# When Behavior Analysis Meets Machine Learning

Formation of Stimulus Equivalence Classes and Adaptive Learning in Artificial Agents

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*Dedicated with love to my parents* **Sedigheh & Mahmoud***,*

*my sister* **Samaneh***,*

*and my husband* **Iman**

# Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* at the University of OsloMet - Oslo Metropolitan University. The research presented here was conducted at OsloMet Artificial Intelligence (AI) Lab as well as the Experimental Studies of Complex Human Behaviour (ESCo HuB) Lab, under the supervision of Prof. Anis Yazidi (main supervisor), Prof. Erik Arntzen, and Prof. Hugo L. Hammer.

The thesis is a collection of four theoretical papers, the common theme of which is computational explanations of behavioral processes with a focus on developing tools for the study of human behavior in simple yet powerful computational simulations. The papers are preceded by an introductory chapter that bonds them together and provides background information and motivation for the work. The candidate is the first author and corresponding author for the first three papers, and the second author with equal contribution with professor Anis Yazidi for the last paper. This work was supported by the OsloMet through grant number 160139.

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Above all, I express my heartfelt gratitude to my parents, for all the support, love, and meaning they have gifted to me throughout my life. Beyond any doubt, they have always been sincere consultants whose wise manners of living continue to inspire me. They have also been an everlasting source of peace and happiness in my life. I feel greatly thankful for having my parents, my brothers Sajad and Khosrow, my sister Samaneh, my sister-in-law Ghazal, and my kind husband Iman, for their unconditional everlasting love, inspiration, support, and encouragement.

I have to express my deepest gratitude to my sister Samaneh for being one of the most influential people in my life. It is a great privilege and source of happiness to have such a smart, kind, courage, thoughtful, motivated, and strong person as a caring sister by my side. I would like to sincerely thank her for being one of the co-authors in Study II of this thesis, which makes the resulted paper stronger, and the writing process more pleasant.

And last but not the least, my special thanks go to my husband Iman who has always been a caring, patient fellow. This was great to simultaneously initiate my OsloMet Ph.D. and marital life with Iman. He encourages me to face difficulties, cross borders of the habitual, and welcome unknown opportunities for academic research and professional life. We were forced to spend most of our holidays on my Ph.D. ambition. Regardless, his cheerfulness to assist me realizing my objectives has always eased the path of hard working. Thanks for your hopeful correspondence, delicate roses, and mindful companionship Iman!

Asieh Abolpour Mofrad

Oslo, January 2021

"The purpose of (scientific) computing is insight, not numbers."

*Richard Hamming*

# Abstract

In this thesis, two well studied subjects in behavior analysis are computationally modeled; formation of stimulus equivalence classes, and adaptive learning. The former is addressed in Study I and Study II, while the latter is addressed in Study III and Study IV.

**Background.** Stimulus equivalence as a behavioral analytic approach studies cognitive skills such as memory and learning. Despite its importance in experimental studies, from a computational modelling point of view, the formation of stimulus equivalence classes has largely been under-investigated. On the other hand, adaptive learning in a broad sense, is a tool to study several cognitive tasks including memory and remembering. An appropriate model can be used as a cognitive level finder, and as a recommendation tool to optimize the training and learning sequence of tasks.

**Aims.** To propose computational models that replicate formation of stimulus equivalence classes and adaptive learning. The models are supposed to be simple, flexible and interpretable in order to be suitable for analysis of human complex behavior.

**Methods.** Agents endowed with Reinforcement learning, more precisely Projective Simulation and Stochastic Point Location, are used to model the interaction between experimenter and the participant through the testing/learning process. Formation of derived relations in Study I is achieved by on demand computation during the test phase trials using likelihood reasoning. In Study II, subsequent to the training phase, an iterative diffusion process called Network Enhancement is used to form derived relations, which turns the test phase into a memory retrieval phase. The solution to Stochastic Point Location in Study III aims to estimate the tolerable task difficulty level in an online and interactive settings. In Study IV, the appropriate task difficulty for training and learning is sought by using a target success rate that is usually defined beforehand by the experimenter using a method called Balanced Difficulty Task Finder.

**Results.** The proposed models for replication of equivalence relations, called Equivalence Projective Simulation (Study I) and Enhanced Equivalence Projective Simulation (Study II) could replicate a variety of settings in a matching-to-sample procedure. The models are quite flexible and appropriate to replicate results from real experiments and simulate different scenarios before performing an empirical experiment involving human subjects. In Study III, we suggest a new method to estimate the unknown point location in the Stochastic Point Location problem domain using the mutual probability flux concept and we prove that the proposed solution outperforms the legacy solution reported in the literature. The probability of receiving correct response from the participant is also estimated as a measure of reliability of participant's performance. In Study IV, we propose a model that is able to suggest a manageable difficulty level to a learner based on online feedback via an asymmetric adjustment technique of difficulty.

**Discussion.** We aimed for models that are flexible, interpretative without a need of extensive pre-training of the model. By resorting to the theory of Projective Simulation, we propose an interpretable simulator for equivalence relations that enjoys the advantage of being easy to configure. By virtue of the Stochastic Point Location model, it is possible to eliminate the need for prior-knowledge about the participant while also avoiding complex modelling techniques. Although not pursued in this thesis, those two lines of modelling could be used in a complementary setting. For instance, adaptive learning can be integrated in the training phase of matching-to-sample or titrated delayed matching-to-sample procedures as suggested in Study IV.

*K*eywords: human complex behavior, learning and memory, stimulus equivalence classes, arbitrary matching-to-sample, titrated delayed matching-to-sample, artificial intelligence, reinforcement learning, adaptive learning, stochastic point location

# Sammendrag

I denne oppgaven er velstuderte emner i atferdsanalyse modellert ved bruk av beregningsmodeller; formasjon av stimulusekvivalensklasser, og adaptiv læring. Det første er diskutert i Studie I og Studie II, og det andre i Studie III og Studie IV.

**Bakgrunn.** Stimulusekvivalens som en atferdsanalytisk tilnærming studerer kognitive ferdigheter som hukommelse og læring. Til tross for sin viktighet i eksperimentelle studier sett fra beregnings og modelleringsperspektivet, har formasjonen av stimulusekvivalensklasser i hovedsakelig vært lite forsket på. På en annen side, adaptiv læring, i vid forstand, er et verktøy for å studere flere kognitive funksjoner, inkludert hukommelse og evne til å huske. En passende modell kan brukes for å finne kognitivt nivå, og som et anbefalingsverktøy for optimalisering av oppgaverssekvenser for trening og læring.

**Mål.** Å foreslå beregningsmodeller som er i stand til å replisere formasjon av stimulusekvivalensklasser og adaptiv læring. Modellene forventes å være enkle, fleksible og tolkbare for å være godt egnet til analysering av menneskelig komplisert atferd.

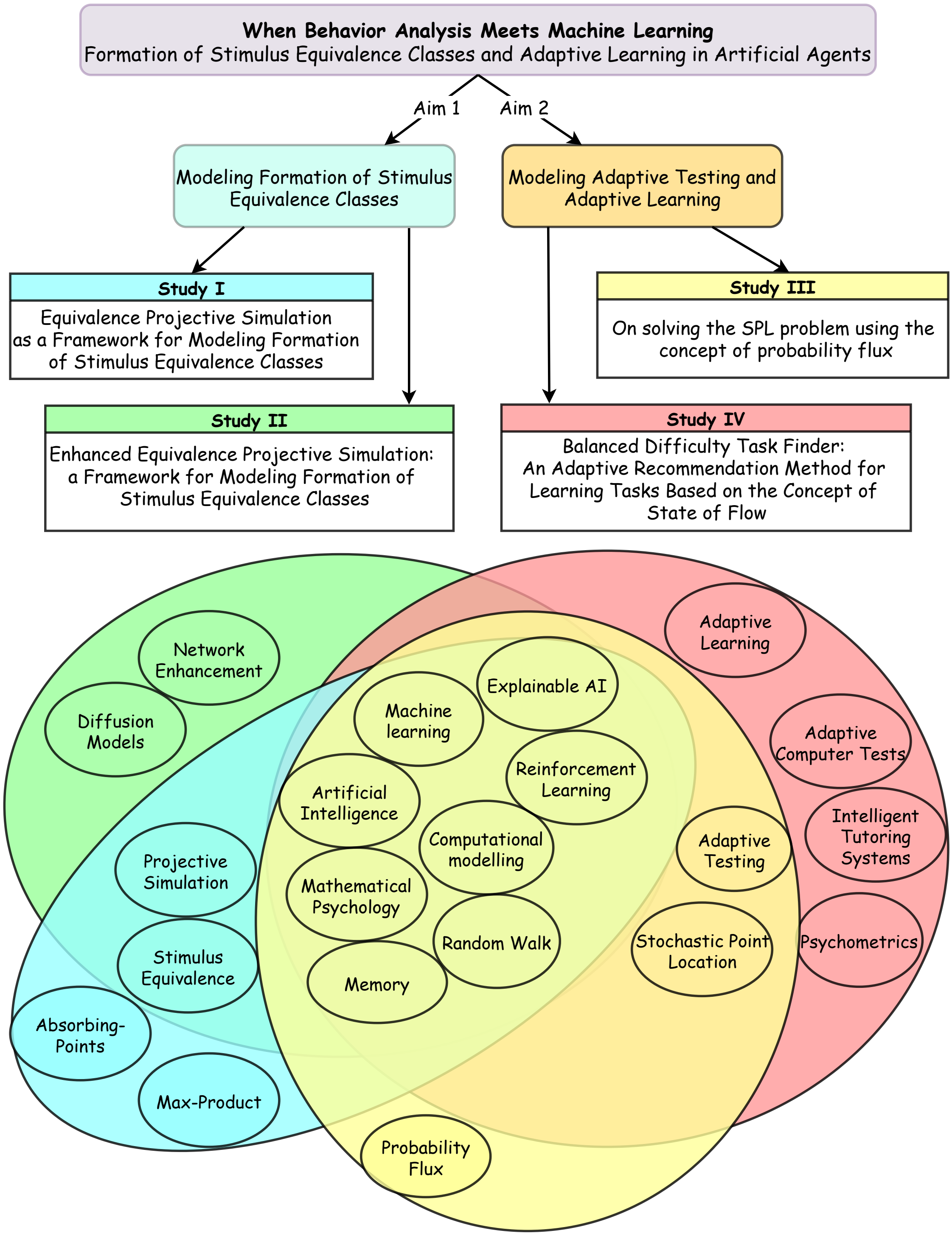
**Metoder.** Agenter utstyrt med forsterkende læring, mer presist projektiv simulering og stokastisk punktlokalisering, er brukt til å modellere samhandling mellom eksperimentator og forsøkspersonen gjennom en prøving og læringsprosess. Formasjonen av deriverte relasjoner i Studie I er oppnådd ved behovsbasert beregning under prøveforsøksfasen ved bruk av sannsynlighetsresonnementer. I Studie II, etter treningsfasen, en iterative diffusjonsprosess kalt nettverkforbedring er brukt til å danne deriverte relasjoner, som omgjør testfasen til en fase for gjenvinning av hukommelse. Stokastisk punktlokalisering i Studie III tar sikte på vurdering av passende vanskelighetsnivå i et interaktivt miljø i reell tid. I Studie IV, søkes passende vanskelighetsgrad på oppgavene ved prøving og læring ved å bruke en viss suksessrate og som vanligvis er definert av eksperimentatoren på forhånd ved bruk av en metode kalt *Balanced Difficulty Task Finder*.

**Resultater.** Foreslåtte modeller for replikasjoner av ekvivalensrelasjoner, som kalles ekvivalens projektiv simulering (Studie I) og forbedret ekvivalens projektiv simulering (Studie II) kan replikere en rekke ulike matching-to-sample-prosedyrer. Modellene er helt fleksible og passende for å replikere resultater fra ekte eksperimenter og simulere ulike scenarioer før gjennomføring av empiriske eksperimenter med mennesker. I Studie III, foreslår vi en ny metode for å vurdere den ukjent posisjonen i stokastisk punktlokaliserings problemdomen ved bruk av konseptet kalt *mutual probability flux* og vi beviser at vår

foreslåtte løsning utkonkurrerer andre løsninger rapportert i litteraturen. Sannsynligheten for å få korrekte responser fra forsøkspersonen er også vurdert som et pålitelighetsmål til forsøkspersonens gjennomføring. I Studie IV, foreslår vi en modell som anbefaler et passende vanskelighetsnivå for en bruker basert på umiddelbare tilbakemeldingen og justering av vanskelighetsgrad gjennom en teknikk kalt *asymmetric adjustment*.

**Diskusjon.** Vårt mål var å lage modeller som er fleksible, fortolkende uten behov for forhåndstrening av modellen. Ved bruk av projektiv simuleringteori, foreslår vi en tolkningsmulig simulator for ekvivalensrelasjoner som i tillegg enkelt kan konfigureres. Ved å bruke Stokastisk punktlokaliseringsmodellen elimineres behovet for tidligere kunnskap om forsøkspersonen og samtidig unngås behovet for kompleks modellering. Selv om at det er ikke fulgt i denne oppgaven, kan disse to retningene for modellering bli brukt i et kompletterende miljø. For eksempel, adaptiv læring kunne bli innlemmet i treningsfasen av matching-to-sample eller titrert forsinket matching-to-sample-prosedyrer, og som er foreslått i Studie IV.

*N*økkelord: kompleks menneskelig atferd, læring og hukommelse, stimulusekvivalensklasser, arbitrær matching-to-sample, titrert forsinket matching-to-sample, kunstig intelligens, adaptiv læring, stokastisk punktlokalisering




Thesis at a glance. Two main objective of this thesis were to computationally model formation of stimulus equivalence classes, and adaptive learning. As illustrated, Study I and Study II, address first aim and Study III and Study IV address the second aim. The related concepts to each paper is depicted in a Venn diagram with color matching to the study ID.

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# Introduction

The study of human behavior and cognition is a complex cross-disciplinary field of research involving various disciplines which include mainly philosophy, psychology, neuroscience, computer science, anthropology, and linguistics. Understanding the learning and memory mechanisms is essential in the effort to understand human cognition. In brief, learning can be understood as a process of acquiring or modifying knowledge and behaviors based on previous interactions over time. Memory is a firmly related concept where the previously learned information is maintained and available to be applied (see, Clark, 2017, for a basic history of research on the phenomenon of learning and memory).

Among scientific disciplines whereby human complex behaviors are addressed, behavior analysis science has its own approach

> ...most philosophers (and psychologists) treat cognition as a phenomenon that is built into the psyche and they ask questions about its role in such other phenomena as perception, communication, reasoning, intellectual activities, and so on. Behavior analysts, however, treat cognition as a name that summarizes a set of activities, mostly learned. Instead of accepting cognition as a built-in phenomenon, they do experiments that demonstrate how to teach the activities that constitute cognition. From the point of view that we ourselves construct cognition by means of specifiable operations, any philosophical treatment of cognition requires an understanding of those operations, that is to say, of how the construction of cognition is designed. (Sidman, 2010, p. 143)

In other words, behavior analysis treats cognition in terms of observable behaviors and activities that are mostly learnt and not built-in in the brain.

Computational models of psychological processes such as connectionist models are often used for modeling human perception, cognition, and behavior, as well as the learning processes underlying the behavior, and the storage and retrieval from memory (see McClelland, 1988, for instance). Modeling, in a general term, can render vague and complex ideas reachable, explicit, and precise enough such that their implications become clear.

> ...the usefulness of a model is not simply a matter of its correctness... models of complex processes should be taken as tools that help us understand the implications of possible assumptions that might be made about the characteristics of information-processing systems. (McClelland, 1988, p. 114)

Therefore, a model might be considered as a theory describing a real-life phenomenon which can be used to gain insights, build hypotheses and make predictions in empirical research. Mathematical psychology, which dates to 1950s, is an important branch and pillar in psychological theory such as learning, memory, classification, choice response time, decision making, attention, and problem solving (see, Busemeyer, Wang, Townsend, & Eidels, 2015). Since mathematical psychology can be used in theory construction, many areas of cognitive and experimental psychology are built on formal mathematical models and theories (Batchelder, 2010).

Artificial Intelligence (AI), although is considered as a field of research in its own right, could enrich the landscape of mathematical psychology methods as AI is concerned with the design of algorithms that mimic human natural intelligence. A definition of AI due to Bellman is:

> The automation of activities that we associate with human thinking, activities such as decision-making, problem solving and learning (Bellman, 1978).

Due to great advancement in AI, machine learning, and reinforcement learning, any effort to study mutual lessons of human brain and AI is worthy.

In order to conduct research in learning and memory, in this thesis two well-studied subjects in behavior analysis are modeled using AI algorithms; formation of stimulus equivalence classes, and adaptive learning in the face of different task difficulty levels that can model the learning experience of a learner.

Sidman (1971) identified and explored the *stimulus equivalence* phenomenon, the term which was co-opted from earlier scientists (e.g, Hull, 1939; Klüver, 1933; Tolman, 1938). Equivalence relations was originally used to study teaching methodologies for children and adults with developmental disabilities like autism spectrum disorder and Down syndrome (e.g., Arntzen, Halstadtro, Bjerke, & Halstadtro, 2010; Sidman, Cresson Jr, & Willson-Morris, 1974). Seen from a broader perspective, equivalence relations is an important research topic worthy of great attention due to its role in language, creativity and inductive inference (Sidman, 2018).

Many cognitive tasks that address remembering and learning, deal with the adjustment between task difficulty and the cognitive level of the task taker. The cognitive level of a participant is important both in studying memory problems, and in designing a sequence of training tasks with suitable difficulty level. For instance, titrated delayed matching-to-sample (TDMTS) method and Spaced Retrieval Training (SRT) (Camp, Gilmore, & Whitehouse, 1989), can be used respectively to study important variables for analyzing short-term memory problems (Arntzen & Steingrimsdottir, 2014a), and to learn and retain target information by recalling that information over increasingly longer intervals; a method which is especially used for people with dementia (Camp, Foss, O'Hanlon, & Stevens, 1996). Although testing and learning by practicing have different aims, by adaptive learning, we refer to a wide range of methods where the participant's performance is central in designing the training or testing procedures.

Despite the fact there are several computational models for both the formation of stimulus equivalence classes, and adaptive learning, in this thesis, reinforcement learning is chosen as the ground for modeling due to the interactive nature of the problems in hand. Even though there are other modeling methods that could have been used in this thesis from the realm of AI, we deliberately choose not to adopt them in this thesis because of the importance of interpretability of models in psychology which makes other black-box AI models inappropriate.

In Study I a novel instance of Projective Simulation (Briegel & De las Cuevas, 2012) which we called Equivalence Projective Simulation (EPS) is proposed for modeling equivalence relations. This model is further enhanced by applying a network enhancement method (Wang et al., 2018) in Study II which we refer to as Enhanced Equivalence Projective Simulation (E-EPS) model. These models successfully simulate the results of some well-known studies in the stimulus equivalence literature. To address the adaptive learning aspect, in Study III, we provide a method by which we can search for the difficulty level that a participant can manage based on his previous performance. This method can propose the most appropriate sequence of tasks for either testing or learning. The proposed search algorithm is based on modeling our problem as an instance of the Stochastic Point Location (SPL) problem (Oommen, 1997). A method to estimate and track the probability of receiving correct response from the participant in tandem with the estimation of tolerable task difficulty level is proposed in Study III. In Study IV, the focus is more on training and learning, and therefore we consider motivational tests fitting the capabilities of the participants in line with the efficiency of the length of the test. The idea of Study IV is closely related to the state of "Flow" in psychology and "balanced-difficulty" in game design.

In the rest of this comprehensive introduction, the importance of equivalence relations, theoretical accounts and parameters in formation of equivalence classes are addressed first. Then, the role of adaptive learning in cognitive tasks and the related concepts from mathematical psychology and psychometrics are discussed. Some essential background information about Artificial Intelligence, computational models in psychology and their role in psychology research, together with the known connectionist models are provided to make the thesis self-contained. The underlying reinforcement learning methods on which we base our model are provided afterwards. The Network Enhancement diffusion based model that is used in Study II is presented before Related works section where we briefly survey the prior models of equivalence relations and adaptive learning. At the end, a summary of the four studies in this thesis is provided and discussed.

# On Stimulus Equivalence

> The problem of equality (or of equivalents, as it has been called) is precisely the problem of finding alternative stimulus configurations for which some attribute of a response remains invariant. This problem shows up in many forms and in many fields of inquiry. As a matter of fact, an inventory would probably show it to be one of the commonest problems tackled by psychologists. (Stevens, 1951, p. 36)

The focus of Sidman (1971) seminal study was on teaching reading comprehension to a young man with intellectual disability. This seminal study made stimulus equivalence prominent in behavior analysis research for about 50 years (e.g, Arntzen, 2012; Critchfield, Barnes-Holmes, & Dougher, 2018). In general, stimuli are in an equivalence relation in the sense that they evoke the same behavioral response. Derived stimulus relations are the new relations that can be deduced from explicitly taught relations and could address aspects of learning that have been characterized as creative or generative.

Sidman and Tailby (1982) later formalized stimulus equivalence through relations in mathematical equivalence sets i.e. the relations between stimuli possess the properties of reflexivity ($A = A$), symmetry (if $A = B$ then $B = A$), and transitivity (if $A = B$ and $B = C$, then $A = C$).

Stimulus equivalence framework as an efficient learning method benefits children and adults with developmental disabilities such as autism spectrum disorder (Arntzen, Halstadtro, Bjerke, & Halstadtro, 2010; Arntzen, Halstadtro, Bjerke, Wittner, & Kristiansen, 2014; Groskreutz, Karsina, Miguel, & Groskreutz, 2010; McLay, Sutherland, Church, & Tyler-Merrick, 2013; Ortega & Lovett, 2018), Down syndrome (Sidman et al., 1974), and children with degenerative visual impairments (Toussaint & Tiger, 2010). Equivalence relations have also been used in teaching new concepts to children (Sidman, Willson-Morris, & Kirk, 1986), young people and adults without developmental disabilities (Arntzen & Eilertsen, 2020; Saunders, Chaney, & Marquis, 2005), and college students (Fienup, Covey, & Critchfield, 2010; Fienup, Wright, & Fields, 2015; Grisante et al., 2013; Hove, 2003; Lovett, Rehfeldt, Garcia, & Dunning, 2011; Placeres, 2014; Walker, Rehfeldt, & Ninness, 2010). Neurocognitive disorders, such as Alzheimer's disease, is one another target research area in equivalence relation studies. For instance, it has been discussed that derived relational responding is deteriorated as the cognitive impairment advances over time (Arntzen & Steingrimsdottir, 2017; Arntzen & Steingrimsdottir, 2014b; Arntzen, Steingrimsdottir, & Brogård-Antonsen, 2013; Bódi, Csibri, Myers, Gluck, & Kéri, 2009; Brogård-Antonsen & Arntzen, 2019, 2; Ducatti & Schmidt, 2016; Gallagher & Keenan, 2009; Seefeldt, 2015; Sidman, 2013; Steingrimsdottir & Arntzen, 2011).

The study by Sidman (1971) is not only a major landmark in the experimental analysis of human behavior, but also in the analysis of language and cognition. Figure 1 represents some of the topics and research related to stimulus relations reported by Critchfield et al. (2018). In the following I review the areas of research most relevant for the topic of this thesis, including equivalence relations and different theoretical accounts for equivalence classes, parameters in the formation of stimulus equivalence, training procedures and structures.

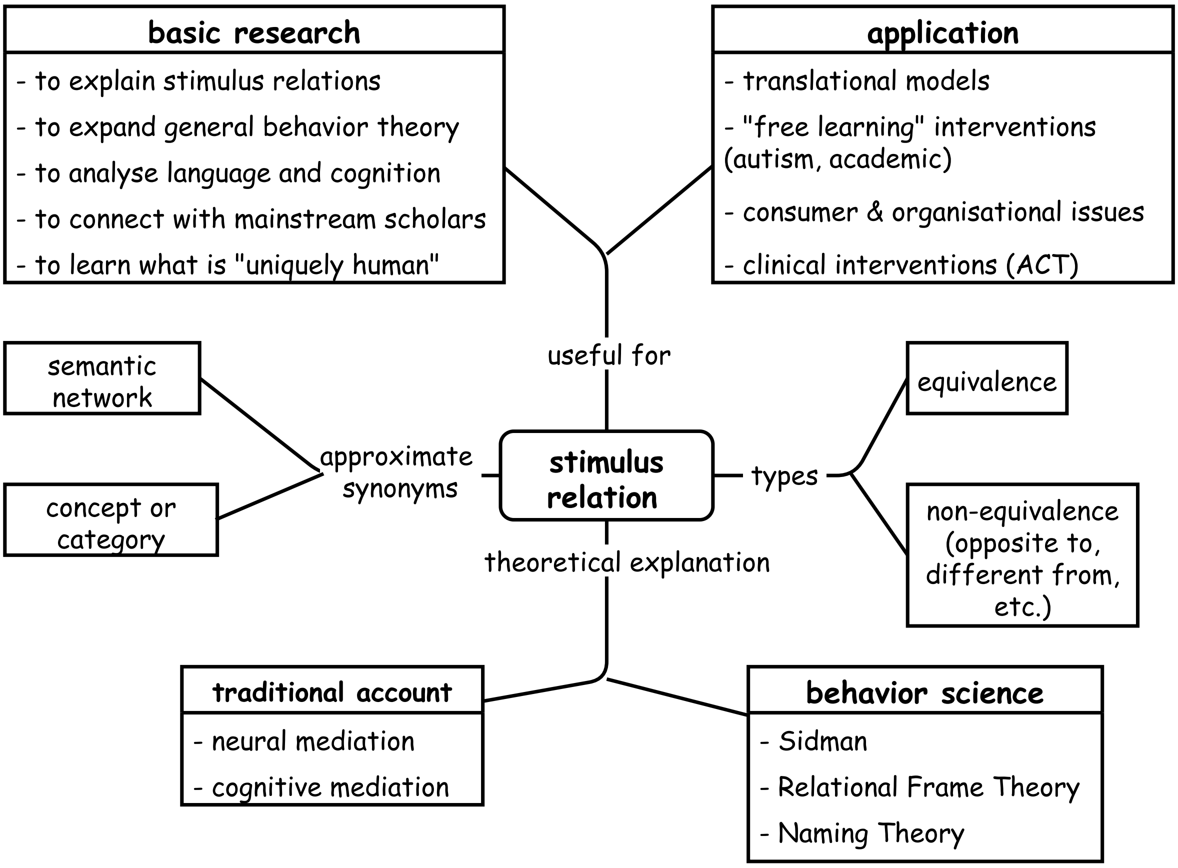


Figure 1: A summary of how the concept of "stimulus relations" emerges in different research areas based on the concept map provided by Critchfield, Barnes-Holmes, and Dougher (2018).

## Equivalence Relations in Real Life

It is fair to claim that the most important role of equivalence relations is making language such a powerful factor in human social life by using words and other symbols. As a matter of fact, words have meanings and the type of word meaning is a symbolic reference according to which the word refers to another thing or event. Sidman (2018), for example, describes this kind of symbolic reference in this way:

> ...one of the most fascinating observations is that we often react to words and other symbols as if they are the things or events they refer to. Even though we do not treat word and referent as equal in all respects, we attribute some of the same properties to both. This treatment of linguistic forms as equivalent to their referents permits us to listen and read with comprehension, to work out

> problems in their absence, to instruct others by means of speech or text, to plan ahead, to store information for use in the future, and to think abstractly–all of these by means of words that are spoken, written, or thought in the absence of the things and events they refer to. (Sidman, 2018, p. 34)

The substitution of words and other symbols with their referents may generate strange behavior though. Words, for instance, are not able to produce direct damage or hurt; but deeply integrating of words with what they refer to transform them to hurtful tools which can be used to inflict pain.

> In fact, words are considered to be hurtful. Witness what has now become commonplace in our daily news: first, killings after the receipt of actual or imagined verbal insults and, second, such killings then being justified even in the courtroom as self-defense. (Sidman, 2018, p. 35)

Examples of treating linguistic forms and nonverbal symbols as equivalent to their referents in reality abound (e.g, Sidman, 1994, 2018).

Equivalence relations by definition require the emergence of new relations from a baseline of explicitly arranged relations. This shows the incredible efficiency of the experimental paradigm as a method of teaching and a potential contribution of the equivalence research to instructional technology (for some instances of this efficiency in teaching and education, see Fienup et al., 2010; Lovett et al., 2011; Placeres, 2014; Saunders et al., 2005; Walker et al., 2010). Indeed, an equivalence class composed of $m$ stimuli, requires only $(m-1)$ stimulus-stimulus pairs to be trained and $(m-1)^2$ relations will emerge (e.g, Arntzen, 1999, 2012). Given that each component of class is used in at least one trained relation, and further none of the trained relations can have the same two stimuli as components[1]. There exist many possible ways for designing training procedures, some of them might be more efficient than the others (Arntzen, Grondahl, & Eilifsen, 2010; Arntzen & Hansen, 2011; Arntzen & Holth, 1997; Fields, Adams, Verhave, & Newman, 1990; Fienup et al., 2015; Hove, 2003; Lyddy & Barnes-Holmes, 2007; O'Mara, 1991).

[1]The calculation is intuitive. If a class has $m$ members, then there are $m(m-1)$ bidirectional relations between class members. By reducing the baseline relations we have $m(m-1)-(m-1) = (m-1)^2$. These values are just multiplied by $c$ (the number of classes), to have the total number of training and emergent relations in a setting; i.e. $c(m-1)$ baseline relations results into $c(m-1)^2$ derived relations (Arntzen, 1999, 2010).

The emergence of new behavior which is captured in equivalence classes, is also a defining feature of creativity. The creative process is largely considered as an unapproachable phenomenon and clearly, creativity entails more than only equivalence relations. However, due to the fact that equivalence relations can underlie creative acts, better understanding of equivalence relations results into better understanding creativity. Sidman (2018) identifies the creativity from equivalence relations point of view as follows:

> To the extent that we can say, "Teach a person that $A$ is related to $B$, and $B$ to $C$, and then, without further teaching, you will find the person relating $C$ to $A$, $A$ to $C$, $B$ to $A$, and $C$ to $A$," we are predicting acts of creativity from a set of specified circumstances. This is exactly what has happened over and over in the research on equivalence. In the very process of testing for equivalence relations, we see creativity being displayed even by people who have been classified as nonlearners. The more we find out about equivalence relations, the better we will understand and thereby become able to generate desirable creative performances. (Sidman, 2018, p. 41-42)

In other words, research on equivalence and the testing phase that underlie the emergence of untaught behavior is a tangible approach to study and understand some aspects of creativity.

## Different Theoretical Accounts of Stimulus Equivalence

In order to explain stimulus equivalence, there have been three main theories in behavior analysis literature; Sidman's theory (e.g, Sidman, 1994), naming theory (e.g, Horne & Lowe, 1996), and Relational Frame Theory (e.g, Hayes, 1994). These different theoretical accounts show that the stimulus equivalence phenomenon is not completely understood yet and there is still room for further investigations. A major difference between Sidman (1994) and the other theories is that Sidman considers establishing equivalence relationships as a basic behavioral process, while both Hayes (1991) and Horne and Lowe (1996) have tried to specify the historical conditions that give rise to responding derived relations. To explain how stimulus equivalence emerged, much of the legacy research focused on identifying the naming of the stimuli.

> We identify naming as the basic unit of verbal behavior, describe the condi-

> tions under which it is learned, and outline its crucial role in the development of stimulus classes and, hence, of symbolic behavior. (Horne & Lowe, 1996, p. 185)

While Horne and Lowe (1996) emphasize on naming as the mediator for equivalence formation, Sidman has repeatedly stated that equivalence relations cannot be derived from more basic principles and, therefore, they are taken for granted:

> An equivalence relation, therefore, has no existence as a thing; it is not actually established, formed, or created. It does not exist, either in theory or in reality. It is defined by the emergence of new - and predictable - analytic units of behavior from previously demonstrated units. (Sidman, 1994, pp. 388-389)

The Relational Frame Theory (RFT) account of stimulus equivalence has been developed by Hayes (1991, 1994), Hayes, Barnes-Holmes, and Roche (2001). The history of reinforcement for bidirectional responding across multiple-exemplar training is the main focus in RFT (Hayes et al., 2001). Clayton and Hayes (1999) explain the main difference between RFT and Sidman's understanding of stimulus equivalence in this way:

> Unlike the position of Sidman (1994), in which stimulus equivalence is reduced to a basic stimulus function, RFT explains equivalence as the result of prolonged exposure to the contingencies of reinforcement operating within a verbal community. (Clayton & Hayes, 1999, p. 150)

Besides an account for formation of stimulus equivalence classes, RFT is a psychological theory of human language which is built upon equivalence theory (Hayes, 1991, 1994; Hayes et al., 2001). Details of RFT as well as comparison of the three accounts is discussed by Clayton and Hayes (1999).

