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Irreducibility of Lagrangian Quot schemes over an algebraic curve

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Abstract
Let C be a complex projective smooth curve and W a symplectic vector bundle of rank $2n$ over C. The Lagrangian Quot scheme $LQ_{-e}(W)$ parameterizes subsheaves of rank n and degree $-e$ which are isotropic with respect to the symplectic form. We prove that $LQ_{-e}(W)$ is irreducible and generically smooth of the expected dimension for all large e, and that a generic element is saturated and stable.

1 Introduction

Let C be a smooth algebraic curve of genus $g \geq 0$ over $\mathbb{C}$. A vector bundle W over C is called *symplectic* if there exists a nondegenerate skew-symmetric bilinear form $\omega\colon W \otimes W \to L$ for some line bundle L. Such an ω is called an L-valued symplectic form. A subsheaf E of W is called *isotropic* if $\omega|_{E\otimes E} = 0$. By linear algebra, a symplectic bundle has even rank $2n$ and any isotropic subsheaf has rank at most n. An isotropic subbundle (resp., subsheaf) of rank n is called a *Lagrangian subbundle* (resp., *Lagrangian subsheaf*). For information on semistability and moduli of symplectic bundles, see [1].

For vector bundles, Popa and Roth proved the following result on the irreducibility of Quot schemes.

Theorem 1.1 ([16, Theorem 6.4]) *For any vector bundle V over C, there is an integer $d(V, k)$ such that for all $d \geq d(V, k)$, the Quot scheme* $\mathrm{Quot}^{k,d}(V)$ *of quotient sheaves of V of rank k and degree d is irreducible.*

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As a corollary, they showed that for sufficiently large d, the Quot scheme $\mathrm{Quot}^{k,d}(V)$ is generically smooth of the expected dimension, and a general point of $\mathrm{Quot}^{k,d}(V)$ corresponds to an extension $0 \to E \to V \to V/E \to 0$ where E and V/E are stable vector bundles. A significant feature of this theorem is that it holds for an arbitrary bundle V, with no assumption of generality or semistability.

The main goal of the present paper is to obtain the analogous result for Lagrangian Quot schemes of symplectic bundles (Theorem 4.1). One may try to argue as in [16], but one vital step does not appear to adapt in an obvious way: Given a symplectic bundle W of rank $2n$ and for a fixed vector bundle E of rank n, the space parameterizing Lagrangian subsheaves $E \subset W$ is a locally closed subset of $\mathbb{P}H^0(C, \mathrm{Hom}(E, W))$, whose irreducibility seems difficult to decide. This is discussed further at the beginning of § 4.

We take instead a different approach: We exploit the geometry of symplectic extensions, together with deformation arguments, as developed in [4] and [11]. In particular, Proposition 4.14 uses a geometric interpretation for the statement that a nonsaturated Lagrangian subsheaf can be deformed to a subbundle. The connection between extensions and geometry is provided by principal parts, and is developed in § 3 and § 4.3. This provides an alternative language to Čech cohomology for bundle extensions over curves, and makes transparent the link between the geometric and cohomological properties of the extensions.

We remark that the same argument applies to the vector bundle case, and we expect that similar results can be obtained by these methods for other principal bundles.

In [3], we use the main result in this paper to solve the problem of counting maximal Lagrangian subbundles of symplectic bundles, as Holla [12] used the irreducibility of Quot schemes to count maximal subbundles of vector bundles. Also we expect that an effective version of the irreducibility result for semistable bundles would yield an effective base freeness (or very ampleness) result on the generalized theta divisors on the moduli of symplectic bundles, as in [16, § 8] for vector bundles. We note that Theorem 4.1 does not give an effective bound on e but only the existence of a bound, due to the existence statement in Proposition 4.4. It would be nice to have an effective and reasonably small uniform bound for semistable symplectic bundles.

In another direction; several techniques in the present paper have also been used in [2] to study isotropic Quot schemes associated to bundles with orthogonal structure.

Notation

Throughout, C denotes a complex projective smooth curve of genus $g \geq 0$. If W is a vector bundle over C and $E \subset W$ a locally free subsheaf, we denote by $\overline{E}$ the saturation, which is a vector subbundle of W.

2 Lagrangian Quot schemes

In this section, we define the Lagrangian Quot scheme of a symplectic bundle and study its tangent spaces.

Given a vector bundle V over C, the Quot scheme $\mathrm{Quot}^{k,d}(V)$ parameterizes quotient sheaves of V of rank k and degree d; alternatively, subsheaves of V of rank $\mathrm{rk}\, V - k$ and degree $\deg V - d$. Let W be a bundle of rank $2n$ which carries an L-valued symplectic form, where $\deg L = \ell$. Then from the induced isomorphism $W \cong W^* \otimes L$, we have $\deg W = n\ell$. We denote by $LQ_{-e}(W)$ the sublocus of $\mathrm{Quot}^{n,e+n\ell}(W)$ consisting of Lagrangian subsheaves of degree $-e$ and call it a *Lagrangian Quot scheme*. The choice of notation "LQ_{-e}" reflects the fact that we will most often consider points of Lagrangian Quot schemes as subsheaves rather than as quotients.

Remark 2.1 Note that $LQ_{-e}(W) \hookrightarrow \mathrm{Quot}^{n,e+n\ell}(W)$ depends on the choice of symplectic form ω. However, by [8, Remarque, p. 130], if ω and ω' are two symplectic forms on W then there exists a bundle automorphism ι of W such that $\iota^*\omega' = \omega$. Then $F \mapsto \iota(F)$ induces an isomorphism $LQ_{-e}(W, \omega) \xrightarrow{\sim} LQ_{-e}(W, \omega')$. In view of this, we shall abuse notation and write simply $LQ_{-e}(W)$.

We recall some other important notions: For each integer e and each $x \in C$ we have the *evaluation map* $\mathrm{ev}_x^e \colon \mathrm{Quot}^{n,e+n\ell}(W) \dashrightarrow \mathrm{Gr}(n, W|_x)$ which sends a subsheaf E to the fiber $E|_x$, when this is defined. Also, let $\mathrm{LG}(W)$ be the *Lagrangian Grassmannian bundle* of W, that is, the subfibration of $\mathrm{Gr}(n, W)$ whose fiber at $x \in C$ is the Lagrangian Grassmannian $\mathrm{LG}(W|_x)$.

Lemma 2.2 *Let W be an L-valued symplectic bundle of rank $2n$ as above. If $g \geq 2$ and $e \geq \frac{n(g-1-\ell)}{2}$, then the locus $LQ_{-e}(W)$ is a nonempty closed subset of* $\mathrm{Quot}^{n,e+n\ell}(W)$.

Proof By [4, Theorem 1.4 and Remark 3.6], any symplectic bundle has a Lagrangian subbundle of degree $-e_0$ for some $e_0 \geq \left\lceil \frac{n(g-1-\ell)}{2} \right\rceil$. For $e > e_0$, we can take an elementary transformation of the Lagrangian subbundle of degree $-e_0$ to get a Lagrangian subsheaf of degree $-e$. This proves the nonemptyness.

For the closedness: Write $\mathrm{Indet}(\mathrm{ev}_x^e)$ for the indeterminacy locus of ev_x^e:

$$\mathrm{Indet}(\mathrm{ev}_x^e) = \{[E \to W] \in \mathrm{Quot}^{n,e+n\ell}(W) : E \text{ is not saturated at } x\},$$

which is a closed subset of $\mathrm{Quot}^{n,e+n\ell}(W)$. It is easy to see that

$$LQ_{-e}(W) = \bigcap_{x \in C} \left((\mathrm{ev}_x^e)^{-1} \left(\mathrm{LG}(W|_x)\right) \cup \mathrm{Indet}(\mathrm{ev}_x^e) \right).$$

As $\mathrm{LG}(W|_x)$ is closed in $\mathrm{Gr}(n, W|_x)$, we see that $LQ_{-e}(W)$ is closed. □

Remark 2.3 The genus assumption $g \geq 2$ is imposed to get the sharp bound $e \geq \frac{n(g-1-\ell)}{2}$ for nonemptyness of $LQ_{-e}(W)$. This bound is proven in [4] for $g \geq 2$, but for the case $g = 0$ or 1, we still have an existence of a bound to guarantee the nonemptyness of $LQ_{-e}(W)$.

We denote by $LQ_{-e}(W)^\circ$ the open sublocus of $LQ_{-e}(W)$ corresponding to vector bundle quotients. The following is a generalization of [5, Lemma 4.3].

Proposition 2.4 *Assume that $LQ_{-e}(W)^\circ$ is nonempty. Let $[j \colon E \to W]$ be a point of $LQ_{-e}(W)^\circ$.*

(a) *Every irreducible component of $LQ_{-e}(W)^\circ$ has dimension at least*

$$\chi(C, L \otimes \mathrm{Sym}^2 E^*) = \frac{n(n+1)}{2}(\ell - g + 1) + (n+1)e.$$

(b) *The Zariski tangent space of $LQ_{-e}(W)^\circ$ at $[j : E \to W]$ is given by*

$$T_j LQ_{-e}(W)^\circ \cong H^0(C, L \otimes \mathrm{Sym}^2 E^*).$$

(c) *If $h^1\left(C, L \otimes \mathrm{Sym}^2 E^*\right) = 0$, then $LQ_{-e}(W)^\circ$ is smooth and of dimension $\chi(C, L \otimes \mathrm{Sym}^2 E^*)$ at j.*

Proof (a) Let Z be an irreducible component of $LQ_{-e}(W)^{\circ}$. Let $[j\colon E \to W]$ be a point of Z contained in no other component. Let $\sigma : C \to \mathrm{LG}(W)$ be the section corresponding to the subbundle $[j\colon E \to W]$. Let P be the Hilbert polynomial of the subscheme $\sigma(C)$ of $\mathrm{LG}(W)$ and Y a component of $\mathrm{Hilb}^P(\mathrm{LG}(W))$ containing the point $[\sigma(C)]$. Write $\pi : \mathrm{LG}(W) \to C$ for the projection. The normal bundle of $\sigma(C)$ in $\mathrm{LG}(W)$ is isomorphic to the restriction of the vertical tangent bundle $T_\pi = \mathrm{Ker}(d\pi)$, which in turn is isomorphic to $L \otimes \mathrm{Sym}^2 E^*$. Hence by the deformation theory of Hilbert schemes, we have

$$\dim_{[\sigma(C)]} Y \;\geq\; \chi\left(C, L \otimes \mathrm{Sym}^2 E^*\right).$$

Since a general member of Y corresponds to a section of π, there is a rational map $Y \dashrightarrow LQ_{-e}(W)^{\circ}$ defined on a nonempty open subset. As $[\sigma(C)]$ is mapped to j, the image of Y lies inside Z. Clearly the map $Y \dashrightarrow Z$ is generically injective, so we see that $\dim Z \geq \chi\left(C, L \otimes \mathrm{Sym}^2 E^*\right)$.

(b) Let $\alpha\colon E \to W/E \cong E^* \otimes L$ represent a tangent vector to the Quot scheme $\mathrm{Quot}^{n,e+n\ell}(W)$ at $[j\colon E \to W]$. For each $x \in C$, the section α defines an element $\alpha(x) \in T_{j(E|_x)}\mathrm{Gr}(n, W|_x)$, and the deformation preserves isotropy of E if and only if $\alpha(x)$ is tangent to the subvariety $\mathrm{LG}(W|_x) \subset \mathrm{Gr}(n, W|_x)$ for all x.

The result now follows from the following description of the tangent space of the Lagrangian Grassmannian:

$$T_{j(E|_x)}\mathrm{LG}(W|_x) = \left(L \otimes \mathrm{Sym}^2 E^*\right)|_x \;\subset\; \left(L \otimes E^* \otimes E^*\right)|_x = T_{j(E|_x)}\mathrm{Gr}(n, W|_x).$$

(c) By (a) and (b), if $h^1\left(C, L \otimes \mathrm{Sym}^2 E^*\right) = 0$ then

$$\dim T_j LQ_{-e}(W)^{\circ} \;=\; \chi\left(C, L \otimes \mathrm{Sym}^2 E^*\right) \;\leq\; \dim_j LQ_{-e}(W)^{\circ}.$$

Thus we have equality and $LQ_{-e}(W)^{\circ}$ is smooth at j. □

3 Symplectic extensions

If F is a Lagrangian subbundle of a symplectic bundle W, then there is a short exact sequence $0 \to F \to W \to F^* \otimes L \to 0$. An extension induced by a symplectic structure in this way will be called a *symplectic extension*. In this section, we recall or prove some facts on symplectic extensions which we will need later.

