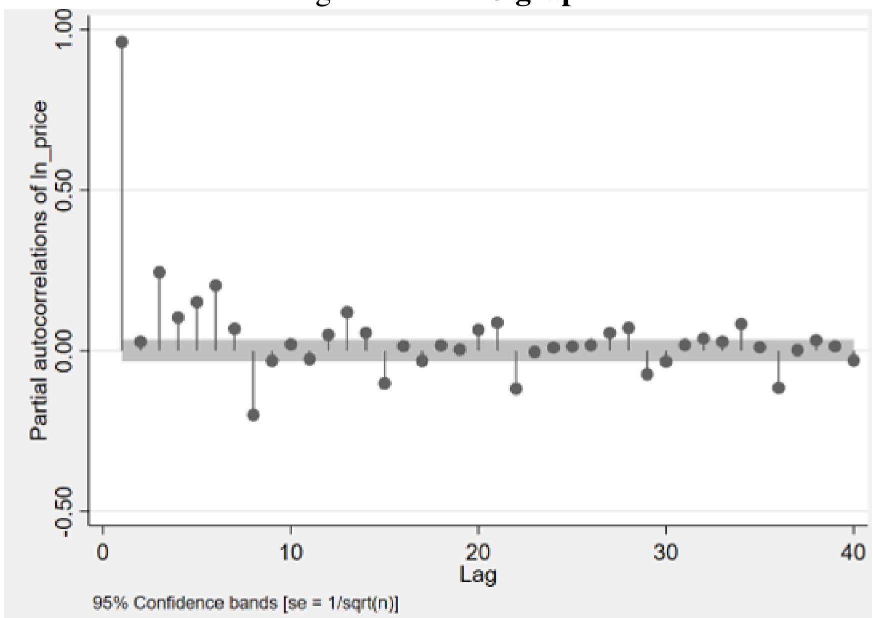
Figure A.3: **PPERRON test**

```
. pperron ln_price

Phillips-Perron test for unit root       Number of obs   = 3,285
Variable: ln_price                       Newey-West lags =     8

H0: Random walk without drift, d = 0

                                       Dickey-Fuller
                   Test        -------- critical value --------
              statistic           1%           5%          10%
---------------------------------------------------------------
 Z(rho)         -71.997      -20.700      -14.100      -11.300
 Z(t)            -6.058       -3.430       -2.860       -2.570
---------------------------------------------------------------
MacKinnon approximate p-value for Z(t) = 0.0000.
```

Augmented dickey fuller and phillips-perron test

A.6 STATA Test for Autocorrelation

Figure A.4: **PAC graph**



A.7 STATA Markov Switching Model

Figure A.5: **Markov Switching Regression**

```
. mswitch ar d.ln_price, switch(l1.ln_price) ar(0) varswitch

Performing EM optimization:

Performing gradient-based optimization:

Iteration 0:   log likelihood =  2387.7368
Iteration 1:   log likelihood =   2389.493
Iteration 2:   log likelihood =  2389.4955
Iteration 3:   log likelihood =  2389.4955

Markov-switching autoregression

Sample:    1/2/2013 thru 12/31/2021                    Number of obs =    3,286
Number of states = 2                                   AIC           = -1.4495
Unconditional probabilities: transition                HQIC          = -1.4442
                                                       SBIC          = -1.4346

Log likelihood = 2389.4955
```

D.ln_price	Coefficient	Std. err.	z	P>|z|	[95% conf.	interval]
State1						
ln_price						
L1.	-.0490557	.0101442	-4.84	0.000	-.068938	-.0291735
_cons	.2617441	.054725	4.78	0.000	.1544851	.3690031
State2						
ln_price						
L1.	-.0208533	.0042126	-4.95	0.000	-.0291098	-.0125967
_cons	.1186061	.0240818	4.93	0.000	.0714066	.1658057
sigma1	.301582	.0083158			.2857159	.3183291
sigma2	.06522	.0016245			.0621125	.0684831
p11	.912385	.0138929			.8810572	.9360604
p21	.0388811	.0058557			.0288966	.0521303