## Parameters in Formation of Stimulus Equivalence

The detailed design of the experiment is essential to assess the validity of the results of the experiment and evaluate the significance of data (Sidman, 2010). Some of the variables that influence equivalence class formation are addressed below.

## Type of Stimuli

Type of stimuli is one of variables that influence the formation of equivalence classes and can increase yields on equivalence tests; which means the ratio of participants who meet criteria for equivalence class formation (see, Arntzen & Mensah, 2020, and references therein). Effect of stimuli type in the formation of equivalence classes is well explored in many research studies including use of pronounceable nonsense syllable (consonant-vowel-consonant (CVC) trigrams, Lyddy, Barnes-Holmes, & Hampson, 2000), familiar color-form compounds (Smeets & Barnes-Holmes, 2005), nameable stimuli (Bentall, Dickins, & Fox, 1993), pronounceable stimuli (Mandell & Sheen, 1994), rhyming stimuli (Randell & Remington, 2006), and meaningful pictures in both children and adults (Arntzen, 2004; Arntzen & Lian, 2010; Arntzen & Lunde Nikolaisen, 2011; Holth & Arntzen, 1998; O'Connor, Rafferty, Barnes-Holmes, & Barnes-Holmes, 2009). Moreover, including familiar color pictures to the stimuli of a class has been recently studied to model meaningfulness in a laboratory setting (e.g., Arntzen & Mensah, 2020; Arntzen & Nartey, 2018; Arntzen, Nartey, & Fields, 2014, 2015, 2018a, 2018b; Fields & Arntzen, 2018; Fields, Arntzen, Nartey, & Eilifsen, 2012; Mensah & Arntzen, 2017; Nartey, Arntzen, & Fields, 2014, 2015a, 2015b; Nedelcu, Fields, & Arntzen, 2015; Travis, Fields, & Arntzen, 2014).

## Training Procedure

The minimum prerequisites for studying formation of equivalence classes are first, to have two trained relations with one common element (explained later as node), and second to test for emergence of reflexivity, symmetry and transitivity relations (Sidman, 1994; Sidman & Tailby, 1982).

Matching-to-sample (MTS) procedure is the traditional, useful and powerful methodology for studying derived stimulus relations (e.g, Sidman, 1994). In the MTS process, a given stimulus, say $A_1$, is matched with $B_1$ among a set of comparison stimuli, say $B_1$, $B_2$, and $B_3$. The discrimination is based on programmed consequence, and not because of perceptual resemblance between the matched stimuli. Arbitrary MTS means there is no conceptual relation between members of an equivalence class. An example of an arbitrary MTS is depicted in Figure 2. When the matching criteria are arbitrary, usually the procedural term, conditional discriminations, is used. This arbitrary match between stimuli, is the key aspect to study the emergence of equivalence relations that are not matched directly (Sidman, 2009). The MTS procedure has two phases, the training phase where baseline relations are trained and the testing phase where derived relations are tested.

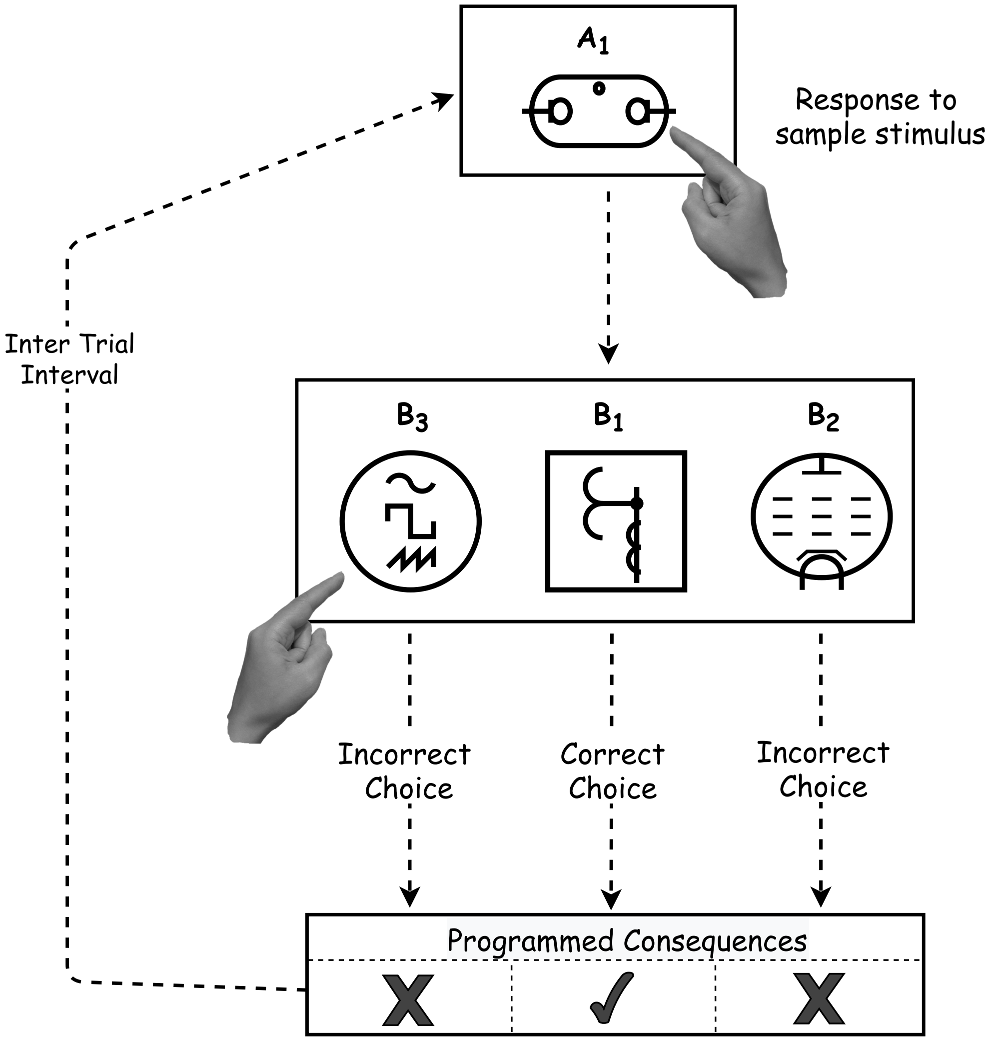


Figure 2: Example of an arbitrary MTS training trial. The discrimination is based on programmed consequence, and not because of resemblance between the sample and comparison stimuli (figure is based on an example by Arntzen & Steingrimsdottir, 2014b).

There have been some variations on standard MTS. For instance, complex or compound stimuli have been used in identity and arbitrary MTS (e.g, Braaten & Arntzen, 2019; Markham & Dougher, 1993; Schenk, 1995; Smeets, Schenk, & Barnes, 1995).

The go/no-go task is another procedure that could be used to train and test for equivalence responding with compound stimuli (e.g, Debert, Huziwara, Faggiani, De Mathis, & McIlvane, 2009; Debert, Matos, & McIlvane, 2007; Grisante et al., 2013). Similar procedures to the go/no-go procedure have been developed, such as go left/go right, or yes/no (D'amato & Worsham, 1974), or same/different (Edwards, Jagielo, Zentall, & Hogan, 1982). In the review by Fields, Reeve, Varelas, Rosen, and Belanich (1997), the wide variety of psychological phenomena that have been addressed using these methodologies is outlined.

By applying MTS procedure, Fields and Verhave (1987) identified four atemporal parameters with which defined the structures of all equivalence classes: class size, number of nodes, training directionality, and nodal density. These parameters are defined briefly below.

### Training structures (training directionality)

In equivalence literature, with MTS procedure three training structures have been used in establishing conditional discriminations: linear series (LS), many-to-one (MTO) also known as comparison-as-node, and one-to-many (OTM), also known as sample-as-node (e.g, Arntzen, 2012). Figure 3 shows the training structures for three-members equivalence classes. The order of training relations are: AB, and BC in LS; AC, and BC in MTO; and AB, AC, in OTM settings. There are several studies on the differences between LS, OTM, and MTO training structures (e.g, Arntzen, 2012; Arntzen, Grondahl, & Eilifsen, 2010; Arntzen & Hansen, 2011).

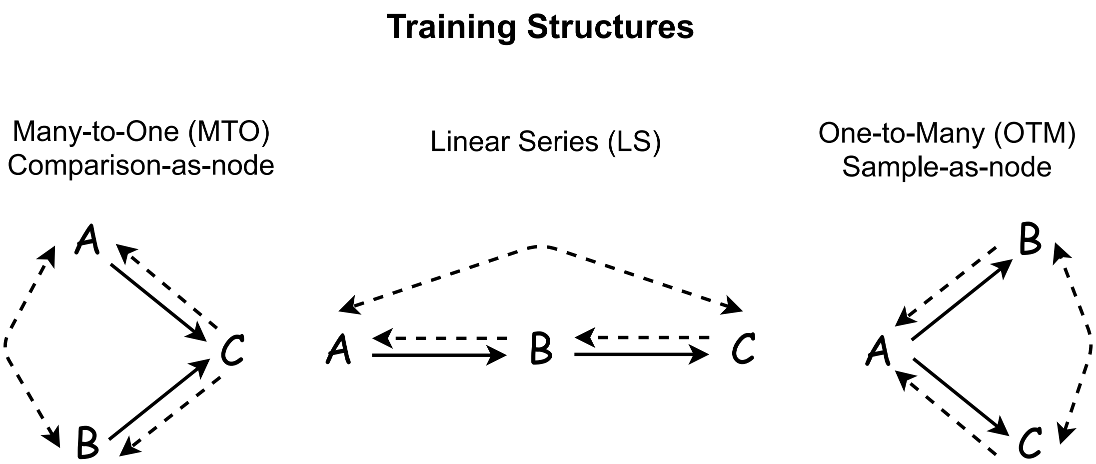


Figure 3: Training structures for an equivalence class with three members indicating by A, B, and C letters. Solid arrows show the training relations and dashed lines show derived relations, or test relations. Training relations are: AB, and BC in LS; AC, and BC in MTO; and AB, AC, in OTM (e.g, Arntzen, 2012).

### Class Size vs. Number of Classes

Class size refers to the number of members within a class. Class size and number of classes may also affect the formation of stimulus equivalence classes. For instance, the reported results of two experiments that have been conducted to study stimulus equivalence as a function of class size and number of classes indicate that the number of nodes disrupts the probability of equivalence class formation more significantly than the number of classes (Arntzen & Holth, 2000).

### Nodal Number

A node in equivalence class terms refers to any stimulus, or class member, which is related to at least two other members in the equivalence class through training. Stimuli that relate to only one other stimulus in a class are referred as singles (Fields & Verhave, 1987). Nodal number (Sidman, 1994) or nodal distance (Fields & Verhave, 1987) is the number of nodes between two members of the equivalence class. Nodal number specifies the smallest number of required baseline nodes for particular new stimulus pairs to become members of a relation. For instance, when AB and BC relations are trained, the nodal number for AC relation is one (see linear series in Figure 3). Nodal density refers to the number of stimuli related to a particular node. In the AB/BC training, the nodal density of B is two for AC/BC training, the nodal density of C is two and for AB/AC the nodal density of A is two, see Figure 3.

### Relatedness in Equivalence Class

In stimulus equivalence literature, it has been postulated that after the baseline relations are trained well in typical humans, all the stimuli in an equivalence class are each equally related to each other (e.g, Barnes, Browne, Smeets, & Roche, 1995; Fields, Adams, Verhave, & Newman, 1993; Sidman & Tailby, 1982). However, evidence from experimental studies show that under some conditions, different stimuli can have different levels of relatedness (see, Doran & Fields, 2012, for more details). Fields and Verhave (1987) address the effect of class size, number of nodes, training directionality, and nodal density, either alone or in conjunction with each other, on the relatedness between stimuli in an equivalence class. The study by Spencer and Chase (1996) addresses the relatedness on equivalence formation by measuring the response speed during equivalence responding and provides a temporal analysis of the responses. A decrease in the relatedness between the members with higher nodal number is observed too in Fields and Verhave (1987). Likewise, Doran and Fields (2012) show that stimuli within an equivalence class are differently related based on nodal distance and relational type. The degree of relatedness between equivalent stimuli has been studied using more sensitive measures than MTS results (e.g., Bortoloti & de Rose, 2011) confirming the notion that equivalent stimuli may differ in their degree of relatedness.

### Delayed Matching-to-Sample

Conditional discriminations procedures might be either simultaneous MTS or delayed MTS (see, Arntzen, 2006, as the first parametric study combining delayed MTS and equivalence performance). In simultaneous MTS, the sample stimulus is presented along with the comparison stimuli, however, in delayed MTS, the sample stimulus appears first and disappears for some programmed time before the comparison stimuli are presented. Simultaneous MTS procedures are applicable when studying learning, while delayed MTS procedure is relevant for the study of memory (Blough, 1959; Palmer, 1991). The delay period could be fixed or could be "titrating". Titrated delayed MTS method, also referred to as adjusting delayed MTS (Cumming & Berryman, 1965; Sidman, 2013), changes the length of the delay as a function of the participants' responses using trials and errors. In this sense, the participants' responses provide additional information about the remembering behavior of the participant. Titrated delayed MTS has been used to study remembering in a variety of settings, including to study important variables in analyzing short-term memory problems (e.g, Arntzen & Steingrimsdottir, 2014a). Delayed MTS procedures often increase the equivalence responding yield, which could be the result of rehearsal to keep the sample information during the delay until comparison stimuli appear (e.g, Arntzen & Vie, 2013).

### Performance Evaluation in Matching-to-Sample Tasks

The establishment of the baseline conditional discriminations during training is usually evaluated through a threshold or mastery criterion (e.g., 95% − 100% of the trials in a training block answered correct). The successful or unsuccessful establishment of the baseline relations provides important information about the learning capabilities of a participant and therefore an experimenter may adjust the training procedure to increase the chance of establishing baseline relations based on the participant performance (see, Arntzen, 2012, for instance). If the participant is able to pass the training criterion the derived relations will be tested. Usually the criterion ratio in the test phase is lower than its equivalent in the training phase and there is no programmed consequence. By passing the criterion for testing in all relations, the equivalence class is considered to be formed (Sidman & Tailby, 1982).

Reaction time (Dymond & Rehfeldt, 2001; Whelan, 2008) or speed (Imam, 2006; Spencer & Chase, 1996) can be considered as a supplementary measure worth analyzing in the stimulus equivalence research. The research on reaction time shows that the response latency for symmetry trials is longer compared to the baseline relations and even longer in transitivity and equivalence trials, moreover, reaction time toward the end of testing is lower compared to the beginning ( e.g., Bentall et al., 1993; Holth & Arntzen, 2000).

Moreover, the stimulus equivalence literature has been expanded by using additional measures, including sorting tests (Arntzen, Granmo, & Fields, 2017), brain imaging such as fMRI (Dickins, 2005; Dickins et al., 2001) and EEG recording (Arntzen & Steingrimsdottir, 2017; Haimson, Wilkinson, Rosenquist, Ouimet, & McIlvane, 2009), or eye-tracking analysis (Dube et al., 1999; Hansen & Arntzen, 2018; Steingrimsdottir & Arntzen, 2016). These additional measures could lead to fine-grained analysis of the conditional discriminations learning and stimulus equivalence responding (Palmer, 2010).

# Adaptive Behavior and Learning

Time delay procedure, first was used by Touchette (1971) in teaching discrete responses to individuals with intellectual disability. From then, time delay and its modifications have been used in the fields of special education, speech-language pathology, and early intervention (Pennington, 2018). As it has been mentioned, delayed MTS and titrated delayed MTS can be used as measurement techniques in short-term memory studies (Arntzen & Steingrimsdottir, 2014a; Sidman, 2013). Spaced Retrieval Training (e.g., Camp et al., 1996; Camp et al., 1989) is another method of learning and retaining target information by recalling the information over longer intervals. These methods can be seen as instances of adaptive learning and the delay time can be addressed by applying theories from the psychometrics field. In the following, a brief overview of the research from behavior analysis scientists in education and adaptive learning is provided. The field of psychometrics and its use in personalized learning is also addressed in this part.

## Behavior Analysis in Education

The history of applying behavior analysis theory in the design and production of hardware and apparatus as well as methods and practices is quite rich (Twyman, 2014). Skinner's Technology of teaching, published in 1968, reflects his theoretical perspective applied to problems in teaching and learning (Skinner, 2016). To surpass the usual classroom experience, Skinner designed and implemented a series of studies to improve teaching methods for spelling, math, and other school subjects using a teaching machine (Skinner, 1954, 1960). A teaching machine could be any device which arranges contingencies of reinforcement. The teaching machine of Skinner was a mechanical device that uses an algorithm which combines teaching and testing items and helps the student to gradually learn the material via a sort of reinforcement learning. Skinner teaching machine aims to provide a problem tailored precisely to a student skill level, not to the class average, and assesses every answer immediately to determine the next learning step. Tailored or personalized learning can not usually achieved in nowadays learning classes given the scarcity and cost of human teachers, which motivates using a teaching machine to handle this type of tailored learning.

Keller formulated the Personalized System of Instruction for college teaching (Keller, 1968). Personalized System of Instruction is a widely used teaching plan composed of small, self-paced, modularized units of instruction with guidance to lead learners over the modules until they achieve mastery (see, Twyman, 2014, for more details on how behavior analysis has contributed to the field of education). It is noteworthy that the importance of tailoring stimuli in learning has also been studied through stimulus equivalence methodology (Arntzen & Eilertsen, 2020).

## Mathematical Psychology and Psychometrics

A model, which is central in scientific research, can be seen as a simplified illustration of a system to reduce its complexity and its behavior quantitatively and also qualitatively. Model types can be conceptual, verbal, diagrammatic, physical, or formal (mathematical). The central function of modeling is to turn vague and complex ideas into accessible, explicit, and precise enough so that their implications become clear (e.g, McClelland, 1988).

Devising models mimicking the brain mechanisms is quite hard in psychological science due to the brain's incredible complexity. Busemeyer et al. (2015) address the benefits of modeling in this complex area as follows:

> Nonetheless, the resources of mathematical modeling, neuroscientific knowledge and techniques, and excellent behavioral and neuropsychological experimental designs offer the best we can hope for. ...Electrical engineering and computer science have long possessed rigorous quantitative bodies of knowledge; we could call them meta-theories, of how to infer the internal mechanisms and dynamics from observable behavior. (Busemeyer et al., 2015, p. 91).

Historically, the application of mathematics to psychology dates back to at least the seventeenth century (Batchelder, 2015); theories for experimental phenomena led to the field of mathematical psychology in the 1950s and statistical methods for measuring individual differences led to the field of psychometrics in the 1930s. Since experimental psychology was dominated by behavioral learning theory in 1950s, mathematical models of the learning process (or mathematical learning theory) became a central topic in mathematical psychology (Bush & Estes, 1959).

Psychometrics is rather concerned more with how psychological constructs, such as intelligence, neuroticism, or depression, can be optimally related to observables, like outcomes of psychological tests, genetic profiles, and neuroscientific information (e.g, Borsboom, Molenaar, et al., 2015). The psychometric model, in a sense is a measurement model that integrates the correspondence between observational and theoretical terms. The measurement model falls under the scope of item response theory (IRT), a subfield of psychometrics, if the observed variables are responses to test items. IRT can be seen as a collection of measurement models which is especially important in the analysis of educational tests and adaptive testing (e.g, Chen & Chang, 2018).

> Briefly, IRT offers several advantages over classical test theory: (1) it provides more in-depth insight at the item level; (2) it facilitates the development of shorter measures (by applying computerized adaptive routines); (3) it detects cross-group variations in item performance (called differential item functioning or DIF); (4) and it permits linking scores from one measure to another (Krabbe, 2016, Chapter 10, p. 171).

In IRT-based models, items have different difficulty levels. Defining or measuring *task difficulty* can be addressed in many ways. IRT models determine the probability of a discrete result, such as a correct response to an item, based on both item and respondee parameters using mathematical functions (Krabbe, 2016, Chapter 10).

Adaptive testing has become increasingly important with the advent of computerized test administration tools. Adaptive testing shortened the test without affecting reliability by administering items based on the previous item responses of the learner. Computerized adaptive testing (CAT), also called tailored testing, is a form of computer-based test that adapts to the examinee's ability level (e.g, Linden & Glas, 2000). It can be seen as a form of computer-administered test in which the next item selected to be administered depends on the correctness of the examinee's responses to the late items administered (see, Embretson, 1992, for some contributions of CAT to psychological research).

Adaptive learning designs could also benefit from psychometrics approaches such as IRT to extract required information for adaptive and personalized learning (Chen, Li, Liu, & Ying, 2018). An adaptive learning system provides instruction based on the current status of a learner and together with advances in technology, provides learners with high-quality and low cost instructions. A recommendation system is a key component of an adaptive learning system that recommends the next item (video lectures, practices, and so on) based on the history of learner (e.g, Chen et al., 2018, for some psychometrics applications in adaptive personalized learning). Adaptive learning in the form of Intelligent Tutoring Systems, that benefits from the application of artificial intelligence techniques, will be presented later.

## Artificial Intelligence - Machine Learning

Artificial intelligence (AI) is the field devoted to build *intelligence* demonstrated in machines, unlike the natural intelligence displayed by humans and animals. Concerning the concept of intelligence makes AI similar to philosophy and psychology, however, AI attempts to build artificial intelligent entities in addition to understanding natural intelligence found in nature.

The concept of *intelligence* has methodically been studied for long times. Scholars from philosophy and psychology have always attempted to study cognitive functions such as vision, learning, remembering, and reasoning theoretically and through real experiments. During 1950s, the advent of computer systems led to a paradigm shift as the abstracts contemplation on those cognitive categories to be shaped as real experimental and theoretical discipline. At the moment, AI is a flourishing field which includes many sub-fields; from perception and logical reasoning, playing games such as chess and go, to diagnosing diseases. Due to the increasing popularity of AI, experts in different research disciplines are getting more interested in AI literature and its applications in their respective research fields. It can also be asserted that AI researchers, who are computer scientists by definition, have applied AI methodologies to other disciplines and that interdisciplinary AI research is gaining a lot of momentum. Therefore, this is fair to claim that the AI as a field of study has expanded with a broad workability (e.g, Russell & Norvig, 2009, for more details).

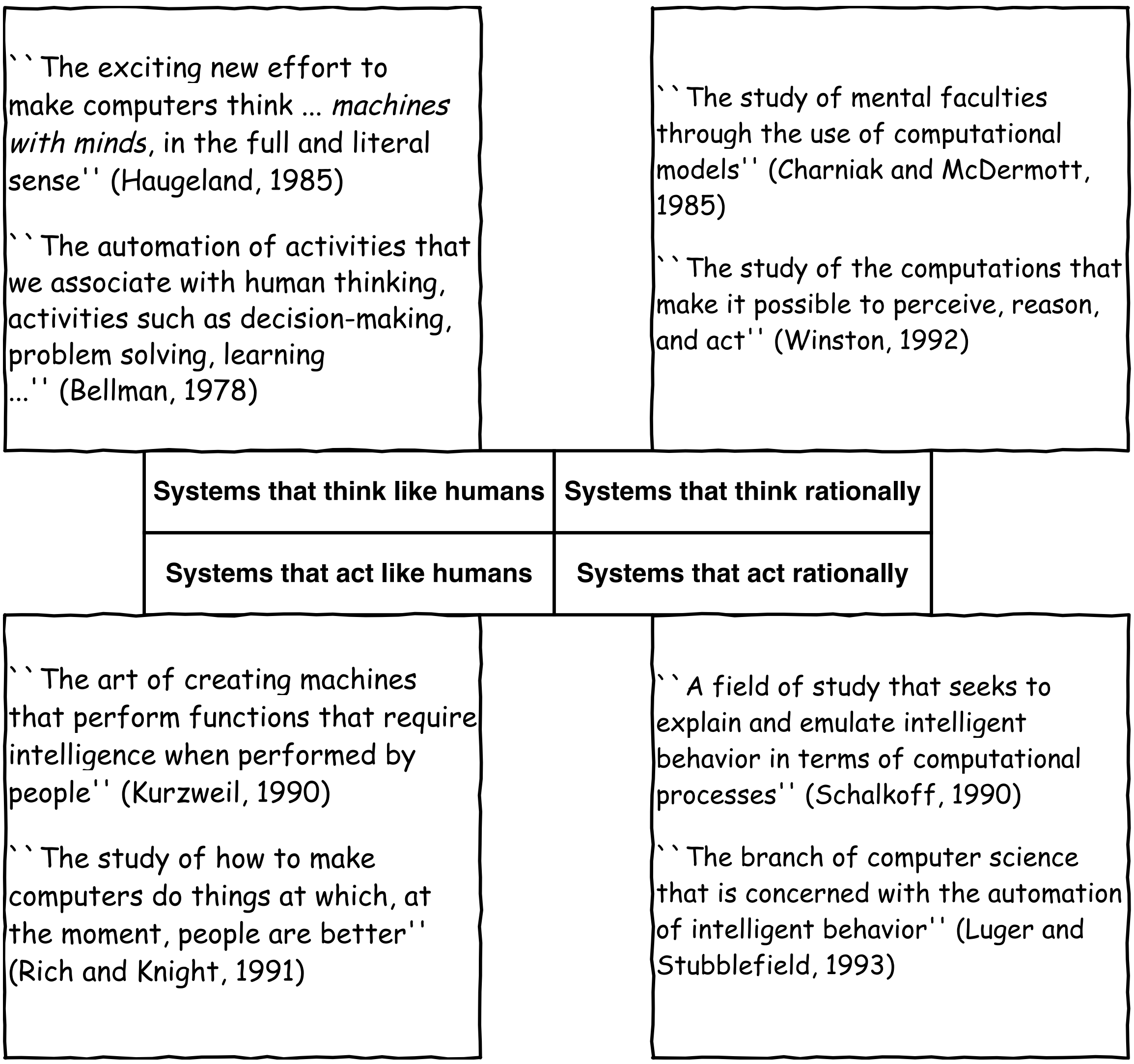


Figure 4: Some definitions of AI, based on the categorization by Russell and Norvig (2009, Chapter 1)

Some definitions of artificial intelligence that are collected by Russell and Norvig

(2009) are reported in Figure 4. Based on these definitions, an AI researcher might concerned with either thinking or behavior and want to model humans, or an ideal concept of intelligence (called rationality). The distinction between a human-centered approach and a rationalist approach comes from the fact that humans often make mistakes (see, Kahneman, Slovic, Slovic, & Tversky, 1982, for some of the systematic errors in human reasoning). Therefore, a human-based approach can be seen as an empirical science, involving hypothesis and experimental confirmation, while a rationalist approach involves a combination of mathematics and engineering (Russell & Norvig, 2009). Regardless of which approach is chosen to the AI, better understanding of brain and intelligent behaviors in humans and animals could play an essential role in building intelligent machines (see, Hassabis, Kumaran, Summerfield, & Botvinick, 2017, for some advances in AI that have been inspired from neuroscience).

In the following, first some artificial neural network models, known as connectionist models are briefly introduced. Then, reinforcement learning models and some specific types that are more relevant to this thesis are reviewed.

## Neural Networks and Connectionist Models of Cognition

A Computational model, typically studies a complex system by running a simulation on a computer with the desired parameters and interpreting the behavior of the model. The computational models in the field of cognitive science are referred to as *computational cognitive models* or *computational psychology* which can be theories of cognition; mostly process based theories (e.g, Sun, 2008).

Historically, the first known artificial unit based on biological neurons is the *McCulloch-Pitts neuron* (McCulloch & Pitts, 1943) which is an abstracted version of a real neuron and functions as a logic gate, which is assumed to be the main function of a neuron. These types of neurons are also known as *threshold logic units* (e.g, Kruse, Borgelt, Braune, Mostaghim, & Steinbrecher, 2016) since a symbolic logic is applied in order to describe what neural nets can do. In McCulloch-Pitts nets, a neuron becomes active and sends a signal to other neurons if it receives enough excitatory input that is not compensated by equally strong inhibitory input.

*Perceptron* (Rosenblatt, 1958) is one of the first major advances from the McCulloch-Pitts neuron with the ability to classify some pattern of input data. Perceptron uses non-binary input and weight connections, and adjust the weights so that network can learn. In such models an artificial neuron is not equivalent to a single biological neuron, but rather a population of neurons performing a particular function. Some of Rosenblatt's predictions for future of perceptrons demonstrated to be surprisingly accurate:

> Later Perceptrons will be able to recognize people and call out their names and instantly translate speech in one language to speech or writing in another language, it was predicted.[2]

The idea of using changes in the weight values of the network for learning in artificial neural networks is based on the biological neural systems and a simple rule of synaptic plasticity which is proposed by Hebb (1949):

> When an axon of cell $A$ is near enough to excite a cell $B$ and repeatedly or persistently takes part in firing it, some growth process or metabolic change takes place in one or both cells such that $A$'s efficiency, as one of the cells firing $B$, is increased. p.62, Hebb (1949)

This type of learning, known as Hebb's rule or associative learning, suggested that connections between neurons with correlated activity must be strengthened; a postulate which is supported by a large body of experimental evidence (see Karaminis & Thomas, 2012; Sommer, 2012).

Adding hidden layers to perceptrons yields multi-layer perceptrons, which are basically the Feedforward neural networks, known also as parallel distributed processing (PDP) models, Connectionist models, and deep learning. Feedforward networks are called so since the signals pass along only one way; forward (see Figure 5 for the general structure of feedforward neural networks). Although Feedforward neural networks are not considered as biologically plausible, these networks can be seen as universal function approximation where many applications of the AI is based on these types of networks; such as object recognition tasks, speech recognition, image processing, self-driving cars (see, Liu et al., 2017, for a survey of on architectures and applications of such networks). The *Backpropagation* algorithm which was introduced in the 1970s, became an important fast learning approach to neural networks after famous paper by Rumelhart, Hinton, and Williams (1985). Feedforward neural networks uses backpropagation algorithm in

[2]The New York Times, "New Navy Device Learns By Doing" (8 July 1958)

training for supervised learning and this algorithm is the workhorse of learning in neural networks. Briefly, the network processes the input and by passing signals through layers produces an output. Since the learning is supervised, the correct output is known and therefore the error can be calculated. The network has to figure out how to change the weights (learning is through adjusting the weights) in order to generate outputs closer to the right answer. This weight adjustment is for all the connections in the network and network must figure out how much of the error is due to a particular connection and its weight; this is also known as *Credit Assignment Problem* (e.g, Schmidhuber, 2015).

Backpropagation algorithm propagates the error backward toward the network and captures the amount of error due to each connection in order to adjust the weights to most efficiently decrease the the errors see, LeCun, Touresky, Hinton, and Sejnowski, 1988, for a theoretical framework of backpropagation algorithm. Convolutional neural networks, which is highly praised models in neural networks, is categorized as a feedforward neural network (Krizhevsky, Sutskever, & Hinton, 2012).

Recurrent neural networks is a more biologically plausible model, in comparison with feedforward nets, which is appropriate to the time-dependent information processing such as memory, sequences and dynamics. In recurrent neural networks, units are not just connected and send signals to the forward layer, but they can project into units in the same layer and the previous layer. Unlike feedforward neural networks where the output is only a function of input (a static function is computed), in recurrent neural network the output is a function of input and internal states or history of learning (a dynamic function is computed). Sometimes it is hard to distinguish and analyze recurrent neural networks as multi-layer network due to the cycles in the structure which provides the system memory. Hopfield neural networks (Hopfield, 1982) which is also known as content addressable memories is one of the first models of the learning and retrieval of memories in the brain. Hopfield neural networks could form a full memory using partial information related to that memory. The learning in Hopfield is based on Hebb's rule and the so-called *one-shot learning,* where the network needs to observe each query and memory only once to learn the association between them (Hopfield, 1982). Boltzmann Machine, also called stochastic Hopfield network with hidden units, is a similar but more powerful recurrent neural network where units are stochastic (Ackley, Hinton, & Sejnowski, 1985). Recurrent neural networks appeared in many applications such as natural language processing (e.g, Collobert et al., 2011), language understanding machine translation, video processing, music composition and financial markets (Medsker & Jain, 1999). Training recurrent neural networks is much more challenging than feedforward networks. Back-propagation through time is a popular solution Which basically transforms a recurrent network into a feedforward network and update the training weights (Lillicrap & Santoro, 2019). Long Short Term Memory (LSTM) networks is a widely used recurrent neural network (Hochreiter & Schmidhuber, 1997) which uses memory cells, that is basically a series of gates. LSTMs' memory ability make it the standard recurrent neural network in AI.

Spiking neural networks is another class of neural networks which is a closer model to the brain, since a real neural network uses spikes, but adding more realism to the model adds more challenge to train and implement the model.