Recall that any locally free sheaf V on C has a flasque resolution

$$0 \;\to\; V \;\to\; \underline{\mathrm{Rat}}\,(V) \;\to\; \underline{\mathrm{Prin}}\,(V) \;\to\; 0,$$

where $\underline{\mathrm{Rat}}\,(V) = V \otimes_{\mathcal{O}_C} \underline{\mathrm{Rat}}\,(\mathcal{O}_C)$ is the sheaf of sections of V with finitely many poles, and $\underline{\mathrm{Prin}}\,(V) = \underline{\mathrm{Rat}}\,(V)/V$ is the sheaf of principal parts with values in V. Taking global sections, we have a sequence of Abelian groups

$$0 \;\to\; H^0(C, V) \;\to\; \mathrm{Rat}\,(V) \;\to\; \mathrm{Prin}(V) \;\to\; H^1(C, V) \;\to\; 0. \tag{3.1}$$

The *support* of a principal part $p \in \mathrm{Prin}(V)$ is the subscheme of C defined by the annihilator of p in $\mathcal{O}_C$. If $\mathrm{Supp}(p)$ is a divisor $D := \Sigma_{i=1}^m d_i x_i$, then p is a global section of $V(D)/V$. Abusing notation, p can be represented by a finite sum

$$\frac{v_1}{z_1^{d_1}} + \cdots + \frac{v_m}{z_m^{d_m}}$$

where z_i is a local parameter at x_i for $1 \leq i \leq m$, and v_i is a local section of V near x_i. Note that the principal part p is determined by the images of the v_i in $V_{x_i} / \left(m_{x_i}^{d_i} \cdot V_{x_i} \right)$.

For $\beta \in \mathrm{Rat}\,(V)$, we denote by $\overline{\beta}$ the principal part $\beta \mod H^0(C, V)$. If $p \in \mathrm{Prin}(V)$, we write $[p]$ for the associated class in $H^1(C, V)$.

3.1 Symmetric principal parts and symplectic extensions

Let F be any bundle of rank n. The sheaf $L^{-1} \otimes F \otimes F$ has a natural involution exchanging the factors of F. The *transpose* ${}^t p$ of a principal part $p \in \mathrm{Prin}(L^{-1} \otimes F \otimes F)$ is defined to be the image of p under this involution. Then p is *symmetric* if ${}^t p = p$, or equivalently $p \in \mathrm{Prin}(L^{-1} \otimes \mathrm{Sym}^2 F)$. Note that this is stronger than the condition $[{}^t p] = [p]$ in $H^1(C, L^{-1} \otimes F \otimes F)$.

Now any $p \in \mathrm{Prin}(L^{-1} \otimes F \otimes F)$ defines naturally an $\mathcal{O}_C$-module map $F^* \otimes L \to \underline{\mathrm{Prin}}\,(F)$, which we also denote p. Suppose p is a symmetric principal part in $\mathrm{Prin}(L^{-1} \otimes \mathrm{Sym}^2 F)$. Following [15, Chapter 6], we define a sheaf W_p by

$$W_p(U) \ := \ \{(f, \varphi) \in \underline{\mathrm{Rat}}\,(F)(U) \oplus (F^* \otimes L)(U) : \overline{f} = p(\varphi)\} \tag{3.2}$$

for each open set $U \subseteq C$. It is not hard to see that this is an extension of $F^* \otimes L$ by F.

Now there is a canonical pairing $\langle \, , \rangle \colon \underline{\mathrm{Rat}}\,(F) \oplus \underline{\mathrm{Rat}}\,(F^* \otimes L) \to \underline{\mathrm{Rat}}\,(L)$. By an easy computation (see the proof of [11, Criterion 2.1] for a more general case), the standard symplectic form on $\underline{\mathrm{Rat}}\,(F) \oplus \underline{\mathrm{Rat}}\,(F^* \otimes L)$ defined on sections by

$$\omega\,((f_1, \phi_1), (f_2, \phi_2)) \ = \ \langle f_2, \phi_1 \rangle - \langle f_1, \phi_2 \rangle \tag{3.3}$$

restricts to a *regular* symplectic form on W_p with respect to which the subsheaf F is Lagrangian. This shows that for each symmetric principal part $p \in \mathrm{Prin}(L^{-1} \otimes \mathrm{Sym}^2 F)$ there is a naturally associated symplectic extension of $F^* \otimes L$ by F. We now give a refinement of [11, Criterion 2.1], showing that every symplectic extension can be put into this form.

Lemma 3.1 *Let W be any symplectic bundle and $F \subset W$ a Lagrangian subbundle.*

(a) *There is an isomorphism of symplectic bundles $\iota \colon W \xrightarrow{\sim} W_p$ for some symmetric principal part $p \in \mathrm{Prin}(L^{-1} \otimes \mathrm{Sym}^2 F)$ such that $\iota(F)$ is the natural copy $W_p \cap \underline{\mathrm{Rat}}\,(F)$ of F in W_p, which is given over each open set $U \subseteq C$ by $\{(f, 0) : f \in F(U)\}$.*
(b) *The class of the extension $0 \to F \to W_p \to F^* \otimes L \to 0$ in $H^1(C, L^{-1} \otimes \mathrm{Sym}^2 F)$ coincides with $[p]$.*

Proof (a) As much of this proof is computational, we outline the main steps and leave the details to the interested reader.

Since F is isotropic, W is an extension $0 \to F \to W \to F^* \otimes L \to 0$. By [10, Lemma 3.1][1], there exists $p' \in \mathrm{Prin}(L^{-1} \otimes F \otimes F)$ such that the sheaf of sections of W is given by

$$U \ \mapsto \ W_{p'}(U) \ = \ \left\{ (f, \phi) \in \underline{\mathrm{Rat}}\,(F)(U) \oplus (F^* \otimes L)(U) : p'(\phi) = \overline{f} \right\}. \tag{3.4}$$

Using the facts that F is isotropic and the form is antisymmetric and nondegenerate, one shows that there exist $A \in \mathrm{Aut}\,(F)$ and $B \in \mathrm{Rat}\,(L^{-1} \otimes \wedge^2 F)$ such that the given symplectic form ω' on the sheaf $W_{p'}$ is given by

$$\omega'((f_1, \phi_1), (f_2, \phi_2)) \ = \ \langle A(f_2), \phi_1 \rangle - \langle A(f_1), (\phi_2) \rangle + \langle B(\phi_2), \phi_1 \rangle \tag{3.5}$$

[1] This is unpublished, but it is the obvious generalization of the rank two case treated in [15, Lemma 6.5].

Using in addition that the restriction of ω' to $W_{p'}$ is regular, one shows that

$$Ap' - {}^t(Ap') + \overline{B} \;=\; \left(Ap' + \frac{\overline{B}}{2}\right) - {}^t\left(Ap' + \frac{\overline{B}}{2}\right) \;=\; 0 \;\in\; \mathrm{Prin}(L^{-1} \otimes F \otimes F).$$

Hence $p := Ap' + \frac{1}{2}\overline{B}$ is a symmetric principal part.

Let now W_p be defined as in (3.2). As mentioned above, the form ω in (3.3) restricts to a regular symplectic form on W_p. A tedious but elementary calculation shows that

$$(f', \phi') \;\mapsto\; \left(A(f') + \frac{B}{2}(\phi'), \phi'\right)$$

defines an isomorphism $\iota\colon W_{p'} \xrightarrow{\sim} W_p$ satisfying $\iota^*\omega = \omega'$ and mapping $F \subset W_{p'}$ to $F \subset W_p$.

Part (b) is proven exactly as for extensions of line bundles in [15, Lemma 6.6]. □

3.2 Lagrangian subbundles in reference to a fixed symplectic extension

From (3.2), we obtain a splitting $\mathrm{Rat}\,(W) = \mathrm{Rat}\,(F) \oplus \mathrm{Rat}\,(F^* \otimes L)$. This is a vector space of dimension $\mathrm{rk}\,(W)$ over the field $K(C)$ of rational functions on C. If $\beta \in \mathrm{Rat}\,(\mathrm{Hom}(F^* \otimes L, F))$, we write Γ_β for the graph of the induced map of $K(C)$-vector spaces $\mathrm{Rat}\,(F^* \otimes L) \to \mathrm{Rat}\,(F)$. Abusing notation, we also denote by Γ_β the associated sub-$\mathcal{O}_C$-module of $\underline{\mathrm{Rat}}\,(F) \oplus \underline{\mathrm{Rat}}\,(F^* \otimes L)$.

Moreover, if F and G are subsheaves of a sheaf H, we write $F \cap G$ for the presheaf $U \mapsto F(U) \cap G(U)$. As this is the kernel of the composed maps $F \to H \to H/G$ and $G \to H \to H/F$, in fact it is a sheaf.

Proposition 3.2 *Let $p \in \mathrm{Prin}(L^{-1} \otimes \mathrm{Sym}^2 F)$ be any symmetric principal part. Let W_p be as in (3.2).*

(a) *There is a bijection between the $K(C)$-vector space $\mathrm{Rat}\,\left(L^{-1} \otimes \mathrm{Sym}^2 F\right)$ and the set of Lagrangian subbundles $E \subset W_p$ with $\mathrm{rk}\,(E \cap F) = 0$. The bijection is given by $\beta \mapsto \Gamma_\beta \cap W_p$. The inverse map sends a Lagrangian subbundle E to the uniquely determined $\beta \in \mathrm{Rat}\,(L^{-1} \otimes \mathrm{Sym}^2 F)$ satisfying $\mathrm{Rat}\,(E) = \Gamma_\beta$.*

(b) *If $E = \Gamma_\beta \cap W_p$ then projection to $F^* \otimes L$ gives an isomorphism of sheaves $E \xrightarrow{\sim} \mathrm{Ker}\,\big((p - \overline{\beta})\colon F^* \otimes L \to \underline{\mathrm{Prin}}\,(F)\big)$. Note that $\left[p - \overline{\beta}\right] = [p]$ is the class of the symplectic extension $\delta(W_p) \in H^1(C, L^{-1} \otimes \mathrm{Sym}^2 F)$.*

(c) *For a fixed $p - \overline{\beta} \in \mathrm{Prin}(L^{-1} \otimes \mathrm{Sym}^2 F)$, the set of Lagrangian subbundles $\Gamma_{\beta'} \cap W_p$ with $\overline{\beta'} = \overline{\beta}$ is a torsor over $H^0(C, L^{-1} \otimes \mathrm{Sym}^2 F)$. In particular, it is nonempty.*

Proof Parts (a) and (b) follow from [11, Theorem 3.3 (i) and (iii)]. Note that as the symplectic form on W is given by (3.3), the α referred to in [11] is zero.

Part (c) is a slight generalization of [11, Corollary 3.5]. From the description (3.2), we see that $(\beta(\phi), \phi) \in \Gamma_\beta(U)$ belongs to $W_p(U)$ if and only if $\phi \in \mathrm{Ker}(p - \overline{\beta})(U)$, so $\Gamma_\beta \cap W_p$ is a lifting of $\mathrm{Ker}(p - \overline{\beta})$. By part (a), it is isotropic and saturated.

Moreover, under the bijection in (a) the set of liftings $\Gamma_{\beta'} \cap W_p$ with $\overline{\beta'} = \overline{\beta}$ is in canonical bijection with the set of β' such that $\overline{\beta'} = \overline{\beta}$. By (3.1), this is a torsor over $H^0(C, L^{-1} \otimes \mathrm{Sym}^2 F)$. □

Remark 3.3 In part (c) above, we characterize different liftings of $\mathrm{Ker}(q)$ for a fixed $q \in \mathrm{Prin}(L^{-1} \otimes \mathrm{Sym}^2 F)$ with $\delta(W) = [q]$. More generally, there can exist also distinct β, β' with $p - \overline{\beta} \neq p - \overline{\beta'}$ such that $\mathrm{Ker}(p - \overline{\beta}) = \mathrm{Ker}(p - \overline{\beta'})$ as subsheaves of $F^* \otimes L$. Such β and β' correspond to distinct liftings $E \hookrightarrow W$. We shall study this phenomenon in Lemma 3.6.

We now give a slight refinement of Lemma 3.1, essentially allowing us to choose convenient coordinates on W.

Lemma 3.4 *Let F and E be Lagrangian subbundles of W such that $\mathrm{rk}\,(F \cap E) = 0$. Then there exists a symmetric principal part $p_0 \in \mathrm{Prin}(L^{-1} \otimes \mathrm{Sym}^2 F)$ and an isomorphism of symplectic bundles $\iota\colon W \xrightarrow{\sim} W_{p_0}$, such that*

$$\iota(E) \;=\; \Gamma_0 \cap W_{p_0} \;=\; 0 \oplus \mathrm{Ker}(p_0),$$

where $\Gamma_0 = 0 \oplus \mathrm{Rat}\,(F^* \otimes L)$ is the graph of the zero map $\underline{\mathrm{Rat}}\,(F^* \otimes L) \to \underline{\mathrm{Rat}}\,(F)$.