A.8 Quadrinomial Trees

Figure A.6: **Price/Value tree**

0	1	2	3	4 ...52	0	1	2	3	4 ...52
761,64	1504,39	2971,46	5869,21	11592,85	82,88	163,70	323,34	638,65	1261,46
NOK	880,57	1739,29	3435,44	6785,66	Million NOK	95,82	189,26	373,82	738,37
	385,60	1018,06	2010,87	3971,86		41,96	110,78	218,81	432,19
	658,78	761,64	1177,03	2324,86		71,68	82,88	128,08	252,98
		445,81	1504,39	1360,81			48,51	163,70	148,08
		195,22	880,57	2971,46			21,24	95,82	323,34
		1301,21	515,42	1739,29			141,59	56,09	189,26
		761,64	385,60	1018,06			82,88	41,96	110,78
		333,52	225,71	595,90			36,29	24,56	64,84
		569,80	98,84	761,64			62,00	10,75	82,88
			2570,15	445,81				279,67	48,51
			1504,39	260,95				163,70	28,39
			880,57	195,22				95,82	21,24
			658,78	114,27				71,68	12,43
			385,60	50,04				41,96	5,44
			168,86	5076,54				18,37	552,40
			1125,47	2971,46				122,47	323,34
			658,78	1739,29				71,68	189,26
			288,48	1018,06				31,39	110,78
			492,85	1301,21				53,63	141,59
				761,64					82,88
				445,81					48,51
				333,52					36,29
				195,22					21,24
				85,49					9,30
				2223,03					241,90
				1301,21					141,59
				761,64					82,88
				569,80					62,00
				333,52					36,29
				146,05					15,89
				973,47					105,93
				569,80					62,00
				249,52					27,15
				426,29					46,39

The price/value tree (U^H, U^L, D^H, D^L) for S (left) and NPV (right) from $n(0 \rightarrow 4)$.

Figure A.7: **Option value electricity tree**

0	1	2	3	4	5 …52	0	1	2	3	4	5 …52
434,20	742,75	2209,82	5107,57	10831,21	22136,49	**263,99**	742,75	2209,82	5107,57	10831,21	22136,49
NOK	503,39	977,65	2673,80	6024,02	12641,36	NOK	270,40	977,65	2673,80	6024,02	12641,36
	330,75	567,66	1249,23	3210,22	7083,56		232,33	287,55	1249,23	3210,22	7083,56
	414,27	429,20	623,40	1563,22	3830,40		253,38	255,89	415,39	1563,22	3830,40
		360,39	742,75	682,86	1926,23			233,59	742,75	599,17	1926,23
		281,46	498,04	2209,82	811,65			211,39	262,46	2209,82	811,65
		668,71	381,95	977,65	5107,57			539,57	235,30	977,65	5107,57
		429,20	324,12	562,02	2673,80			255,89	223,86	280,29	2673,80
		320,54	297,35	393,98	1249,23			224,12	210,76	236,94	1249,23
		397,42	251,68	424,15	617,51			243,39	195,07	247,71	415,39
			1808,51	353,42	403,22				1808,51	225,12	238,87
			742,75	305,85	742,75				742,75	209,92	742,75
			498,04	274,08	492,65				262,46	202,69	254,47
			407,05	261,36	374,67				245,07	193,26	226,81
			324,12	230,54	306,71				223,86	180,84	208,81
			273,77	4314,90	317,41				204,01	4314,90	215,30
			606,41	2209,82	289,67				363,83	2209,82	202,03
			407,05	977,65	264,57				245,07	977,65	191,19
			313,80	562,02	243,86				216,50	280,29	186,20
			375,17	663,45	236,89				233,29	539,57	178,66
				424,15	213,70					247,71	165,74
				353,42	9265,52					225,12	9265,52
				312,58	5107,57					215,51	5107,57
				274,08	2673,80					202,69	2673,80
				244,72	1249,23					188,30	1249,23
				1461,39	617,51					1461,39	415,39
				663,45	1808,51					539,57	1808,51
				424,15	742,75					247,71	742,75
				390,01	492,65					234,97	254,47
				312,58	374,67					215,51	226,81
				269,98	399,76					197,14	236,68
				542,25	317,41					274,02	215,30
				390,01	289,67					234,97	202,03
				302,84	265,43					208,83	195,21
				344,09	243,86					223,15	186,20
					223,73						174,02
					3629,28						3629,28
					1808,51						1808,51
					742,75						742,75
					492,65						254,47
					600,54						363,83
					399,76						236,68
					317,41						215,30
					305,74						207,80
					265,43						195,21
					241,90						181,83
					1161,16						1161,16
					600,54						363,83
					399,76						236,68
					367,98						224,80
					305,74						207,80
					262,99						190,42
					471,18						251,08
					367,98						224,80
					284,39						200,79
					309,44						213,61