Both feedforward and recurrent networks use continuous activation functions and back-propagation which depends on this continues function. In other words, the activity of a unit can take any value in a range and that value passes through all the connections. However, real neurons send action potentials, i.e. the neuron send either the spike or nothing. Spiking neural networks are good models to study how brain works and there is significant research attention around these networks as an inspiration to the algorithms in AI (see, Taherkhani et al., 2020, for a review on learning in spiking neural networks).

From cognitive psychology point of view, connectionist models attempt to explain and replicate intellectual abilities by assuming that neural systems pass activation among simple, interconnected processing units (McClelland, Rumelhart, Group, et al., 1987). Connectionist models, or neural network models, are the most commonly used cognitive model today which similar to all cognitive models, need some building blocks and some organization for them. The primitives in PDP or connectionist models are units and connections; with emphasis on the parallel nature of neural processing, and the distributed nature of neural representation. To define the architecture of a connectionist model, one decides on the number of units, connectivity pattern between the units, and the interactions with the environment. The network term typically refers to the set of units and their connections (McClelland, 1988).

Although Connectionist models have been primarily studied in cognitive science and cognitive psychology, they have been also applied in behavior analysis studies (Barnes &

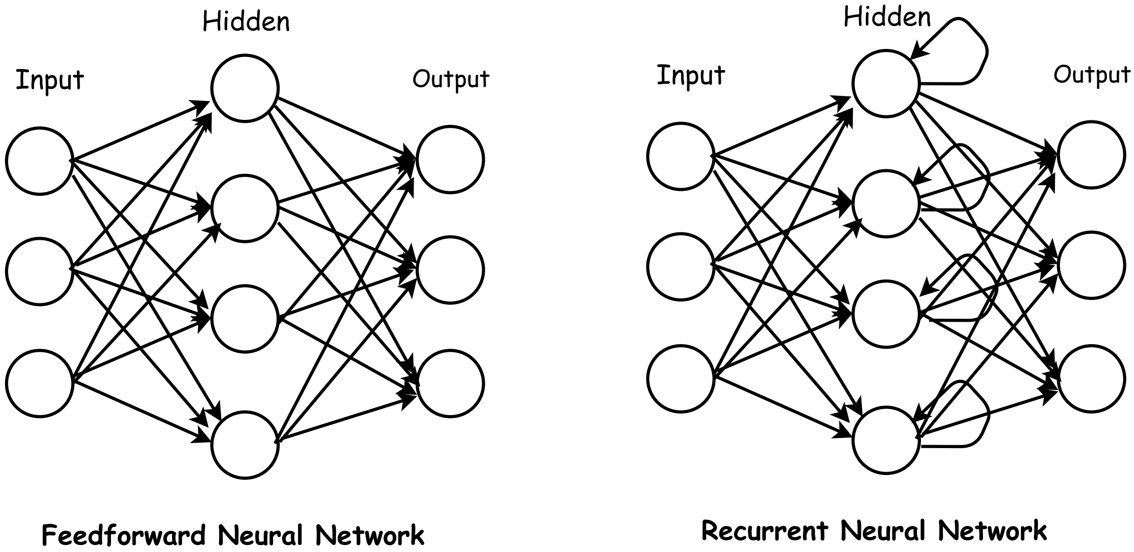


Figure 5: The general structure of two basic neural networks; feedforward and recurrent neural networks. Deep neural networks usually have many hidden layers. In the learning process, the knowledge is stored in the connections by changing the connection weights which justifies the connectionism name for these models.

Hampson, 1993; Barnes & Holmes, 1991; Fodor & Pylyshyn, 1988; Staddon & Bueno, 1991).

## Computational Reinforcement Learning

Reinforcement learning (RL) refers to the scientific study of how animals, humans, and machines use the experience to adapt their behavior in order to maximize the received total reward from an environment (Busemeyer et al., 2015, Chapter 5).

RL is a highly interdisciplinary field of research that lies at the intersection between computer science, machine learning, psychology, and neuroscience. RL deals with learning from evaluative feedback (see Figure 6 for a schematic view) and not corrective feedback bearing similarities to supervised learning which is dominant in the field of machine learning (Sutton & Barto, 2018). In supervised learning, an external supervisor provides a set of labeled samples and the objective is to generalize or extrapolate the kind of responses that are taught during training. Supervised learning, unlike reinforcement learning, is an inadequate choice for reactive learning from interaction with the environment (see, Sutton & Barto, 2018, for details of reinforcement learning schemes).

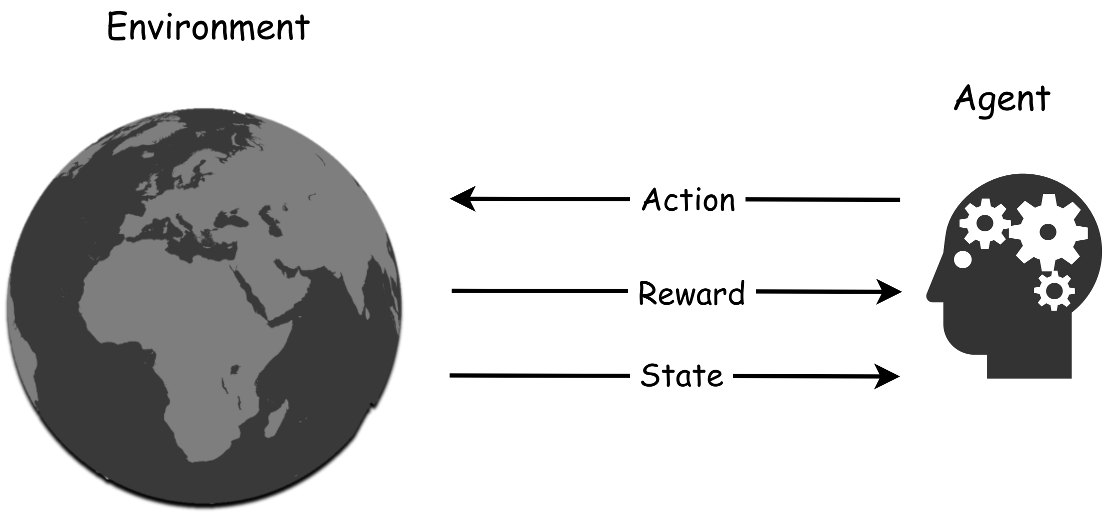


Figure 6: The general structure of an RL interacting with environment. At each time step, the agent perceives the state of environment, performs an action and receives a feedback. The feedback is the source of learning for the agent which tries to revise its future actions to maximize its total received reward over time.

**Projective Simulation**

A newly developed particular type of RL agent, called Projective Simulation (PS) agents (Briegel & De las Cuevas, 2012) is presented in this section (for detailed comparisons of RL and PS, see Bjerland, 2015; Mautner, Makmal, Manzano, Tiersch, & Briegel, 2015).

PS model, similar to other reinforcement learning algorithms, can be embodied in an environment, to perceive stimuli, execute actions, and learn through trial and error. The PS episodic memory is perhaps the most important difference of PS with other standard RL algorithms which facilitates modeling more complex features. Learning in standard RL algorithms is based on estimation of value functions, whilst in PS learning is through the re-configuration of episodic memory. This re-configuration could be either by updating transition probabilities or by adding/creating new clips (Bjerland, 2015; Melnikov, Makmal, Dunjko, & Briegel, 2017).

PS agent interaction with the environment is similar to other RL agents, so that the agent receives a precept, chooses an action among the possible actions which usually leads to a reward (positive or negative) from environment. Note that usually negative reward is referred to as penalty. The agent then tries to bias its actions towards the actions with positive rewards. The internal mechanism of PS by which agents act is based on a directed, weighted network of clips, so called Episodic and Compositional Memory (ECM), where each clip could represent a remembered percept, action, or sequences of them. It is noteworthy that the term episodic memory has been coined by Tulving et al. (1972) to refer to the ability to vividly remember specific episodes of one's life. EMC memory in PS agents can be described as a probabilistic network of clips (see Figure 7 for a schematic view of a network clip in PS agents).

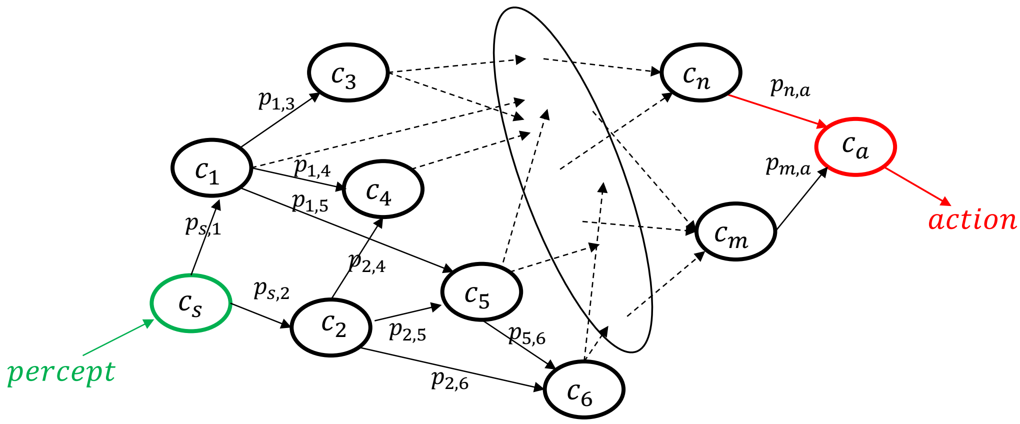


Figure 7: A memory network in PS model and a random walk on the clip space which starts with activation of clip $c_s$ and reaches the action clip $c_a$ that coupled-out the real action. The clips and transition probabilities between them can evolve based on the environment feedback.

As demonstrated in Figure 7, once a percept is observed by agent, a clip corresponding to the percept is activated and a random walk on the clip network triggers. When an action clip is reached, agent realises the corresponding action. The probabilities of going from one clip to the next are based on the connection weights of the relevant edges, so called $h$-value. If agent receives reward for the chosen action, the edges traversed to reach that decision are reinforced and as a result the probability of repeating the same behavior will be increased. The learning in PS agents is not limited to updating the connection weights through Bayesian rules. The structure of network clip can also be altered by creating and adding new clips. New clips, can be added either by composing existing ones under certain compositional principles, or by adding blank clips that can represent novel content (Melnikov et al., 2017).

**Stochastic Point Location (SPL)**

Stochastic Point Location (SPL) problem, also synonymously known as Stochastic Search on the Line (SSL), deals with searching for an unknown point in an interval under faulty feedback. SPL is a fundamental optimization problem, pioneered by Oommen (1997) which has received increasing research interest (e.g, Huang & Jiang, 2012; Mofrad, Yazidi, & Hammer, 2019; Yazidi, Granmo, Oommen, & Goodwin, 2014). The searching algorithm, also called a *learning mechanism*, receives information regarding the direction of the search by interacting with a stochastic environment, which means the provided information for the direction towards the point location is noisy and could be erroneous.

The first solution to the SPL problem proposed by Oommen (1997), relies on the strategy of discretizing the search interval and performing a controlled random walk on it using a discretization parameter $N$ which is called the resolution. The state of learning mechanism at each step represents the estimation of point location. The convergence of this strategy is proved for an infinite resolution (i.e., infinite memory), but this strategy yields rather poor accuracy for low resolutions. See, Figure 8 for a general view of SPL solution with discretizing the interval.

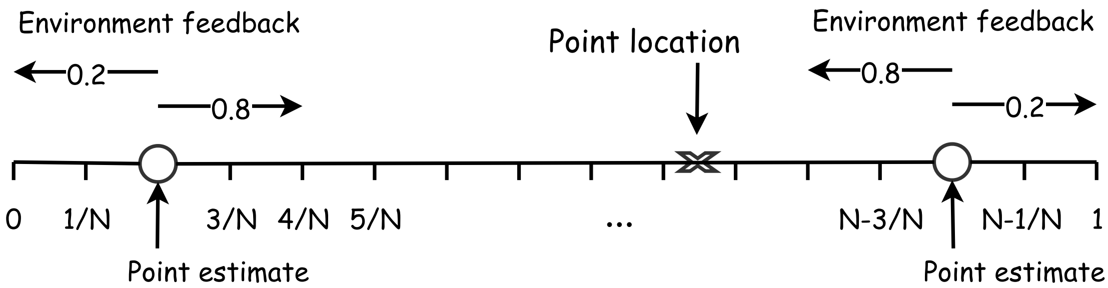


Figure 8: The proposed solution for SPL is to discretize the interval and inquiry environment the direction to the location of point. Since the feedback is faulty, there is a chance with probability 0.2 that environment points the learner towards the wrong direction. In the figure, two locations of point estimates in the right and left of searching point are shown together with the probability of receiving each direction form Environment.

The SPL problem and the proposed solutions (see, e.g. Huang & Jiang, 2012; Jiang, Huang, & Li, 2016; Oommen, 1997; Oommen & Calitoiu, 2008; Oommen, Kim, Samuel, & Granmo, 2008; Yazidi et al., 2014) can be seen as a general optimization framework that can model a wide range of scientific and real-life problems (see Yazidi & Oommen, 2017, for a survey of the solutions that has been reported to the SPL).

## Diffusion and Random Walk Processes

Diffusion models have been widely used in cognitive and neural processes of decision making representation and choice response time in psychology (see Ratcliff, Smith, Brown,

& McKoon, 2016, for a survey of diffusion models in psychology). Decision making in diffusion models are based on accumulating samples of noisy evidence to a response criterion, where new information is used to update beliefs and choose future actions. Decision making is considered as a statistical process, where successive samples of noisy evidence or stimulus information are accumulated until a response criterion is meet. The noisy evidence assumption comes from either the fact that the stimulus itself is composed of a sequence of noisy events or from the noisy coding in the neural system. Random walk models can be seen as the discrete-time counterpart of diffusion models (Smith & Ratcliff, 2015).

Network enhancement is based on (Zhou, Bousquet, Lal, Weston, & Schölkopf, 2004) to solve semi-supervised labeling problem where some of the data points are labels where the majority was unlabeled. A diffusion process is used to label the rest of the data points by diffusion of information globally over local graph structure in an iterative manner. In semi-supervised learning problems the prior assumption of consistency is essential, that is nearby units most probably have the same label and units on the same structure are expected to have the same label. The former assumption is local, and the latter assumption is global.

As Zhou et al. (2004) explicitly mention, the Network enhancement is inspired by "spreading activation network" in experimental psychology.

> This algorithm can be understood intuitively in terms of spreading activation networks (Anderson, 1983; Shrager, Hogg, & Huberman, 1987) from experimental psychology.

Spreading activation refers to a class of algorithms that propagate activation levels in a network in order to select the most closely related nodes to the activation source. The idea of tracing chains of connections addressed in experimental psychology research, and in the theories of psychologists such as Freud and Pavlov (Anderson, 1983). Quillian (1967) introduced spreading activation as a computational process that can accomplish search in semantic networks. Collins and Loftus (1975) proposed a version of the semantic network model which activation values simultaneously spread through all network links to account for semantic priming phenomena. In psychological models spreading activation is used for selecting among several possible related memories or actions when the situation is ambiguous (Shrager et al., 1987). Zhou et al. (2004) algorithm which is inspired by spreading activation networks (Shrager et al., 1987) updates, let every unit in the network iteratively spread its label information to its neighbors until a global stable state is achieved. The algorithm controls the relative amount of the information from unit neighbors and its initial label information using a regularization parameter. Network Enhancement model (Wang et al., 2018) introduced an iterative algorithm for denoising weighted networks with inspiration from the local and global consistency proposed by Zhou et al. (2004). Specifically, Wang et al. (2018) introduced a localized network to capture the information of all paths of length three or less connecting any given two nodes which is used for spreading the information and updating the network weights.

**Applications of Network Enhancement** Network Enhancement has found multiple applications which include enforcing similarity links between individual based on genomics data (Wang et al., 2014), image segmentation in computer vision (Wang, Jiang, Wang, Zhou, & Tu, 2012), community detection (Hu, Wang, Chen, & Dai, 2020).

### Network Enhancement

Network Enhancement (NE) (Wang et al., 2018) is a diffusion-based computational approach that has been proposed for denoising weighted biological networks. NE converts a noisy, undirected, weighted network into a network with the same nodes but different connections and weights. The basic assumption is that nodes which are connected through paths with high weight edges, most probably are directly connected with a high weight edge. In this regards, the diffusion process in NE uses random walks of length three or less and a regularized interaction flow in order to revise edge weights (See Figure 9 for an illustration).

The NE algorithm, takes as an input a weighted network, and iteratively updates its associated weighted adjacency matrix using the NE diffusion process. NE can be used for a more accurate detection of modules/clusters in the network.

## Related Works

In this section the existing computational models of equivalence classes in behavior analysis domain as well as Intelligent Tutoring Systems as adaptive learning models are reviewed.

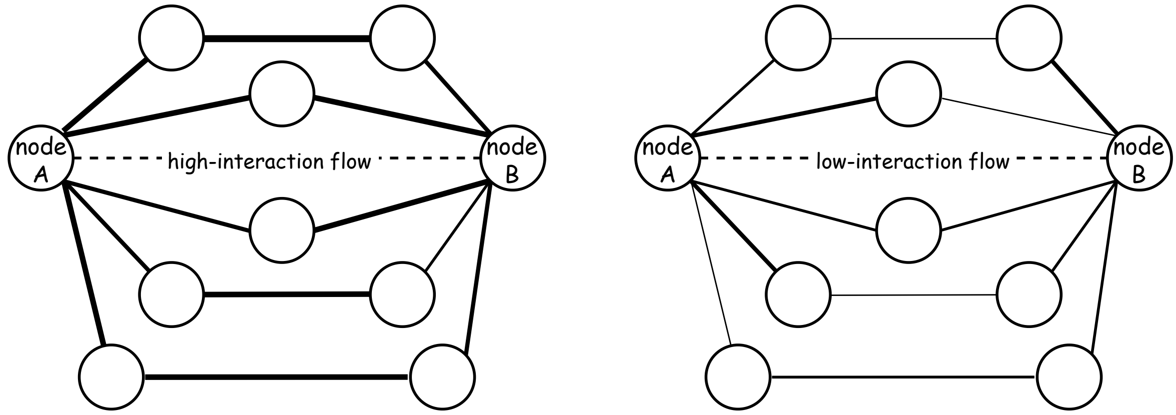


Figure 9: Network enhancement updates the network structure by updating the connection weights based on the neighboring connections and paths of length two and three. In the left panel, there are strong paths between node $A$ and node $B$ and therefore, network enhancement increases the direct connection weight between them. The opposite scenario is in the right panel, where due to the low-weight connections network enhancement process decreases the weight of direct connection. This figure is a modification of proposed illustration by Wang et al. (2018).

## Computational Models of Equivalence Classes

Artificial neural networks could advance the understanding of way derived stimulus relations are formed, by either simulating MTS procedures (Barnes & Hampson, 1993; Cullinan, Barnes, Hampson, & Lyddy, 1994; Lyddy, Barnes-Holmes, & Hampson, 2001; Tovar & Westermann, 2017) or training stimulus relations through compound stimuli and alternative procedures to MTS (Tovar & Chávez, 2012; Vernucio & Debert, 2016).

A well-known behavior-analytic approach to modelling with neural networks is the network for relational responding, called RELNET (Barnes & Hampson, 1993) which was designed to simulate complex human behaviors, from the RFT perspective. The model is a feed-forward neural network that uses standard backward-error propagation (Rumelhart et al., 1985) for learning. Three modular stages of RELNET are an *encoder* that preprocesses the stimuli for the *relational responding machine* (central system) as the second stage, and the third stage is a *decoder* that decodes the output of the central system. The simulation of learning task is performed through the central system and the encoder and decoder act like a simple pattern association. Barnes and Hampson (1993) replicated the empirical study by Steele and Hayes (1991) which is a contextual control of derived stimulus relations with MTS procedure. Modified versions of RELNET have been used to simulate other examples of derived relational responding (see, e.g. Cullinan et al., 1994). RELNET is also used to study the effect of training protocols in equivalence class formation (Lyddy & Barnes-Holmes, 2007) through modeling the experiment by Arntzen and Holth (1997). However, Barnes and Hampson (1997) discuss that lack of neural plausibility is the major weaknesses of RELNET model. Tovar and Chávez (2012), also criticise the RELNET model by arguing that the transitive relations are partially trained during encoding which means the model purpose to form untrained relations is not completely meet.

Another computational model of formation of equivalence classes with MTS procedure is presented by (Tovar & Westermann, 2017). The proposed fully interconnected neural network model links stimulus equivalence field to Hebbian learning, associative learning and categorization. In the model, each neuron accounts for a stimulus represented through activation. The weighted connections among neurons spread activation in the network, and coactivation of neurons based on Hebbian learning, updates the connection weights. Three high impact studies (Devany, Hayes, & Nelson, 1986; Sidman & Tailby, 1982; Spencer & Chase, 1996) have been simulated with this model. To validate the model, the connection weights in the neural network were compared with the results of real experiments.

Tovar and Chávez (2012) used a three-layer feed-forward neural network to train stimulus equivalence relations with compound stimuli procedures. The network inputs are stimulus pairs (e.g., $A_1B_1$, $A_1B_3$) and the outputs are yes/no responses. In order to derive relations in desired classes, this model requires a previous learning of all possible relations of an equivalence class, say $XYZ$. Vernucio and Debert (2016) considers a replication of this model with a go/no-go procedure, i.e. just considering a yes responses. Although RELNET, yes/no, and go/no-go models successfully replicate the formation of equivalence classes, they are not addressing relatedness between members of stimulus classes, and therefore they are not considered to be biologically plausible (O'Reilly & Munakata, 2000; Tovar & Westermann, 2017).

The real time neural network model presented by Lew and Zanutto (2011) is based on biological mechanisms that are able to learn different tasks such as operant conditioning and delayed MTS. The first layer of this three layers network receives sensory input from the environment and produces *short-term memory traces* of them. The second layer's job is further filtering of task relevant stimuli which will in the third layer be associated with the proper response (for an application of the model, see Rapanelli, Frick, Fernández, &

Zanutto, 2015).

An overview of existing connectionist models of formation of equivalence relations is provided by (Ninness, Ninness, Rumph, & Lawson, 2018) along with a neural network framework called emergent virtual analytics (EVA), where the process of applying neural network simulations in behavior-analytic research is demonstrated (see Ninness, Rehfeldt, & Ninness, 2019, for more simulations with EVA).

## Computational Theories for Adaptive Testing and Learning

Several studies show that personalized learning is the key to increased fulfillment of potential (e.g, Miliband, 2004). A possible solution to the latter problem is resorting to the advances in $AI$ in order to personalize the teaching process. Achieving computer tutoring systems that are as effective as human tutors can be traced back to the earliest days of computers (Smith & Sherwood, 1976). Recent research indicate that computer tutoring systems can raise student performance beyond the level of traditional classes and even beyond the level of students who learn from human tutors see, Kulik and Fletcher, 2016, for a survey. Chirikov, Semenova, Maloshonok, Bettinger, and Kizilcec (2020) show that online education platforms could scale high-quality science, technology, engineering, and mathematics (STEM) education through national online education platforms at universities. This means that such instruction can produce similar learning outcomes for students as traditional, in-person classes with a much lower cost see also, VanLehn, 2011, for a review of relative effectiveness of human tutoring, intelligent tutoring systems, and no tutoring.

### Intelligent Tutoring System (ITS)

Intelligent systems (also known as intelligent tutoring systems), is a broad term and consists of any computer program or educational software containing an artificial-intelligence component (Freedman, 2000). ITS aims to provide personalized sophisticated instructional advice which outperforms conventional computer-aided instruction systems which is comparable with human tutors. ITS can be classified according to their underlying algorithm. Model-tracing tutor, is a well-known category where the algorithm tracks the progress of student and controls the performance to be in an acceptable interval by adjusting feedback and providing guidance along the way (e.g, Freedman, 2000; Shute & Zapata-

Rivera, 2010). Sleeman and Brown (1982) pioneered the idea behind ITS for designing systems assisting students reach their full potential in a limited amount of time. There are different artificial intelligence approaches to create an ITS including multi-armed bandits (e.g, Clement, Roy, & Oudeyer, 2015), Bayesian-networks (e.g, Millán, Loboda, & Pérez-de-la-Cruz, 2010) and neural-networks (e.g, Zatarain Cabada, Barron Estrada, Gonzalez Hernandez, & Oramas Bustillos, 2015). Many ITSs are based on Computerized Adaptive Testing (CAT) which as mentioned earlier is an efficient computer-based testing (see for instance Hatzilygeroudis, Koutsojannis, Papavlasopoulos, & Prentzas, 2006; Jansen, Hofman, Savi, Visser, & van der Maas, 2016; Kozierkiewicz-Hetmańska & Nguyen, 2010). In adaptive testing, the aim is to estimate the participant's ability and administration of problems should provide as much information as possible (Birnbaum, 1968; Eggen & Verschoor, 2006). Adaptive learning or training, on the other hand needs to consider other factors which are the motivation and involvement of participant (Jansen et al., 2016). In most adaptive testing and CATs, selection of problems is due to the participant's current estimated ability (see, Eggen & Verschoor, 2006, for instance) which means when the probability of success equals to 0.5. This condition provides highest information and minimizes the test length. This level of challenge could be frustrating for many of participants and trade-off between the length of the test and motivation and pleasure of students is needed. For instance, by using a higher success rate, CAT principles have been successfully applied for practicing math skills (e.g, Jansen et al., 2016; Jansen et al., 2013; Klinkenberg, Straatemeier, & van der Maas, 2011). The optimal strategy for motivating the student has been investigated by Lumsden (1994) which is backed up by Clement et al. (2015) where the tasks should be slightly beyond the participant's current abilities, concurring with theories of intrinsic motivation.

Despite that research on ITS has produced many interesting theoretical insights, using ITS regularly in schools is not common and there is more effort that is needed to deploy ITS in real-life learning settings (Shute & Zapata-Rivera, 2010).

Works on *Learning Automata (LA)* and ITSs e.g, Oommen and Hashem, 2013 can be addressed here. In simple terms, LA is a stochastic machine attempting to find the optimal strategy from a set of actions in a random environment. LA is particularly important in decision making under uncertainty see, Narendra and Thathachar, 2012, for an introduction to LA. The term tutorial-like systems refers to study tutorial systems while no entity needs to be a real-life individual. So, all the component of the model are basically an algorithm that simulate the real teacher, student, knowledge-domain, etc (Oommen & Hashem, 2013).

For a few design and analysis of tutorial-like system models using LA, consider modeling of a student (Oommen & Hashem, 2009b), modeling of a classroom of students where artificial students can interact and learn from each other as well as the teacher (Oommen & Hashem, 2009a), modeling of a (stochastic) teacher (Hashem & Oommen, 2007), modeling the domain knowledge (Oommen & Hashem, 2010), and modeling how teaching abilities of a teacher can be improved (Oommen & Hashem, 2013).

# Studies Conducted for the Dissertation

The four studies included in this thesis propose computational models of formation of stimulus equivalence classes and adaptive testing and adaptive learning. A summary of the papers is provided first, followed by ethical considerations and discussion.

## Summary of the Studies

### Study I

In this study, formation of stimulus equivalence classes through a MTS procedure has been modeled based on a reinforcement learning framework called projective simulation (PS) (Briegel & De las Cuevas, 2012). In order to make the PS model appropriate for stimulus equivalence, we modify the model and name it, equivalence PS (EPS). To the best of our knowledge, EPS is the first study which proposes a computational model in stimulus equivalence based on machine learning. The extra features that EPS has, in comparison with PS, can be further used in machine learning research. The proposed model is as a transparent model as opposed to black-box models in AI and is able to control various factors such as learning rate, forgetting rate, symmetry and transitivity relation formation and stimulus relatedness through nodal number. The model is able to mimic both typical participants and participants with some disabilities. The influential experimental studies in behavior analysis literature due to Sidman and Tailby (1982), Devany et al. (1986), and Spencer and Chase (1996) have been simulated with our EPS framework in order to validate our model. Alternatives by which a hypothetical experiment in stimulus equivalence research can be studied through EPS have also been discussed in this study. The learning algorithm, similar to empirical settings, has two phases, the training phase where the clip network will be shaped and the testing phase where the clip network is used to cope with new, derived relations. In the proposed model (EPS) a directed graph is used to differentiate between all types of relations. The symmetry relations are formed during training, and we assume transitivity and equivalence relations are also formed during training. However, transitivity and equivalence transition probabilities are calculated on demand upon trials in the MTS test, which is based on the findings that the response latencies in transitivity and equivalence tests are typically longer than trained relations or symmetry tests (Bentall et al., 1993). We propose several methods to address the test phase and derived relations, including max-product, memory sharpness, and random walk on the memory network with absorbing action sets. The transitive and equivalence relations in EPS model are not a part of clip network and they are derived upon request, i.e. when they appear in a MTS trial during the testing phase. The formation of symmetry relations, by virtue of the flexibility of PS, can be postponed until the testing phase. On the other hand, it is possible to establish all the connections in the training phase and gradually update them during MTS training phase, or during the MTS test.

**Study II**

Enhanced EPS (E-EPS) model that builds upon EPS proposed a new way of modeling derived relations which possesses many advantages in comparison with EPS. The E-EPS has almost the same training phase as the model in Study I, but derived relations are formed after training phase using an iterative diffusion process called Network Enhancement (Wang et al., 2018). As a result, in testing phase E-EPS retrieves relations from memory, unlike EPS where derived relations are computed on demand by using some type of likelihood reasoning. During the network enhancement phase, the structure of clip network changes and indirect relations get enhanced.

We can regard the clip network at the end of training phase of EPS as a noisy version of the agent's memory network that is supposed to contain all trained and derived relations strong. Using a denoising approach such as network enhancement could produce a new less noisy clip network with information regarding the equivalence class formation. We propose directed NE (DNE) diffusion process by which we can control the formation or non-formation of derived relations. In order to study the role of parameters on the agent performance in the E-EPS model, various experimental settings are simulated and discussed. Moreover, training procedures LS, MTO, and OTM are simulated where similar to the main stream literature in behavior analysis (see, e.g. Arntzen, 2012; Arntzen, Grondahl, & Eilifsen, 2010; Arntzen & Hansen, 2011), the model yields better performance in OTM and MTO cases in comparison with LS.

E-EPS has less parameters in comparison with EPS and is a much simpler computational method and yet accurate. Theoretical analysis of the updating process and convergence guarantees are established.

**Study III**

The proposed model in this study is a solution to the SPL problem which is a search problem for a point location in an interval based on the feedback's from environment. One of the applications of this model is to find the tolerable difficulty level by the participant in an adaptive test. To make the model more realistic, the feedback from participants is assumed to be inaccurate and the workable difficulty level is assumed to be non-stationary over time. In this study, two major contributions to the SPL problem are proposed. First, we employ the concept of mutual probability flux between neighboring states along the line to improve the estimation of the point location. Next, we estimate the error probability characterizing the environment while tracking the position of the optimal point. This study is partially based on our previous work (Mofrad, Yazidi, & Hammer, 2017) by which we show that the SPL problem can be solved by tracking two multinomially distributed random variables using the Stochastic Learning Weak Estimator (SLWE) method (Oommen & Rueda, 2006). We proposed to integrate the SLWE, which figures among the most prominent estimators for non-stationary distributions, as the inherent part of a more sophisticated solution for the SPL. The estimation strategy at each time step revolves around tracking the distribution and estimating the point location based upon it. Different statistical operators namely maximum, expectation, and median have been applied on the estimated probability vectors to obtain point location estimates. Simulation results indicate that, the proposed methods produce smoother estimates than those obtained from other SPL solutions and can track the changes more efficiently. The probability of receiving correct feedback from environment, called environment effectiveness, is usually unknown and might change over time. As the second contribution of this study, we estimate the feedback error probability in tandem with the unknown location estimation. In Study III, the participant performance is modeled using a stair function with two levels: a high performance for difficulties under the optimal "manageable" difficulty level and a low performance for difficulties just above the same level, i.e., the "manageable" optimal difficulty level. However, if we rather use a more realistic performance function according to which the performance is continuous and monotonically decreases as a function of the difficulty level, the approach in Study III will basically converge to difficulty level for which the participant performance is at 50% under some mild conditions. Such remark motivated Study IV in which we resort to the latter realistic performance model, for efficiently finding a higher rates of performance that are motivating enough for the learner, usually above 50% such as 70%. For a schematic description of approaches to find task difficulty based on success probability in Study Study III and Study IV see Figure 10.