Proof From Lemma 3.1 and Proposition 3.2, we may assume that W is an extension

$$0 \;\to\; F \;\to\; W_p \;\to\; F^* \otimes L \;\to\; 0$$

for a symmetric $p \in \mathrm{Prin}(L^{-1} \otimes F \otimes F)$, and that

$$E \;=\; \Gamma_\beta \cap W_p \;\cong\; \mathrm{Ker}(p - \overline{\beta})$$

for some $\beta \in \mathrm{Rat}\,(L^{-1} \otimes \mathrm{Sym}^2 F)$. Then $(f, \phi) \mapsto (f - \beta(\phi), \phi)$ defines an isomorphism $\iota\colon W_p \xrightarrow{\sim} W_{p-\overline{\beta}}$ sending $E = \Gamma_\beta \cap W_p$ to $\Gamma_0 \cap W_{p-\overline{\beta}}$. Set $p_0 := p - \overline{\beta}$. If ω and ω_0 are the standard symplectic forms (3.3) on W_p and W_{p_0} respectively, then an easy computation using the symmetry of β shows that $\iota^* \omega_0 = \omega$. □

Remark 3.5 Apropos Lemma 3.4 and (3.2): As $\mathrm{Ker}(p_0)$ is only a subsheaf of $F^* \otimes L$, it may be of interest to indicate how it lifts to a saturated subsheaf, or a subbundle of W_{p_0}. For simplicity, suppose $L = \mathcal{O}_C$ and $\mathrm{Im}\,(p_0) \cong \mathbb{C}_x$, so p_0 is represented by $\frac{\eta_1 \otimes \eta_1}{z}$ where z is a local parameter at x on a neighborhood U and η_1 is some regular section of $F|_U$ which is nonzero at x.

Complete η_1 to a frame $\{\eta_i\}$ for F on U and let $\{\phi_i\}$ be the dual frame for F^*. Then the principal part $p_0(\phi_1) \in \underline{\mathrm{Prin}}\,(F)(U)$ is represented by

$$\frac{\eta_1 \otimes \eta_1}{z}(\phi_1) \;=\; \frac{\langle \eta_1, \phi_1 \rangle \cdot \eta_1}{z} \;=\; \frac{\eta_1}{z}.$$

Hence in view of (3.2), a frame for W_p on U is given by

$$(\eta_1, 0), \ldots, (\eta_n, 0), \left(\frac{\eta_1}{z}, \phi_1 \right), (0, \phi_2), \ldots, (0, \phi_n). \tag{3.6}$$

Now a frame over U for the subsheaf $0 \oplus \mathrm{Ker}(p_0)$ of W_{p_0} is given by

$$(0, z \cdot \phi_1), (0, \phi_2), \ldots, (0, \phi_n). \tag{3.7}$$

Writing $(0, z \cdot \phi_1)$ in terms of the frame (3.6), we have

$$(0, z \cdot \phi_1) \;=\; z \cdot \left(\frac{\eta_1}{z}, \phi_1 \right) - (\eta_1, 0).$$

From this we see that the images of (3.7) in $W_p|_x$ are independent. Hence $0 \oplus \mathrm{Ker}(p_0) \hookrightarrow W_{p_0}$ is a vector bundle inclusion at x. This computation also shows that the intersection of the subbundles $\Gamma_0 \cap W_{p_0}$ and F at x is the line spanned by $\eta_1(x)$ in $F|_x$. □

3.3 Isotropic liftings of an elementary transformation

Let W be a symplectic extension $0 \to F \to W_p \to F^* \otimes L \to 0$, and let $0 \to E \xrightarrow{\gamma} F^* \otimes L \to \tau \to 0$ be an elementary transformation where τ is some torsion sheaf. Assume that there is a lifting $j \colon E \to W$. By Proposition 3.2, there exists a rational map $\beta \colon \underline{\mathrm{Rat}}\,(F^* \otimes L) \to \underline{\mathrm{Rat}}\,(F)$ such that $E \subseteq \Gamma_\beta \cap W_p \cong \mathrm{Ker}(p - \overline{\beta})$. The following result, generalizing Proposition 3.2 (c), provides the main idea to "linearize" the space of Lagrangian subsheaves of W which respects the fixed symplectic extension and elementary transformation.

Lemma 3.6 *The set of liftings of $\gamma \colon E \to F^* \otimes L$ to Lagrangian subsheaves of $W = W_p$ is a torsor over $H^0\left(C, \mathrm{Hom}(E, F) \cap \underline{\mathrm{Rat}}\,(L^{-1} \otimes \mathrm{Sym}^2 F)\right)$.*

Before starting the proof, let us indicate how the intersection of $\mathrm{Hom}(E, F)$ and $\underline{\mathrm{Rat}}\,(L^{-1} \otimes \mathrm{Sym}^2 F)$ is well defined. Since $L^{-1} \otimes F \xrightarrow{{}^t\gamma} E^*$ is an elementary transformation, E^* is a subsheaf of $\underline{\mathrm{Rat}}\,(L^{-1} \otimes F)$. Hence $\mathrm{Hom}(E, F) = E^* \otimes F$ and $\underline{\mathrm{Rat}}\,(L^{-1} \otimes \mathrm{Sym}^2 F)$ are both sub-$\mathcal{O}_C$-modules of $\underline{\mathrm{Rat}}\,(L^{-1} \otimes F \otimes F)$.

Proof Suppose that $j_1 \colon E \to W$ and $j_2 \colon E \to W$ are two liftings of γ to Lagrangian subsheaves. Then the saturations $\overline{j_i(E)}$ are Lagrangian subbundles. By Proposition 3.2 (a), there exist uniquely defined $\beta_1, \beta_2 \in \mathrm{Rat}\,(L^{-1} \otimes \mathrm{Sym}^2 F)$ such that for $i = 1, 2$ the map $j_i \colon E \to W_p$ is given by

$$v \ \mapsto \ (\beta_i(v), \gamma(v)) \ \in \ W_p \ \subset \ \underline{\mathrm{Rat}}\,(F) \oplus (F^* \otimes L).$$

Then we calculate

$$j_1(v) - j_2(v) \ = \ (\beta_1(v), \gamma(v)) - (\beta_2(v), \gamma(v)) \ = \ ((\beta_1 - \beta_2)(v), 0).$$

Hence $j_1 - j_2$ defines an element of $H^0\left(C, \mathrm{Hom}(E, F) \cap \underline{\mathrm{Rat}}\,(L^{-1} \otimes \mathrm{Sym}^2 F)\right)$.

Conversely, suppose $v \mapsto (\beta(v), \gamma(v))$ is a lifting of γ as above. If $\alpha \in \mathrm{Rat}\,(L^{-1} \otimes \mathrm{Sym}^2 F)$ is regular on $\gamma(E) \subset F$, then $v \mapsto (\beta(v) + \alpha(v), \gamma(v))$ uniquely determines another rank n subsheaf of W_p lifting $\gamma(E)$. Since $\beta + \alpha$ is symmetric, by Proposition 3.2 (a), this subsheaf is isotropic. □

Motivated by Lemma 3.6, we make a definition.

Definition 3.7 Let $0 \to E \xrightarrow{\gamma} F^* \otimes L \to \tau \to 0$ be as above. From $L^{-1} \otimes F \xrightarrow{{}^t\gamma} E^*$ we deduce an inclusion $L^{-1} \otimes F \otimes F \to E^* \otimes F$. We define S_γ to be the saturation of $L^{-1} \otimes \mathrm{Sym}^2 F$ in $E^* \otimes F$.

Note that S_γ depends only on γ, not on an extension $0 \to F \to W \to F^* \otimes L \to 0$.

Lemma 3.8

(a) *There is an exact sequence* $0 \to L^{-1} \otimes \mathrm{Sym}^2 F \to S_\gamma \to \tau_1 \to 0$*, where τ_1 is a torsion sheaf. In particular, S_γ is locally free of rank $\frac{1}{2}n(n+1)$ and degree* $\deg(L^{-1} \otimes \mathrm{Sym}^2 F) + \deg(\tau_1)$.

(b) *There is an exact sequence* $0 \to S_\gamma \to L \otimes \mathrm{Sym}^2 E^* \to \tau_2 \to 0$*, where τ_2 is a torsion sheaf.*

(c) *Suppose that τ is isomorphic to $\mathcal{O}_D$ for a reduced divisor D. Then τ_1 is isomorphic to τ. In particular, in this case* $\deg(S_\gamma) = \deg(L^{-1} \otimes \mathrm{Sym}^2 F) + \deg(\tau)$.

Proof (a) This follows from the definition of S_γ.

(b) Consider the following diagram (which is not exact), where the horizontal arrows are induced by ${}^t\gamma$.

$$\begin{tikzcd} L^{-1} \otimes \mathrm{Sym}^2 F \arrow[r] \arrow[d, hook] & S_\gamma \arrow[r] \arrow[d, hook] & (\mathrm{Sym}^2 E^*) \otimes L \arrow[d, hook] \\ L^{-1} \otimes F \otimes F \arrow[r] & E^* \otimes F \arrow[r] & E^* \otimes E^* \otimes L \end{tikzcd}$$

As the horizontal arrows are isomorphisms at the generic point, we obtain (b).

(c) If $\tau \cong \mathcal{O}_D$ for a reduced divisor D, then we have also

$$\frac{E^*}{{}^t\gamma(L^{-1} \otimes F)} \cong \mathcal{O}_D.$$

At each $x \in D$, a local basis for $E^* \subset \underline{\mathrm{Rat}}\,(L^{-1} \otimes F)$ is given by

$$\frac{\lambda \otimes \eta_1}{z}, \lambda \otimes \eta_2, \ldots, \lambda \otimes \eta_n,$$

where $\{\eta_1, \ldots, \eta_n\}$ is a suitable local basis of F and λ a local generator of L^{-1}, and z is a local parameter at x. Then a local basis of $E^* \otimes F$ is given by

$$\left\{ \frac{\lambda \otimes \eta_1 \otimes \eta_k}{z} : 1 \le k \le n \right\} \cup \left\{ \lambda \otimes \eta_m \otimes \eta_k : \begin{array}{c} 2 \le m \le n; \\ 1 \le k \le n \end{array} \right\}.$$

Thus a local basis of S_γ is given by

$$\left\{ \frac{\lambda \otimes \eta_1 \otimes \eta_1}{z} \right\} \cup \left\{ \frac{1}{2} (\lambda \otimes \eta_k \otimes \eta_m + \lambda \otimes \eta_m \otimes \eta_k) : \begin{array}{c} 1 \le k, m \le n; \\ (m, k) \ne (1, 1) \end{array} \right\}.$$

Therefore, in this case τ_1 is a sum of torsion sheaves of degree 1, each supported at one of the points $x \in D$. The statement follows. $\square$

Remark 3.9 Lemma 3.8 (c) is false if τ_1 is not of the form $\mathcal{O}_D$ for a reduced D. For example, suppose that $L = \mathcal{O}_C$ and $\tau = \mathcal{O}_x \oplus \mathcal{O}_x$, so E^* is spanned near x by $\frac{\eta_1}{z}, \frac{\eta_2}{z}, \eta_3, \ldots, \eta_n$ for suitable η_i. Then S_γ is spanned near x by

$$\frac{\eta_1 \otimes \eta_1}{z}, \quad \frac{\eta_1 \otimes \eta_2 + \eta_2 \otimes \eta_2}{z}, \quad \frac{\eta_2 \otimes \eta_2}{z}, \quad \begin{array}{l} \frac{1}{2}(\eta_i \otimes \eta_j + \eta_j \otimes \eta_i) : 1 \le i \le j \le n; \\ (i, j) \notin \{(1, 1), (1, 2), (2, 2)\}. \end{array}$$

Thus $\tau_1 = \mathcal{O}_x^{\oplus 3} \ncong \tau$.

3.4 Geometry in extension spaces

Let $F \to C$ be a bundle of rank n, and consider the scroll $\pi : \mathbb{P}F \to C$. Throughout this subsection, we shall assume that $h^1(C, L^{-1} \otimes \mathrm{Sym}^2 F) \ne 0$.

By Serre duality and the projection formula, there is an isomorphism

$$\mathbb{P}H^1(C, L^{-1} \otimes \mathrm{Sym}^2 F) \xrightarrow{\sim} |\mathcal{O}_{\mathbb{P}F}(2) \otimes \pi^*(K_C L)|^*.$$

Thus we obtain a natural map $\psi : \mathbb{P}F \dashrightarrow \mathbb{P}H^1(C, L^{-1} \otimes \mathrm{Sym}^2 F)$ with nondegenerate image.

We shall use an explicit description of ψ, given in [4, § 2]. For each $x \in C$, there is a sheaf sequence

$$0 \to L^{-1} \otimes \mathrm{Sym}^2 F \to L^{-1}(x) \otimes \mathrm{Sym}^2 F \to \frac{L^{-1}(x) \otimes \mathrm{Sym}^2 F}{L^{-1} \otimes \mathrm{Sym}^2 F} \to 0. \tag{3.8}$$

Taking global sections, the associated long exact sequence is a subsequence of (3.1) for $V = L^{-1} \otimes \mathrm{Sym}^2 F$. The following is easy to check by explicit computation.

Lemma 3.10 *The map ψ can be identified fiberwise with the projectivization of the coboundary map in the associated long exact sequence of* (3.8)*, restricted to the image of the Segre embedding $\mathbb{P}F|_x \hookrightarrow \mathbb{P}(L^{-1} \otimes \mathrm{Sym}^2 F)|_x$. In particular, the image of $\eta \in \mathbb{P}F|_x$ is defined by the cohomology class of a principal part of the form $\frac{\lambda \otimes \eta \otimes \eta}{z}$, where z is a local parameter at x and λ a local generator of L^{-1}.*

Remark 3.11 Although we do not use this fact, we mention that ψ is an embedding if F is stable and $\deg(F) < n(\frac{\ell}{2} - 1)$ (see [4, Lemma 2.6] for the case where $L = \mathcal{O}_C$). The important property of ψ for us will be that the image is nondegenerate. This is central to Proposition 4.14.