$C(i,n)^{Electricity}$: $\hat{\sigma}^H$(left) and $\hat{\sigma}^L$(right) for n(0→5). Estimated with matlab code from (A.1)

Figure A.8: **Option value investment tree**

0	1	2	3	4	5 ...52
47,25	80,82	240,46	555,77	1178,59	2408,76
NOK	54,78	106,38	290,95	655,50	1375,56
	35,99	61,77	135,93	349,32	770,79
	45,08	46,70	67,83	170,10	416,80
		39,21	80,82	74,30	209,60
		30,63	54,19	240,46	88,32
		72,77	41,56	106,38	555,77
		46,70	35,27	61,16	290,95
		34,88	32,36	42,87	135,93
		43,24	27,39	46,15	67,19
			196,79	38,46	43,88
			80,82	33,28	80,82
			54,19	29,82	53,61
			44,29	28,44	40,77
			35,27	25,09	33,37
			29,79	469,52	34,54
			65,99	240,46	31,52
			44,29	106,38	28,79
			34,15	61,16	26,54
			40,82	72,19	25,78
				46,15	23,25
				38,46	1008,22
				34,01	555,77
				29,82	290,95
				26,63	135,93
				159,02	67,19
				72,19	196,79
				46,15	80,82
				42,44	53,61
				34,01	40,77
				29,38	43,50
				59,00	34,54
				42,44	31,52
				32,95	28,88
				37,44	26,54
					24,34
					394,92
					196,79
					80,82
					53,61
					65,35
					43,50
					34,54
					33,27
					28,88
					26,32
					126,35
					65,35
					43,50
					40,04
					33,27
					28,62
					51,27
					40,04
					30,95
					33,67

0	1	2	3	4	5 ...52
28,73	80,82	240,46	555,77	1178,59	2408,76
NOK	29,42	106,38	290,95	655,50	1375,56
	25,28	31,29	135,93	349,32	770,79
	27,57	27,84	45,20	170,10	416,80
		25,42	80,82	65,20	209,60
		23,00	28,56	240,46	88,32
		58,71	25,60	106,38	555,77
		27,84	24,36	30,50	290,95
		24,39	22,93	25,78	135,93
		26,48	21,23	26,95	45,20
			196,79	24,50	25,99
			80,82	22,84	80,82
			28,56	22,06	27,69
			26,67	21,03	24,68
			24,36	19,68	22,72
			22,20	469,52	23,43
			39,59	240,46	21,98
			26,67	106,38	20,80
			23,56	30,50	20,26
			25,39	58,71	19,44
				26,95	18,03
				24,50	1008,22
				23,45	555,77
				22,06	290,95
				20,49	135,93
				159,02	45,20
				58,71	196,79
				26,95	80,82
				25,57	27,69
				23,45	24,68
				21,45	25,75
				29,82	23,43
				25,57	21,98
				22,72	21,24
				24,28	20,26
					18,94
					394,92
					196,79
					80,82
					27,69
					39,59
					25,75
					23,43
					22,61
					21,24
					19,79
					126,35
					39,59
					25,75
					24,46
					22,61
					20,72
					27,32
					24,46
					21,85
					23,24