**Study IV**

The Balanced Difficulty Task Finder (BDTF) proposed in this study is a method for adequate difficulty task assignment based on the principles adaptive learning. The BDTF method is general and can be integrated as a part of an ITS system. BDTF has applications in learning and remembering techniques in behavior analysis methods such as titrated delayed MTS and online learning environments. The idea behind BDTF is to select tasks with appropriate difficulty to the learner so that ensure that the level of motivation and enjoyment during the learning is maintained. The strategy is similar to Elo's chess skill rating (Glickman, 1995) and TrueSkill (Herbrich, Minka, & Graepel, 2006) for matching game players, where players with similar capabilities and skills are matched. Choosing an appropriate opponent or appropriate game level in BDTF is analogous to choosing automatically an appropriate level of the learning task. In this regard, each participant starts with a predefined difficulty level of tasks (Clement et al., 2015), and based on the performance of the learner the difficulty of future tasks is adjusted in a trial and error manner.

The difficulty of any given task is mapped to a number between zero and one where zero denotes the lowest possible difficulty and one denotes the highest possible difficulty. In BDTF, we assume the chance of success in a given task decreases monotonically as the

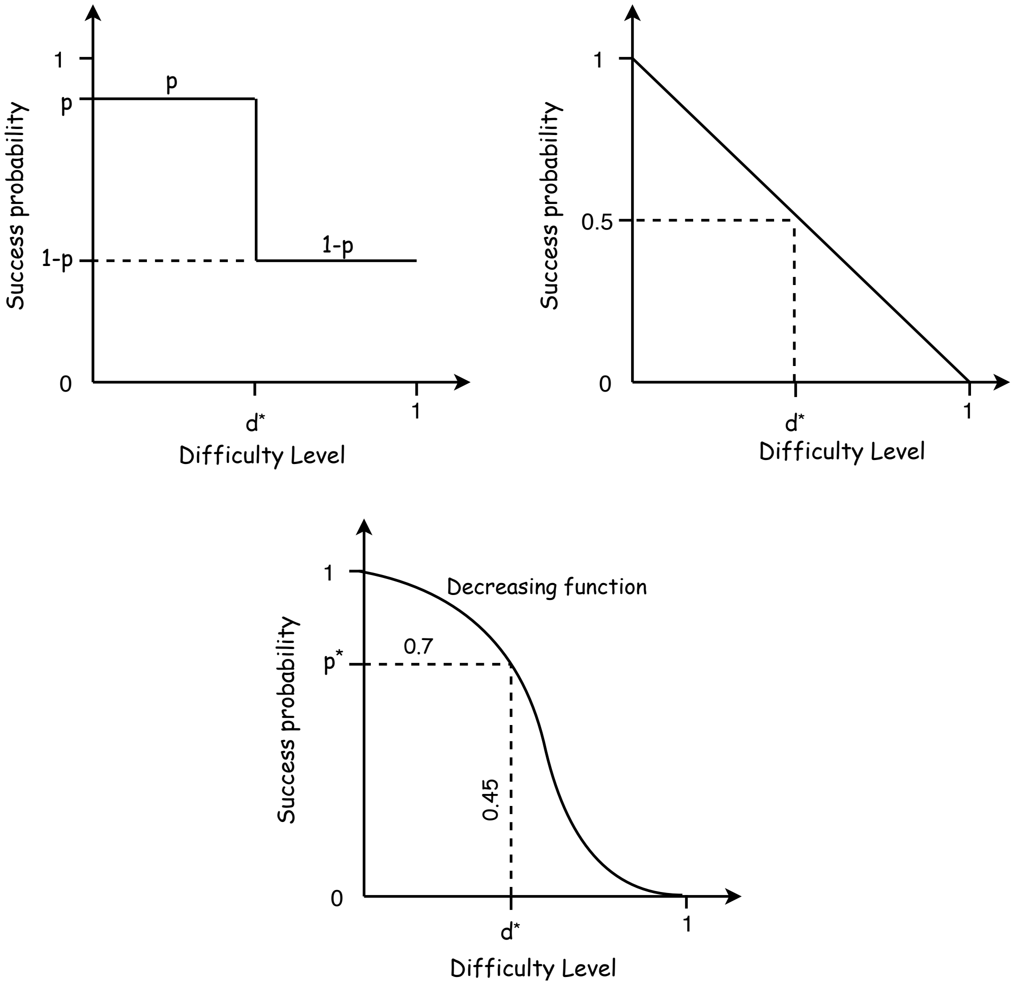


Figure 10: Comparison of Study III and Study IV. The top-left figure represents the stair function with two levels which we used in Study III. $p$ represents high performance for difficulties under the optimal "manageable" difficulty level $d^*$, and $1 - p$ is a low performance for difficulties above $d^*$. The top-right figure, shows the performance of proposed model if we instead using a more realistic performance function according to which the performance is continuous and monotonically decreases as a function of the difficulty level. The approach in Study III will converge to difficulty level for which the participant performance is at 0.5 under some mild conditions. The bottom figure, depicts the approach in Study IV with a more realistic performance model, for efficiently finding a higher rates of performance that are motivating enough for the learner, usually above 0.5 such as 0.7.

difficulty level increases. The recommended task difficulty gets increased upon success and decreased upon failure in an asymmetric manner so that we can adjust the difficulty of the given tasks and consequently drive the system towards a state of flow (Chen, 2007).

## Ethical Considerations

Since this work reports no original empirical research, it is not subject to the ethical considerations regarding participants (humans and animals) that psychology research usually deals with.

On the other hand, ethical concerns regarding AI algorithms appear mostly in the cases with social dimensions where AI algorithms act as autonomous agents and team-mates. For instance, avoiding cognitive biases, seeking reliability and applicability in decision-making domains such as healthcare, and preventing possible harms of AI to humans and other morally relevant beings (Bostrom & Yudkowsky, 2014).

Computational modelling in basic research though draw different ethical views. Resorting to a modeling approach rather than to an experimental approach, as an alternative methodology in studies involving human and animal subjects, may allow behavior-analytic researchers to formulate, explore, and examine ideas prior to full experimental testing. These models usually are highly controllable and precise and can be promising candidates for research. Using cognitive and computational models can reduce the possible harm to the human and nonhuman subjects by limiting the need to test all the ideas on them. Moreover, in some cases, such as in the case of Relational Frame Theory, that relies on a long and often complex histories of explicit reinforcement, it would be so difficult, and even unethical, to test hypothesises directly on subjects in a behavioral laboratory (Barnes & Hampson, 1997).

An important feature in developing AI algorithms in general and cognitive models in particular is transparency and interpretability of the models (e.g, Bostrom & Yudkowsky, 2014; Fleischmann & Wallace, 2017). Some modeling paradigms such as neural networks are not considered practical in many settings even though they yield accurate predictions as the predictions of those models are opaque and the logic behind them can not be explained clearly. The area of Explainable Artificial Intelligence (XAI) (see Biran & Cotton, 2017, for a survey on XAI) has been attracting more attention recently due to the need to increase the trust and transparency of intelligent agents (e.g, Miller, 2019), including the design of models that are inherently interpretable instead of black-box models (see Rudin, 2019, for instance).

In modeling equivalence formation, it is vital to understand the model and to also design a biologically plausible model. We found Projective Simulation model as an appropriate candidate for this aim. For adaptive learning, we could apply more complex models but we rather opted for a fairly simple probabilistic model in favor of interpretability and explainability. Therefore, all the proposed models in the studies for this thesis can be categorized as XAI models with the capability of easy interpretation.

## Discussion

The purpose of this dissertation was to propose flexible and interpretable computational models that replicate formation of stimulus equivalence classes and adaptive learning. Reinforcement learning agents, in the form of Projective Simulation and Stochastic Point Location, were the chosen candidates to model the interaction between experimenter and the participant through the processes.

Although neural networks is one of the most powerful simulation techniques, designing models that are inherently interpretable instead of black-box models (see Rudin, 2019, for instance) is an advantage. Projective simulation (Briegel & De las Cuevas, 2012; Mautner et al., 2015) is a fairly simple graphical model which provides a flexible paradigm that can be easily extended. Based on projective simulation, we proposed a general simulator for equivalence relations. In Study I formation of derived relations are computed on demand through some type of likelihood reasoning. In Study II, Network Enhancement (Wang et al., 2018) updates the agent memory after training phase and as a result the test trials become like a memory retrieval phase. The simulators could replicate a variety of settings in a matching-to-sample procedure. There are different mechanistic accounts on the formation of stimulus equivalence classes and derived relations. Galizio, Stewart, et al. (2001) discuss that some degree of equivalence class formation occurs during the MTS training, which is further enhanced during the testing. In many studies, on the other hand, the emergence of equivalence relations is considered to be only the result of testing lower-stage relations (see, part E of Dickins, 2015, for a discussion). Different approaches to the test phase proposed in Study I and Study II can be interpreted in accordance with different views on mechanism of deriving relations. Modified versions of the models can address other types of training procedures, such as compound stimuli. A possible approach for modeling compound stimuli is to use the generalized projective simulation (Melnikov et al., 2017) that considers clips composed of different categories. The proposed models, on the other hand, can be considered as an extension of PS model which might be interesting solely from a machine learning point of view. For instance, symmetry connections and variable action sets could be used in more general applications or the use of diffusion methods to update the clip network.

To compare E-EPS with EPS, we found the implementation of NE more adequate and useful, in the sense that it can be interpreted as a denoising procedure that happens within the agent's memory. The employed procedures such as the *max-product rule* described in Study I had been introduced as an external, ad-hoc computational tool rather than an intrinsic feature of the model. The use of NE, on the other hand, fits more naturally into the semantics of PS/EPS. On the more technical side, the E-EPS model also avoids the use of the *memory sharpness* parameter and provides a more elegant process to model and control the transitivity relations and other features such as the nodal effect. From computational point of view, the presented method in Study II has less parameters in comparison with Study I where the testing phases involved more processing than memory.

In modeling stimulus equivalence, we propose an interpretative reinforcement learning models and focus on the computational models proposed in behavior analysis domain. However, the concept of equivalence relations is modeled in other domains more frequently, say in cognitive psychology and cognitive science (e.g, Kumaran & McClelland, 2012) and a more comprehensive research is needed to include all the models.

Finding the cognitive level of a participant in a learning task in order of designing suitable level of training is one of the key challenges faced by many learning methods. This problem is modeled in Study III by SPL with certain conditions. For instance, the point location is set to be non-stationary, since the manageable difficulty level will change as time goes for trained participant. Additionally, certainty/probability of the results is unknown, because there are many factors that might affect the response to a training which might be irrelevant to the real ability of the participant. In Study III, a new solution for estimation of point location in the SPL problem is proposed by using the mutual probability flux concept. The participant performance in this study is modeled through a stair function with two levels: a high performance for difficulties under the optimal *manageable* difficulty level and a low performance for difficulties just above the same level, i.e., the *manageable* optimal difficulty level. The proposed solution outperforms the original method and estimates the tolerable task difficulty level as fast as possible. As a measure of reliability of participant performance, the probability of receiving correct response from the participant is also estimated. However, if we rather use a more realistic performance function according to which the performance is continuous and monotonically decreases as a function of the difficulty level, the approach in Study III will basically converge to difficulty level for which the participant performance is at 50% under some mild conditions.

Such remark motivated Study IV in which we resort to the latter realistic performance model, for efficiently finding a higher rates of performance that are motivating enough for the learner, usually above 50% such as 70%. In Study IV the Balanced Difficulty Task Finder model searches for appropriate difficulty level in adaptive learning setting via an asymmetric adjustment technique. The appropriate task difficulty for training and learning is sought by targeting a higher rate of success in the task sequence in order to to maintain the level of motivation and enjoyment during the learning, for instance with 70% chance of success. Unlike neural network and Bayesian-network models that rely on comprehensive student models to be reliable and effective (e.g, Clement et al., 2015), the proposed model makes a weaker link between the student and the cognitive model and therefore is much simpler, yet efficient.

As it has been mentioned in Study IV, the task difficulty techniques and adaptive learning models can be used in a complementary manner with EPS and E-EPS models for formation of stimulus equivalence classes in the training phase of MTS or titrated delayed MTS procedures.

It is discussed that there are many parameters to design a training phase. If a difficulty level can be assigned to each trial, or each block of trials, then the abilities of participants could be taken into consideration and an adaptive training can be proposed. A handy example is adjusting the time delay between sample stimulus and the comparison stimuli which is referred to as tirtrated delayed MTS. The EPS and E-EPS models can easily have a delay factor by rendering the forgetting factor delay dependent. Note that in our studies the delay is assumed to be very negligible and so the forgetting factor is fixed.

## Concluding Remarks

Understanding learning and memory mechanisms is crucial in the effort to understand human behavior and cognition. Computational models of cognition and behavior, as simplified models of a complex system, might be useful tools to study brain activity and to analyze experimental data as well as exploring new ideas through simulation. Computational models can examine variables that are challenging to examine on humans or animals due to time constrains or ethical issues. Due to the more control over the experimental variables, components of the computational models can be manipulated, disrupted, impaired, and removed to study the effect of those components on the results. (Barnes & Hampson, 1993; McClelland, 2009; Ninness et al., 2018).

In this thesis, to address the complex behaviors such as learning and memory, computational reinforcement learning algorithms are used to model formation of stimulus equivalence classes and adaptive learning in the face of different task difficulty levels.

In Study I, we proposed to apply Projective Simulation learning agent to the field of behavior analysis. Our modified version of PS, which we called *Equivalence Projective Simulation (EPS)*, enables the agent to learn and form stimulus equivalence classes in matching-to-sample (MTS) experiments. The proposed reinforcement learning agent, EPS, has a directed, weighted memory, clip network where each clip represents a remembered stimulus that is added to the clip network during the training phase. During MTS training, the associated connections to the baseline relations are reinforced. In order to replicate the test phase and study the agent ability to form new relations, EPS relies on some type of likelihood reasoning whenever tested via a MTS trial. In other words, in EPS model, derived relations are calculated on demand in the test phase trials externally using a computational method such as *max-product*, *memory sharpness*, or *absorbing action sets*. The flexibility and interpretability of the PS/EPS model allows us to model a broad range of behaviors in matching-to-sample (MTS) experiments.

In Study II we significantly expand on and improve the model by combining EPS with a new method, called Network Enhancement (NE), which corresponds to some post-processing of the agent's episodic memory. Network Enhancement (Wang et al., 2018). changes the structure of clip network by enhancing indirect relations with strong paths. Network enhancement is applied after the training process in order to de-noise the memory network and distill derived relations. This approach to the testing phase and derived relations can be seen as an offline approach, relies rather on memory retrieval during test phase than on complex logical processing.

Network enhancement changes the structure of clip network by enhancing indirect relations with strong paths. A modification of network enhancement diffusion method is proposed by which the updated network remains directed and we can control the agent's ability to derive transitivity and also control its ability to derive symmetry.

The proposed method for post-processing of the episodic memory allows one to study and control the main relations that define an equivalence class, namely reflexivity, symmetry and transitivity. The main advantage of being able to control these relations through the model parameters in a clear and interpretable way is that realistic hypotheses can be easily simulated and new behavior experiments with humans can be designed to test these hypotheses and gain new insights.

The simulation results of real experiments show that the proposed models are appropriate candidates for replication of formation and non-formation of stimulus equivalence classes. We also compare the main training structures; LS, MTO, OTM, and notice better outcomes of MTO and OTM training structures in comparison with LS which supports reported evidences from behavioral analysis literature. E-EPS implementation, in comparison with EPS is more adequate and useful, where NE method fits naturally into the semantics of PS/EPS.

To study the learning and remembering, as the second direction of thesis we propose a task difficulty recommendation system which can be applied for either testing the level of participants or providing training administration. In Study III, we propose methods to yield a difficulty level that a participant can manage in an online manner. The proposed algorithm is a solution to the stochastic point location that can models many dynamic and interactive optimization problems. We show that the algorithm is faster and more accurate than legacy solutions.

The algorithm is also able to estimate the reliability of participant responses which later can be the decision basis whether participant is in a reliable condition to continue the training or not.

In Study IV, we benefit from concept of flow in psychology and game-balance in the game field to propose a task recommender algorithm. The argument is that to maintain an efficient learning experiment, we need to find tasks that are both challenging and motivating for the participant, say the tasks that 70% of the time are correctly answered by the learner

Overall, to keep the proposed models in the thesis transparent and interpretable, we avoid the more common approach of neural network models that are often considered as black-box models and instead use interpretable machine learning models. It is noteworthy as the final point that although the studies in this thesis aim to propose models for specific behavior analysis research areas, the proposed models are not limited to behavior analysis and could embrace broader research areas.

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**Study I**

# Equivalence Projective Simulation as a Framework for Modeling Formation of Stimulus Equivalence Classes

**Asieh Abolpour Mofrad, Anis Yazidi, Hugo L. Hammer and Erik Arntzen**

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# Equivalence Projective Simulation as a Framework for Modeling Formation of Stimulus Equivalence Classes

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**Stimulus equivalence (SE) and projective simulation (PS) study complex behavior, the former in human subjects and the latter in artificial agents. We apply the PS learning framework for modeling the formation of equivalence classes. For this purpose, we first modify the PS model to accommodate imitating the emergence of equivalence relations. Later, we formulate the SE formation through the matching-to-sample (MTS) procedure. The proposed version of PS model, called the equivalence projective simulation (EPS) model, is able to act within a varying action set and derive new relations without receiving feedback from the environment. To the best of our knowledge, it is the first time that the field of equivalence theory in behavior analysis has been linked to an artificial agent in a machine learning context. This model has many advantages over existing neural network models. Briefly, our EPS model is not a black box model, but rather a model with the capability of easy interpretation and flexibility for further modifications. To validate the model, some experimental results performed by prominent behavior analysts are simulated. The results confirm that the EPS model is able to reliably simulate and replicate the same behavior as real experiments in various settings, including formation of equivalence relations in typical participants, nonformation of equivalence relations in language-disabled children, and nodal effect in a linear series with nodal distance five. Moreover, through a hypothetical experiment, we discuss the possibility of applying EPS in further equivalence theory research.**

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## 1 Introduction

In this letter, we present a novel machine learning model that is able to efficiently replicate human behavior in equivalence experiments. The main stream of research in modeling equivalence behavior for humans using connectionist models involves neural networks. Despite being far less complex than neural network–based models, our model is easy to interpret and flexible enough to model a wide range of behaviors in a matching-to-sample (MTS) experiment.

Sidman (1971) introduced the stimulus equivalence term, which Sidman and Tailby (1982) later characterized through mathematical relations in equivalence sets: reflexivity, symmetry, and transitivity between members of an equivalence class. By training some relations in a class, experimenters could test the emergence of new relations or derived relations on the trained relations. As a general rule, a class composed of $n$ stimuli needs only $(n-1)$ stimulus-stimulus matches to be trained. Each component of these relations must be used in at least one trained relation, and none of the trained relations can have the same two stimuli as components. Given these constraints, there exist many ways for selecting training relation sets, some possibly more efficient than the others (Fields, Adams, Verhave, & Newman, 1990; O'Mara, 1991; Arntzen & Holth, 1997; Hove, 2003; Lyddy & Barnes-Holmes, 2007; Arntzen, Grondahl, & Eilifsen, 2010; Arntzen & Hansen, 2011; Fienup, Wright, & Fields, 2015).

Stimulus equivalence framework as a learning method was originally used to teach children and adults with developmental disabilities like autism and Down's syndrome (Sidman, Cresson, & Willson-Morris, 1974; Groskreutz, Karsina, Miguel, & Groskreutz, 2010; Toussaint & Tiger, 2010; Arntzen, Halstadtro, Bjerke, & Halstadtro, 2010; McLay, Sutherland, Church, & Tyler-Merrick, 2013; Arntzen, Halstadtro, Bjerke, Wittner, & Kristiansen, 2014; Ortega & Lovett, 2018). However, the equivalence theory can be used in teaching new concepts to normal children and adults, including college students (Sidman, Willson-Morris, & Kirk, 1986; Hove, 2003; Saunders, Chaney, & Marquis, 2005; Fienup, Covey, & Critchfield, 2010; Walker, Rehfeldt, & Ninness, 2010; Lovett, Rehfeldt, Garcia, & Dunning, 2011; Grisante et al., 2013; Placeres, 2014; Fienup et al., 2015). Some neurocognitive disorders like Alzheimer's disease are also a research area that equivalence theory deals with where it is discussed that derived relational responding is deteriorated as the cognitive impairment advances over time (Bódi, Csibri, Myers, Gluck, & Kéri, 2009; Gallagher & Keenan, 2009; Steingrimsdottir & Arntzen, 2011; Sidman, 2013; Arntzen, Steingrimsdottir, & Brogård-Antonsen, 2013; Arntzen & Steingrimsdottir, 2014, 2017; Seefeldt, 2015; Ducatti & Schmidt, 2016; Brogård-Antonsen & Arntzen, 2019).

One interesting feature of stimulus equivalence is its efficiency and the fact that just a small fraction of relations has to be explicitly taught. This could make a faster intervention in disorders. By training on only a few relations, fewer training trials are needed, as the rest of relations can be simply deduced.

The stimulus equivalence relationship to verbal behavior is another interesting research topic in equivalence literature. For instance, Hall and Chase (1991) wrote that all equivalence classes could be defined as verbal behavior, but all verbal behavior cannot be fit into equivalence classes. Moreover, the evidence shows that stimulus equivalence relations are not formed properly in nonverbal humans (Devany, Hayes, & Nelson, 1986) and animals (Nissen, 1951; Sidman et al., 1982; Hayes, 1989). Furthermore, the relational frame theory (RFT) is a psychological theory of human language built on equivalence theory (Hayes, 1991, 1994; Barnes-Holmes & Roche, 2001). This theory describes stimulus equivalence research in relation to Skinner's verbal behavior (see, e.g., Barnes, 1994; Clayton & Hayes, 1999; Barnes-Holmes, Barnes-Holmes, & Cullinan, 2000; Hayes & Sanford, 2014; Hayes, Sanford, & Chin, 2017, for more details on RFT research).

Investigations in the area of stimulus equivalence traditionally, have employed humans or animals as experimental participants. However, artificial neural network (ANN) models of cognition, often referred to as *connectionist models* (CMs) (see McClelland & Rumelhart, 1987; Bechtel & Abrahamsen, 1991; Commons, Grossberg, & Staddon, 2016, for CMs) have been developed to simulate the behavior of human participants in stimulus equivalence experiments. Connectionism tries to explain and replicate intellectual abilities using ANNs (McClelland & Rumelhart, 1987). Many researchers have been exploring methods in which ANNs could develop the understanding of derived stimulus relations by using simulated MTS procedures (Barnes & Hampson, 1993; Cullinan, Barnes, Hampson, & Lyddy, 1994; Lyddy, Barnes-Holmes, & Hampson, 2001; Tovar & Westermann, 2017) or by training stimulus relations through compound stimuli and alternative procedures to MTS (Tovar & Chávez, 2012; Vernucio & Debert, 2016). A connectionist model of RFT is presented in Barnes and Hampson (1997).

Connectionism brings a common conceptual and empirical domain for both behavior analysis and cognitive science (Fodor & Pylyshyn, 1988; Staddon & Bueno, 1991; Barnes & Holmes, 1991; Barnes & Hampson, 1993). Developing connectionist models of equivalence formation could be a tool to study the limitations and power of connectionism. For instance, modeling the formation of stimulus equivalence classes shows that semantic and syntactic relations can be modeled through connectionist networks (Barnes & Hampson, 1993) as opposed to discussion within frequently cited studies (Fodor & Pylyshyn, 1988).

The development of computational models makes it possible to examine variables that are challenging to examine on humans or animals due to time constraints or ethical issues. For instance, components of the computational model can be easily manipulated, disrupted, impaired, and removed to see the effect of those components on the results. Having more control over the experimental variables, including a controllable environment, is a major advantage of these models over experiments with human and animal subjects (Barnes & Hampson, 1993; McClelland, 2009; Ninness, Ninness, Rumph, & Lawson, 2018).

Computational models could be used for exploring the implications of new ideas through simulation (McClelland, 2009). Behavior-analytic researchers can apply ANNs to understand, simulate, and predict derived stimulus relations made by human participants. Furthermore, a good model of complex behaviors, like the formation of stimulus equivalence classes, will lead to a better understanding of the disorders that applied behavior analysis deals with and might enable us to suggest new interventions for patients (Murre, Graham, & Hodges, 2001; Baddeley, Kopelman, & Wilson, 2003).

On the other hand, the experimental data from humans could enhance the model of brain function in an efficient way. Similar to studies with human subjects, patients' data are valuable for making the model more realistic. For instance, knowing that people with dementia might not be able to derive transitive relations (Arntzen, Steingrimsdottir, & Brogård-Antonsen, 2013), would be an aid to advance the model.

Although neural networks are one of the most powerful simulation techniques, their black box nature makes interpreting their models hard (Zhang et al., 2018), and there are serious discussions for designing models that are inherently interpretable instead of black box models (see Rudin, 2019, for instance).[1] Moreover, in general, the computational power comes from a complex network that replicates the complex behavior appropriately, but does not help in understanding the underlying mechanisms of the brain (see, e.g., Silver et al., 2016, deep neural network model) and (see, e.g., Mnih et al., 2015, deep reinforcement learning model). Among different types of machine learning schemes, reinforcement learning (RL; Sutton & Barto, 2018) is the closest computational model to actual learning in humans and other animals, and many RL algorithms are inspired by biological learning systems such as stimulus-response theory from behavioral psychology.

The newly developed idea of projective simulation (PS) agents (Briegel & De las Cuevas, 2012) can be seen as an RL algorithm. Projective simulation (Briegel & De las Cuevas, 2012; Mautner, Makmal, Manzano, Tiersch, & Briegel, 2015) provides a flexible paradigm that can be easily extended, a feature that makes it a suitable framework for equivalence class formation. PS is not a black box, and although it is a fairly simple graphical model, we will demonstrate that it is powerful enough to model equivalence class formation.

[1] By "black box," we mean that although we can get accurate predictions from the model, we cannot explain or identify the logic behind the predictions in a clear way.

We propose a modified version of PS in order to make the model appropriate for equivalence modeling. The modification of the PS model not only makes it suitable for producing equivalence emergence, but also adds extra features to the PS model that can be used in machine learning research. Indeed, by studying how the brain works in equivalence theory, we can devise more intelligent algorithms that mimic human nature and can be applied in other fields.

The outline of the letter is as follows. In section 2, we provide the required background from stimulus equivalence and projective simulation. The state-of-the-art computational models of equivalence formation are discussed and compared to the newly presented model. Moreover, PS is compared with standard reinforcement learning methods, and the motivation behind choosing PS as the basis of our model is provided. In section 3, we present the modified version of PS (called EPS hereafter). Section 4 reports the artificial model results from EPS and compares them to the results of real experiments in order to demonstrate that the model can produce realistic results despite its simplicity. In section 5 we provide concluding remarks and further suggestions.

## 2 Background and Related Works

To address the required background of this work, in section 2.1, we explain the concept of stimulus equivalence and some methods that are used to learn and test the relations in behavior analysis. In section 2.2, we discuss some computational models and connectionist models of stimulus equivalence class formation. We explain the projective simulation as a model of intelligence machines in section 2.3. The standard reinforcement learning (RL) models are compared with PS in section 2.4 and the reasons behind selection of PS framework are discussed.

**2.1 Stimulus Equivalence.** Stimulus equivalence research is about complex human behavior research, including research on memory and problem solving, that in the past was studied only by cognitive psychology (Sidman, 1990). The stimulus-equivalence methodology, introduced by Sidman (1994), uses MTS procedures to train arbitrary relations between unfamiliar stimuli and deals with testing some derived relations through reflexivity, symmetry, transitivity, and equivalence.[2]

The MTS or conditional discrimination procedure occurs when a stimulus, say $A_1$, is given, and it must be paired with $B_1$ among a set of comparison stimuli, say $B_1$, $B_2$, and $B_3$. The discrimination is done through feedback or rewards provided by the experimenter, not because of resemblance between the matched stimuli. This arbitrary match between stimuli

[2] "Arbitrary MTS" means there is no conceptual relation between an equivalence class members.

is a key aspect for studying the emergence of equivalence relations that are not matched directly (Sidman, 2009).

Two main procedures in behavior analysis for training the relations are MTS, which uses simple stimuli (Sidman, 1971; McDonagh, McIlvane, & Stoddard, 1984; Sidman et al., 1986; Arntzen, 2012), and the go/no-go procedure or successive matching-to-sample (S-MTS) that uses compound (or complex) stimuli (Markham & Dougher, 1993; Debert, Matos, & McIlvane, 2007; Debert, Huziwara, Faggiani, De Mathis, & McIlvane, 2009; Grisante et al., 2013; Lantaya, Miguel, Howland, LaFrance, & Page, 2018). In MTS, the traditional procedure, a sample stimulus is paired with one of the given choices; in compound stimuli, a match is shown, and the participant learns if it is a correct match or not through trial and error (see Grisante et al., 2013; Lantaya et al., 2018, for comparison of the procedures).

In equivalence literature, three training structures have been used to establish conditional discrimination with an MTS procedure: linear series (LS), many-to-one (MTO), and one-to-many (OTM) (Arntzen, 2012). For instance, if any of equivalence classes have four members, each from one of *A*, *B*, *C*, and *D*, categories, the order of training relations would be: *AB*, *BC*, and *CD* in LS; *AD*, *BD*, and *CD* in MTO; and *AB*, *AC*, and *AD* in OTM settings. However, a mixture of these methods is also a possibility—for example, *AB*, *BC*, and *DC*.

Conditional discrimination procedures might also be either simultaneous MTS or delayed MTS. In simultaneous MTS, a sample stimulus is presented that might require response.[3] Subsequent to the response, the comparison stimuli will appear. Both sample and comparisons remain on the screen until one of the comparisons is selected. However, in delayed MTS, the sample stimulus appears and disappears first. Then the comparison stimuli appear after a certain time delay, which could be fixed (called fixed delayed MTS) or changing (called titrated delayed MTS).

The performance evaluation of participant is usually done according to the criterion that the participant must pass in order to have mastered the training phase. After mastery of the training relations, the testing phase is done. Note that the mastery criterion ratio should be placed higher in training (e.g., 0.95–1) than in testing (e.g., 0.9–1), and that in the testing phase, there is no feedback from the experimenter.

The equivalence class is considered to be formed whenever the evidence (passing the criterion for testing) shows that all these relations are established (Sidman & Tailby, 1982) (for more details about MTS training

[3] The standard MTS procedure requires that the sample stimulus receives a response by the participant before the comparison stimuli appear (say, by clicking on the sample stimulus in computer setting experiments or by touching it in a physical setting). This guarantees that the sample stimulus has been observed. Sometimes there is no need to respond, but a delay between the appearance of sample stimuli and responses (usually 1 to 2 s) is considered.

and testing procedures and parameters in formation of stimulus equivalence classes, see Arntzen, 2012).

**2.2 Computational Models of Formation of Stimulus Equivalence Classes.** There are two main families of equivalence simulation methods: the first familiy simulates MTS procedures that consider simple stimuli (Barnes & Hampson, 1993; Cullinan et al., 1994; Lyddy et al., 2001; Tovar & Westermann, 2017), and the second familiy simulates equivalence formation through compound stimuli (Tovar & Chávez, 2012; Vernucio & Debert, 2016).

One of the well-known behavior-analytic approaches to neural network is RELNET, the network for relational responding, which is a feedforward neural network with backpropagation learning (Barnes & Hampson, 1993). The model consists of three modular stages: the first stage is an encoder that preprocesses the stimuli for the second stage, called relational responding machine (central system), and the third stage is a decoder that decodes the output of the relational responding machine. The three stages are separate modules. The encoder and decoder act like a simple pattern association, while the simulation of the learning task is done through the central system. RELNET simulates the MTS procedure for training and testing trials of conditional relations. It is used (Barnes & Hampson, 1993) to replicate a contextual control of derived stimulus relations in a real experiment (Steele & Hayes, 1991) and to study the effect of training protocols in equivalence class formation (Lyddy & Barnes-Holmes, 2007) by modeling the experiment in Arntzen and Holth (1997). One of the criticisms of to the RELNET model is that the transitive relations were partially trained during encoding and therefore not derived from directly trained relations, in accordance with the formation of equivalence classes (Tovar & Chávez, 2012).

Another computational model that uses MTS procedure is presented in Tovar and Westermann (2017). The model is a fully interconnected neural network that links the equivalence class field to Hebbian learning, associative learning, and categorization. The model assumptions are threefold. First, each neuron accounts for a stimulus that is represented through activation. Second, the weighted connections between different neurons spread activation in the network, and third, the coactivation of neurons based on Hebbian learning updates the connection weights, and, as a result, the network learns the relatedness of relations, both trained and derived. The model simulates three high-impact studies (Sidman & Tailby, 1982; Devany et al., 1986; Spencer & Chase, 1996), and the connection weights in the model were compared with the results of real experiments, which validates the model in various scenarios, such as the replication of failures in transitive responding for the experiment with disabilities (Devany et al., 1986).