Now let $\eta_1, \ldots, \eta_t$ be points of $\mathbb{P}F|_{x_1}, \ldots, F|_{x_t}$ for distinct $x_1, \ldots, x_t \in C$, and $\gamma : E \to F^* \otimes L$ be the corresponding elementary transformation. Let S_γ be as in Definition 3.7.

Lemma 3.12 *We have $h^1(C, S_\gamma) = 0$ if and only if the points $\psi(\eta_1), \ldots, \psi(\eta_t)$ span $\mathbb{P}H^1(C, L^{-1} \otimes \mathrm{Sym}^2 F)$.*

Proof The proof of Lemma 3.8 (c) shows that S_γ is an elementary transformation

$$0 \longrightarrow L^{-1} \otimes \mathrm{Sym}^2 F \longrightarrow S_\gamma \longrightarrow \bigoplus_{k=1}^{t} \mathbb{C} \cdot \frac{\lambda_k \otimes \eta_k \otimes \eta_k}{z_k} \longrightarrow 0,$$

where the z_k and λ_k are defined analogously as in Lemma 3.8 (c). In view of Lemma 3.10, the lemma follows from the associated long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(C, L^{-1} \otimes \mathrm{Sym}^2 F) &\longrightarrow H^0(C, S_\gamma) \longrightarrow \mathbb{C}^t \\ \longrightarrow H^1(C, L^{-1} \otimes \mathrm{Sym}^2 F) &\longrightarrow H^1(C, S_\gamma) \longrightarrow 0. \end{aligned} \tag{3.9}$$

□

Remark 3.13 Suppose that $h^0(C, L^{-1} \otimes \mathrm{Sym}^2 F) = 0$. By exactness of (3.9), we see that $H^0(C, S_\gamma)$ is the vector space of linear relations of the points $\psi(\eta_k)$ in $\mathbb{P}H^1(C, L^{-1} \otimes \mathrm{Sym}^2 F)$.

4 Irreducibility of Lagrangian Quot schemes

Let W be an L-valued symplectic bundle of rank $2n$, where $\deg L = \ell$. In general, the Lagrangian Quot schemes $LQ_{-e}(W)$ can be reducible, and also there may be irreducible components whose points all correspond to non-saturated subsheaves. In this section, we shall prove the following theorem, showing that for sufficiently large e, these phenomena disappear.

Theorem 4.1 *Let W be an arbitrary L-valued symplectic bundle over C. Then there exists an integer $e(W)$ such that for $e \geq e(W)$, the Lagrangian Quot scheme $LQ_{-e}(W)$ is irreducible and generically smooth of dimension $(n+1)e + \frac{1}{2}n(n+1)(\ell - g + 1)$, and a general point of $LQ_{-e}(W)$ corresponds to a Lagrangian subbundle. Moreover, when $g \geq 2$, a sufficiently general point of $LQ_{-e}(W)$ defines a stable vector bundle.*

Remark 4.2 Recall that $LQ_{-e}(W)^{\circ}$ denotes the open sublocus of $LQ_{-e}(W)$ corresponding to vector bundle quotients. Theorem 4.1 shows in particular, for large e, that $LQ_{-e}(W)$ is a compactification of $LQ_{-e}(W)^{\circ}$ which is in fact the closure of $LQ_{-e}(W)^{\circ}$ in $\mathrm{Quot}^{n,e+n\ell}(W)$. Other compactifications of $LQ_{-e}(W)^{\circ}$ have also been studied; more generally, generalizations of Quot schemes to principal G-bundles: Hilbert schemes of sections of $\mathrm{LG}(W)$ as in [14] and moduli of stable maps to $\mathrm{LG}(W)$ as in [13] and [16]. One attractive feature of $LQ_{-e}(W)$ is that it naturally supports a universal family of sheaves, inherited from $\mathrm{Quot}^{n,e+n\ell}(W)$.

Before embarking on the proof of Theorem 4.1, let us compare our approach with the proof of the analogous statement for $\mathrm{Quot}^{n,e+n\ell}(W)$ in [16]. Replacing the Grassmannian bundle $\mathrm{Gr}(n, V)$ with the Lagrangian Grassmannian bundle $\mathrm{LG}(W)$, the argument of [16, § 3] shows the dimension bound

$$\dim LQ_{-e}(W) \leq \binom{n+1}{2} + (n+1)(e - e_0)$$

where $-e_0$ is the degree of a maximal Lagrangian subbundle of W. However, the following difficulty arises.

If V and E are vector bundles of rank N and n respectively with $n < N$, then sheaf injections $E \to V$ are parameterized by an open subset of the linear space $H^0(C, \mathrm{Hom}(E, V))$. One can then construct the irreducible space of stable rank n subsheaves of V as in [16, Proposition 6.1]. However, for a symplectic bundle W, isotropic subsheaves $[j : E \to W]$ define a locally closed subset of $H^0(C, \mathrm{Hom}(E, W))$. This seems to be a nonlinear subvariety, for which there is no guarantee of irreducibility.

To overcome this difficulty, we introduce a collection of auxiliary Lagrangian subbundles F of W. It will emerge in view of Lemma 3.6 that Lagrangian subsheaves can be parameterized in a linear way if one also records how they are related to a fixed such F.

We proceed to the first ingredient of the proof, which is a result on the evaluation maps $\mathrm{ev}_x^e : LQ_{-e}(W)^{\circ} \to \mathrm{LG}(W|_x)$.

4.1 Surjectivity of evaluation maps

Recall that if $\mathcal{F} \to B \times C$ is a family of objects over C, we denote by $\mathcal{F}_b$ the restriction $\mathcal{F}|_{\{b\} \times C}$. Let $\mathcal{W} \to B \times C$ be a family of bundles of rank $r \geq 2$ and degree w parameterized by an irreducible base B. For each integer d, we have the relative Quot scheme $\mathcal{Q}uot^{r-1,w+d}(\mathcal{W}) \to B$ parameterizing invertible subsheaves of degree $-d$ of the $\mathcal{W}_b$.

Lemma 4.3 *Let $\mathcal{W}$ be as above. Then there exists an integer $m_0(\mathcal{W})$ such that for $d \geq m_0(\mathcal{W})$, the evaluation map $\mathrm{ev}_x^d : \mathrm{Quot}^{r-1,w+d}(\mathcal{W}_b)^{\circ} \to \mathbb{P}\mathcal{W}_b|_x$ is surjective for all $(b, x) \in B \times C$.*

Proof Fix $p \in C$, and let $\mathcal{O}_C(p)$ be the corresponding effective line bundle of degree 1 on C. By a semicontinuity argument, there is an integer $m_0 = m_0(\mathcal{W})$ such that

- for all $d \geq m_0$ and for all $b \in B$, the evaluation map on sections
$$\phi_d \colon H^0(C, \mathcal{W}_b(dp)) \otimes \mathcal{O}_C \longrightarrow \mathcal{W}_b(dp)$$
is a surjective bundle map, and
- no $\mathcal{W}_b(dp)$ admits a trivial quotient bundle.

Thus the evaluation map $\mathrm{Quot}^{r-1,w+d}(\mathcal{W}_b) \dashrightarrow \mathcal{W}_b|_x$ is surjective for all d and all (b, x). We now show that the restriction to the locus $\mathrm{Quot}^{r-1,w-d}(\mathcal{W}_b)^\circ$ of saturated subsheaves is surjective for all b.

Choose a generating subspace $V \subseteq H^0(C, \mathcal{W}_b(dp))$ of dimension $r + 1$. Let $\psi \colon C \to \mathbb{P}V$ be the map sending x to the point defined by the one-dimensional subspace $V \cap H^0(C, \mathcal{W}_b(dp - x))$.

Now if $\psi(C)$ is contained in a $\mathbb{P}^{r-1} \subset \mathbb{P}V$, then the set
$$\{s \in V : s \text{ has a zero}\} = \bigcup_{y \in C} \left(V \cap H^0(C, \mathcal{W}_b(dp - y))\right)$$
is contained in an r-dimensional subspace $V' \subset V$. Let W' be the subbundle generated by V'. As V' contains a one-dimensional space of sections vanishing at any given point, $\mathrm{rk}\,(W') = r - 1$. Then any one-dimensional subspace of V complementary to V' generates a trivial line subbundle of $\mathcal{W}_b(dp)$ intersecting W' everywhere in zero. Thus $\mathcal{W}_b(dp)$ contains a trivial direct summand. But this is excluded by our choice of d. Hence $\psi(C)$ is nondegenerate in $\mathbb{P}V$.

Now let λ be any line in $\mathcal{W}_b|_x$. Write $V_\lambda := \phi_d^{-1}(\lambda)$. As $\mathbb{P}V_\lambda$ is a $\mathbb{P}^1$ in $\mathbb{P}V$, by the previous paragraph $\mathbb{P}V_\lambda \cap \psi(C)$ is a finite set (containing $\psi(x)$). Any $s \in V_\lambda$ not lying over this finite set is a nowhere vanishing section of $\mathcal{W}_b(dp)$ which spans λ at x. The lemma follows. □

Proposition 4.4 *Let $\mathcal{W} \to B \times C$ be a family of L-valued symplectic bundles of rank $2n$ parameterized by an irreducible base B. Then there exists an integer $f_0(\mathcal{W})$ such that if $f \geq f_0(\mathcal{W})$, then for all $(b, x) \in B \times C$, the evaluation map $LQ_{-f}(\mathcal{W}_b)^\circ \to \mathrm{LG}(\mathcal{W}_b|_x)$ is surjective.*

Proof We shall prove the lemma by induction on n. If $\mathrm{rk}\,(\mathcal{W}) = 2$ then, as any line subbundle is isotropic, it suffices to set $f_0(\mathcal{W}) = m_0(\mathcal{W})$ as in Lemma 4.3.

Now suppose $2n \geq 4$. Set $m_0 = m_0(\mathcal{W})$ as in Lemma 4.3 and consider the relative Quot scheme and universal line bundle
$$\mathcal{P} \to \mathcal{Q}uot^{r-1,w+m_0}(\mathcal{W})^\circ \times C \to B \times C$$
parameterizing degree $-m_0$ line subbundles of all the $\mathcal{W}_b$. As any line subbundle is isotropic, $\mathcal{P}^\perp/\mathcal{P}$ is a family of L-valued symplectic bundles of rank $2n - 2$ parameterized by the total space of $\mathcal{Q}uot^{r-1,w+m_0}(\mathcal{W})^\circ \to B$. Since $\mathcal{Q}uot^{r-1,w+m_0}(\mathcal{W})^\circ$ is quasi-projective over B, it has finitely many irreducible components. By induction, we may assume there exists an integer $f_0(\mathcal{P}^\perp/\mathcal{P})$ such that for any $b \in B$ and any $P \in \mathrm{Quot}^{r-1,w+m_0}(\mathcal{W}_b)^\circ$, if $a \geq f_0(\mathcal{P}^\perp/\mathcal{P})$ then the evaluation map
$$LQ_{-a}(P^\perp/P)^\circ \to \mathrm{LG}\left(P^\perp/P\right)|_x$$
is surjective for all $x \in C$.

Now we return to the original family $\mathcal{W} \to B \times C$. For any (b, x), let Λ be a Lagrangian subspace of a fiber $\mathcal{W}_b|_x$. Choose any line $\lambda \subset \Lambda$. By Lemma 4.3, we may choose a line subbundle $P \subset \mathcal{W}_b$ of degree $-m_0$ with $P|_x = \lambda$. Then Λ/λ is a Lagrangian subspace of

$(P^\perp/P)|_x$. By the previous paragraph, for any $a \geq f_0(\mathcal{P}^\perp/\mathcal{P})$ we may assume there exists $\tilde{E} \in LQ_{-a}(P^\perp/P)^\circ$ such that $\tilde{E}|_x = \Lambda/\lambda$. The inverse image of $\tilde{E}$ in $\mathcal{W}_b$ is a Lagrangian subbundle E of $\mathcal{W}_b$ of degree $\deg(\tilde{E}) + \deg(P) = -a - m_0$ satisfying $E|_x = \Lambda$. Setting $f_0(\mathcal{W}) = m_0 + f_0(\mathcal{P}^\perp/\mathcal{P})$, we have proven the proposition. $\square$

4.2 Proof of Theorem 4.1

Fix now an arbitrary L-valued symplectic bundle W of rank $2n \geq 2$. We write f_0 for $f_0(W)$ as defined in Proposition 4.4, where W is regarded as a family with one element. We now introduce the "auxiliary" Lagrangian subbundles F mentioned at the start of § 4.