$C(i,n)^{Investment}$: $\hat{\sigma}^H$(left) and $\hat{\sigma}^L$(right) for n(0→5). Estimated with MATLAB code from (A.1)

Figure A.9: **Early exercise boundaries for Investment**

1	2	3	4	5	...52	1	2	3	4	5	...52
80,82	**240,46**	**555,77**	**1178,59**	**2408,76**		**80,82**	**240,46**	**555,77**	**1178,59**	**2408,76**	
54,78	**106,38**	**290,95**	**655,50**	**1375,56**		29,42	**106,38**	**290,95**	**655,50**	**1375,56**	
35,99	61,77	**135,93**	**349,32**	**770,79**		25,28	31,29	**135,93**	**349,32**	**770,79**	
45,08	46,70	67,83	**170,10**	**416,80**		27,57	27,84	**45,20**	**170,10**	**416,80**	
	39,21	**80,82**	74,30	**209,60**			25,42	**80,82**	**65,20**	**209,60**	
	30,63	54,19	**240,46**	**88,32**			23,00	28,56	**240,46**	**88,32**	
	72,77	41,56	**106,38**	**555,77**			**58,71**	25,60	**106,38**	**555,77**	
	46,70	35,27	61,16	**290,95**			27,84	24,36	30,50	**290,95**	
	34,88	32,36	42,87	**135,93**			24,39	22,93	25,78	**135,93**	
	43,24	27,39	46,15	67,19			26,48	21,23	26,95	**45,20**	
		196,79	38,46	43,88				**196,79**	24,50	25,99	
		80,82	33,28	**80,82**				**80,82**	22,84	**80,82**	
		54,19	29,82	53,61				28,56	22,06	27,69	
		44,29	28,44	40,77				26,67	21,03	24,68	
		35,27	25,09	33,37				24,36	19,68	22,72	
		29,79	**469,52**	34,54				22,20	**469,52**	23,43	
		65,99	**240,46**	31,52				**39,59**	**240,46**	21,98	
		44,29	**106,38**	28,79				26,67	**106,38**	20,80	
		34,15	61,16	26,54				23,56	30,50	20,26	
		40,82	72,19	25,78				25,39	**58,71**	19,44	
			46,15	23,25					26,95	18,03	
			38,46	**1008,22**					24,50	**1008,22**	
			34,01	**555,77**					23,45	**555,77**	
			29,82	**290,95**					22,06	**290,95**	
			26,63	**135,93**					20,49	**135,93**	
			159,02	67,19					**159,02**	**45,20**	
			72,19	**196,79**					**58,71**	**196,79**	
			46,15	**80,82**					26,95	**80,82**	
			42,44	53,61					25,57	27,69	
			34,01	40,77					23,45	24,68	
			29,38	43,50					21,45	25,75	
			59,00	34,54					29,82	23,43	
			42,44	31,52					25,57	21,98	
			32,95	28,88					22,72	21,24	
			37,44	26,54					24,28	20,26	
				24,34						18,94	
				394,92						**394,92**	
				196,79						**196,79**	
				80,82						**80,82**	
				53,61						27,69	
				65,35						**39,59**	
				43,50						25,75	
				34,54						23,43	
				33,27						22,61	
				28,88						21,24	
				26,32						19,79	
				126,35						**126,35**	
				65,35						**39,59**	
				43,50						25,75	
				40,04						24,46	
				33,27						22,61	
				28,62						20,72	
				51,27						27,32	
				40,04						24,46	
				30,95						21,85	
				33,67						23,24	

Early exercise boundaries Investment: $\hat{\sigma}^H$(left) and $\hat{\sigma}^L$(right) for n(1→5). Bold text = optimal to exercise. Estimated with MATLAB code from (A.1)