Another promising alternative to MTS is to train stimulus equivalence relations with compound stimuli procedures (Tovar & Chávez, 2012). The network input in this case is stimulus pairs (e.g., $A_1B_1$, $A_1B_3$) and the output is yes/no responses. This model requires previous learning of all possible relations of an equivalence class, say, $XYZ$, in order to be able to make derived relations in desired classes. A replication of Tovar and Chávez (2012) using a go/no-go procedure (i.e., just considering a yes responses) is presented in Vernucio and Debert (2016). Both connectionist models are capable of simulating humans' formation of derived stimulus relations without the assistance of sample marking duplicators that RELNET needs. Although RELNET, go/no-go, and yes/no models are promising models, they are criticized for their inability to describe the relatedness of members of stimulus classes and are not considered to be biologically plausible (Tovar & Westermann, 2017; O'Reilly & Munakata, 2000).

The neural network presented in Lew and Zanutto (2011) is a real-time neurocomputational model based on biological mechanisms that is able to learn various tasks, including operant conditioning and DMTS. The network has three layers. The first layer receives sensory input from the environment and produces short-term memory traces for them. The second layer allows further filtering of task-relevant stimuli, which will then be associated with the proper response in the third layer (see Rapanelli, Frick, Fernández, & Zanutto, 2015, for an application of the model).

A good overview of existing CMs is provided in Ninness et al. (2018) along with a working example of a neural network called emergent virtual analytics (EVA; see Ninness, Rehfeldt, & Ninness, 2019, for more simulations with EVA). Through EVA, the process of applying neural network simulations in behavior-analytic research is demonstrated.

In our current study, we model the MTS procedure and use simple stimuli based on PS as a machine learning framework. The proposed model is not a connectionist model but a reinforcement learning agent that is biologically plausible and uses Hebbian learning principles.

**2.3 Projective Simulation.** Projective simulation, introduced recently (Briegel & De las Cuevas, 2012), is a machine learning model built on principles from physics and relies on stochastic processing of experience. The model can be seen as a reinforcement learning algorithm that can be embodied in an environment to perceive stimuli, execute actions, and learn through trial and error.

PS has a neural network–type structure that is considered to be its physical basis, where any initial experience can activate other patterns in a spatiotemporal manner. The memory type in PS denoted as episodic and compositional memory (ECM), which literally is a directed, weighted network of clips, where each clip represents a remembered percept, action, or sequences of them.[4] Episodic and compositional memory can be described

[4] Episodic memory, introduced in psychology by Tulving (1985) and Ingvar (1985), has gained increasing attention in the cognitive neuroscience and other scientific fields.

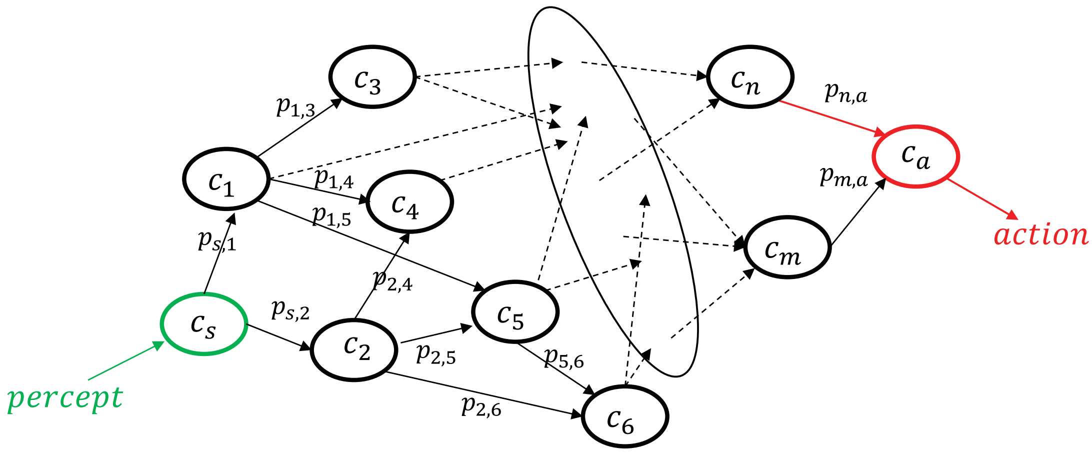



Figure 1: A memory network in PS model and a random walk on the clip space that starts with the activation of clip $c_s$ and reaches the action clip $c_a$ that coupled out the real action. The clips and transition probabilities between them can evolve based on feedback from the environment.

as a probabilistic network of clips. In the following, we use the terms *episode* and *clip* interchangeably.

Once a percept is observed, its coupled clip is activated and a random walk on the clip network is triggered, until an action clip is reached and coupled out as a real action that the agent does. In other words, any recall of memory is understood as a dynamic replay of an excitation pattern, which gives rise to episodic sequences of memory (see Figure 1).

Indeed, in the PS model, a random walk in the network of clips happens before the action is excited. An interpretation is that the agent projects itself to the future (imagines what will happen if an action is chosen) and therefore complex decisions might be taken, including choices that were not in the training phase (like stimulus equivalence).

The main part of the agent is usually considered to be its learning program, which depends on the nature of the agent and its environment. The learning in PS is realized by updating of weights and structure through adding new clips. The connection weights between the clips are updated through Bayesian rules. New clips will be created and added via interaction with the environment as perceptions or from existing clips under certain compositional principles (Melnikov, Makmal, Dunjko, & Briegel, 2017).

**2.4 Comparison of PS with Reinforcement Learning.** Here, we provide a very brief comparison between PS and other well-studied RL algorithms (see Sutton & Barto, 2018, for details of reinforcement learning schemes, and Bjerland, 2015, and Mautner et al., 2015, for detailed comparisons of RL and PS). We also discuss our reasons for selecting PS over other RL methods.

It is worth mentioning that RL is different from supervised learning, which is dominant in the field of machine learning. Supervised learning is performed through a set of labeled samples provided by an external supervisor. The objective of this important kind of learning is to generalize or extrapolate the kind of responses that are taught during training. In this way, it can handle situations that were not present in training trials. A reinforcement learning model bears close similarity to human and animal learning. The development of RL algorithms has benefited from advancements within other fields, especially psychology and neuroscience. Supervised learning, however, is not an adequate choice for learning from interaction with environment.

The notion of PS can be used as an RL algorithm, since, like RL, it is an independent embodied agent that interacts with the environment and learns by trial and error through feedback. However, PS is a more general framework that is able to use quantum mechanics and solve larger tasks than those possible with RL (Paparo, Dunjko, Makmal, Martin-Delgado, & Briegel, 2014; Mautner et al., 2015).

The most important difference between PS and other standard RL algorithms is its episodic memory, which allows modeling more complex features. More specifically, learning in RL is based on estimation of value functions, while in PS, learning is through the reconfiguration of the memory network. This reconfiguration could be simply the update of transition probabilities or by adding or creating new clips. Standard RL has no counterpart for this dynamic change in structure (new clips), which makes the PS model more flexible (Melnikov et al., 2017; Bjerland, 2015).

The fact that PS distinguishes between real percepts and actions by using their internal representation makes it more similar to the functioning of the brain, such as the idea of cognitive maps (Tolman, 1948; Behrens et al., 2018), the role of internal manipulation of representations (Piaget, Chilton, & Inhelder, 1971), and brain mechanisms for episodic memory (Hasselmo, 2011).

## 3 Formation of Stimulus Equivalence Classes in Projective Simulation Setting

We present the standard model of PS formalism in section 3.1, where the notations are mostly from Melnikov et al. (2017). In section 3.2 we present EPS through algorithms.

**3.1 The Formalism of PS.** First, the agent's policy is defined as an external view of agent's way of behaving at a given time $t$. The policy is denoted by $P^{(t)}(a|s)$, which represents the probabilities for selecting each possible action $a \in \mathcal{A}$, when percept $s \in \mathcal{S}$ is received.

Let $\mathcal{C} = \{c_1, \cdots, c_p\}$ be the set of possible internal states of the agent. In the clip network of memory, the transition probabilities from clip $c_i \in \mathcal{C}$ to clip $c_j \in \mathcal{C}$ at time step $t$ are defined as

$$p^{(t)}(c_j|c_i) = \frac{h^{(t)}(c_i, c_j)}{\sum_k h^{(t)}(c_i, c_k)}, \tag{3.1}$$

where the weight $h^{(t)}(c_i, c_j)$, called the $h$-value, is updated as follows at time step $t$:

$$h^{(t+1)}(c_i, c_j) = h^{(t)}(c_i, c_j) - \gamma(h^{(t)}(c_i, c_j) - 1) + \begin{cases} \lambda^{(t)} & \text{if traversed} \\ 0 & \text{else} \end{cases}, \tag{3.2}$$

where $0 \leq \gamma \leq 1$ is a damping parameter and $\lambda \in \Lambda$ is a nonnegative reward given by the environment. Equation 3.2 shows that $h^{(t+1)}(c_i, c_j)$ will be affected by the reward at previous time $t$ only if the $(c_i, c_j)$ connection was traversed during the random walk at time $t$. $\Lambda$ could be a subset of real numbers, in accordance with the learning task and environment type. In the simplest case, $\Lambda = \{0, 1\}$, where $\lambda = 1$ means a reward and $\lambda = 0$ means no reward. $h$-values are initialized with $h_0 = 1$ as soon as a transition link (an edge) is established. A positive damping parameter enables the agent to weaken and even totally forget what it has been learned until time step $t$ (i.e., $h^{(t)}(c_i, c_j) - h_0$). As discussed in the PS literature (Melnikov et al., 2017, for instance), the damping term is not necessary for stationary environments as in contextual bandit tasks (Wang, Kulkarni, & Poor, 2005). The SE task that we model has a stationary environment in which the desired percept-action relations do not change over time. However, since we aim at modeling the brain and because gradual forgetting is an important characteristic of human memory, we keep it in the model.[5]

In order to keep conditional probabilities in equation 3.1 well defined, equation 3.2 guarantees that $h$-values are lower bounded by $h_0$ when reward $\lambda$ is not negative. An alternative expression for the transition probability, known as the softmax (or Boltzmann) distribution function, can handle the negative rewards and keeps the transition probabilities nonnegative,

$$p^{(t)}(c_j|c_i) = \frac{e^{\beta h^{(t)}(c_i, c_j)}}{\sum_k e^{\beta h^{(t)}(c_i, c_k)}}, \tag{3.3}$$

[5] To apply the PS agent in modeling the contextually controlled equivalence classes, the environment might be considered nonstationary since the established relations could change to new relations. In spite of that, a contextually controlled equivalence class experiment can be considered stationary if one argues that the relations will not change under a specific context.

where $\beta$ can be used for tuning the learning rate as well. Lower values of $\beta$ increase the chance of choosing an edge with a larger $h$-value.[6]

Before moving to the next section, we briefly introduce emotion tags and reflection time in the PS model. Emotion tags belong to an emotion space that has arbitrary emotion states. The tags are attached to the transition links between clips and indicate the associated feedback that was stored in the evaluation system of memory. The role of these tags is similar to a short-term memory of the previous rewards for previous actions. So the agent might avoid an action if a negative tag is attached. The state of the emotion tag attached to transition links changes based on the feedback, so if the environment changes, the agent could update its short-term memory quickly. It is important to consider emotion tags as internal memory of rewards, distinct from external real rewards by environment.[7]

The emotion tags can be used by the agent in order to avoid immediate action when the reflection time is greater than one ($R > 1$). Reflection time is the frequency that the agent can reflect on its action. More specifically, if the random walk on the memory space ended in an action where the agent remembers that the previous reward for this action was not desirable, the agent reexcites the percept clip and gives other action clips the chance to be selected rather than coupling out the action clip to the real clip.

**3.2 Equivalence Projective Simulation Model.** Some desired features of a beneficial model in equivalence formation through MTS could be:

1. The ability to form equivalence classes—that is, correctly match derived relations (i.e., symmetry, transitivity and equivalence) in MTS trials
2. The ability to show different relatedness factors between stimuli in an equivalence class—for instance, show that relatedness is an inverse function of the nodal distance (Fields, Adams, & Verhave, 1993)
3. Endowment with the forgetting ability similar to that humans
4. The ablility to model memory and learning disabilities by manipulating tuning parameters
5. Possible use as a hypothesis testing tool before conducting a real experiment

There are different views on the mechanism of deriving relations, that is, either during the training phase and before the testing phase or during the MTS test. For instance, Galizio, Stewart, and Pilgrim (2001) noted

[6] Note that there is no tuning parameter in equation 3.2 for $h$-values. Moreover using $\beta$ computationally means that instead of a natural logarithm, a different base (i.e., $e^{\beta}$) is used.

[7] We do not use emotion tags in this letter. For modeling more advanced scenarios of equivalence formation such as contextually controlled stimulus equivalence formation (Bush, Sidman, & Rose, 1989), one can apply emotion tags to improve the model.

that some degree of equivalence class formation occurs during MTS training and that it is enhanced during testing. However, in many other studies, the emergence of equivalence relations is considered to be the result of testing lower-stage relations (see Dickins, 2015, part E, for a discussion). As explained in Dickins (2015) explained, evidence is established through brain-imaging studies.

At this juncture, we provide the assumptions in our EPS model:

- Appropriate training of baseline relations is necessary for the formation of an equivalence class, but it is not sufficient.
- Any symmetry relation is a function of its entailed baseline relation. $K_2$ attempts to model mechanisms in the brain that can influence the formation of symmetry.
- Formation of transitivity is a function of well-trained baseline relations. However, a memory sharpness ($\theta$) less than one can weaken the effect of baseline relations. Memory sharpness ($\theta$) plays a similar role to $K_2$ and controls derived relations with nodal distance greater than one (i.e., transitivity and equivalence relations).
- $\theta$ could be chosen as a constant independent of the nodal distance or could vary according to it.
- Equivalence formation is a function of both symmetry and transitivity formation, so $K_2$ and $\theta$ could be seen as other mechanisms in the brain, along with the reinforcement of baseline relations, that might affect the emergence of equivalence relations.
- In EPS, whenever symmetry and transitivity relations emerge, equivalence relations will emerge as well.
- EPS does not model reflexivity, since in many experiments with human adults, the ability to perform a reflexivity task is usually taken for granted (see Dickins et al., 2001, for instance).[8]

The proposed model (EPS) aims at modeling human behavior when all the stimuli in an equivalence class are expected to be equal. One option is to consider an undirected graph as the memory clip, define all the stimulus-stimulus connections as bidirectional, and drop the $K_2$ parameter. However, evidence from experimental studies shows that derived relations are sometimes weaker than baseline relations, and sometimes not formed at all. In order to cover more general cases, such as humans who are not able to derive new relations, we consider a directed graph and differentiate the types of relations. Moreover, EPS can be extended to model other derived relations in line with equivalence or sameness, as it is in RFT. In such a case, a

[8] PS has the capability to add features to the stimuli and define transitions between clips within each category, including self-loops at each clip. The formation of reflexivity can be achieved using high $h$-values for self-loops and low $h$-values within different stimuli in the same category.

directed graph, similar to the current selection in EPS, is needed to differentiate relation types.

In the proposed EPS model, symmetry relations are formed during training, with the assumption that transitivity and equivalence are also formed during training. However, since the response latencies in transitivity and equivalence tests at the beginning are typically longer than trained relations or symmetry tests (Bentall, Dickins, & Fox, 1993), transitivity and equivalence transition probabilities are calculated for each trial in the MTS test. However, by virtue of the flexibility of PS, the model can be modified so that the formation of symmetry relations is postponed to the testing phase. Also, one can establish connections in the training phase and gradually update them during the MTS testing phase or during the MTS test.

In the following, we model an arbitrary MTS experiment independent of the training structures (LS, OTM, MTO). The agent has no memory at the beginning (the memory space $\mathcal{C}$ is empty); however, all the stimuli potentially belong to the set of percepts ($\mathcal{S}$) and actions ($\mathcal{A}$), as well as remembered clips $\mathcal{C}$. This initialization will be shown with $\mathcal{S} = \mathcal{C} = \mathcal{A} = \emptyset$. The percept and possible actions are provided by the environment at each time step.

The sample stimuli will make the percept clips, and the comparison stimuli will make the action clips. A policy corresponds to a set of stimulus-response rules or associations where $\mathcal{S}$ is the set of stimuli and $\mathcal{A}$ is the set of responses. The memory space will be updated and enlarged through the training phase. Clips are added the first time that the agent perceives them.

The algorithm has two phases. In the training phase, the memory network is shaped, and in the testing phase, no new memory clip is created but new connections can be added and initialized.[9]

*3.2.1 Training Phase.* At each time step in general and at the beginning more specifically, the agent might create new clips, add new transition links, and update them based on the reward value. In the model, a memorized clip could simultaneously play the role of either percept clip or action clip.

Since the training structure is through MTS, the possible actions in each trial are limited to a subset of all actions—the set of comparison stimuli.[10] The action space at time $t$ is denoted by $\mathcal{A}_t$. The probability that action $a^{(t)}$ is chosen by the agent when percept $s^{(t)}$ is presented may depend on the history of experiment. Indeed, the agent learns through changing its internal network, which determines the agent future policy.

In the PS model and in general form, the clips as the building blocks of memory are defined as sequences of remembered percepts and actions. In

[9] The agent can be provided with the possibility of creating new, or "fictitious," clips during the testing phase. We do not resort to fictitious clips in this letter.

[10] For simplicity, we consider that the location (order) of comparison stimuli is not important.

modeling the SE, each memory clip represents a remembered stimulus, as either a sample stimulus or a comparison stimulus.

Note that the sample stimulus (percept $s \in \mathcal{S}$) and the comparison stimuli (actions $a \in \mathcal{A}_t$) belong to different categories, like Greek letters, nature pictures, or colored balls. As a result, each of the class members belongs to a different category (say, category $A$ or $B$) and there is no connection (paired relation) within elements of a category.

Moreover, in stimulus equivalence, there is no redundancy in the training phase, so the only information that could assist during the learning comes from the members of a category. Consequently, when a new category appears in trials, the agent creates connections with equal weights since there is no prior information from previous connections. The agent's operation cycle can be summarized as follows:

1. Stimulus $s \in \mathcal{S}$ with probability $P^{(t)}(s)$ is perceived from the environment.
2. A fixed input-coupler probability function $\mathcal{I}(c|s)$ activates the memory clip $c \in \mathcal{C}$, denoted by $c_s$. This typically maps the real stimulus $s$ to its internal representation clip with probability 1. When a stimulus is perceived for the first time, a clip is created and added to the network.
3. Action set $\mathcal{A}_t$ is perceived from the environment. If any of the actions $a \in \mathcal{A}_t$ do not have an internal image, a clip $c_a$ will be created.
4. If there exist connections among the sample and comparisons, the agent computes the $p^{(t)}(c_a|c_s), a \in \mathcal{A}_t$ based on the $h$-values. If such connections do not exist, the agent establishes and initializes them and then computes the probabilities $p^{(t)}(c_a|c_s)$.
5. The agent selects one of the possible actions based on the computed probability distribution. Then excitation of the selected action clip maps to a real action $a \in \mathcal{A}$ through a fixed output-coupler function $\mathcal{O}(a|c_a)$. Similar to the input coupler function in general, this function maps the internal action to the real action with probability one.
6. The agent receives a positive or negative reward from the environment. The connection weights, $h$-values, will be updated due to this feedback such that the desired match be reinforced.

An important issue in modeling the SE is that the percepts and actions could play the same role. For instance, $B_1$ is a possible action in $AB$ relation training, and in the original PS memory, it is remembered as an action clip, but it would play the role of percept in $BC$ training. Subsequently, the role of clips will be changed based on the trial. This double role of clips makes the network slightly different from PS. Another distinction between the models is derived relations. Handling symmetry relations in the model is taken care of by establishing the opposite transition links whenever a specific MTS is presented for the first time.

Therefore, initialization of the transition links and $h$-values for newly added clips is done simply by establishing two direct connections for each possible new match and initializing them with $h_0$. So if the newly added clip is a percept clip, the number of new connections would be $2|\mathcal{A}_t|$. If it is an action clip, just two connections will be established.

To complete the process, the updating rules for $h$-values based on the environment feedback must be added. Recall that we consider negative reward in the model as well: $\Lambda = \{-1, 0, 1\}$. The reason is that in MTS methods, the participants are usually notified whether the chosen stimulus was correct or incorrect.

We suggest two methods for updating $h$-values. The first method, similar to PS, keeps positive the $h$-values that are lower bounded by $h_0$. Therefore, this method is suitable for being used in both equations 3.1 and equation 3.3. The second method cannot be used in equation 3.1. The difference between methods occurs when the agent received a negative reward. We explain the positive reward first and then the two alternatives for negative reward.

Suppose the percept is $s \in \mathcal{S}$, coupled into $c_s \in \mathcal{C}$, and the chosen action by the agent is $a \in \mathcal{A}_t$, which is coupled out from clip $c_a \in \mathcal{C}$. Let $\lambda^{(t)} = 1$: the agent chooses the correct option, which must be reinforced. The $h$-value updates will be calculated like a PS model,

$$h^{(t+1)}(c_s, c_a) = h^{(t)}(c_s, c_a) - \gamma(h^{(t)}(c_s, c_a) - 1) + K_1\lambda^{(t)}, \tag{3.4}$$

where $K_1$ is a positive value and equals one based on PS. The opposite link, $(c_a, c_s)$, will be updated in a similar way, but with parameter $0 < K_2 \leq K_1$ (see equation 3.5). We could consider a simpler model with bidirectional connections representing typical humans who are trained well. This is analogous to setting $K_2 = K_1$:

$$h^{(t+1)}(c_a, c_s) = h^{(t)}(c_a, c_s) - \gamma(h^{(t)}(c_a, c_s) - 1) + K_2\lambda^{(t)}. \tag{3.5}$$

If $\lambda^{(t)} = -1$, the agent chooses a wrong option that must be inhibited.

- *First scenario for updating h-values.* This negative reward reinforces all the actions in $\mathcal{A}_t$ except the one that the agent has chosen. Let $c_{a'} \in \mathcal{O}^{-1}(\mathcal{A}_t) - \{c_a\}$, where $\mathcal{O}()$ is the output coupler function that transforms a set of clips into real actions. Since $\mathcal{O}()$ is one-to-one, its inverse is well defined,[11] the updates rule is

$$h^{(t+1)}(c_s, c_{a'}) = h^{(t)}(c_s, c_{a'}) - \gamma(h^{(t)}(c_s, c_{a'}) - 1) - K_3\lambda^{(t)}, \tag{3.6}$$

[11] We abuse notation since $\mathcal{O}()$ coupled-out an action clip to its real counterpart; however, for simplicity, we use the same notation for the function that sends a set of clips to the set of real actions.

where $K_3 \leq \frac{K_1}{m-1}$ and $m = |\mathcal{A}_t|$ is the number of options in the action space at time $t$.[12] Note that since $\lambda^{(t)} = -1$, the term $-K_3\lambda^{(t)}$ is positive. The symmetry connections are updated in the same way, that is, the transition weight from clip $c_{a'}$ to clip $c_s$ will be increased by an additive factor $K_4$ where $0 < K_4 \leq \frac{K_2}{m-1}$:

$$h^{(t+1)}(c_{a'}, c_s) = h^{(t)}(c_{a'}, c_s) - \gamma(h^{(t)}(c_{a'}, c_s) - 1) - K_4\lambda^{(t)}. \tag{3.7}$$

- *Second scenario for updating h-values.* The second scenario is similar to the positive reward. The $h$-values of the transitions will be updated by a negative factor. In this case, only the soft-max method can be used for conditional probabilities.

When all clips are created and all possible relations are added and initiated, further training trials are updating the $h$-values as explained above until the desired relations meet the criterion, so we will be able to move to the testing phase.

*3.2.2 Testing Phase.* The testing phase starts when all training relations meet the mastery criterion. In this phase, we test the emergent relations that are not trained explicitly. During the test, basically there is no feedback, and we can consider that the evolution of the network based on external feedback is finished. However, one can consider the feedback $\lambda = 0$ and let the forgetting factor work with dissipation rate $\gamma$. The experimenter can consider various testing procedures, such as a random selection of mixture-of-the-learn relations and the emergent ones, or testing symmetry relations first, then transitivity relations, and equivalence relations (a combination of symmetry and transitivity) afterward. At the end of experiment, usually the percentage of correct choices in a specific relation will be calculated and analyzed.

In the artificial model, however, one can use the final policy $P(a|s), a \in \mathcal{A}, s \in \mathcal{S}$ for analysis instead of running a testing phase. The agent's functioning during the testing phase can be summarized as follows:

1. Stimulus $s \in \mathcal{S}$ with probability $P^{(t)}(s)$ is perceived.
2. A fixed input-coupler probability function $\mathcal{I}(c_s|s)$ activates the memory clip $c_s \in \mathcal{C}$.
3. Action set $\mathcal{A}_t$ is perceived from the environment.
4. If connections exist among the sample and comparisons, the agent computes the $p^{(t)}(c_a|c_s), a \in \mathcal{A}_t$ based on the $h$-values. If such connections do not exist, the agent establishes imaginary connections and computes the

[12] The reason that we define $K_3$ this way is intuitive. The information we got from the negative reward reinforces other options; moreover, it is an indirect process so the expectation is that it will be less effective than the direct ones.

probabilities $p^{(t)}(c_a|c_s)$. The connections in this case represent the transitivity or equivalence relations.[13] This is the case when nodal distance (Fields & Verhave, 1987) or, equivalently, nodal number (Sidman, 1994) is positive, and there is at least a path with length $L \geq 2$ between the possible matches.[14]

There might be several options and policies to compute the probability of derived connections. For instance, one might consider the most probable paths between $c_s$ and each action $c_a$, $a \in \mathcal{A}_t$, which is

$$p^{(t)}(c_a|c_s) = \max_{\mathcal{P}_L \in \mathcal{P}(c_s, c_a)} \prod_{i=0}^{L-1} p^{(t)}(c_{l_{i+1}}|c_{l_i}), \tag{3.8}$$

where $\mathcal{P}(c_s, c_a)$ is the set of all possible paths from $c_s$ to $c_a$, and $\mathcal{P}_L \in \mathcal{P}(c_s, c_a)$ is a specific one with $L \geq 2$. $l_i$; $i = 1, 2, \ldots (L-1)$ shows the indices of intermediate clips, while $c_{l_0} = c_s$ and $c_{l_L} = c_a$. In section 4.1, the max-product scenario for computing derived probabilities is addressed.

Memory sharpness, $0 \leq \theta \leq 1$, functions as a mechanism to control the formation of transitivity relations, in line with baseline relations training. Memory sharpness is analogous to the deliberation time in the PS model.

If $\theta = 1$, meaning it is simply omitted from the model, the well-trained baseline relations result in strong transitivity connections. This fact is not always true for all real experiments. Therefore, we introduce memory sharpness in the model to control transitivity, equivalence relations, and the effect of the nodal distance. Memory sharpness can also represent the effect of comparison stimuli and to what extent the agent recalls its memory (memory sharpness is addressed in section 4.2).

Instead of max-product policy, equation 3.8, one might consider a random walk in $\mathcal{C}$, starting from $c_s$ and ending with a clip in $\mathcal{A}_t$. In other words, instead of finding the most probable path from $c_s$ to each of possibilities in $\mathcal{A}_t$, the probability of reaching each action from $c_s$ can be considered. These probabilities, as explained in detail in section 4.3, can be computed easily when actions $c_a \in \mathcal{A}_t$ are set to be absorbing states of the underlying Markov chain, at time $t$.

5. The agent selects one of the possible actions based on probabilities $p^{(t)}(c_a|c_s)$, and the activation of the action clip maps to a real action $a \in \mathcal{A}$ through a fixed output-coupler function $\mathcal{O}(a|c)$.

Since, the aim is to compare the performance of this artificial agent with human results, we could have considered these probabilities without

[13] In this case, if one does not establish and update the inverse links during the training phase, symmetry connections must be calculated.

[14] A node in equivalence class terms refers to any stimulus, or class member, that connects at least two other members in the equivalence class through training. The nodal distance or nodal number is the number of nodes between the two members.

**Algorithm 1:** Environment Operation in the EPS Model: Training Phase.

**input:** Experiment Protocol

initialization

$\mathcal{S} = \emptyset,\ \mathcal{A} = \emptyset,\ t = 1$

**begin**

  **while** *All training relations meet the criterion* **do**

    Show the sample stimulus $s$ to the agent

    **if** $s \notin \mathcal{S}$ **then**

      $\mathcal{S} = \mathcal{S} \cup \{s\}$

    Show the comparison stimuli $a \in \mathcal{A}_t$ to the agent

    **if** $\mathcal{A}_t \not\subseteq \mathcal{A}$ **then**

      $\mathcal{A} = \mathcal{A} \cup \mathcal{A}_t$

    Feedback (reward) to the agent based on its action

    $t = t + 1$

  Show the termination message to the agent

**output:** $\mathcal{S},\ \mathcal{A}$

running the testing phase. However, we prefer to keep it this way to show similar functioning of the agent in the testing phase.

Algorithms 1 and 2, respectively, summarize the environment and the agent operations in the training phase. Note that the protocol gives all the information that the experimenter (and the environment in the artificial model) needs to perform the experiment, including all the stimuli, the training structure (say, LS, OTM, or MTO), learning, and the mastery criterion.

It is worth mentioning that the training loop in algorithm 1 might have other stopping criteria along with the mastery of training relations. For instance, an upper bound for the number of trials $t$ might be specified in the protocol or a limitation on the time period that the participant can spend before choosing an option. The experimenter might exclude such participants from analysis. However, in the artificial model, there is no need to consider such cases; instead, it is more beneficial to put some restrictions on the memory evolution and tuning parameters to avoid undesired scenarios.

The environment and agent algorithms during the testing phase (i.e., no feedback) are presented in algorithms 3 and 4, respectively.

A sample protocol sheet that the experimenter has is presented in protocol 1, and a description of how EPS models this experiment is provided in detail in appendix A.

**Algorithm 2:** Agent Operation in the EPS Model: Training Phase.

**input:** Parameters and updating rule
initialization
$\mathcal{C} = \emptyset,\ t = 1$
**begin**
    **while** *Not receiving the termination message* **do**
        **if** $\mathcal{I}(s) \notin \mathcal{C}$ **then**
            create $c = \mathcal{I}(s)$
            $\mathcal{C} = \mathcal{C} \cup \{c\}$
        **if** $\mathcal{A}_t^c = \{a | a \in \mathcal{A}_t \text{ and } \mathcal{O}^{-1}(a) \notin \mathcal{C}\} \neq \emptyset$ **then**
            **for** $a \in \mathcal{A}_t^c$ **do**
                create $c = \mathcal{O}^{-1}(a)$
                $\mathcal{C} = \mathcal{C} \cup \{c\}$

- Create new connections if any new clip is added; initialize $h$-values
- Compute the probability distribution for $c_a \in \mathcal{A}_t$, then choose an action based upon that
- Update $h$-values
- $t = t + 1$

**output:** $\mathcal{C}$

**Algorithm 3:** Environment Operation in EPS Model: Testing Phase.

**input:** Experiment Protocol, $\mathcal{S},\ \mathcal{A}$
initialization
$t = 1$
**begin**
    **while** *All testing relations presented* **do**
        Show the sample stimulus $s$ to the agent
        Show the comparison stimuli $a \in \mathcal{A}_t$ to the agent
        Record the results
        $t = t + 1$
    Show the termination message to the agent
**output:** Test results

**Algorithm 4:** Agent Operation in EPS Model: Testing Phase.

**input:** $\mathcal{C}$

initialization

$t = 1$

**begin**

  **while** *Not receiving the termination message* **do**

    Receive $c_s$ and $c_a \in \mathcal{A}_t$

    **if** *Connections exist between them* **then**

      Compute probabilities based on $h$-values

    **else**

      Compute the probabilities between $c_s$ and $c_a \in \mathcal{A}_t$ with an appropriate algorithm

    Choose an option based on the probability distribution over $\mathcal{A}_t$

**Protocol 1**:

- Three four-member classes $\{A_1, B_1, C_1, D_1\}$, $\{A_2, B_2, C_2, D_2\}$, and $\{A_3, B_3, C_3, D_3\}$ are going to be trained with an arbitrary MTS procedure.
- Let the order of training relations be AB, BC, and DC.
- The set of comparison stimuli will appear simultaneously after a 1 s delay.
- The training is in blocks of 30 trials, a mixture of the possible three relations, each 10 times. Each answer will be followed by feedback: correct ($\lambda = 1$) or incorrect ($\lambda = -1$).[15]
- The training mastery criterion is to answer 90% of the trials in the block correctly.
- If the participant leaves the experiment or could not learn a set of relation after $T = 1000$ steps, terminate the experiment by notifying the participant.
- If the training mastery criterion is met, the testing phase consists of, respectively, four blocks; baseline, symmetry, transitivity, and equivalence. The baseline block is composed of $AB$, $BC$, and $DC$ relations, each 9 times. Symmetry is a block of $BA$, $CB$, and $CD$ relations, each repeated 9 times. A transitivity block with size 9 contains the $AC$ relation. The equivalence block contains $CA$, $BD$, $DB$, $AD$, and $DA$ relations, 9 times each.