Definition 4.5 Let F be a Lagrangian subbundle of W. We define

$$Q_F^e \;:=\; \{E \in LQ_{-e}(W) : \mathrm{rk}\,(E \cap F) \;=\; 0\}.$$

Remark 4.6

(a) In view of the exact sequence $0 \to F \to W \to F^* \otimes L \to 0$, if $E \in Q_F^e$, then E is an elementary transformation of $F^* \otimes L$. Therefore, Q_F^e is nonempty only if $e \geq \deg(F) - n\ell$.
(b) For any Lagrangian subsheaf $F \subset W$ and any $e \geq f_0(W)$, by Proposition 4.4 we can find $[j \colon E \to W] \in LQ_{-e}(W)^\circ$ such that $E|_x \cap F|_x = 0$ for some and hence for general $x \in C$. Thus for any Lagrangian subbundle F and any $e \geq f_0(W)$, the locus Q_F^e contains a component whose general member is saturated.
(c) Clearly Q_F^e is open in all components of $LQ_{-e}(W)$, although it may be empty in some.

In what follows, we shall always assume the "auxiliary" bundles F have degree $-f_0(W) =: -f_0$. This will give the best bound $e(W)$ in Theorem 4.1 available with these methods.

Proposition 4.7

(a) *Any Lagrangian subsheaf $E \subset W$ belongs to Q_F^e for some Lagrangian subbundle F of degree $-f_0$, where $\deg(E) = -e$. In particular, for any e, as F varies in $LQ_{-f_0}(W)^\circ$ the loci Q_F^e form an open covering of $LQ_{-e}(W)$.*
(b) *Suppose now that $e \geq f_0$. Then for $F, F' \in LQ_{-f_0}(W)^\circ$, the intersection $Q_F^e \cap Q_{F'}^e$ is nonempty.*

Proof (a) Let E be any Lagrangian subsheaf of W. By Proposition 4.4, we can find a Lagrangian subbundle F of degree $-f_0$ intersecting $E|_x$ in zero at some $x \in C$. Thus $[E \to W]$ belongs to Q_F^e, where $\deg(E) = -e$.

(b) We must find a Lagrangian subsheaf E of degree $-e$ intersecting both F and F' generically in rank zero. For some $x \in C$, choose $\Lambda \in \mathrm{LG}(W|_x)$ intersecting both $F|_x$ and $F'|_x$ in zero. As by hypothesis $e \geq f_0$, by Proposition 4.4 we can find a Lagrangian subbundle E of degree $-e$ satisfying $E|_x = \Lambda$. Then $[E \to W]$ is a point of $Q_F^e \cap Q_{F'}^e$. $\square$

Next, for any bundle G, we denote by $\mathrm{Elm}^t(G)$ the Quot scheme $\mathrm{Quot}^{0,t}(G)$ parameterizing torsion quotients of degree t; that is, elementary transformations $G' \subset G$ with $\deg(G/G') = t$.

Now let F be any degree $-f_0$ Lagrangian subbundle of W. For any e, given an element $[j\colon E \to W]$ of Q^e_F, by composing with the surjection $W \to \frac{W}{F} = F^* \otimes L$ we get an elementary transformation $\widetilde{j}\colon E \to F^* \otimes L$. The association $j \mapsto \widetilde{j}$ defines a morphism

$$\Psi\colon Q^e_F \ \to \ \mathrm{Elm}^{e+f_0+n\ell}(F^* \otimes L).$$

We now study a certain subset of Q^e_F with some desirable properties. To ease notation, we set $t = t(e) := e + f_0 + n\ell$.

Definition 4.8 For each F as above, let $(Q^e_F)^\circ$ be the subset of Q^e_F of subsheaves $[j\colon E \to W]$ such that

(i) E is saturated in W; that is, j is a vector bundle injection;
(ii) The torsion sheaf $(F^* \otimes L)/\widetilde{j}(E) \in \mathrm{Elm}^t(F^* \otimes L)$ is of the form $\mathcal{O}_D$ for a reduced divisor $D \in C^{(t)}$; and
(iii) $h^1(C, S_{\widetilde{j}}) = 0$.

Remark 4.9

(a) Note that conditions (ii) and (iii) depend only on the map $\widetilde{j}\colon E \to F^* \otimes L$, and not a priori on W.
(b) A point $[j\colon E \to W] \in Q^e_F$ satisfies (ii) if and only if $\widetilde{j}$ belongs to the open subset

$$U_{\mathrm{red}} \ := \ \{[\gamma\colon E \to F^* \otimes L] : (\det \gamma) \text{ reduced in } C^{(t)}\} \ \subset \ \mathrm{Elm}^t(F^* \otimes L)$$

(compare with [6, § 2]). Furthermore, one can construct a family $\mathcal{S} \to U_{\mathrm{red}} \times C$ of elementary transformations of $\mathrm{Sym}^2 F \otimes L^{-1}$ such that $\mathcal{S}|_{\{\widetilde{j}\}\times C} \cong S_{\widetilde{j}}$ for each $\widetilde{j} \in U_{\mathrm{red}}$. By the semicontinuity theorem, the locus of $\widetilde{j}$ satisfying condition (iii) is an open subset of U_{red}. Therefore, conditions (ii) and (iii) together define an open subset of Q^e_F. As condition (i) is open in Q^e_F, it follows that $(Q^e_F)^\circ$ is open in Q^e_F.
(c) If $h^1(C, L^{-1} \otimes \mathrm{Sym}^2 F) = 0$ then (iii) follows from Lemma 3.8 (a). Otherwise, by Lemma 3.12, if $\left[\widetilde{j}\colon E \to F^* \otimes L\right]$ satisfies (ii), then (iii) is equivalent to the points $\eta_1, \ldots, \eta_t \in \mathbb{P}F$ corresponding to the elementary transformation $E \subset F^* \otimes L$ spanning $\mathbb{P}H^1(C, L^{-1} \otimes \mathrm{Sym}^2 F)$.

Remark 4.10 In view of conditions (ii) and (iii) and (3.9), the locus $(Q^e_F)^\circ$ is nonempty only if $f_0 + n\ell + e \geq h^1(C, L^{-1} \otimes \mathrm{Sym}^2 F)$. By Riemann–Roch, this becomes

$$e \ \geq \ nf_0 + \frac{n(n+1)}{2}(g-1) + \frac{n(n-1)}{2}\ell + h^0(C, L^{-1} \otimes \mathrm{Sym}^2 F).$$

Now for any $F \subset W$ we have $h^0(C, L^{-1} \otimes \mathrm{Sym}^2 F) \leq h^0(C, L^{-1} \otimes \mathrm{Sym}^2 W)$. In order to obtain later a bound which will apply to $(Q^e_F)^\circ$ for all F, **in what follows, we shall always assume that**

$$e \ \geq \ e_1(W) \ := \ nf_0 + \frac{n(n+1)}{2}(g-1) + \frac{n(n-1)}{2}\ell + h^0(C, L^{-1} \otimes \mathrm{Sym}^2 W) + 1. \tag{4.1}$$

(The final $+1$ term is required for technical reasons in Proposition 4.14.)

Proposition 4.11 *For any $[j\colon E \to W] \in (Q^e_F)^\circ$, the following holds.*

(a) *We have $h^1(C, L \otimes \mathrm{Sym}^2 E^*) = 0$.*
(b) *The locus $(Q^e_F)^\circ$ is smooth and of the expected dimension $\chi(C, L \otimes \mathrm{Sym}^2 E^*)$ at E.*

Proof By definition of $(Q_F^e)^\circ$, the subsheaf $j(E)$ is saturated in W. From Lemma 3.8 (b) it follows that $H^1(C, L \otimes \mathrm{Sym}^2 E^*)$ is a quotient of $H^1(C, S_{\tilde{j}})$. As the latter is zero by definition of $(Q_F^e)^\circ$, we obtain statement (a). Part (b) now follows from Proposition 2.4 (c). $\square$

Proposition 4.12 *Let X be a nonempty irreducible component of $(Q_F^e)^\circ$. Then for $t = e + f_0 + n\ell$, the map $\Psi\colon X \to \mathrm{Elm}^t(F^* \otimes L)$ is dominant and has irreducible fibers.*

Proof For any $[j\colon E \to W] \in X$, by Proposition 4.11, we have

$$\dim(X) = \chi\left(C, L \otimes \mathrm{Sym}^2 E^*\right).$$

Moreover, by Lemma 3.6, the fiber $\Psi^{-1}\left(\widetilde{j}\right)$ is an open subset of a torsor over $H^0(C, S_{\tilde{j}})$. Hence it is irreducible, and of dimension $h^0(C, S_{\tilde{j}})$. As $h^1(C, S_{\tilde{j}}) = 0$ by definition of $(Q_F^e)^\circ$, in fact $\dim(\Psi^{-1}(\widetilde{j})) = \chi(C, S_{\tilde{j}})$. Thus $\Psi(X)$ has dimension at least

$$\chi\left(C, L \otimes \mathrm{Sym}^2 E^*\right) - \chi(C, S_{\tilde{j}}) = \deg(L \otimes \mathrm{Sym}^2 E^*) - \deg(S_{\tilde{j}}),$$

the last equality using Lemma 3.8 (b). Now by definition of $(Q_F^e)^\circ$, the torsion sheaf $(F^* \otimes L)/\widetilde{j}(E)$ is of the form $\mathcal{O}_D$ for a reduced $D \in C^{(t)}$. Using Lemma 3.8 (c), we compute that

$$\deg\left(L \otimes \mathrm{Sym}^2 E^*\right) - \deg(S_{\tilde{j}}) = nt,$$

which is exactly $\dim \mathrm{Elm}^t\,(F^* \otimes L)$. Therefore, $\Psi(X)$ is dense in $\mathrm{Elm}^t\,(F^* \otimes L)$ as the latter is irreducible. $\square$

Proposition 4.13 *For any $F \in LQ_{-f_0}(W)^\circ$, the locus $(Q_F^e)^\circ$ is irreducible.*

Proof Suppose X_1 and X_2 were distinct irreducible components of $(Q_F^e)^\circ$. By Proposition 4.12, the restriction of Ψ to either component is dominant with irreducible fibers of dimension $\chi(C, S_{\tilde{j}})$ for any $[j\colon E \to W] \in (Q_F^e)^\circ$. But by Lemma 3.6 the fiber of the whole map $\Psi\colon (Q_F^e)^\circ \to \mathrm{Elm}^t(F^* \otimes L)$ is an open subset of a principal homogeneous space for $H^0(C, S_{\tilde{j}})$. Therefore, X_1 and X_2 would have to intersect along a dense subset of a generic fiber. But this would contradict the smoothness of $(Q_F^e)^\circ$ proven in Proposition 4.11. Thus $(Q_F^e)^\circ$ is irreducible. $\square$

The following key result shows the density of the well-behaved sublocus $(Q_F^e)^\circ \subset Q_F^e$ for sufficiently large e.

Proposition 4.14 *Let $e_1(W)$ be as defined in (4.1). For $e \geq e_1(W)$, the locus $(Q_F^e)^\circ$ is nonempty and dense in Q_F^e.*

As the proof of this proposition is somewhat involved, we postpone it to § 4.3. The following is immediate from Propositions 4.13 and 4.14.

Corollary 4.15 *For $e \geq e_1(W)$, the locus Q_F^e is nonempty and irreducible.*

Now we can prove Theorem 4.1.

Proof of Theorem 4.1 By Proposition 4.7 (a), the loci Q_F^e are nonempty and cover $LQ_{-e}(W)$. By Corollary 4.15, for $e \geq e_1(W)$, each Q_F^e is dense in exactly one component of $LQ_{-e}(W)$. By Proposition 4.7 (b), if $e \geq f_0(W)$, this must be the same component for all F. Therefore, $LQ_{-e}(W)$ has only one irreducible component.

Furthermore, by Proposition 4.14, each $(Q^e_F)^\circ$ is in fact dense in $LQ_{-e}(W)$. Hence, a general point of $LQ_{-e}(W)$ represents a vector subbundle, and by Proposition 4.11 is smooth.

Finally, we show that general $E \in LQ_{-e}(W)$ is stable as a vector bundle. For fixed $F \in LQ_{-f_0}(W)^\circ$, if $t = e + f_0 + n\ell \geq n^2(g-1)+1$, then a general stable bundle E of degree $-e$ occurs as an elementary transformation of $F^* \otimes L$. By Proposition 4.12, if we assume that $e \geq \max\{e_1(W), n^2(g-1)+1-f_0-n\ell\}$ then a general element of $\mathrm{Elm}^t(F^* \otimes L)$ lifts to W. Hence, since $LQ_{-e}(W)$ is irreducible, a general $E \in LQ_{-e}(W)$ is a stable vector bundle.

In summary, setting

$$e(W) \ = \ \max\{f_0(W), e_1(W), n^2(g-1)+1-f_0(W)-n\ell\},$$

we obtain Theorem 4.1. □

In analogy with [16, Proposition 6.3], Theorem 4.1 implies immediately the following:

Corollary 4.16 *If $g \geq 2$, then every symplectic bundle W of rank $2n \geq 2$ can be fitted into a symplectic extension $0 \to E \to W \to E^* \otimes L \to 0$ where E is a stable bundle.*

4.3 Proof of Proposition 4.14

We shall prove Proposition 4.14 by showing that for any $[E \to W] \in Q^e_F \setminus (Q^e_F)^\circ$, there exists a one-parameter deformation of E in Q^e_F of which a general member belongs to $(Q^e_F)^\circ$. As Zariski closed sets are analytically closed, it will suffice to construct an analytic deformation. We shall use principal parts to construct this deformation explicitly. The proof will be given in § 4.3.5, after we assemble some results on families of principal parts and extensions. It will be convenient to work in slightly greater generality than in § 3.1.