Figure A.10: **Intrinsic Value Electricity price**

1	2	3	4	5	...52	1	2	3	4	5	...52
742,75	**2209,82**	**5107,57**	**10831,21**	**22136,49**		**742,75**	**2209,82**	**5107,57**	**10831,21**	**22136,49**	
118,93	**977,65**	**2673,80**	**6024,02**	**12641,36**		118,93	**977,65**	**2673,80**	**6024,02**	**12641,36**	
-376,04	256,42	**1249,23**	**3210,22**	**7083,56**		-376,04	256,42	**1249,23**	**3210,22**	**7083,56**	
-102,86	0,00	415,39	**1563,22**	**3830,40**		-102,86	0,00	**415,39**	**1563,22**	**3830,40**	
	-315,83	**742,75**	599,17	**1926,23**			-315,83	**742,75**	**599,17**	**1926,23**	
	-566,42	118,93	**2209,82**	**811,65**			-566,42	118,93	**2209,82**	**811,65**	
	539,57	-246,22	**977,65**	**5107,57**			**539,57**	-246,22	**977,65**	**5107,57**	
	0,00	-376,04	256,42	**2673,80**			0,00	-376,04	256,42	**2673,80**	
	-428,12	-535,93	-165,74	**1249,23**			-428,12	-535,93	-165,74	**1249,23**	
	-191,84	-662,80	0,00	415,39			-191,84	-662,80	0,00	**415,39**	
		1808,51	-315,83	-72,69				**1808,51**	-315,83	-72,69	
		742,75	-500,69	**742,75**				**742,75**	-500,69	**742,75**	
		118,93	-566,42	118,93				118,93	-566,42	118,93	
		-102,86	-647,37	-246,22				-102,86	-647,37	-246,22	
		-376,04	-711,60	-459,95				-376,04	-711,60	-459,95	
		-592,78	**4314,90**	-376,04				-592,78	**4314,90**	-376,04	
		363,83	**2209,82**	-535,93				**363,83**	**2209,82**	-535,93	
		-102,86	**977,65**	-629,53				-102,86	**977,65**	-629,53	
		-473,16	256,42	-662,80				-473,16	256,42	-662,80	
		-268,79	539,57	-703,79				-268,79	**539,57**	-703,79	
			0,00	-736,31					0,00	-736,31	
			-315,83	**9265,52**					-315,83	**9265,52**	
			-428,12	**5107,57**					-428,12	**5107,57**	
			-566,42	**2673,80**					-566,42	**2673,80**	
			-676,15	**1249,23**					-676,15	**1249,23**	
			1461,39	415,39					**1461,39**	**415,39**	
			539,57	**1808,51**					**539,57**	**1808,51**	
			0,00	**742,75**					0,00	**742,75**	
			-191,84	118,93					-191,84	118,93	
			-428,12	-246,22					-428,12	-246,22	
			-615,59	-102,86					-615,59	-102,86	
			211,83	-376,04					211,83	-376,04	
			-191,84	-535,93					-191,84	-535,93	
			-512,12	-592,78					-512,12	-592,78	
			-335,35	-662,80					-335,35	-662,80	
				-718,36						-718,36	
				3629,28						**3629,28**	
				1808,51						**1808,51**	
				742,75						**742,75**	
				118,93						118,93	
				363,83						**363,83**	
				-102,86						-102,86	
				-376,04						-376,04	
				-473,16						-473,16	
				-592,78						-592,78	
				-687,70						-687,70	
				1161,16						**1161,16**	
				363,83						**363,83**	
				-102,86						-102,86	
				-268,79						-268,79	
				-473,16						-473,16	
				-635,31						-635,31	
				80,36						80,36	
				-268,79						-268,79	
				-545,82						-545,82	
				-392,93						-392,93	

Intrinsic value electricity price: $\hat{\sigma}^H$(left) and $\hat{\sigma}^L$(right) for n(1→5). Bold text = optimal to exercise. Estimated with MATLAB code from (A.1)