[15] For instance, the first block would be a random shuffling of $A_1$, $A_2$, and $A_3$, as a sample stimulus, 10 times each. $B_1$, $B_2$, and $B_3$ make the comparison stimuli or action set in a random order.

- Compute the percentage of correct answers for the emergent relations and determine if the equivalence relation is formed.
- The mastery criterion ratio for the test part is 0.9.

## 4 Simulation of Stimulus Equivalence

Although investigation of various parameters' assembly is not in the scope this letter, in order to validate the model and explain its functionality, some real experiments from the literature, including experiments with patients, are provided and simulated. We have to figure out how parameters must be tuned in order to get similar results for healthy people or patients. We fix $\theta = 1$ in section 4.1 and simulate the sample experiment provided in protocol 1 using the max-product method for computing the probability distributions. Next, similar to Tovar and Westermann (2017), we simulate some high-impact experimental studies in Sidman and Tailby (1982), Devany et al. (1986), and Spencer and Chase (1996). The training is in the standard format in which $h$-values get positive values. A replication of Spencer and Chase (1996) with softmax policy, with both positive and negative $h$-values, is reported at the end of this section. In section 4.2, we explain the concept of memory sharpness in detail. We discuss similarities between the deliberation length in the PS model and nodal distance or nodal number in equivalence theory and model studies in Devany et al. (1986) as well as Spencer and Chase (1996). The third case, in section 4.3, is to compute the transition probabilities between a sample stimulus clip and comparison stimuli clips through a random walk, that is, as if the action clips are the absorbing states of the network. In this setting, similar to that of Spencer and Chase (1996), we explain how the reaction time might be increased with nodal distance.

The concluding experiment in section 4.4, is a replication of that of Devany et al. (1986) with a different training setting. The aim is to show that EPS is a suitable model to investigate a new hypothesis in equivalence study.

The following reported simulation results are the average over 1000 simulations.

**4.1 Case 1: Max-Product Algorithm for Probability of Transitive and Equivalence Relations.** In the testing phase, when there is no direct connection between percept $c_s$ and actions $c_a \in \mathcal{A}_t$, in order to find the path from $c_s$ to $c_a$ with the highest probability, one way is to convert the max-product problem into a min-sum problem by using the negative logarithm value.

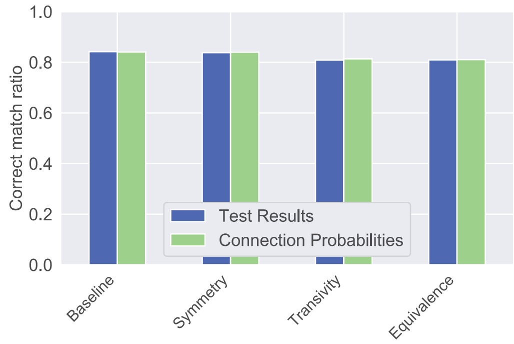
This is similar to the maximum likelihood algorithms where likelihoods are converted to log likelihoods. In this manner, products are converted to sums, and max-products are converted to max-sums. Similarly, the negative log likelihood converts max-product to min-sum. These variations are all trivially equivalent. Through this conversion, finding the path with the highest probability will be the lowest-cost path problem. The lowest-cost path problem then can be solved with Dijkstra's (1959) often-cited and well-known algorithm or the min-sum/Viterbi algorithm (see MacKay, 2003). The final values as the probability product of the path with the highest probabilities are normalized to obtain the probability distribution over the action set that an agent uses to select actions (for more details on how the model computes the max-product of testing phase, see appendix B).

In this scenario, the relative value of the probabilities is important, and the nodal distance that affects the probability values might be ignored during the normalization. In section 4.2 we deal with this issue using the memory sharpness parameter.

*4.1.1 Experiment 1: Simulation of Protocol 1.* Consider an example based on protocol 1, where the training phase is *AB*, *BC*, and *DC*, respectively. The mastery criterion is set to 0.9, and each block contains 30 trials. For instance, a block for training *AB* contains 10 trials with correct match $A_1B_1$, 10 trials with correct match $A_2B_2$, and 10 trials with correct match $A_3B_3$. In the results reported in Figure 2, the blue bars show the outcome of testing phase (the counterpart of what experimenter receive), and the green bars show the connection weights of the memory network at the end of experiment (a representative of the internal state). The baseline is composed of a block of relations $AB$, $BC$, and $DC$ each nine times, which means each relation is repeated three times in the block. Symmetry is a block of $BA, CB,$ and $CD$, each repeated nine times in a similar way. The transitivity contains a block of *AC* relations of size 9. Finally, the equivalence shows the results for a block of $CA, BD, DB, AD,$ and $DA$ relations, nine times each.

In the simulation represented in Figure 2a the parameters are $\gamma = 0.001$, $K_1 = 1$, $K_2 = 0.9$, $K_3 = 0.5$, and $K_4 = 0.45$. Figure 2a shows that all the relations in equivalence classes are formed. The baseline relations ratio is about .85, and for transitivity and equivalence, the ratio is about 0.8. In Figure 2b, the forgetting factor changes to $\gamma = 0.01$, which means that the agent forgets faster. We see that a higher forgetting factor can affect the results severely. The baseline relations ratio falls to 0.6, and transitivity and equivalence relations ratios fall to about 0.4. Figure 2b also shows a difference between the connection weights at the end of experiment and the test results. This explains why experimenters usually repeat the relations during training even after mastery and test the relations as a mixture of all relations to cancel the effect of forgetting.

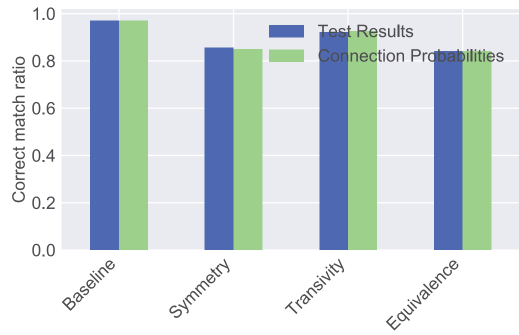
In Figure 2c, $\gamma = 0.001$, $K_1 = 5$, $K_2 = 4$, $K_3 = 2.5$, $K_4 = 2$, we aim at examining how an agent can learn or derive faster by tuning $K_i, i = 1, 2, 3, 4$ parameters. We see that the results shown in Figures 2a and 2c are similar, and the only difference is the time for mastery relations, reported in Table 1. With the setting in Figure 2c, each training block has to be repeated around 3.5 times on average, whereas this is about 6.5 times in Figure 2a setting. So we can tune the block repetition in training by manipulating parameters. In




(a) The results for Experiment 1, when $\gamma = 0.001$, $K_1 = 1$, $K_2 = 0.9$, $K_3 = 0.5$, $K_4 = 0.45$


(b) The results for Experiment 1, when $\gamma = 0.01$, $K_1 = 1$, $K_2 = 0.9$, $K_3 = 0.5$, $K_4 = 0.45$


(c) The results for Experiment 1, when $\gamma = 0.001$, $K_1 = 5$, $K_2 = 4$, $K_3 = 2.5$, $K_4 = 2$




(d) The results for Experiment 1, when $\gamma = 0.001$, $K_1 = 20$, $K_2 = 1$, $K_3 = 3$, $K_4 = 0.3$

Figure 2: Simulation results derived from experiment 1 with different parameters. The blue bar is the outcome of experiment (analogous to what experimenter receives), and the green bar is the connection weight of the memory network at the end of experiment (representing the internal state).

Table 1: Training Time in Various Settings.

| Training | Time (Figure 2a) | Time (Figure 2b) | Time (Figure 2c) | Time (Figure 2d) |
|---|---|---|---|---|
| (AB, 30) | 6.558 | 7.221 | 3.470 | 1.846 |
| (BC, 30) | 6.662 | 7.299 | 3.476 | 1.868 |
| (DC, 30) | 6.471 | 7.188 | 3.350 | 1.845 |

Figure 2d, we study the behavior of an agent when the symmetry relations are not constructed properly by setting $\gamma = 0.001$, $K_1 = 20$, $K_2 = 1$, $K_3 = 3$, and $K_4 = 0.3$. We observe that a higher value of $K_1$ makes training faster—about 1.8 times repetition of blocks. As a consequence, the forgetting factor is less effective, and the baseline relations ratio is about 0.97. We see that the difference between $K_1 = 20$ and $K_2 = 1$ values resulted in weaker symmetry formation and weaker equivalence relations consequently.

Table 2: Training Order in Experiment 2, a Replication of Sidman and Tailby (1982).

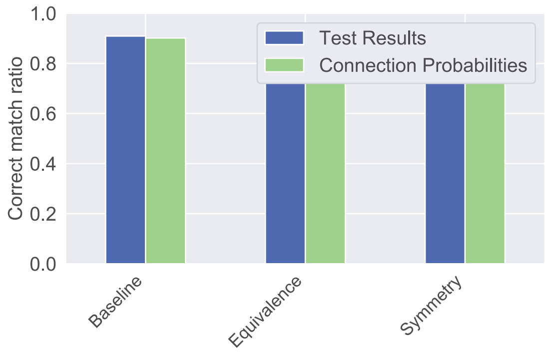
| Training | Block Size | Time | Mastery |
|---|---|---|---|
| 1. Training *AB* | | | |
| $A_1B_1, A_2B_2$ | 20 | 4.146 | 0.927 |
| $A_1B_1, A_3B_3$ | 20 | 3.253 | 0.930 |
| $A_2B_2, A_3B_3$ | 20 | 2.077 | 0.932 |
| $A_1B_1, A_2B_2, A_3B_3$ | 30 | 1.641 | 0.936 |
| 2. Training *AC* | | | |
| $A_1C_1, A_2C_2$ | 20 | 4.241 | 0.927 |
| $A_1C_1, A_3C_3$ | 20 | 3.244 | 0.930 |
| $A_2C_2, A_3C_3$ | 20 | 2.075 | 0.934 |
| $A_1C_1, A_2C_2, A_3C_3$ | 30 | 1.682 | 0.935 |
| 3. Training *AB* and *AC* | | | |
| $A_1B_1, A_2B_2, A_3B_3,$ $A_1C_1, A_2C_2, A_3C_3$ | 30 | 1.497 | 0.936 |
| 4. Training *DC* | | | |
| $D_1C_1, D_2C_2$ | 20 | 4.215 | 0.929 |
| $D_1C_1, D_3C_3$ | 20 | 3.182 | 0.931 |
| $D_2C_2, D_3C_3$ | 20 | 1.991 | 0.934 |
| $D_1C_1, D_2C_2, D_3C_3$ | 30 | 1.628 | 0.932 |
| 5. Training *AB*, *AC*, and *DC* | | | |
| $A_1B_1, A_2B_2, A_3B_3,$ $A_1C_1, A_2C_2, A_3C_3,$ $D_1C_1, D_2C_2, D_3C_3$ | 45 | 1.721 | 0.935 |

Note: The average number of training blocks before reaching the mastery criterion ratio, 0.9, in addition to the results in the last block are reported in the Time and Mastery columns, respectively.

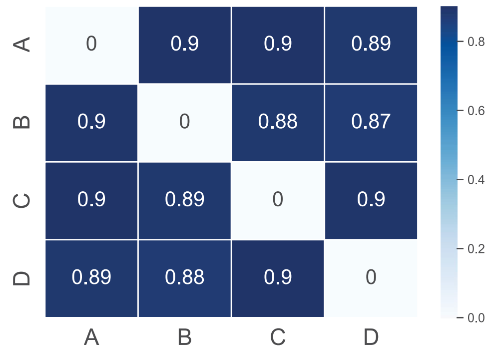
As reported in Table 1, a greater value of $K_1$ makes the training faster in general. The forgetting factor also affects the training time. For instance, if $K_1 = 1$ and $\gamma = 0.001$, each block must be repeated about 6.5 times on average. This will be about 7.3 blocks for $K_1 = 1$ and $\gamma = 0.01$ and will be about 1.8 blocks for $K_1 = 20$ and $\gamma = 0.001$.

Next, similar to Tovar and Westermann (2017) we replicate the studies of Sidman and Tailby (1982), Devany et al. (1986), and Spencer and Chase (1996).

*4.1.2 Experiment 2: Sidman and Tailby (1982).* In this study, stimulus classes with four members are studied in order to analyze the power of equivalence relations in generating larger networks. Eight children with typical development were trained with three four-member stimulus classes. In stimuli set *A* were spoken Greek letter names; the other stimuli sets (*B*, *C*, and *D*) were sets of different printed Greek letters. The training order was *AB* and *AC* relations first and then *DC* relations (see Table 2 for the




(a) The agent's results in baseline and derived relations in Experiment 2.




(b) The final probability of correct response between categories.

Figure 3: The replication of Sidman and Tailby (1982) when $\gamma = 0.001$, $K_1 = 2$, $K_2 = 1.8$, $K_3 = 1$, $K_4 = 0.9$.

order of training and the blocks of MTS trials). The time column shows how many blocks on average were used to achieve mastery in the simulation. We put the mastery criterion ratio at 0.9. The number of necessary blocks was reduced as the relations repeated. The testing phase in Sidman and Tailby (1982) was a combination of some baseline and some derived relations, but we test each relation, say $AB$, in a block of 30 trials. The results presented in Figure 3 show the similar results as the experiment, i.e. the formation of relations.

*4.1.3 Experiment 3: Devany et al. (1986).* The results for replicating the experiment in Devany et al. (1986) are presented here. This is to model the case of language-disabled children who cannot manage the equivalence relations. In Devany et al. (1986), three groups of children learned $AB$ and $AC$ relations from two classes and the tested for formation of $BC$ and $CB$.[16] The training order is presented in Table 3. The test results and the transition probabilities of the network at the end of experiment are presented in Figure 4. In the testing phase, each block consists of 20 trials—say, $BC$ consists of 10 $B_1C_1$ and 10 $B_2C_2$. As Figure 4 shows, the symmetry and equivalence relations are not formed properly. While the ratio of the baseline relations is about 0.9, the $BC$ and $CB$ ratio is about 0.6.

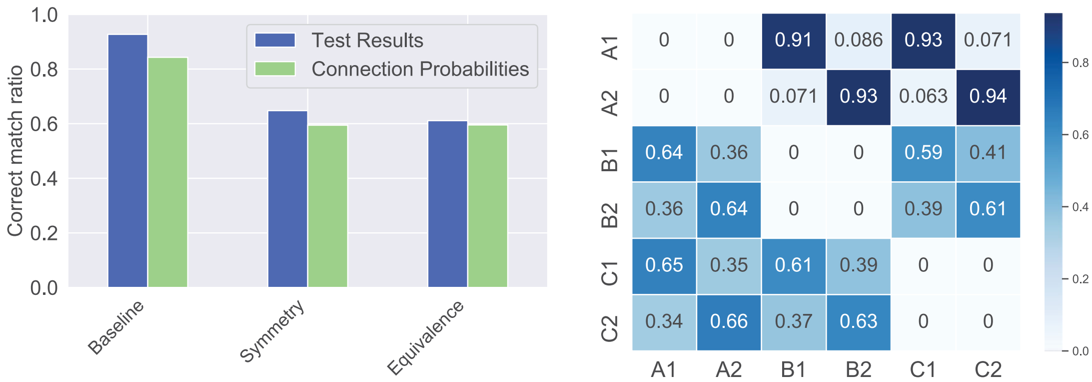
In this experiment, we weaken the formation of equivalence relations with a lower $K_2$ parameter, which controls the formation of symmetry relation. However, as can be seen in experiment 6, even with strongly formed symmetry relations, the formation of equivalence relations is not

[16] The three groups were typically developing children, children with a learning disability with some language skills, and children with a learning disability without language skills.

Table 3: The Training Order in Experiment 3, a Replication of Devany et al. (1986) for Children with a Learning Disability without Language Skills.

| Training | Block Size | Time | Mastery |
|---|---|---|---|
| $A_1B_1$ | 10 | 1.825 | 0.944 |
| $A_2B_2$ | 10 | 1.821 | 0.947 |
| $A_1B_1, A_2B_2$ | 10 | 1.141 | 0.960 |
| $A_1C_1$ | 10 | 1.871 | 0.950 |
| $A_2C_2$ | 10 | 1.841 | 0.949 |
| $A_1C_1, A_2C_2$ | 10 | 1.118 | 0.959 |
| $A_1B_1, A_2B_2, A_1C_1, A_2C_2$ | 8 | 1.746 | 1.000 |

Notes: The Time column shows the average repetition of the training block before reaching the mastery criterion ratio (0.9), when $\gamma = 0.01$, $K_1 = 1$, $K_2 = 0.1$, $K_3 = 0.2$, and $K_4 = 0.05$. The Mastery column refers to the results in the last block.




(a) Trained and derived results

(b) Results based on pairs

Figure 4: The results for experiment 3, the replication of Devany et al. (1986) when $\gamma = 0.01$, $K_1 = 1$, $K_2 = 0.1$, $K_3 = 0.2$, and $K_4 = 0.05$.

guaranteed as nonformation of transitivity induces nonformation of equivalence relations. See experiment 9 as well to see the effect of $K_2$ and $\theta$ in the model.

*4.1.4 Experiment 4: Spencer and Chase (1996).* The experiment in Spencer and Chase (1996) addresses the relatedness (or nodal distance effect) of equivalence formation, with the expectation of observing a decrease in the relatedness between the members with higher nodal distance. Spencer and Chase (1996) measure the response speed during equivalence responding and provide a temporal analysis of the responses. Similar to Tovar and Westermann (2017), we try to replicate the standard group, formed by college students. However, we measure the relatedness by the ratio of correct answers and the transition probabilities of the network. In the

Table 4: The Training Order in Experiment 4, a Replication of Spencer and Chase (1996), to Study the Nodal Effect.

| | Number of Trials per Relation | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Training | *AB* | *BC* | *CD* | *DE* | *EF* | *FG* | Time | Mastery |
| *AB* | 48 | | | | | | 2.864 | 0.944 |
| *BC* | 24 | 24 | | | | | 2.925 | 0.941 |
| *CD* | 12 | 12 | 24 | | | | 3.139 | 0.939 |
| *DE* | 9 | 9 | 9 | 24 | | | 2.737 | 0.928 |
| *EF* | 6 | 6 | 6 | 6 | 24 | | 3.294 | 0.937 |
| *FG* | 3 | 3 | 3 | 6 | 9 | 24 | 3.438 | 0.937 |
| Baseline maintenance | 3 | 3 | 3 | 3 | 3 | 3 | 1.850 | 0.964 |

Notes: The average time before reaching the mastery criterion ratio (0.9), when $\gamma = 0.005$, $K_1 = 5$, $K_2 = 2$, $K_3 = 2$, and $K_4 = 1$. The Mastery column refers to the results in the last block.

Table 5: The Testing Block Order in Experiment 4, a Replication of Spencer and Chase (1996) to Study the Nodal Effect.

| Label | Testing Block | Block Size |
|---|---|---|
| Baseline | *AB*, *BC*, *CD*, *DE*, *EF*, *FG* | $6 \times 9$ |
| Symmetry | *BA*, *CB*, *DC*, *ED*, *FE*, *GF* | $6 \times 9$ |
| Transitivity | *AC*, *AD*, *AE*, *AF*, *AG*, *BD*, *BE*, *BF*, *BG*, *CE*, *CF*, *CG*, *DF*, *DG*, *EG* | $15 \times 9$ |
| Equivalence | *CA*, *DA*, *EA*, *FA*, *GA*, *DB*, *EB*, *FB*, *GB*, *EC*, *FC*, *GC*, *FD*, *GD*, *GE* | $15 \times 9$ |

Note: The results are depicted in Figures 5a and 5b.

experiment, three seven-member stimulus classes consisting of nonsense figures are trained in six sets of relations (*AB*, *BC*, *CD*, *DE*, *EF*, and *FG*) for the three classes) via MTS with three response options per trial. Training consists of seven stages with 48 trials per stage. The training order and the simulation time to learn them are presented in Table 4. The mastery criterion ratio was 0.9. We use three different orders for the testing phase; the first two are provided in Tables 5 and 6, and the third one is a mixture of all the relations with a random order. Figure 5 shows that the model, similar to real experiments, is sensitive to the order of testing. We have better results for baseline relations, around 0.92, when these relations are tested first in Figures 5a to 5d, compared to results in Figures 5e and 5f, which is about 0.87. Generally the forgetting factor affects relations during both the training and testing phases; therefore, using a shuffled mix of all relation types in the testing phase can weaken the forgetting effect. For instance, in Figure 5f, we see that the strongest relation results are about 0.87 and the

Table 6: The Testing Block Order in Experiment 4, a Replication of Spencer and Chase (1996) to Study the Nodal Effect.

| Label | Testing Block | Block Size |
|---|---|---|
| Baseline | $AB, BC, CD, DE, EF, FG$ | $6 \times 9$ |
| Symmetry | $BA, CB, DC, ED, FE, GF$ | $6 \times 9$ |
| $1 - Tr$ | $AC, BD, CE, DF, EG$ | $5 \times 9$ |
| $2 - Tr$ | $AD, BE, CF, DG$ | $4 \times 9$ |
| $3 - Tr$ | $AE, BF, CG$ | $3 \times 9$ |
| $4 - Tr$ | $AF, BG$ | $2 \times 9$ |
| $5 - Tr$ | $AG$ | $1 \times 9$ |
| $1 - Eq$ | $CA, DB, EC, FD, GE$ | $5 \times 9$ |
| $2 - Eq$ | $DA, EB, FC, GD$ | $4 \times 9$ |
| $3 - Eq$ | $EA, FB, GC$ | $3 \times 9$ |
| $4 - Eq$ | $FA, GB$ | $2 \times 9$ |
| $5 - Eq$ | $GA$ | $1 \times 9$ |

Note: The results are depicted in Figures 5c and 5d.

weakest relation results are about 0.71, but in Figure 5d, these values are, respectively, 0.92 and 0.6.

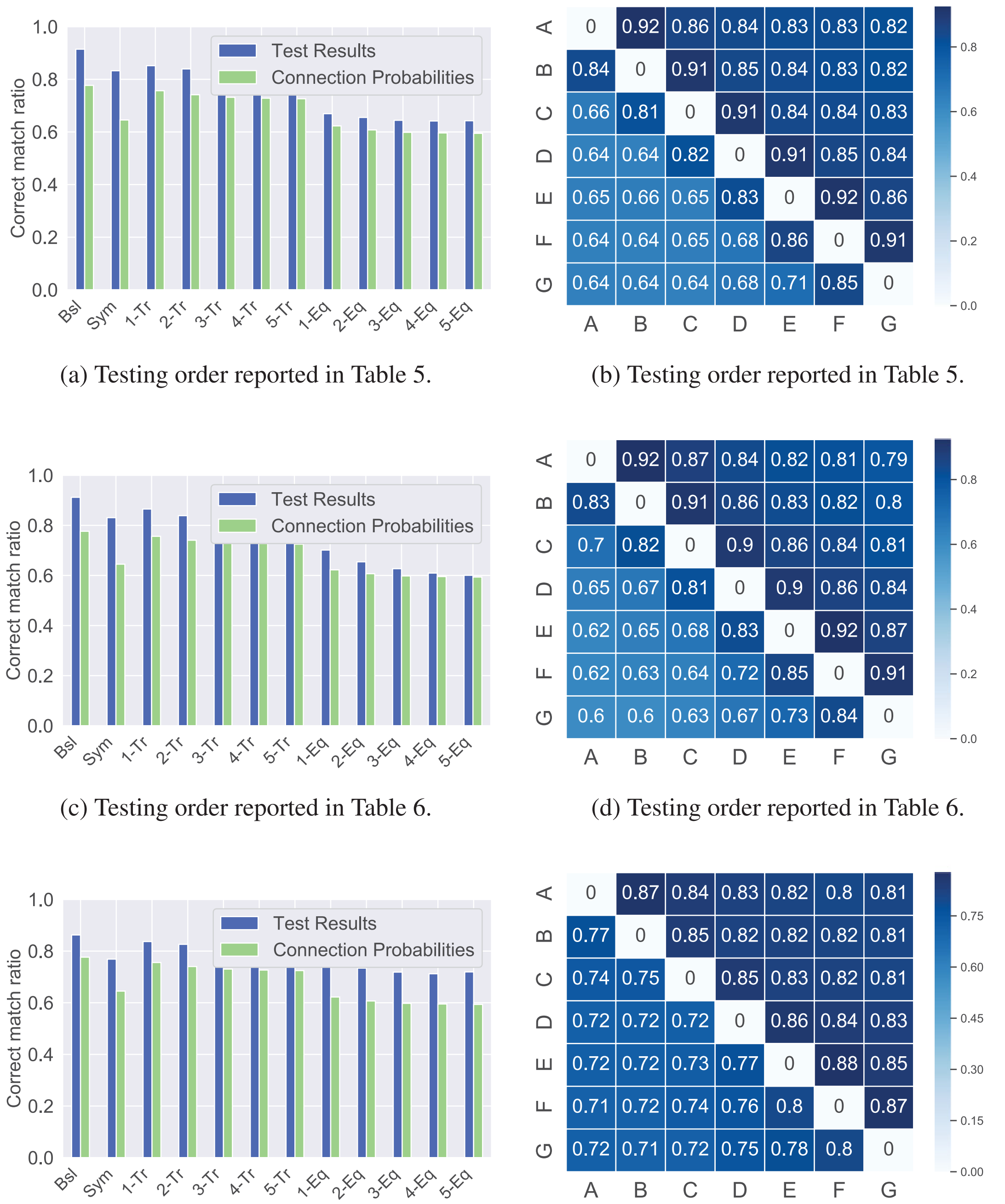
Despite the order of testing, the results in Figure 5 show that the model is sensitive to the nodal distance and can show a reverse effect. However, in order to achieve a better nodal effect, we simulate this experiment with other methods of computing probability transitions in the testing phase. In the following, we report only the results for when the testing is a mixture of all relations.

*4.1.5 Experiment 5: Using softmax to compute the probabilities.* For this experiment, we apply the softmax function for transforming $h$-values into probabilities. In this case, there are two options: first, we keep $h$-values positive and use equations 3.4 to 3.7 for updates; second, we allow $h$-values to be negative using equation 4.1 for updates:

$$\begin{aligned} h^{(t+1)}(c_i, c_j) &= h^{(t)}(c_i, c_j) - \gamma(h^{(t)}(c_i, c_j) - 1) + K_1\lambda^{(t)}, \quad \text{direct} \\ h^{(t+1)}(c_j, c_i) &= h^{(t)}(c_j, c_i) - \gamma(h^{(t)}(c_j, c_i) - 1) + K_2\lambda^{(t)}, \quad \text{symmetry} \end{aligned} \tag{4.1}$$

where $\lambda^{(t)} = +1, -1$.

Again, we replicate the experiment in Spencer and Chase (1996). In Figure 6a (positive $h$-values), we see that the higher nodal distance causes weaker results—that is, the nodal distance and relatedness have a reverse relation—compare 0.97 for $AB$ with nodal distance zero to 0.83 for $AG$ with nodal distance five. In Figure 6b, we update the $h$-values using equation 4.1.




(a) Testing order reported in Table 5.

(b) Testing order reported in Table 5.

(c) Testing order reported in Table 6.

(d) Testing order reported in Table 6.

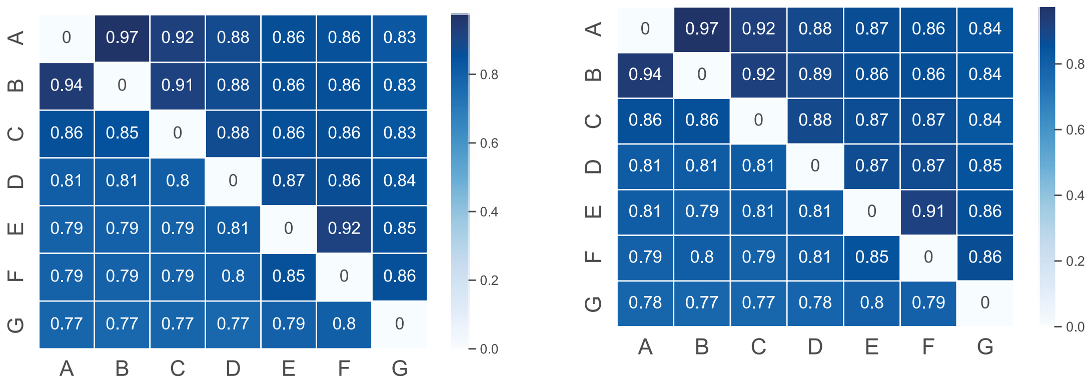
(e) Testing order is a mixture of all relations.

(f) Testing order is a mixture of all relations.

Figure 5: Simulation results for experiment 4, the replication of the Spencer and Chase (1996) experiment when $\gamma = 0.005$, $K_1 = 5$, $K_2 = 2$, $K_3 = 2$, $K_4 = 1$.

Figure 6 shows that in the case of using softmax, both proposed strategies for updating the $h$-values work well.

**4.2 Case 2: Different Deliberation Length (Nodal Distance Effect).** One of the scenarios in the PS model (Briegel & De las Cuevas, 2012) is to have different deliberation times, where $D = 0$ means direct edges,




(a) The results for Experiment 5 when $\gamma = 0.001$, $\beta = 0.02$, $K_1 = 5$, $K_2 = 4$, $K_3 = 2.5$, $K_4 = 2$, when $h$-values are positive.

(b) The results for Experiment 5 when $\gamma = 0.001$, $\beta = 0.02$, $K_1 = 5$, $K_2 = 4$, when $h$-values could get negative values.

Figure 6: Simulation results of study (Spencer & Chase, 1996) using the softmax function to calculate the transition probabilities.

as we have in baseline and symmetry relations, and $D \geq 1$ for sequences with $D$ clips between the percept clip and action clip. Then, after activation of the percept clip, the agent can either go directly to an action clip (called direct) or reach an action clip after some intermediate clips (called compositional)—(the detailed account of updating connection weights ($h$-values) is in Briegel and De las Cuevas, 2012). We slightly twist the concept in order to use it in the testing phase of the EPS model. The deliberation length could be the counterpart for nodal distance in equivalence literature.

In this scenario, during the testing phase and whenever there is no edge between sample stimulus and comparison stimuli, the agent acts as follows:

- Similar to the training phase, if there is no connection between percept and action clips, the agent establishes direct edges and initializes them with $h_0$.
- A memory sharpness parameter, $0 \leq \theta \leq 1$, could control transitivity. It quantifies how much the agent uses the memory, that is, navigates through the memory clips and reaches an action indirectly. The more intact the memory, the higher is the value of $\theta$, and the less intact the memory, the smaller the value of $\theta$.
- In the PS model, an action is chosen through either direct connection or compositional clips; the direct connection will be rewarded, so the chance to go for direct connections will increase. However, we do not have any reward in this stage and alternate between $D = 0$ and $D \geq 1$ using $\theta$. What we do here is perform a two-factor selection. First, either $D = 0$ or $D \geq 1$ is chosen based on the binomial probability with $p = \theta$; then the action will be chosen based on the uniform probabilities ($D = 0$ or no memory) or the max-product scenario.

- The real probabilities can be simply expressed as a biased sum of the two probabilities:[17]

$$P = \theta P_{D\geq 1} + (1-\theta)P_{D=0}. \tag{4.2}$$

As we have mentioned, the case 1 scenario (see section 4.1) is a special case of the scenario proposed here. If the memory sharpness factor achieves its maximum value, $\theta = 1$, the direct connections and $D = 0$ have no effect on the chosen option. The reason for differentiation between the forgetting factor and memory sharpness is that in reality, one might not be able to derive new relations even though direct relations are not forgotten.

Since $\theta$ is expected to somehow control the nodal effect, it could be defined as a function of $D$. Otherwise, it affects various transitivity and equivalence relations in the same manner without taking the nodal distance into account. This nodal effect could be fulfilled in several ways. For instance, an effective memory sharpness, say $\theta'$, can be defined as a function of both $D$ and $\theta$. In this way, we have the ordinary forgetting factor ($\gamma$), general memory sharpness ($\theta$) that relates to usage of memory and transitivity in general, and effective memory sharpness ($\theta'$), which is a sort of memory sharpness under the influence of nodal distance.

An effective memory sharpness definition could be $\theta' = \theta - D(\gamma')$, where $\theta$ is a fixed value, which has already been described. In this case, in order to have $\theta' \geq 0$, we need $\gamma' \leq \frac{\theta}{D}$.