4.3.1 Families of principal parts and extensions

For any $d \geq 0$, there is a vector bundle $\mathcal{T}_d(V)$ over the symmetric product $C^{(d)}$ with fiber $H^0(C, V(D)|_D)$ at $D \in C^{(d)}$. The total space of $\mathcal{T}_d(V)$ parameterizes pairs (p, D) where $D \in C^{(d)}$ and $p \in \mathrm{Prin}(V)$ has poles bounded by D.

Remark 4.17 Note that for $d \geq 1$ the map $\mathcal{T}_d(V) \to \mathrm{Prin}(V)$ is not injective everywhere, as for $d' < d$ there are infinitely many inclusions $\mathcal{T}_{d'}(V) \hookrightarrow \mathcal{T}_d(V)$. More precisely, for any effective D_1 of degree $d - d'$, the canonical inclusion $V \hookrightarrow V(D_1)$ defines an inclusion $\mathcal{T}_{d'}(V) \hookrightarrow \mathcal{T}_d(V)$.

Now let F_1 and F_2 be bundles over C, and suppose that V is a subbundle of $\mathrm{Hom}(F_2, F_1)$. Let $\Pi\colon \mathcal{T}_d(V) \times C \to C$ be the projection, and let Δ_d be the pullback to $\mathcal{T}_d(V) \times C$ of the universal divisor on $C^{(d)} \times C$. There is a natural map

$$P\colon \Pi^* F_2 \ \to \ \Pi^* F_1(\Delta_d)|_{\Delta_d}$$

of sheaves over $\mathcal{T}_d(V) \times C$ whose restriction to $(p, D) \times C$ is identified with $p\colon F_2 \to F_1(D)|_D$. Notice that P factorizes into the composition $\Pi^* F_2 \to \Pi^* F_2|_{\Delta_d} \to \Pi^* F_1(\Delta_d)|_{\Delta_d}$.

Using P, we can globalize the construction (3.2) to a "Poincaré bundle" over $\mathcal{T}_d(V) \times C$. Let $\rho\colon \Pi^* F_1(\Delta_d) \to \Pi^* F_1(\Delta_d)|_{\Delta_d}$ be the restriction map, which may be viewed as a "globalized principal part" map. Let $\mathcal{W} \subset \Pi^* F_1(\Delta_d) \oplus \Pi^* F_2$ be the subsheaf given on open subsets $U \subseteq \mathcal{T}_d(V) \times C$ by

$$\mathcal{W}(U) \ = \ \{(f_1, f_2) \in \left(\Pi^* F_1(\Delta_d) \oplus \Pi^* F_2\right)(U) : P(f_2) = \rho(f_1)\}. \tag{4.2}$$

It is not hard to see that this is an extension of $\Pi^* F_2$ by $\Pi^* F_1$.

We can construct a *family* of V-valued principal parts parameterized by a scheme or analytic space S by giving a map $p\colon S \to \mathcal{T}_d(V)$ for some d. For $s \in S$, we write (p_s, D_s) or simply p_s for the image in $H^0\left(C, V(D_s)|_{D_s}\right) \subset \mathcal{T}_d(V)$ of the point $s \in S$.

Set $\Delta_S := (p \times \mathrm{Id}_C)^* \Delta_d$, and let $\pi : S \times C \to C$ be the projection. By functoriality of pullback, we have the map

$$p_S \ := \ (p \times \mathrm{Id}_C)^* P\colon \pi^* F_2 \ \to \ \pi^* F_1(\Delta_S)|_{\Delta_S},$$

and an extension $0 \to \pi^* F_1 \to \mathcal{W}_S \to \pi^* F_2 \to 0$ given on open sets $U \subseteq S \times C$ by

$$\mathcal{W}_S(U) \ = \ \{(f_1, f_2) \in \left(\pi^* F_1(\Delta_S) \oplus \pi^* F_2\right)(U) : p_S(f_2) = \rho_S(f_2)\}, \tag{4.3}$$

where ρ_S is the natural map $\pi^* F_1(\Delta_S) \to \pi^* F_1(\Delta_S)|_{\Delta_S}$. We shall now study functoriality properties of $\mathcal{W}_S$.

Proposition 4.18 *Let S be a scheme and $p\colon S \to \mathcal{T}_d(V)$ a family of principal parts. Then $(p \times \mathrm{Id}_C)^* \mathcal{W}$ coincides with the extension $\mathcal{W}_S$ defined above. In particular, for each $s \in S$, by Lemma 3.1 (b) we have*

$$\delta\left(\mathcal{W}|_{(p_s, D_s) \times C}\right) \ = \ [p_s] \ \in \ H^1(C, V).$$

Proof By construction, the sheaf $\mathcal{W}$ constructed in (4.2) fits into the exact sequence

$$0 \ \to \ \mathcal{W} \ \to \ \pi^* F_1(\Delta_d) \oplus \pi^* F_2 \ \xrightarrow{(-\rho) \oplus P} \ \pi^* F_1(\Delta_d)|_{\Delta_d} \ \to \ 0. \tag{4.4}$$

Now $\pi^* F_1(\Delta_d)|_{\Delta_d}$ is flat over $\mathcal{T}_d(V)$, as the restriction to any $(p, D) \times C$ is the torsion sheaf $V(D) \otimes \mathcal{O}_D$, which has Hilbert polynomial $d \cdot \mathrm{rk}\,(V)$. Therefore, by Lemma 4.19 below, the pullback of (4.4) remains exact after the base change $S \times C \to \mathcal{T}_d(V) \times C$. Hence $(p \times \mathrm{Id}_C)^* \mathcal{W}$ coincides with the extension $\mathcal{W}_S$ defined above. ◻

Lemma 4.19 *Let $X \to Z$ and $Y \to Z$ be morphisms, and let $\phi\colon X \times_Z Y \to X$ the projection. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence of $\mathcal{O}_X$-modules where $\mathcal{F}_3$ is flat over Z. Then the sequence $0 \to \phi^* \mathcal{F}_1 \to \phi^* \mathcal{F}_2 \to \phi^* \mathcal{F}_3 \to 0$ is exact over $X \times_Z Y$.*

Proof The sheaf sequence $0 \to \phi^* \mathcal{F}_1 \to \phi^* \mathcal{F}_2 \to \phi^* \mathcal{F}_3 \to 0$ is exact if and only if for all $(x, y) \in X \times_Z Y$, the sequence of stalks $0 \to (\mathcal{F}_1)_x \to (\mathcal{F}_2)_x \to (\mathcal{F}_3)_x \to 0$ remains exact after tensoring with $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y}$, where x and y lie over $z \in Z$. We shall prove a more general statement.

Let $C \to A$ and $C \to B$ be ring homomorphisms. Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence of A-modules, where F_3 is flat over C. Consider the sequence

$$0 \ \to \ F_1 \otimes_A (A \otimes_C B) \ \to \ F_2 \otimes_A (A \otimes_C B) \ \to \ F_3 \otimes_A (A \otimes_C B) \ \to \ 0.$$

As for any A-module M we have $M \otimes_A (A \otimes_C B) \cong M \otimes_C B$, the above sequence is identified with

$$0 \ \to \ F_1 \otimes_C B \ \to \ F_2 \otimes_C B \ \to \ F_3 \otimes_C B \ \to \ 0.$$

As F_3 is flat over C, we have $\mathrm{Tor}_1^C(F_3, B) = 0$, so this sequence is exact.

Now we obtain the desired statement by setting $F_i = (\mathcal{F}_i)_x$ for $1 \leq i \leq 3$ and

$$A = \mathcal{O}_{X,x} \quad \text{and} \quad B = \mathcal{O}_{Y,y} \quad \text{and} \quad C = \mathcal{O}_{Z,z}.$$

□

Next, we discuss lifting of elementary transformations in families. Let $\pi : S \times C \to C$ be the projection. To a family of principal parts $p \colon S \to \mathcal{T}_d(V)$ we can also associate the torsion free sheaf

$$\mathrm{Ker}\left(p_S \colon \pi^* F_2 \to \pi^* F_1(\Delta_S)\right) \subseteq \pi^* F_2.$$

Note that in general this does not coincide with $(p \times \mathrm{Id}_C)^* \mathrm{Ker}(P)$. The following generalizes the statement on saturatedness proven in [11, Theorem 3.3 (i)] (see also Proposition 3.2).

Proposition 4.20 *The inclusion* $\mathrm{Ker}(p_S) \to \pi^* F_2$ *lifts to* $\mathcal{W}_S$*, and the image is a saturated subsheaf* $\mathcal{E}_S \subset \mathcal{W}_S$*.*

Proof There is a natural composed map

$$\mathcal{W}_S \to \pi^* F_1(\Delta_S) \oplus \pi^* F_2 \to \pi^* F_1(\Delta_S)$$

where the second map is the projection. The kernel of this map is a subsheaf $\mathcal{E}_S$ of $\mathcal{W}_S$ given over open sets $U \subseteq S \times C$ by

$$\mathcal{E}_S(U) = \{(f_1, f_2) \in \left(\pi^* F_1(\Delta_S) \oplus \pi^* F_2\right)(U) : f_1 = 0 \text{ and } p_S(f_2) = 0\},$$

which clearly is isomorphic to $\mathrm{Ker}(p_S)(U)$. As $\mathcal{W}_S / \mathcal{E}_S$ injects into the torsion free sheaf $\pi^* F_1(\Delta_S)$, it follows that $\mathcal{E}_S$ is saturated in $\mathcal{W}_S$. □

Remark 4.21 We are grateful to the referee for supplying the arguments of Propositions 4.18 and 4.20, and for pointing out several mistakes in our original argument.

4.3.2 General principal parts

We shall now construct explicit families of principal parts with useful properties. Let V be a vector bundle over C. A principal part $p \in \mathrm{Prin}(V)$ will be called *general* if it can be represented by a sum

$$\sum_{i=1}^{m} \frac{\sigma_i}{z_i} \tag{4.5}$$

where $z_1, \ldots, z_m$ are local parameters at distinct points $x_1, \ldots, x_m$ of C respectively, and σ_i is a local section of V which is nonzero at x_i. If $h^1(C, V) \neq 0$, then by an argument similar to that in Lemma 3.10, the cohomology class $\left[\frac{\sigma_i}{z_i}\right]$ defines the image of the point $\sigma_i(x_i)$ in $\psi(\mathbb{P}V) \subseteq \mathbb{P}H^1(C, V)$.

We recall that a finite set of points $x_1, \ldots, x_r \in \mathbb{C}^{N+1}$ (resp., $\mathbb{P}^N$) is said to be *in general position* if for $1 \leq k \leq r$, the span of any k of the x_i has dimension $\min\{k, N+1\}$ (resp., $\min\{k-1, N\}$).

In what follows, Y will denote a nonempty closed subfibration of $\mathbb{P}V \to C$ which is Zariski locally trivial, and $\widehat{Y} \subset V$ the relative cone over Y, which is a Zariski locally trivial closed subfibration of V invariant under fiberwise scalar multiplication.

Definition 4.22 We shall say that $p \in \mathrm{Prin}(V)$ is a *general* $\widehat{Y}$*-valued principal part* if the following conditions are satisfied.

- p is general in the sense of (4.5).
- For each i, the value $\sigma_i(x_i)$ lies in $\widehat{Y}|_{x_i}$.
- If $h^1(C, V) \neq 0$, then the classes $\left[\frac{\sigma_i}{z_i}\right]$ are in general position in $H^1(C, V)$.

Note that we do not directly define "$\widehat{Y}$-valued principal parts", but only "general $\widehat{Y}$-valued principal parts".

In the case of interest to us, $V = L^{-1} \otimes \mathrm{Sym}^2 F$ and Y is the relative Segre embedding $\mathbb{P}F \hookrightarrow \mathbb{P}\mathrm{Sym}^2 F$. However, the proofs in this more general setting are identical and cover other interesting situations, and are in fact notationally less cumbersome.

4.3.3 Deforming to general principal parts

Lemma 4.23 *Let V and Y be as above. If $h^1(C, V) \neq 0$, then assume moreover that $\psi(Y)$ is nondegenerate in $\mathbb{P}H^1(C, V)$, and in particular not contained in the indeterminacy locus of ψ. Let $p \in \mathrm{Prin}(V)$ be a principal part which can be represented by a sum $\sum_{i=1}^m \frac{\sigma_i}{z^{d_i}}$, where each z_i is a local parameter at a point $x_i \in C$ and σ_i is a local section of the subfibration $\widehat{Y}$ with $\sigma_i(x_i) \neq 0 \in \widehat{Y}|_{x_i} \subseteq V|_{x_i}$. Then there exists an analytic family of principal parts*

$$p_S \in H^0\left(S \times C, \pi^* V(\Delta_S)|_{\Delta_S}\right)$$

parameterized by an open disk S around $0 \in \mathbb{C}$, where $\deg(\Delta_s) = \sum_{i=1}^m d_i =: d$ for all $s \in S$, and such that $p_0 = p$ and p_s is a general $\widehat{Y}$-valued principal part in the sense of Definition 4.22 for $s \neq 0$.