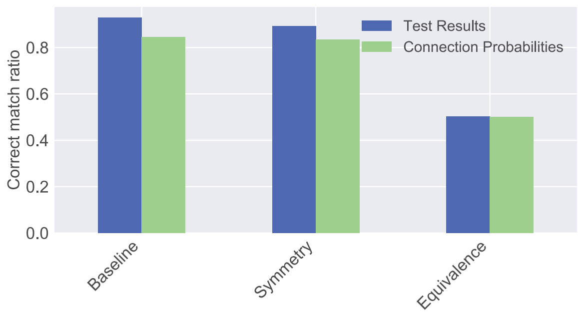
A method similar to the power-law model of psychological Memory can be used as well (Donkin & Nosofsky, 2012)—say,

$$\theta' = \theta D^{-\gamma'}, \text{ for } D \geq 1$$

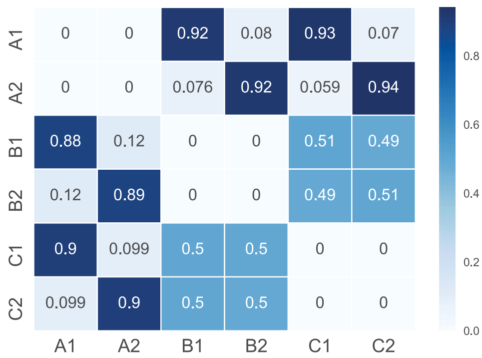
where $0 \leq \gamma' \leq 1$ and the larger the $\gamma'$, the more intense the nodal effect. In the following and for simplicity, we use the memory sharpness term and $\theta$ symbol for effective memory sharpness as well, unless it is ambiguous.

*4.2.1 Experiment 6: Devany et al. (1986) in Case 2 Setting.* Here, similar to experiment 3, the results for replication of the experiment in Devany et al. (1986) is presented. In experiment 3, nonformation of symmetry relations causes nonformation of equivalence relations. We show that nonformation of transitivity relations can result in the same case.

[17] One might look at this as the effect of memory sharpness on the $h$-values. Instead of initializing the direct $h$-values with $h_0$, they might be initialized with a value $K_\theta$ where a smaller $\theta$ is proportional to a larger value of $K_\theta$ (lowering the memory impact and indirect paths). Likewise, a bigger $\theta$ is proportional to a smaller value of $K_\theta$, to scale in favor of using memory and longer paths. Then the outgoing probabilities will be computed in a similar way as PS.




(a) Trained and derived results




(b) Results based on pairs

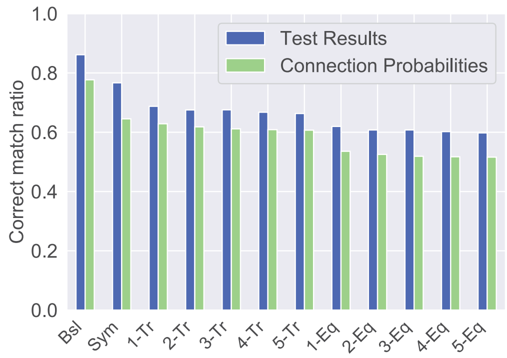
Figure 7: The results for experiment 6: replication of Devany et al. (1986) when $\gamma = 0.01$, $K_1 = 1$, $K_2 = 0.9$, $K_3 = 0.5$, $K_4 = 0.45$, and $\theta = 0.5$.

The training order is presented in Table 3. The test results and the transition probabilities of the network at the end of experiment are presented in Figure 7. As it can be seen from Figure 7, symmetry relations are derived, but transitivity relations are not formed properly. The ratio of the baseline relations is about 0.93, the ratio of the symmetry relations for $BA$ and $CA$ is about 0.9, and the $BC$ and $CB$ relations ratio is about 0.5.

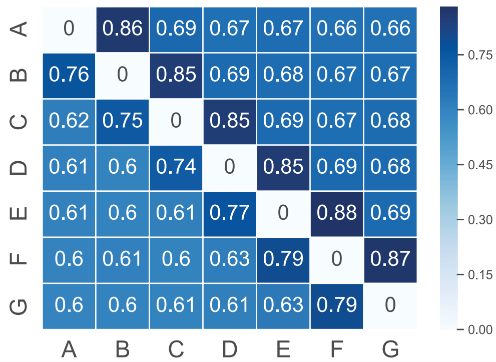
Therefore, in the EPS model, the formation of equivalence relations is a consequence of the formation of both symmetry and transitivity relations.

*4.2.2 Experiment 7: Spencer and Chase (1996) in Case 2 Setting.* We replicate the experiment in Spencer and Chase (1996) to address relatedness (or nodal distance) using memory sharpness. The training order is presented in Table 4; the testing phase is a mixture of all relations. In Figures 8a and 8b, the memory sharpness is fixed to $\theta = 0.7$. In Figures 8c and 8d, the memory sharpness is adjusted in a linear form ($\theta = 0.7 - D(0.1)$), and finally, in Figures 8e and 8f, the memory sharpness is adjusted in a power law form ($\theta = 0.7 \times D^{(-0.8)}$). Comparing the three cases, we observe that in the case of fixed memory sharpness, the indirect relations are influenced in the same way. This can be seen if we compare $AC$ relations with ratio 0.69 to $AG$ relations with ratio 0.66 in Figure 8b. Through adjusting the scenarios, we can model the nodal effect better (see Figures 8c and 8e and compare them to Figure 8a). In Figure 8d, compare the ratio for $AC$, which is 0.63, to the ratio for $AG$, which is 0.42. This rate of changes with nodal distance is much more than fixed memory sharpness. The same comparison in Figure 8f gives 0.69 and 0.43 for nodal distance one at $AC$ and nodal distance five at $AG$ respectively.

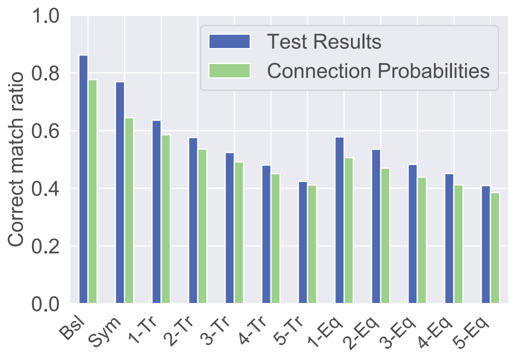
Although the model with memory sharpness (see case 2) is more complex due to extra parameters, it seems that using an adjusting memory



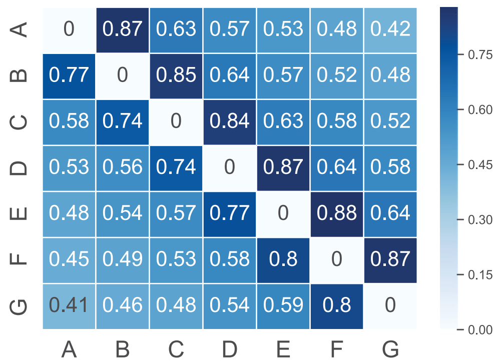

(a) Perceived results by the environment when memory sharpness is fixed: $\theta = 0.7$.



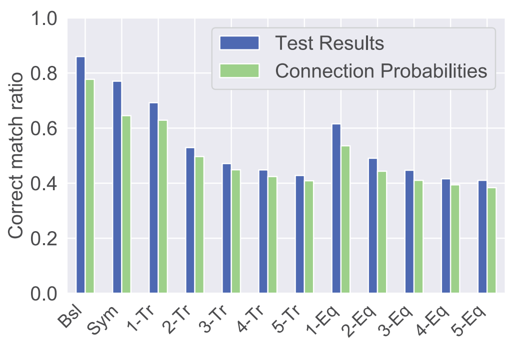

(b) Perceived results by the environment when memory sharpness is fixed: $\theta = 0.7$.



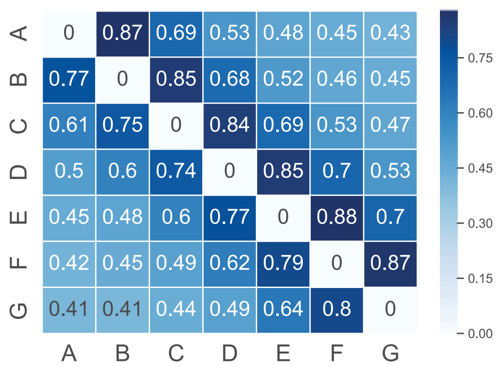

(c) Perceived results by the environment when memory sharpness is linearly adjusting with nodal distance: $\theta = 0.7 - D(0.1)$.




(d) Perceived results by the environment when memory sharpness is linearly adjusting with nodal distance: $\theta = 0.7 - D(0.1)$.




(e) Perceived results by the environment when memory sharpness is adjusting with nodal distance using power law: $\theta = 0.7 \times D^{(-0.8)}$.




(f) Perceived results by the environment when memory sharpness is adjusting with nodal distance using power law: $\theta = 0.7 \times D^{(-0.8)}$.

Figure 8: Simulation results for experiment 7, a replication of a study in Spencer and Chase (1996), when $\gamma = 0.005$, $K_1 = 5$, $K_2 = 2$, $K_3 = 2$, and $K_4 = 1$ with different memory sharpness values.

sharpness could control the nodal distance, and case 2 sounds more promising.

**4.3 Case 3: Action Set as the Set of Absorbing States.** In the standard PS model, an action is coupled out whenever the relevant action clip is reached. If a unit transition probability is assigned from each action clip to itself, then the action clips will be absorbing states of the Markov chain (or memory clip network). Briefly, in an absorbing Markov chain, it is impossible to leave some states once visited. Those states are called absorbing states. Moreover, any state has a path to reach such a state. The nonabsorbing states in an absorbing Markov chain are transient. In our equivalence PS, since a clip can be used as percept clip and action clip interchangeably, the network does not have absorbing states in its general form. However, for simplicity, at each trial where the agent perceives the percept and the action set, we ignore the output connections and assign a unit transition probability for the clips in the action clip. As a result, the clips in the action set temporarily become the absorbing states.

This way, instead of using transition probabilities, we consider the probability of being absorbed by an action clip in $\mathcal{A}_t$, starting at percept stimulus. This is closer to the logic of PS memory clip and the random walk.[18]

If the size of nonabsorbing or transient clips in the network is $n_t$ and the number of absorbing states is $m = |\mathcal{A}_t|$, the transition matrix of the network can be written as

$$P = \begin{pmatrix} Q & R \\ \mathbf{0} & I_m \end{pmatrix},$$

where $Q$ is an $n_t \times n_t$ matrix, $R$ is an $n_t \times m$ matrix, $\mathbf{0}$ is the $m \times n_t$ zero matrix, and $I_m$ is the identity matrix of size $m \times m$. The fundamental matrix is defined as

$$N = (I_{n_t} - Q)^{-1} = \sum_{k=0}^{\infty} Q^k.$$

If one starts at clip $i$, the expected number of nodes before entering an action clip is the $i^{\text{th}}$ component of the vector $N\mathbf{1}$. This could be used to address the answering time that is mentioned in Spencer and Chase (1996). The

[18] One might bias the random walk according to the action set, since it would be different if the actions are present simultaneously or are given with a delay (in the delay case, the random walk will start without any bias from actions presence). In other words, the presence of the action set plays a reinforcing role. One possibility is to consider a parameter similar to memory sharpness that controls the effect of the action set.

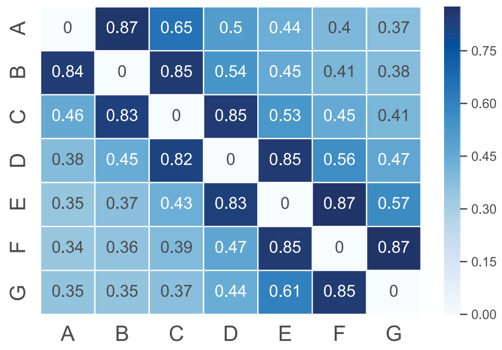
probability starting at $i$ and ending at absorbing state $j$ is the $(i, j)^{\text{th}}$ entry of matrix $M = NR$.

Note that as mentioned in the original PS model, it is possible that the random walk on the clip space falls in a loop and, for instance, goes back and forth between two clips that have a high transition probability to each other. As we will see in the simulation, this results into larger expected steps. However, various mechanisms could control this undesired situation. A method from Briegel and De las Cuevas (2012) is to put a limitation on the random walk time, called the maximum deliberation time, $D_{\max}$. If the agent could not manage to reach an action before $D_{\max}$, whatever the ultimate action would be, it will not be rewarded. But since we are using the random walk for the testing phase, this is not applicable. Even if we use the absorbing Markov chain to compute the probabilities during the training, instead of just relying on direct connections, $D_{\max}$ is not a compatible strategy with real experiments since in the standard SE protocols, too much delay does not have the penalty of not receiving feedback from experimenter (here, the environment). One might use the concept of gating in the model, which is used, for instance, in long short-term memories (Hochreiter & Schmidhuber, 1997) or a kind of local emotional tags similar to PS. Another option to avoid revisiting clips could be self-avoiding walks (SAWs).[19]

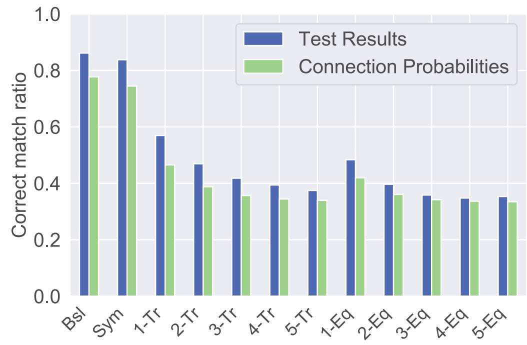
*4.3.1 Experiment 8: Spencer and Chase (1996) in Case 3 Setting (Absorbing States).* We replicate the experiment in Spencer and Chase (1996) to address the relatedness in absorbing state setting. The training order is presented in Table 4. Note that in this experiment, we use a second measurement of nodal distance, which is the expected number of transitions between the sample stimulus and an action.

Figures 8a and 8b show that computing probabilities in an absorbing Markov chain setting has the capability of showing a sort of nodal effect. Compare $AB$ with 0.87 to $AG$ with 0.37. Figure 8c shows that in general, greater nodal distance causes higher expected steps. However, based on the results, nodal distance is not the only factor affecting the number of expected steps, and the probability distribution plays a stronger role. First, we note that for $AB$ and $GF$, the expected number of steps is 1. That's because $A$ and $G$ are located at the two end sides of the learning series. The expected number of steps for $BA$ with zero nodal distance is around 23, which is more than the expected number of steps for $BE$ with nodal distance two (around 12). So we observe that nodal distance is not the only effect; the input and output probabilities and the location of a category in the learning order are also important. Compare $BA$ and $FE$, where both are symmetry relations and one node away from one end of the series, but the expected number of steps for $FE$ is around 4, which is many fewer than 23 steps of $BA$, so

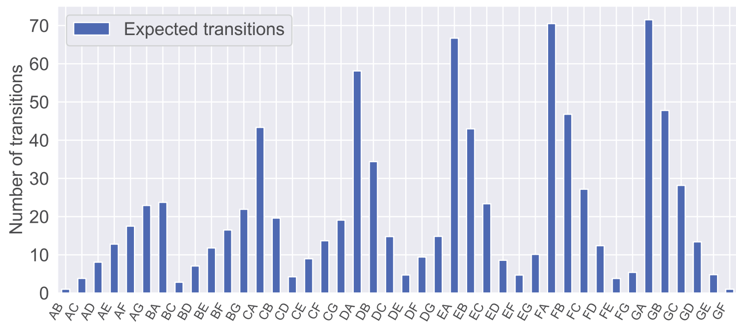
[19] Note that unlike the random walk, the SAW is not a Markovian stochastic process.




(a) The ratio of correct matches perceived by environment.




(b) The results based on nodal distance.




(c) The expected number of steps between the sample stimulus and a match in comparison stimuli.

Figure 9: Simulation results of the Spencer and Chase (1996) study using absorbing Markov chain (see experiment 8) when $\gamma = 0.005$, $K_1 = 5$, $K_2 = 4$, $K_3 = 2.5$, and $K_4 = 2$.

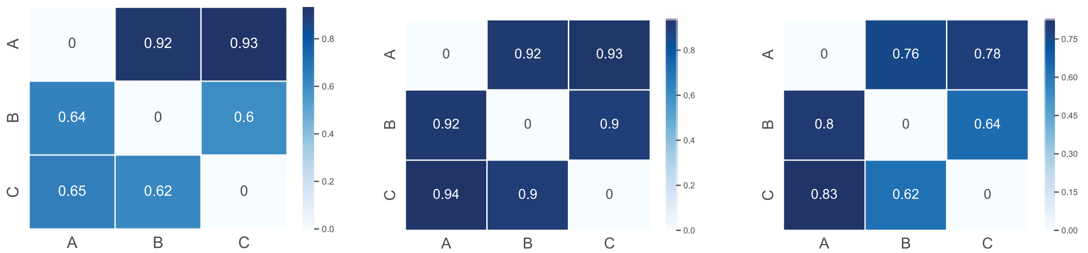
the general form of the network must be taken into account as there are many studies on differences of LS, OTM, and MTO training structures (see Arntzen, Grondahl et al., 2010; Arntzen & Hansen, 2011; Arntzen, 2012).

In Figure 9c, we see that the expected number of steps is higher than the shortest path that shows back-and-forth transitions between the clips.

**4.4 Experiment 9: Devany et al. (1986) with a New Training Setting.** Computational models can be used to gain insight, build hypotheses, make predictions, and formulate questions that lead in new directions for empirical research. Experiment 9 could be considered a hypothetical experiment to see how the proposed EPS model can interact with practical experiments. The question is whether it is possible to gain better results in Devany et al. (1986) with the same number of trials but with a different training order. We propose a new training order that we present in Table 7, along with the original one in Devany et al. (1986).

Table 7: The Proposed Training Order in Experiment 9, an Alternative Training Order to Devany et al. (1986).

| Original Training | Suggested Training | Block Size |
|---|---|---|
| $A_1B_1$ | $A_1B_1$ | 10 |
| $A_2B_2$ | $\mathbf{B_2A_2}$ | 10 |
| $A_1B_1, A_2B_2$ | $A_1B_1, \mathbf{B_2A_2}$ | 10 |
| $A_1C_1$ | $A_1C_1$ | 10 |
| $A_2C_2$ | $\mathbf{C_2A_2}$ | 10 |
| $A_1C_1, A_2C_2$ | $A_1C_1, \mathbf{C_2A_2}$ | 10 |
| $A_1B_1, A_2B_2, A_1C_1, A_2C_2$ | $A_1B_1, \mathbf{B_2A_2}, A_1C_1, \mathbf{C_2A_2}$ | 8 |




(a) Results with original training order, when $\theta = 1$

(b) Results with new training order, when $\theta = 1$

(c) Results with new training, when $\theta = 0.5$

Figure 10: The results for experiment 9 when we use similar parameters as experiment 3—$\gamma = 0.01$, $K_1 = 1$, $K_2 = 0.1$, $K_3 = 0.2$, and $K_4 = 0.05$—but a different memory sharpness in panel c.

Throughout experiment 9, we suggest that if one chooses a mixture of trials between the given categories, the symmetry relations will be stronger, and, as a consequence, the equivalence relations might be formed. Intuitively, by reinforcing one of the pairs (say, $A_1B_1$), the other one will be inhibited, $A_1B_2$, and so $A_2B_2$ gets a higher chance to be selected. The idea is to train $B_2A_2$, which is not formed well (with the chosen parameter values), instead of training $A_2B_2$, which might be derived more easily. The same argument shows that $B_2A_2$ training accelerates deriving $B_1A_1$.

A comparison between Figures 10a and 10b, shows that the agent with similar parameters and training time can achieve better results in $BC$ and $CB$ relations (compare 0.6 in Figure 10a for these relations to 0.9 in Figure 10b). Based on these results, EPS could suggest that experimenters consider a different combination of trials in the baseline training blocks.

To complete the circle, suppose an experimenter tests this hypothesis in practice and observes that the equivalence relations still are not formed. This means that the problem in equivalence formation does not emanate solely from symmetry relation formation, but maybe from the transitivity formation that we referred to in experiment 6. In Figure 10c, we put $\theta = 0.5$

in order to model the new results. This time, even though the symmetry relations are formed well, the equivalence relations are not formed (around 0.64). So first we study the effect of a new training procedure in this experiment and then emphasize the fact that equivalence relation formation in the EPS model is based on both symmetry and transitivity relations. In experiment 3, the lack of strong symmetry relations results in weak equivalence relations (see Figure 10a). We suggest a possible solution by redesigning the training setting in experiment 9. However, if the transitivity relations are not formed similar to experiment 6, equivalence relations cannot be derived as well (see Figure 10c).

Experiment 9 is an example of the possibility of generating and vetting an idea in equivalence theory prior to full experimental testing. Note that to study a behavior, the most important thing is to tune the parameters of the model and then use them to study new settings.

## 5 Conclusion

Although computational models of cognition and behavior are simplified versions of brain activity, they might be a useful tool to study brain activity and analyze experimental data. In this regard, the model must be interpretable and biologically plausible so that psychologists can rely on it.

In our study, we propose a machine learning scheme for modeling the equivalence formation. To the best of our knowledge, it is the first study that approaches computational modeling in stimulus equivalence through machine learning. We consider a specific reinforcement learning model, projective simulation, as the foundation of our model, since we found this model flexible and adaptable to equivalence class formation. The model has an internal episodic memory that could easily be interpreted and extended to replicate various stimulus equivalence experimental settings. Our proposed model is not a black box model, which makes it more appropriate for researchers in behavior analysis to accept and apply it.

As discussed in the simulation results, the model can control various factors such as learning rate, forgetting rate, symmetry, and transitivity formation. Nodal effect, an important topic in equivalence formation, is simulated and explained with EPS. Through simulation of some real experiments in the behavior analysis literature, we display the model capability to behave like typical participants or participants with special disabilities. Moreover, we show how a research idea in equivalence theory can be studied through EPS.

The proposed simulations can be considered a proof of concept, but studying the parameters, optimal tuning for a specific behavior, and comparing the proposed calculation of probabilities require separate study. For instance, one might tune different parameters to model a specific behavior in MTS trials or study the optimal number of members and categories, comparing LS, OTM, and MTO, and so on. Using a softmax function to calculate probabilities in an absorbing Markov chain model, as well as adding a memory sharpness effect, are straightforward steps. Furthermore, it is possible to add direct edges, initialize them (with $h_0$ or an adjusting $h$-value that is proportional to other output $h$-values), and then compute the absorbing probabilities. Alternative options are using emotional tags, gating, or self-avoiding random walks. The main advantage of using PS as the foundation model is that it is quite flexible and easy to interpret. It can be modified to address other types of training procedures, such as compound stimuli, instead of MTS. A possible approach for modeling compound stimuli is to use the generalized projective simulation (Melnikov et al., 2017) that considers clips composed of different categories. The EPS model can be considered an extension of the PS model that might be interesting solely from a machine learning point of view. For instance, symmetry connections and variable action sets could be used in more general applications. Overall, we believe that the PS framework in general, and the introduced EPS model specifically, could be a powerful and flexible tool for computational modeling in equivalence theory that has many advantages over the existing connectionist models.

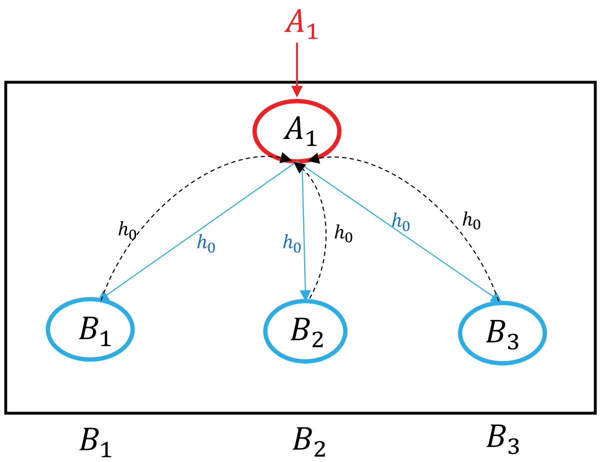
## Appendix A: A Detailed Example on How the Model Works

We explain an experiment through modeling protocol 1. First, one of $A_1, A_2$, and $A_3$ is chosen with probability $P^{(t)}(s) = 1/3$ to be shown as the sample stimulus, where the comparison stimuli (or actions) will be $B_1$, $B_2$, and $B_3$. Hereafter, for simplicity, we use the same notations for actual stimuli and the remembered clips of the stimuli, say, $A_1 = \mathcal{I}(A_1)$, unless this would be ambiguous. In Figures 11 to 15, the inside of the rectangle shows the agent memory (clip network), and the outside shows the environment and actual stimuli. Moreover, red is used for the sample stimuli and its internal clip at current trial, while blue is used for the comparison stimuli at the same trial. Solid links are used for baseline relations, and dashed links represent symmetry links.

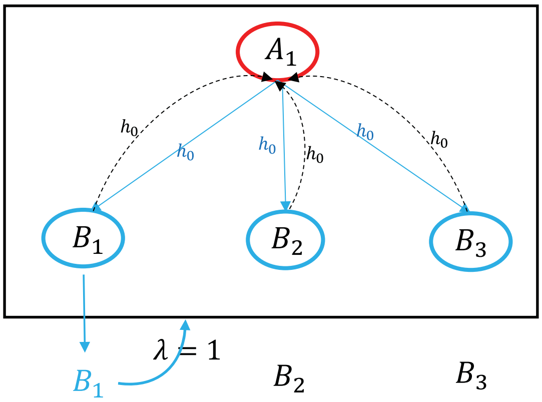
• Consider that at time $t = 1$, sample stimulus $A_1$ is presented to the agent, so $A_1$ is added to the percept set: $\mathcal{S} = \mathcal{S} \cup \{A_1\}$. Also, a memory clip representing $A_1$ is created and added to the memory space: $\mathcal{C} = \mathcal{C} \cup \{A_1\} = \{A_1\}$.

• Based on the learning protocol, the set of comparison stimuli $B_1, B_2, B_3$ will appear after a 1 s delay.[20] Then three memory clips for $B_1$, $B_2$, and $B_3$

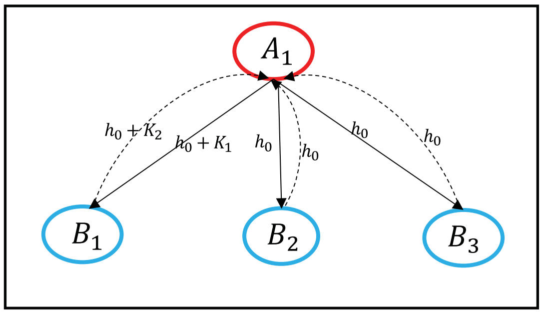
[20] When the relation $A_1B_1$ is the desired relation to be reinforced, the comparison stimuli are chosen from the $B$ category. The number of them could be different, but there will be at least two. In this case, each category contains three members, so we just have one option of three-member comparison, which the location of the stimuli shown does not take into account. If there are more members in categories, we have more options.



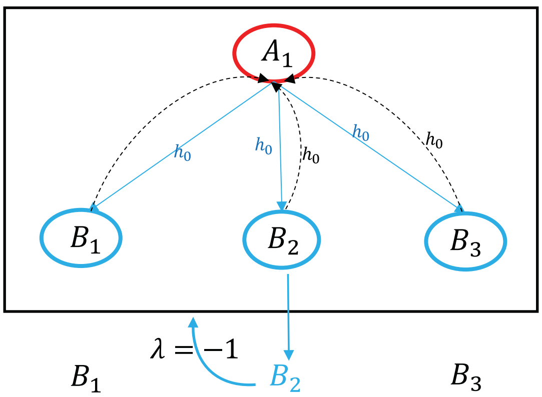

(a) The first stimulus sample $A_1$, followed by the three comparison samples $B_1, B_2, B_3$ (outside the rectangle). The clips are added to the memory, and initialized with $h_0$ (inside the rectangle).



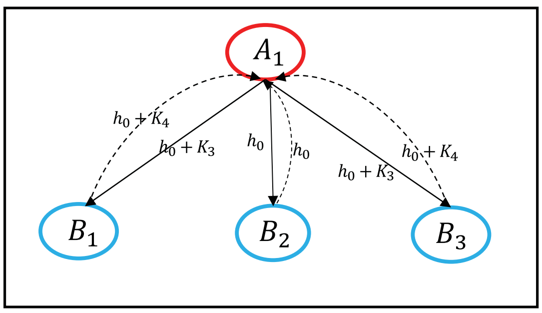

(b) The correct pair is chosen, so agent receives positive reward $\lambda = 1$.




(c) The $h$-value for $A_1B_1$ connection will be added by $K_1$ and as a symmetry effect the $h$-value of $B_1A_1$ will be added by $K_2$, where $K_2 \leq K_1$.




(d) If a wrong option is chosen, negative reward $\lambda = -1$ will be return to the agent.




(e) Negative reward, will amplify other options. So $h$-values for $A_1B_1$ and $A_1B_3$ will be added by $K_3$. $h$-values for symmetry connection, i.e. $B_1A_1$ and $B_3A_1$ will be added by $K_4$.

Figure 11: The first trial for $A_1B_1$ through positive and negative rewards at time step $t = 1$. The agent creates clips for all the perceived stimuli (a) and updates the connection weights based on the environment feedback. The updating rule in positive reward (b,c) and negative reward (d,e), is presented. The percept clip (sample stimulus) is shown in red and the action clips (comparison stimuli) in blue.

are created by the agent and added to the $\mathcal{C}$ space. The action space now has three members as well, so $\mathcal{A} = \mathcal{A} \cup \mathcal{A}_1 = \{B_1, B_2, B_3\}$.

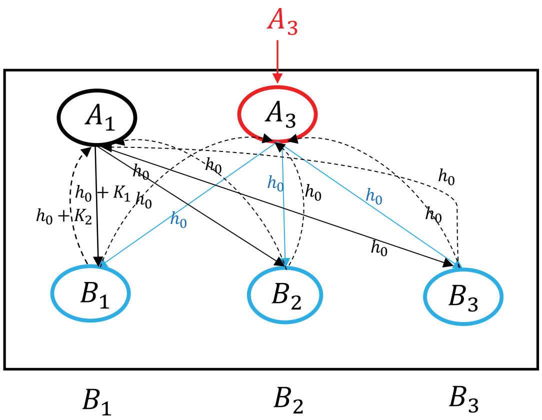
• The new connections and $h$-values must be initiated, since this sample stimulus and comparison stimuli are presented for the first time. At this stage, six edges will be established. Their initial $h$-values are $h_0$: $h^{(1)}(A_1, B_1) = h^{(1)}(A_1, B_2) = h^{(1)}(A_1, B_3) = h_0$, and $h^{(1)}(B_1, A_1) = h^{(1)}(B_2, A_1) = h^{(1)}(B_3, A_1) = h_0$. As a result, the conditional probability distribution $\{p^{(1)}(a|s)\}_{\mathcal{A}_1}$ is uniform for all possible actions in the memory space (see Figure 11a).

• Consider that the agent chooses $a^{(1)}$ where:

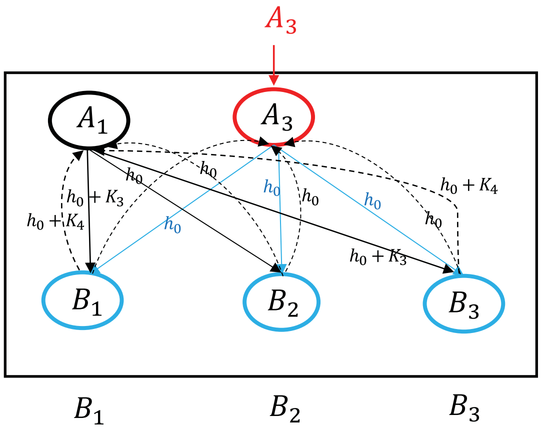
1. $a^{(1)} = B_1$, that is, the agent chooses the correct option, which must be reinforced by $\lambda = 1$. In this case, $h^{(2)}(A_1, B_1)$ will be increased by $K_1$ due to equation 3.4. $K_1$ is set to one based on PS. Moreover, we expect that strengthening $A_1B_1$, affects the formation of $B_1A_1$ (symmetry relation in SE), so $h^{(2)}(B_1, A_1) = h_0 + K_2$. Other transitions remain unchanged (i.e., equal to $h_0$). See Figures 11b and 11c.
2. $a^{(1)} = B_2$, that is, the agent chooses a wrong option (exactly the same for $a^{(1)} = B_3$ at this stage), so $\lambda = -1$. This negative reward reinforces other options, but not the negatively rewarded one (see Figures 11d and 11e). In this example, the transition weight from clip $A_1$ to clips $B_1$ and $B_3$ will be increased by $K_3$ where $K_3 \leq \frac{K_1}{2}$ (see equation 3.6). The symmetry updates are similar: the transition weight from clip $B_2$ to clip $A_1$ will not change, and the transition weights from clips $B_1$ and $B_3$ to clip $A_1$ will be increased by an additive factor $K_4$, where $0 < K_4 \leq \frac{K_2}{2}$, due to equation 3.7.