Note that the x_i in the above statement need not be distinct.

Proof We follow the approach of [4, § 2]. Choose $d = \sum_{i=1}^m d_i$ distinct complex numbers

$$\tau_{i,j} : \quad 1 \leq i \leq m, \quad 1 \leq j \leq d_i .$$

Let S be a disk around $0 \in \mathbb{C}$. For each $s \in S$, let p_s be the principal part

$$\sum_{i=1}^m \frac{\sigma_i}{(z_i - s\tau_{i,1}) \cdots (z_i - s\tau_{i,d_i})}.$$

Clearly $p_0 = p$, while for $s \neq 0$ the support of p_s consists of $\sum_{i=1}^m d_i = d$ distinct points. If $h^1(C, V) = 0$ then we are done.

Otherwise; using partial fraction decomposition, we see that for $s \neq 0$ we have

$$p_s = \sum_{i,j} \frac{\rho_{i,j}}{s} \frac{\sigma_i}{(z_i - s\tau_{i,j})}$$

for nonzero scalars $\rho_{i,j}$ (note that this sum has a removable discontinuity at $s = 0$). Hence, to complete the proof we must show that for $s \neq 0$, the cohomology classes

$$\left[\frac{\sigma_i}{z_i - s\tau_{i,j}}\right] : \quad 1 \leq i \leq m, \quad 1 \leq j \leq d_i \tag{4.6}$$

are in general position, possibly after shrinking S.

By hypothesis, the representative σ_i is a section of the subfibration $\widehat{Y} \subseteq V$ over some Zariski neighborhood U_i of x_i. Thus it defines a quasiprojective algebraic curve $\sigma_i(U_i)$ in the total space of $\widehat{Y}$, and also in $Y \subseteq \mathbb{P}V$. Consider the image $C_i \subseteq \mathbb{P}H^1(C, V)$ of this curve via the map $\psi : \mathbb{P}V \dashrightarrow \mathbb{P}H^1(C, V)$. We claim that the representative σ_i may be changed if necessary such that C_i is nondegenerate in $\mathbb{P}H^1(C, V)$.

To see this; firstly, recall that the principal part p is determined by the values $\sigma_i \mod m_{x_i}^{d_i} \cdot V_{x_i}$. Shrinking U_i, we may assume that $\widehat{Y}|_{U_i}$ is trivial. Since U_i and the fiber of $\widehat{Y}$ may be assumed to be affine, and since by hypothesis $\psi(Y)$ is nondegenerate in $\mathbb{P}H^1(C, V)$, we can if necessary replace σ_i with another section σ_i' of $\widehat{Y}|_{U_i}$ with $\sigma_i' \equiv \sigma_i \mod m_{x_i} \cdot V_{x_i}$ such that the image $C_i' \subset \mathbb{P}H^1(C, V)$ of $\sigma_i'(U_i)$ is nondegenerate.

Since C_i is nondegenerate, clearly so is the image in C_i of any open analytic neighborhood of x_i in U_i. By construction, the classes (4.6) lie inside a union of such nondegenerate analytic curves in $\mathbb{P}H^1(C, V)$. Thus, shrinking S if necessary, we can assume that these classes are in general position for $s \neq 0$. □

4.3.4 Different representatives for a fixed cohomology class

Let $p = \sum_{i=1}^m \frac{\sigma_i}{z_i^{d_i}}$ be as in the previous subsubsection, and consider again the deformation p_S constructed in Lemma 4.23. We shall now show that if we add more points of $\mathbb{P}Y$, we can construct a further deformation p_S' of p such that p_s' is a general Y-valued principal part for $s \neq 0$, and in addition $[p_s'] \equiv [p] \in H^1(C, V)$.

For any $r \geq 1$, choose nonzero $y_1, \ldots, y_r \in \widehat{Y}$ lying over distinct points $u_1, \ldots, u_r$ of C respectively. For $1 \leq k \leq r$, let ν_k be a section of $\widehat{Y}$ near u_1 such that $\nu_k(u_k) = y_k$. For each k, let w_k be a local parameter at u_k. By Lemma 3.10, if $h^1(C, V) \neq 0$ then the cohomology class $\left[\frac{\nu_k}{w_k}\right]$ defines the image of y_k in $\psi(Y) \subseteq \psi(\mathbb{P}V) \subseteq \mathbb{P}H^1(C, V)$.

Let p and p_S be as above. Since $\psi(Y) \subseteq \psi(\mathbb{P}V)$ is nondegenerate in $\mathbb{P}H^1(C, V)$, after perturbing the y_k if necessary, by Lemma 4.23 we may assume that for each $s \neq 0$, the $d + r$ cohomology classes

$$\left[\frac{\sigma_i}{z_i - s\tau_{i,j}}\right] : 1 \leq i \leq m;\ 1 \leq j \leq d_i \quad \text{and} \quad \left[\frac{\nu_k}{w_k}\right] : 1 \leq k \leq r \tag{4.7}$$

are in general position.

We shall require the following easy lemma, whose proof is left to the reader.

Lemma 4.24 *Let H be a vector space. Suppose $t \geq \dim(H) + 1$, and let $v_1, \ldots, v_t \in H$ be in general position. Then any element of H can be written as a linear combination of $v_1, \ldots, v_t$ in which every coefficient is nonzero.* □

Lemma 4.25 *Assume $d + r > h^1(C, V)$. Let p and p_S and $\frac{\nu_1}{w_1}, \ldots, \frac{\nu_r}{w_r}$ be as above. Then there exist nowhere zero analytic functions $a_{i,j}(s)$ and $b_k(s)$ on S such that the family of principal parts*

$$p_s' := p_s + \sum_{i=1}^m \sum_{j=1}^{d_i} s \cdot a_{i,j}(s) \cdot \frac{\sigma_i}{z_i - s\tau_{i,j}} + \sum_{k=1}^r s \cdot b_k(s) \cdot \frac{\nu_k}{w_k}$$

satisfies $[p_s'] \equiv [p]$ for all $s \in S$, and for $s \neq 0$, the principal part p_s' is general $\widehat{Y}$-valued in the sense of Definition 4.22.

Proof We define a map $\Phi \colon S \times \mathbb{C}^{d+r} \to S \times H^1(C, V)$ of affine bundles over S, by

$$\Phi\left(s, (a_{i,j} : 1 \le i \le m; 1 \le j \le d_i), (b_k : 1 \le k \le r)\right)$$
$$= \left(s, [p_s] + \sum_{i,j} s \cdot a_{i,j} \left[\frac{\sigma_i}{z - s\tau_{i,j}} \right] + \sum_{k=1}^{r} s \cdot b_k \left[\frac{\nu_k}{w_k} \right] \right).$$

For $s = 0$, this is the constant map $\mathbb{C}^{d+r} \to H^1(C, V)$ with value $[p_0] = [p]$. On the other hand, if $s \neq 0$ then $\Phi|_s$ is surjective, since the classes (4.7) are nonzero and in general position. Therefore, since $d + r > h^1(C, V)$, by Lemma 4.24 we can choose nowhere zero analytic functions $a_{i,j}(s)$ and $b_k(s)$ such that

$$\Phi\left(s, (a_{i,j}(s) : 1 \le i \le m; 1 \le j \le d_i), (b_k(s) : 1 \le k \le r)\right) \equiv [p]$$

for all $s \in S$. Hence, defining the family of principal parts p'_S by

$$p'_s := p_s + \sum_{i=1}^{m} \sum_{j=1}^{d_i} s \cdot a_{i,j}(s) \cdot \frac{\sigma_i}{z_i - s\tau_{i,j}} + \sum_{k=1}^{r} s \cdot b_k(s) \cdot \frac{\nu_k}{w_k},$$

the lemma follows. □

4.3.5 Proof of Proposition 4.14

Let E be a point of $Q^e_F \setminus (Q^e_F)^\circ$. We shall prove Proposition 4.14 by showing that there exists a deformation $\mathcal{E}_S$ of E in Q^e_F parameterized by a neighborhood S of 0 in $\mathbb{C}$, which satisfies $(\mathcal{E}_S)_0 = E$ and $(\mathcal{E}_S)_s \in (Q^e_F)^\circ$ for $s \neq 0$.

The saturation $\overline{E}$ of E is a Lagrangian subbundle of W of degree $-\bar{e} \ge -e$. By Lemma 3.4, we may assume that W is an extension $0 \to F \to W_{p_0} \to F^* \otimes L \to 0$ as in (3.2) for some principal part $p_0 \in \mathrm{Prin}(L^{-1} \otimes \mathrm{Sym}^2 F)$ such that $\overline{E} = \Gamma_0 \cap W_{p_0} \cong \mathrm{Ker}(p_0)$. By the proof of [4, Lemma 2.7] (essentially a diagonalization procedure for symmetric matrices over $\mathcal{O}_C$), the principal part p_0 can be represented by

$$\sum_{i=1}^{m} \frac{\lambda_i \otimes \eta_i \otimes \eta_i}{z_i^{d_i}}$$

where z_i is a local parameter at a point $x_i \in C$ and λ_i a generator for L^{-1} near x_i, and η_i is a suitable section of F nonzero near x_i; and moreover if $x_{i_1} = \cdots = x_{i_h}$ then $\eta_{i_1}, \dots, \eta_{i_h}$ are independent at x_{i_1}. Furthermore, as $\overline{E} = \mathrm{Ker}(p_0)$, we have

$$\sum_{i=1}^{m} d_i = \deg(F^* \otimes L) - \deg(\overline{E}) = f_0 + n\ell + \bar{e}.$$

If $\bar{e} \neq e$, then since $\mathrm{rk}\,(\overline{E} \cap F) = 0$, the image of $\mathrm{Elm}^{e-\bar{e}}(\overline{E}) \to LQ_{-e}(W)$ is completely contained in Q^e_F. As $\mathrm{Elm}^{\bar{e}-e}(\overline{E})$ is irreducible, if a general point belongs to the closure of $(Q^e_F)^\circ$ in Q^e_F, then in fact every point does. Therefore, if E is nonsaturated, we may assume that E is general in $\mathrm{Elm}^{e-\bar{e}}(\overline{E})$ in the sense that

$$\overline{E}/E \cong \bigoplus_{k=1}^{e-\bar{e}} \mathcal{O}_{u_k},$$

for $e - \bar{e}$ distinct and general points $u_k \in C$ lying outside $\mathrm{Supp}(p_0)$. Thus for $1 \leq k \leq e - \bar{e}$, there exists a local coordinate w_k centered at u_k and local sections ζ_k and μ_k of F and L^{-1} respectively which are nonzero at u_k, such that the element $[E \to F^* \otimes L] \in \mathrm{Elm}^{f_0+n\ell+e}(F^* \otimes L)$ satisfies

$$E = \mathrm{Ker}(p_0) \cap \mathrm{Ker}\left(\sum_{k=1}^{e-\bar{e}} \frac{\mu_k \otimes \zeta_k \otimes \zeta_k}{w_k}\right) = \mathrm{Ker}\left(p_0 + \sum_{k=1}^{e-\bar{e}} \frac{\mu_k \otimes \zeta_k \otimes \zeta_k}{w_k}\right).$$

Here, as usual, we view the principal parts as $\mathcal{O}_C$-linear maps $F^* \otimes L \to \underline{\mathrm{Prin}}\,(F)$.

We now specialize the results of the previous subsections to the present situation. Set $F_1 = F$ and $F_2 = F^* \otimes L$, and $V = L^{-1} \otimes \mathrm{Sym}^2 F$. Let Y be the relative Segre embedding $\mathbb{P}F \hookrightarrow \mathbb{P}\mathrm{Sym}^2 F$, and set $\sigma_i = \lambda_i \otimes \eta_i \otimes \eta_i$ and $\nu_k = \mu_k \otimes \zeta_k \otimes \zeta_k$. Also, $d = \bar{e} + f_0 + n\ell$ and $r = e - \bar{e}$, so $d + r = e + f_0 + n\ell$.

Continuing with this input, for each $s \in S$, we set

$$\begin{aligned} p'_s \; := \; & \sum_{i=1}^{m} \frac{\lambda_i \otimes \eta_i \otimes \eta_i}{(z_i - s\tau_{i,1}) \cdots (z_i - s\tau_{i,d_i})} + \sum_{i=1}^{m} \sum_{j=1}^{d_i} s \cdot a_{i,j}(s) \cdot \frac{\lambda_i \otimes \eta_i \otimes \eta_i}{z_i - s\tau_{i,j}} \\ & + \sum_{k=1}^{r} s \cdot b_k(s) \cdot \frac{\mu_k \otimes \zeta_k \otimes \zeta_k}{w_k}, \end{aligned}$$

precisely as constructed in Lemma 4.25. (Note that if $\bar{e} = e$, then the last sum does not appear.) This defines an analytic map $p' \colon S \to \mathcal{T}_{d+r}(L^{-1} \otimes \mathrm{Sym}^2 F)$. We construct a family of extensions $\mathcal{W}_S$ as in (4.3), except that as S is only a complex manifold, we define $\mathcal{W}_S(U)$ for U open in the analytic topology on $S \times C$. However, considering the inclusion of each point $s \hookrightarrow \mathcal{T}_{d+r}(L^{-1} \otimes \mathrm{Sym}^2 F)$ in turn, we see by Lemma 4.25 and Proposition 4.18 that the extension W_{p_s} is isomorphic to $W = W_{p_0}$ for all $s \in S$. Therefore, replacing S if necessary by a smaller neighborhood of 0 in $\mathbb{C}$, we may assume that $\mathcal{W}_S \cong \pi^* W$, where $\pi : S \times C \to C$ is the projection.

Next, as each p'_s is symmetric, by [11, Criterion 2.1] we see as before that the standard symplectic form (3.3) restricts to a symplectic structure on each $W_{p'_s}$. Furthermore, by an argument similar to that in Proposition 4.20, the sheaf

$$\mathrm{Ker}\left(p'_S \colon \pi^*(F^* \otimes L) \to \pi^* F(\Delta_S)\right) \subset \pi^*(F^* \otimes L)$$

lifts to an analytic subsheaf $\mathcal{E}_S$ of $\pi^* W$. As $\mathcal{E}_S$ is contained in the subsheaf

$$\pi^*(F^* \otimes L) \cap \mathcal{W}_S \subset \pi^* F(\Delta_S) \oplus \pi^*(F^* \otimes L)$$

and $F^* \otimes L$ is isotropic with respect to the symplectic structure (3.3), in particular each $(\mathcal{E}_S)_s$ is isotropic in W. As C is projective, by the GAGA principle the restriction $(\mathcal{E}_S)_s$ admits an algebraic structure for each s. Thus we obtain an analytic map $S \to LQ_{-e}(W)$.

We now make the following claim.

(I) For $s \neq 0$, we have $(\mathcal{E}_S)_s = \mathrm{Ker}(p'_s)$.
(II) $(\mathcal{E}_S)_0 = E$.

By (I), using Proposition 3.2 (c) or Proposition 4.20, for $s \neq 0$ the subsheaf $(\mathcal{E}_S)_s$ is saturated in $\mathcal{W}_S|_{s \times C} \cong W_{p_s} \cong W$. By construction of p_s, it also has properties (ii) and (iii). Hence for $s \neq 0$, the sheaf $(\mathcal{E}_S)_s$ defines a point of $(Q^e_F)^\circ$. By (II), we conclude that $\mathcal{E}_S$ is a deformation of $[E \to W]$ in Q^e_F, satisfying $(\mathcal{E}_S)_0 = E$ and $[(\mathcal{E}_S)_s \to W] \in (Q^e_F)^\circ$ for

$s \neq 0$. This shows that $(Q_F^e)^\circ$ is dense in Q_F^e, as desired.

It remains to prove the claim. We shall do this in the special case where $m = 2$ and $x_1 = x_2$, and $d_1 = d_2 = 2$, and $r = 1$, and assuming also that $L = \mathcal{O}_C$. The general case is only notationally more complicated. (Note that in general, one expects $x_1, \ldots, x_m$ to be distinct, but by assuming $x_1 = x_2$ here, the example illustrates all relevant aspects which can arise in the most general case.) In this case, p_S is represented by a rational section of the form

$$\begin{aligned}(1 + s \cdot a_1(s)) \cdot & \frac{\eta_1 \otimes \eta_1}{(z_1 - s\tau_{1,1})(z_1 - s\tau_{1,2})} \\ + (1 + s \cdot a_2(s)) \cdot & \frac{\eta_2 \otimes \eta_2}{(z_1 - s\tau_{2,1})(z_1 - s\tau_{2,2})} + s \cdot b_1(s) \cdot \frac{\zeta_1 \otimes \zeta_1}{w_1},\end{aligned} \tag{4.8}$$

We shall describe the sheaves $\mathcal{E}_S \cong \mathrm{Ker}(p'_S)$ and $(\mathcal{E}_S)_s \cong \mathrm{Ker}(p'_s)$ using analytic trivializations. Let us make some definitions.

- Let $U \subset C$ be an open disk around $x_1 = x_2$. Shrinking U if necessary, we may assume that the function $z_1 : U \to \mathbb{C}$ is injective and that for each (i, j) and for each s, the function $z_1 - s\tau_{i,j}$ has exactly one zero in U. We denote this zero by $x_{i,j}(s)$.
- Let U' be a disk around u_1 not intersecting U.
- Let U_0 be an open subset of C intersecting each of U and U' in an annulus not containing $x_1 = x_2$ or u_1. Shrinking S, we may assume that U_0 does not contain $x_{i,j}(s)$ for any i, j or s.
- Complete η_1, η_2 to a frame $\eta_1, \eta_2, \ldots, \eta_n$ for F over U. Let $\phi_1, \ldots, \phi_n$ be a frame for F^* over U such that $\phi_\ell(\eta_{\ell'}) = \delta_{\ell,\ell'}$, where $\delta_{\ell,\ell'}$ is the Kronecker delta.
- Let $\varphi_1, \ldots, \varphi_n$ be a frame for F^* over U' satisfying $\zeta_1(\varphi_\ell) = \delta_{1,\ell}$.
- Define $\widetilde{U} := S \times U = \pi^{-1}(U)$, and $\widetilde{U}'$ and $\widetilde{U}_0$ similarly.

Let us now fix $s \in S$ and compute the sheaf $\mathrm{Ker}(p'_s) \subset F^*$ over C. (We shall describe $\mathrm{Ker}(p'_s)$ as a subsheaf of F^*, but it can be embedded in $W_{p'_s}$ as in the example in Remark 3.5.) Over U_0, the sheaves F^* and $\mathrm{Ker}(p'_s)$ coincide for all s. Inspecting the expression (4.8) we see that a frame for $\mathrm{Ker}\left(p'_s\right)$ over U is given by

$$(z - s\tau_{1,1})(z - s\tau_{1,2}) \cdot \phi_1,\ (z - s\tau_{2,1})(z - s\tau_{2,2}) \cdot \phi_2,\ \phi_3,\ \ldots,\ \phi_n. \tag{4.9}$$

Over the remaining open set U', for $s \neq 0$ a frame for $\mathrm{Ker}(p'_s)$ is given by

$$w_1 \cdot \varphi,\ \varphi_2,\ \ldots,\ \varphi_n, \tag{4.10}$$

whereas if $s = 0$, then $\mathrm{Ker}(p'_0)$ is trivial over U'.

We proceed now to compute $\mathrm{Ker}(p'_S) \subset \pi^* F^*$ over $S \times C$, which we will then restrict to $\{s\} \times C$ and compare with $\mathrm{Ker}(p'_s)$. The frames above for $F^*|_U$ and $F^*|_{U'}$ pull back to frames for $\pi^* F^*$ over $\widetilde{U}$ and $\widetilde{U}'$ respectively. Abusing notation, we also denote these frames by $\phi_1, \ldots, \phi_n$ and $\varphi_1, \ldots, \varphi_n$.

Let $h_1\phi_1 + h_2\phi_2 + \cdots + h_n\phi_n$ be a section of $\pi^* F^*$ over $\widetilde{U}$, where the h_ℓ are analytic functions on $\widetilde{U}$. Then

$$p'_S\left(\sum_{\ell=1}^{n} h_\ell \phi_\ell\right)$$

is represented by the rational section of $\pi^* F$ over $\widetilde{U}$ given by

$$\frac{h_1 \cdot (1 + s \cdot a_1(s))}{(z_1 - s\tau_{1,1})(z_1 - s\tau_{1,2})} \cdot \eta_1 + \frac{h_2(1 + s \cdot a_2(s))}{(z_1 - s\tau_{2,1})(z_1 - s\tau_{2,2})} \cdot \eta_2.$$

This is regular at all $(s, x) \in \widetilde{U}$ if and only if for $\ell = 1, 2$ we have

$$h_\ell \cdot (1 + s \cdot a_\ell(s)) \;=\; (z_1 - s\tau_{\ell,1})(z_1 - s\tau_{\ell,2}) \cdot \gamma_\ell$$

for some analytic function γ_ℓ on $\widetilde{U}$. For $j = 1, 2$, clearly $(z_1 - s\tau_{\ell,j})$ is irreducible and does not divide $1 + s \cdot a_\ell(s)$. As $\widetilde{U}$ is a polycylinder, by [7, p. 10] the ring of analytic functions on $\widetilde{U}$ is a UFD. Therefore, $(z_1 - s\tau_{\ell,1})(z_1 - s\tau_{\ell,2})$ divides h_ℓ. It follows that a frame for $\mathrm{Ker}(p'_S)$ over $\widetilde{U}$ is given by

$$(z_1 - s\tau_{1,1})(z_1 - s\tau_{1,2}) \cdot \phi_1,\; (z_1 - s\tau_{2,1})(z_1 - s\tau_{2,2}) \cdot \phi_2,\; \ldots,\; \phi_n. \tag{4.11}$$

By a similar argument, a frame for $\mathrm{Ker}(p'_S)$ over $\widetilde{U}'$ is given by

$$w_1 \cdot \varphi_1,\; \varphi_2,\; \ldots,\; \varphi_n. \tag{4.12}$$

A key point is that this is independent of s.

For a fixed $s \neq 0$, comparing (4.9) with (4.11) and (4.10) with (4.12), we see that $\mathrm{Ker}(p'_S)_s$ and $\mathrm{Ker}(p'_s)$ coincide. This gives part (I) of the claim. As for $s = 0$: By (4.11) and (4.12) and since $x_1 \neq u_1$, we see that

$$\mathrm{Ker}(p'_S)_0 \;=\; \mathrm{Ker}(p_0) \cap \mathrm{Ker}\left(\frac{\zeta_1 \otimes \zeta_1}{w_1}\right) \;=\; E,$$

so we obtain part (II). □

Remark 4.26 The deformation p'_S is most naturally understood from the point of view of secant geometry. For simplicity, assume again that $L = \mathcal{O}_C$ and $\psi \colon \mathbb{P}F \dashrightarrow \mathbb{P}H^1(C, \mathrm{Sym}^2 F)$ is generically an embedding and that $E \subset F^*$ is a general elementary transformation corresponding to $e + f > h^1(C, \mathrm{Sym}^2 F)$ general points of $\mathbb{P}F$. By [4, Lemma 2.10 (i)], if E is nonsaturated in W then $\delta(W)$ lies on the secant spanned by $(\bar{e} + f) < (e + f)$ of these points. Moving inside the family p'_s corresponds to perturbing the linear combination of the points defining $\delta(W)$ to be nonzero at all $e + f$ points, so as to obtain a principal part with support a reduced divisor of degree exactly $e + f$, so defining a saturated isotropic subsheaf of degree $-e$.

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References

1. Biswas, I., Gómez, F., Tomás, L.: Hecke correspondence for symplectic bundles with application to the Picard bundles. Int. J. Math. **17**(1), 45–63 (2006)
2. Cheong, D., Choe, I., Hitching, G.H.: Isotropic Quot schemes of orthogonal bundles over a curve. Int. J. Math. (2021). https://doi.org/10.1142/S0129167X21500476
3. Cheong, D., Choe, I., Hitching, G.H.: Counting Lagrangian subbundles over an algebraic curve. J. Geom. Phys. **167**, 5 (2021). https://doi.org/10.1016/j.geomphys.2021.104288
4. Choe, I., Hitching, G.H.: Lagrangian subbundles of symplectic vector bundles over a curve. Math. Proc. Camb. Phil. Soc. **153**, 193–214 (2012)
5. Choe, I., Hitching, G.H.: A stratification on the moduli spaces of symplectic and orthogonal bundles over a curve. Int. J. Math. **25**, 1450047 (2014). https://doi.org/10.1142/S0129167C14500475
6. Gangopadhay, C., Sebastian, R.: Fundamental group schemes of some Quot schemes on a smooth projective curve. J. Algebra **562**, 290–305 (2020)
7. Griffiths, P., Harris, J.: Principles of Algebraic Geometry. Wiley, New York (1978)
8. Grothendieck, A.: Sur la classification des fibrés holomorphes sur la sphère de Riemann. Am. J. Math. **79**, 121–138 (1957)
9. Hartshorne, R.: Algebraic Geometry. Graduate Texts in Mathematics, p. 52. Springer, Berlin (1977)
10. Hitching, G.H.: Moduli of symplectic bundles over curves. Ph. D. thesis, University of Durham, England (2005)
11. Hitching, G.H.: Subbundles of symplectic and orthogonal vector bundles over curves. Math. Nachr. **280**(13–14), 1510–1517 (2007)
12. Holla, Y.I.: Counting maximal subbundles via Gromov-Witten invariants. Math. Ann. **328**(1–2), 121–133 (2004)
13. Holla, Y.I.: Parabolic reductions of principal bundles. arXiv:math.AG/0204219 (2019)
14. Holla, Y.I., Narasimhan, M.S.: A generalisation of Nagata's theorem on ruled surfaces. Compos. Math. **127**(3), 321–332 (2001)
15. Kempf, G.: Abelian integrals. Monografías del Instituto de Matemáticas, 13. México: Universidad Nacional Autónoma de México. VII (1983)
16. Popa, M., Roth, M.: Stable maps and Quot schemes. Invent. Math. **152**(3), 625–663 (2003)