• Let $t = 2$, and the sample stimulus be $A_3$, while the comparison stimuli are the same as in the previous time step, $\mathcal{A}_2 = \mathcal{A}_1$, so no new action is added to the action space $\mathcal{A}$. $A_3$ is added to the percept space, now $\mathcal{S} = \mathcal{S} \cup \{A_3\} = \{A_1, A_3\}$. Note that the percept and action spaces are not shown in the figures and that we depict only how an agent updates its memory clips during training. Since the trial setting is new, all the transitions will be established, initialized, and updated like the previous time step. This is similar to the first time that the $A_2B_2$ pair is supposed to be learned. See the clip network $\mathcal{C}$ (inside the rectangle) in Figures 12a and Figure 12b when clip $A_3$ is added.

• Now consider that the experiment repeated the trials until all the desired $AB$ relations are trained and 90% of agent's choices within the last 30 trials are correct (see Figure 13a for a schematic representation). The thick solid links between $A_1B_1$, $A_2B_2$, and $A_3B_3$ show the mastery of these baseline relations. The thick dashed links show symmetry formation. The weak links illustrate that although the agent is well trained, there is still a low chance of a wrong choice in an MTS trial. Based on protocol 1, the environment trains $BC$ relation next.




(a) A new sample stimulus $A_3$ is presented by environment, and an agent subsequently creates a new clip and initializes its connections to the action clips with $h_0$. This network is based on a correct choice in the first trial (Figure 11b).



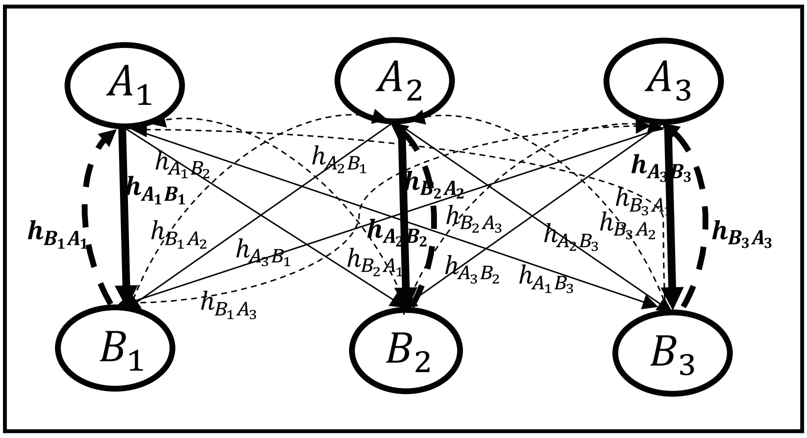

(b) A new sample stimulus $A_3$ presented by environment, and an agent subsequently creates a new clip and initializes its connections to the action clips with $h_0$. This network is based on a wrong choice in the first trial (Figure 11d).

Figure 12: The second trial ($t = 2$) where $A_3$ is the sample stimulus. The agent creates a new clip for $A_3$ and updates the $h$-values based on the learning history, that is, if the network has been updated with a chosen correct pair (see Figure 11c) or with a chosen wrong pair (see Figure 11e). Only $h$-values between the current sample stimulus ($A_3$ in red) and comparison stimuli ($B_1, B_2, B_3$ in blue) will be updated at this trial.

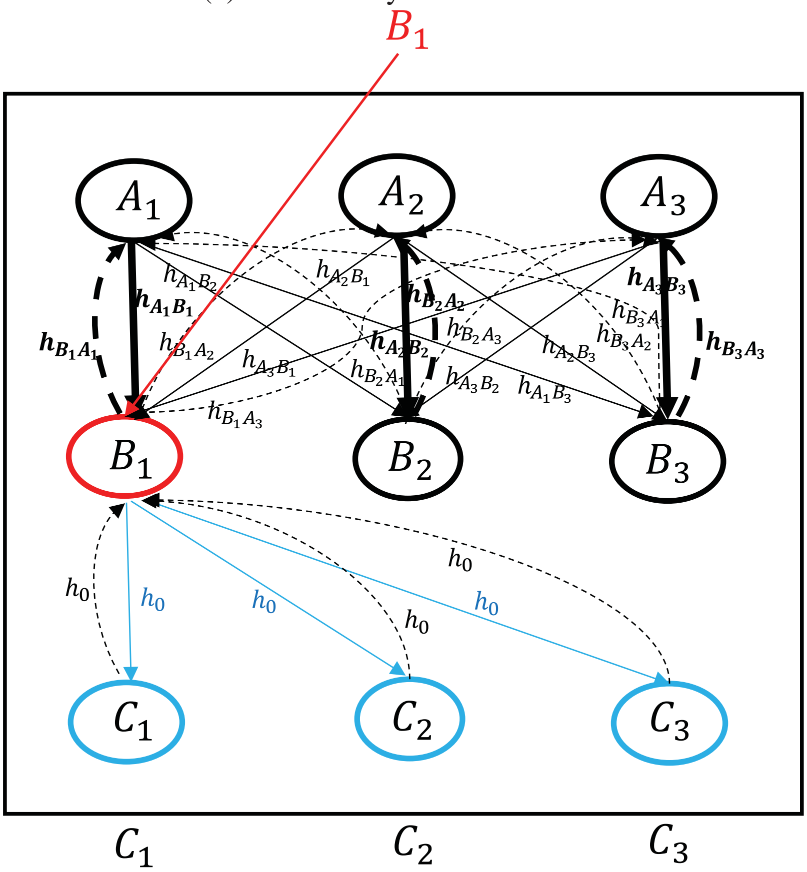
At time $t'$, let the sample stimulus be $B_1$ and the comparison stimuli be $\mathcal{A}_{t'} = \{C_1, C_2, C_3\}$ (see Figure 13b, outside the rectangle). At this point, the percept space is $\mathcal{S} = \mathcal{S} \cup \{B_1\} = \{A_1, A_2, A_3, B_1\}$; the action space would be $\mathcal{A} = \mathcal{A} \cup \mathcal{A}_{t'} = \{B_1, B_2, B_3, C_1, C_2, C_3\}$; and the clip space would be $\mathcal{C} = \{A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3\}$. For clip space representation, see Figure 13b, inside the rectangle. Note that clip $B_1$ in the agent memory represents both a percept clip and an action clip.

At time $t'$, three input and three output links will be established from $B_1$ and initialized with $h_0 = 1$ (see Figure 13b). The probabilities for all comparison stimuli are then uniform: $p^{(t')}(C_1|B_1) = p^{(t')}(C_2|B_1) = p^{(t')}(C_3|B_1) = 1/3$. Similar to the *AB* training step, the environment reinforces the desired relation, and by accomplishing this training phase, we expect a network like the one presented in Figure 14a. Thick solid links show the well-trained baseline relations, thick dashed relations represent the formation of symmetry relations, and weak links show a weak possibility for wrong option in MTS trials.

Suppose that *BC* relation is also trained and passed the criterion (see Figure 14a, thick connections). The final step in the training phase is the *DC* relation. Let $D_2$ be the sample stimulus at time $t''$ and $\mathcal{A}_{t''} = \{C_1, C_2, C_3\}$ (see Figure 14b, outside the rectangle). $D_2$ will be

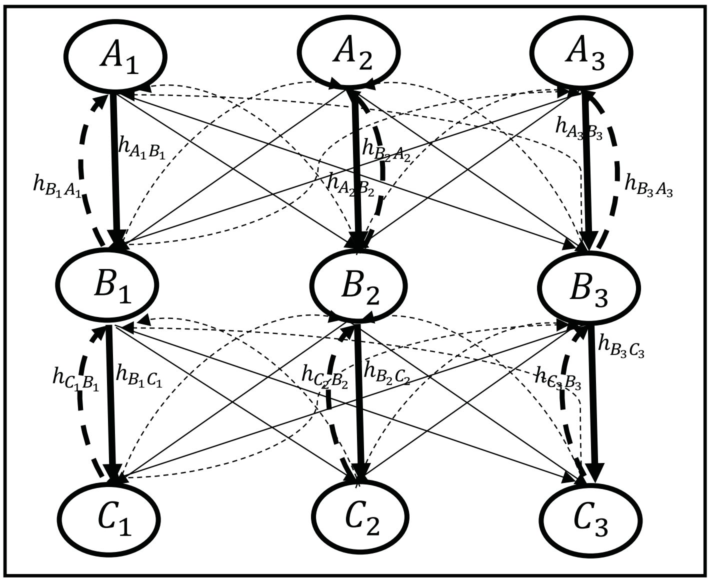



(a) The mastery of $AB$ relation.

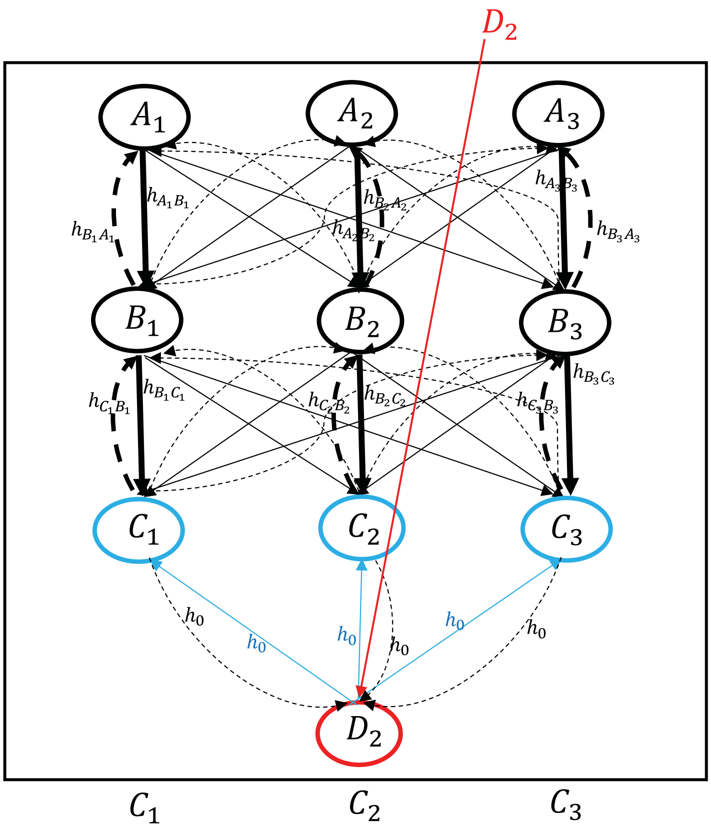



(b) $B_1$ as sample stimulus (outside the rectangle) activate the existed clip $B_1$ (in the rectangle, in red). $C_1$, $C_2$, and $C_3$ are the comparison stimuli (outside the rectangle). Agent creates new clips for them and initializes the links between percept clip $B_1$ and action clips $C_1$, $C_2$, and $C_3$ (inside the rectangle).

Figure 13: When the $AB$ relation is trained (a) and the $B$ category members appear as the sample stimulus, clips in the $B$ category will be activated as the percept clips (b) and $C_1$, $C_2$, and $C_3$ will be action clips.




(a) The mastery of $AB$ and $BC$ relations.




(b) $D_2$ as sample stimulus (outside the rectangle) makes the agent to create a new clip $D_2$ (in the rectangle, in red). $C_1$, $C_2$, and $C_3$ are the comparison stimuli (outside the rectangle). Agent does not create new clips for them, but only initializes the links between new clip $D_2$ and existed clips $C_1$, $C_2$, and $C_3$ (inside the rectangle).

Figure 14: When $AB$ and $BC$ relations are trained (a), and training the relations $DC$ is the next step. $D_2$ appears as the sample stimulus, and its connections are initialized to the $C$ category, which plays the action set role (b).

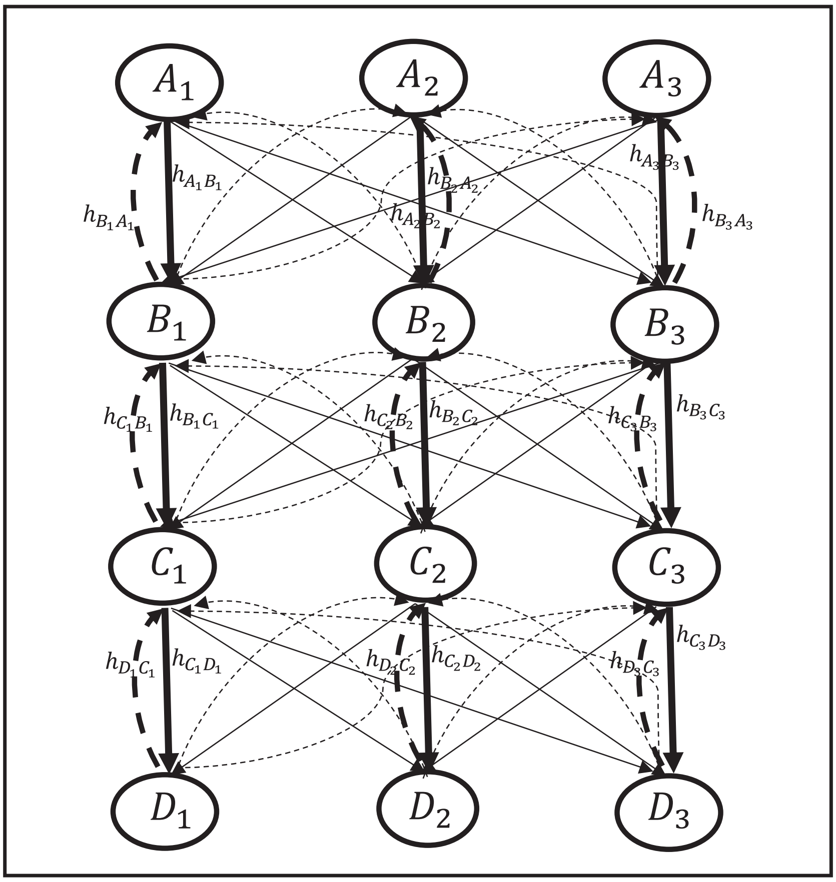



Figure 15: A representation of the memory clip network after the training phase. We show the symmetry connections with the dashed lines in order to clarify that they are not reinforced directly during the MTS procedure.

added to $\mathcal{S}$, but the action space does not change. A clip for $D_2$ would be added to $\mathcal{C}$, and the initial links will be established and initialized with $h_0$ (see Figure 14b, inside the rectangle). The first choice is uniformly selected with probability 1/3, but after enough MTS trials, the probability of the desired pair meets the criterion.

Figure 15 shows the memory network after a successful training phase, where thick connections are the trained relations and weak connections show the wrong unfavorable possible choices. For the testing phase, we can compute the agent's policy and see if the conditional probabilities for symmetry, transitivity, and equivalence relations are according to the protocol. If the desired one passes the criterion, we will say that the equivalence relations are formed for the agent. In the simulation part, section 4, we address the testing phase using different methods to compute probability distribution for an action set when the relation is a derived one. We replicate the testing phase similar to real experiments, along with computation of probabilities.

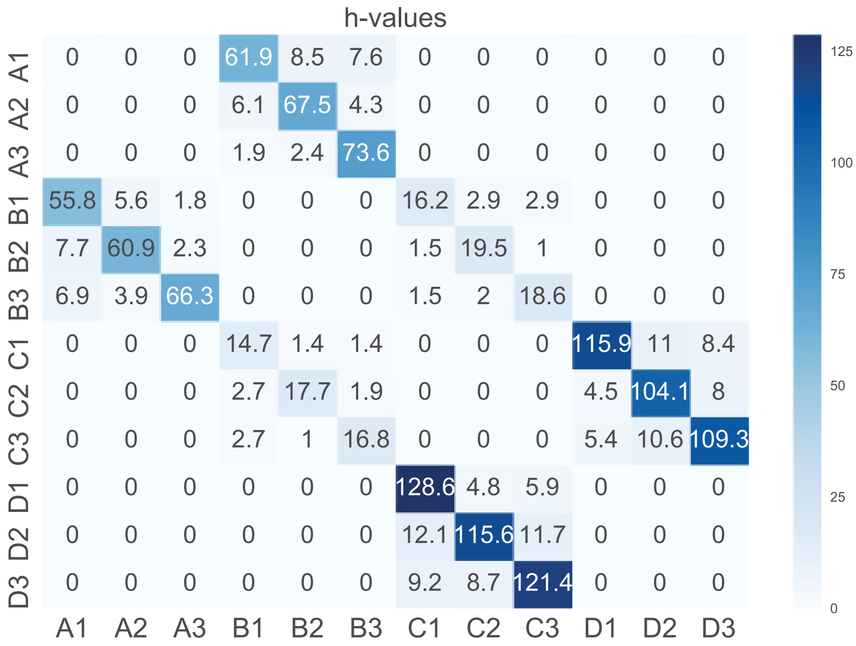
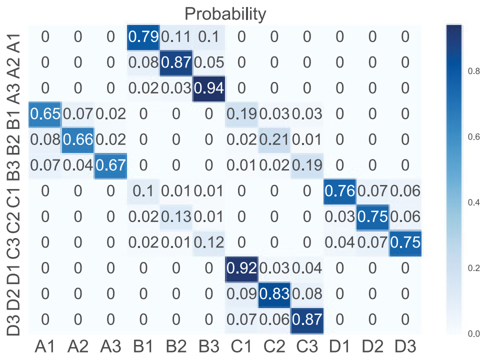



Figure 16: A sample configuration of network $h$-values after training $AB$, $BC$, and $DC$ based on protocol 1 when $\gamma = 0.0001$, $K_1 = 1$, $K_2 = 0.9$, $K_3 = 0.5$, and $K_4 = 0.45$.

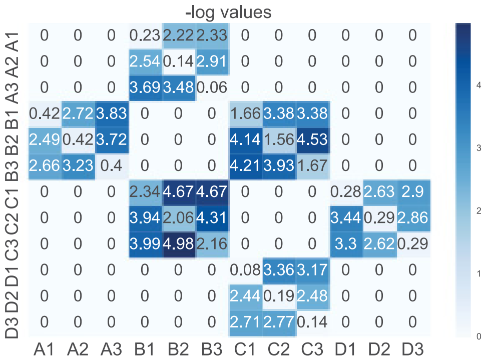
## Appendix B: Calculation of Probability Distribution over an Action Set with Max-Product

Computation of probabilities from $h$-values is an important phase, since the agent updates $h$-values but actions are taken based on probabilities. Two general methods used in the original PS and in EPS are the standard model and softmax model. They respectively use simple normalization and softmax function over $h$-values. However, this could be more challenging when the direct connections do not exist. In this case, one might consider other conditions that might change the computed probabilities. Here, we explain a few possibilities for computing probabilities in a max-product scenario, which we addressed in equation 3.8 and section 4.1.

In Figure 16, a sample structure of the agent's memory clip is presented where the $h$-values are positive and probabilities during training are computed by normalization of $h$-values. In Figure 16, we see that the range of $h$-values for different categories could be quite diverse. For instance, $h$-values between stimuli in category $D$ and $C$ are about six times bigger than $h$-values between stimuli in category $B$ and $C$. This means that the agent is selected more efficiently in $BC$ training trials and passes the criterion more quickly, but behaves less efficiently in $DC$ training trials and needs more blocks of training to meet the criterion. This will affect the probabilities, as represented in Figure 17.




(a) Transition probability of network in Figure 16, using normalization.



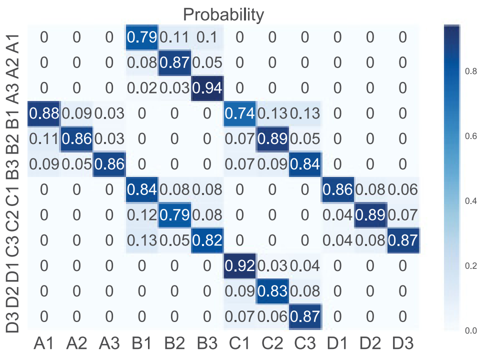

(b) Negative log values of the probabilities to convert max-product problem into a min-sum problem.

Figure 17: Transition probabilities and negative log of probabilities of the sample network in Figure 16.

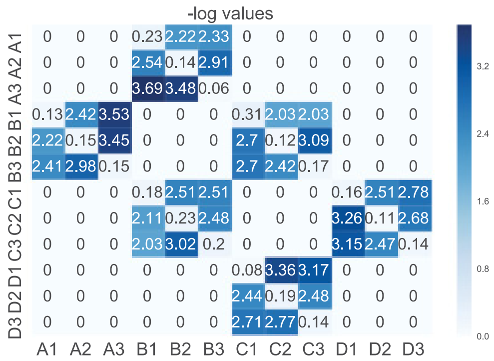
Table 8: Details of Computing Derived Probabilities from the Sample Network in Figure 16.

| Computation Method | $A_2C_3$ | $A_2C_1$ | $A_2C_2$ |
|---|---|---|---|
| No condition (Figure 17) | | | |
| Lowest-cost path | $A_2, B_2, C_2, D_2, C_3$ | $A_2, B_1, C_1$ | $A_2, B_2, C_2$ |
| Min-sum value | 4.4697 | 4.2016 | 1.7019 |
| Calculated probability | 0.0115 | 0.015 | 0.1823 |
| Normalized probability | 0.0549 | 0.0717 | 0.8734 |
| $h$-values | 1.0 | 1.3075 | 15.9229 |
| Category based (Figure 18) | | | |
| Lowest-cost path | $A_2, B_2, C_2, D_2, C_3$ | $A_2, B_2, C_2, B_1, C_1$ | $A_2, B_2, C_2$ |
| Min-sum value | 2.8554 | 2.6740 | 0.2625 |
| Calculated probability | 0.0575 | 0.069 | 0.7692 |
| Normalized probability | 0.0642 | 0.0770 | 0.8588 |
| $h$-values | 1.0 | 1.1989 | 13.3693 |
| Viterbi (Figure 19) | | | |
| Lowest-cost path | $A_2, B_3, C_3$ | $A_2, B_2, C_1$ | $A_2, B_2, C_2$ |
| Min-sum value | 3.0743 | 2.8471 | 0.2625 |
| Calculated probability | 0.0462 | 0.0580 | 0.7692 |
| Normalized probability | 0.0529 | 0.0664 | 0.8807 |
| $h$-values | 1.0 | 1.2551 | 16.6406 |

Suppose that the testing trial has $A_2$ as the sample stimulus and $C_3, C_1, C_2$ as the action stimuli. As reported in Table 8, the path with the lowest cost could pass through a category more than once, say, $A_2, B_2, C_2, D_2, C_3$. Note that the reported simulation results, which we referred to as Dijkstra's algorithm, are similar to Figure 17, that is, without any extra conditions.




(a) Transition probability of network in Figure 16, normalization based on category.



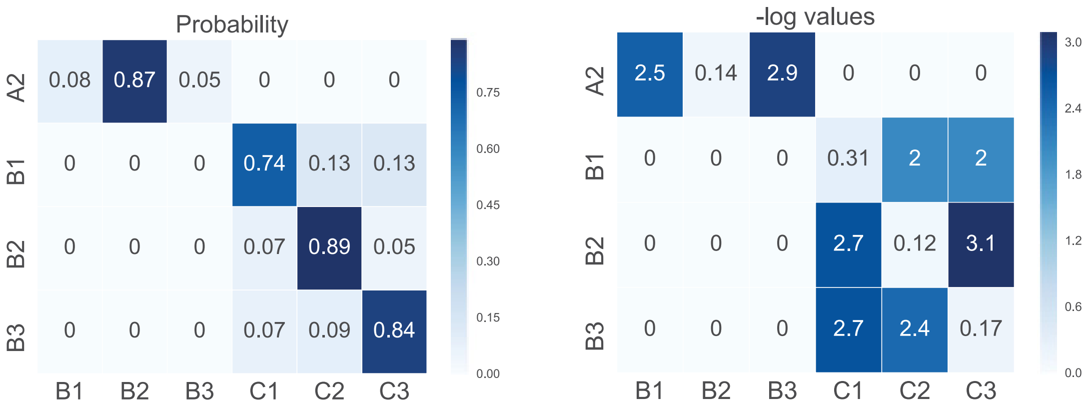

(b) Negative log values of the probabilities to convert max-product problem into a min-sum problem.

Figure 18: Transition probabilities and negative log of probabilities of the sample network in Figure 16 when category is taken into account.

One might argue that the probabilities must be marginalized based on the categories. In other words, the agent first targets a specific category, then, at the second level, chooses a member of that category. Therefore, the probability must be normalized for each category. In Figure 18, a category-based computation in which probabilities are marginalized is presented.

From Table 8, we observe that the calculated probability vector of category-based computation is higher than the previous case in general, but comparison of the normalized vectors shows that the probability of the correct choice, $A_2C_2$, in category-based computation is slightly less than its counterpart. The explanation is that in a category-based version, a multiplicative factor, which represents the probability of choosing each different category, is removed. This affects the probabilities and therefore produces different final distributions. Consider that if the $h$-values for different categories are in the same range, which is what we expect, this multiplicative factor would be the same for all the actions and the normalized probabilities would be the same. In the category-based calculation of probabilities, the lowest-cost path could pass through a category more than once, similar to the first case.

The third scenario, which we refer to as a Viterbi algorithm, avoids passing a category more than once and is based on a trellis diagram from the network. The diagram is an ordered graph from a starting point to the destination layer. The trellis diagram for EPS is configured for each trial and has $c_s$ as the starting point; the layers consist of members of passing categories, and the destination layer is $\mathcal{A}_t$. The strategy to find the passing categories from $c_s$ to $c_a$ is simply to find the shortest path from $c_s$ to $c_a$, keep the members of categories with at least a member in the path, and remove other nodes and edges that have the opposite direction. Figure 19 shows the probabilities on the trellis diagram and negative log values.




(a) Transition probability of network in Figure 16, normalization based on trellis diagram for percept and actions.

(b) Negative log values of the probabilities to convert max-product problem into a min-sum problem.

Figure 19: Transition probabilities and negative log of probabilities of the sample network in Figure 16 when a trellis diagram based on the trial is made first, before computing the probabilities.

The probability of a correct match, $A_2C_2$, from the Viterbi scenario, is slightly higher than the previous methods. The explanation is that by removing some edges and not allowing passage through a category twice, the lowest-cost path in wrong options might be removed.

After finding the probabilities for all the possible actions $a \in \mathcal{A}_t$, we can compute $h$-values of the connections using equation B.1.

$$h^{(t)}(c_s, c_a) = \frac{p^{(t)}(c_a|c_s)}{p_{\min}} h_0, \tag{B.1}$$

where $p_{\min}$ is the minimum of achieved probability where we set its $h$-value equal to $h_0$. Note that if we use the softmax function to compute probabilities, converting probabilities to the $h$-values is through equation B.2,

$$\begin{aligned} \mathbf{h}^{(t)}(c_s, \mathcal{A}_t) = \frac{1}{\beta} \left[ \log(p^{(t)}(c_{a_1}|c_s)) \cdots \log(p^{(t)}(c_{a_m}|c_s)) \right] \\ - \left[ \mathbf{h}_{\min} + h_0, \cdots \mathbf{h}_{\min} + h_0 \right], \end{aligned} \tag{B.2}$$

where $\mathbf{h}_{\min}$ is the minimum value of the computed $h$-values, which is used to put the minimum value of $h$-values to $h_0$.

It is worth mentioning that the final results in the max-product scenario, in spite of the chosen strategy for calculation of probabilities, are quite similar. However, the selected method affects the interpretation of the mechanism of the agent's memory in order to make a decision on a derived relation, which might be of interest in the EPS model.

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## Study II

# Enhanced Equivalence Projective Simulation: a Framework for Modeling Formation of Stimulus Equivalence Classes

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## Study III

# On solving the SPL problem using the concept of probability flux

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# On solving the SPL problem using the concept of probability flux

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**Abstract**

The Stochastic Point Location (SPL) problem Oommen is a fundamental learning problem that has recently found a lot of research attention. SPL can be summarized as searching for an unknown point in an interval under faulty feedback. The search is performed via a Learning Mechanism (LM) (algorithm) that interacts with a stochastic Environment which in turn informs it about the direction of the search. Since the Environment is stochastic, the guidance for directions could be faulty. The first solution to the SPL problem, which was pioneered two decades ago by Oommen, relies on discretizing the search interval and performing a controlled random walk on it. The state of the random walk at each step is considered to be the estimation of the point location. The convergence of the latter simplistic estimation strategy is proved for an infinite resolution, i.e., infinite memory. However, this strategy yields rather poor accuracy for low discretization resolutions. In this paper, we present two major contributions to the SPL problem. First, we demonstrate that the estimation of the point location can significantly be improved by resorting to the concept of *mutual probability flux* between neighboring states along the line. Second, we are able to accurately track the position of the optimal point and simultaneously show a method by which we can estimate the error probability characterizing the Environment. Interestingly, learning this error probability of the Environment takes place in tandem with the unknown location estimation. We present and analyze several experiments discussing the weaknesses and strengths of the different methods.

**Keywords** Stochastic Point Location (SPL) · Mutual probability flux · Flux-based Estimation Solution (FES) · Last Transition-based Estimation Solution (LTES) · Stochastic Learning Weak Estimation (SLWE) · Estimating environment effectiveness

## 1 Introduction

Stochastic Point Location (SPL) is a fundamental optimization problem that was pioneered by Oommen [20] and ever since has received increasing research interest [11, 28]. A Learning Mechanism (LM) attempts to locate a unique point $\lambda^*$ in an interval while the only assistance comes from the information provided by a random Environment ($E$) which informs it, possibly erroneously, whether the location is to the left or to the right of the point. The probability of receiving the correct response from Environment is basically fixed and unknown. The SPL problem, which was addressed by Oommen and a few others [11, 13, 20–22, 28], is indeed a general optimization framework where a large class of optimization problems could be modeled as an instantiation of it, see [31] for a survey of all the reported solutions to the SPL.

The assumption that the parameter or point location in the SPL setting does not change over time is not the case in many real-life dynamic systems such as web-based applications [10]. Indeed, the probability of receiving the correct response from Environment might be unknown and even non-stationary. Sliding window [12] is a traditional strategy for estimation in non-stationary Environments. However, choosing the appropriate window size would be crucial. When the window size is too small, the estimation will be poor. Contrarily, if the window size is rather large, the estimation accuracy will be degraded.

It is worth mentioning that Continuous Point Location with Adaptive Tertiary Search (CPL-ATS) strategy [23] is another method of solving SPL which systematically and

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recursively searches for sub-intervals that $\lambda^*$ is guaranteed to reside in, with an arbitrarily high probability. A series of guessing which starts with the mid-point of the given interval estimates the point location and repeats until the requested resolution is achieved. The given interval is partitioned into three sub-intervals where three LA work in parallel in each sub-interval and at least one of them will be eliminated from further search. So, it is crucial in CPL-ATS to construct the partition and elimination process. This method is further developed into the CPL with Adaptive $d$-ary Search (CPL-AdS) Strategy [24] where the current interval is partitioned into $d$ sub-intervals, instead of three. A larger $d$ results in faster convergence, as a consequence the decision table of elimination process becomes more complicated. An extension of the CPL-AdS scheme, which could also operate in non-stationary environments, is presented in [11]. The decision formula is proposed to modify the decision table in [24] to resolve certain issues of original CPL-AdS scheme.

In [35] an SPL algorithm based on Optimal Computing Budget Allocation (OCBA), named as SPL-OCBA, is proposed. SPL-OCBA employs OCBA and the historical sample information to find the location of a target point. Zhang et al. [33] integrated SPL with Particle Swarm Optimization (PSO)- which is a popular swarm intelligence algorithm- in a noisy Environment, in order to alleviate the impacts of noise on the evaluation of true fitness and increase the convergence speed.

In order to fasten the SPL scheme, the work reported in [26] proposes to use the last two transitions of the SPL to decide whether to increase or decrease the step size. Intuitively, two suggestions from the Environment in a row for going left or right will increase the step size. On the other hand, the step size is decreased whenever the SPL oscillates between two sates; this might be an indication that the optimal point is located between those two states.

In [7], SPL is modified in accordance with the classical Random Walk-based Triple level Algorithm (RWTA), where Environment provides three kinds of responses, i.e, right, left or unmoved.

A generalization of the hierarchical SPL scheme [28] to the case of deceptive Environment was proposed in [34]. In order to deal with the deceptive nature of the Environment and still be able to estimate the optimal location, the original tree structure found in [28] was extended by a symmetric tree rooted at the root node and it was shown that the SPL will converge to a leaf node in that symmetric tree in case the Environment is deceptive, while it will converge to the leaf node in the original tree if the Environment is informative i.e., not deceptive.

There is a wide range of scientific and real-life problems that can be modeled as the instances of SPL problem, such as adaptive data encoding, web-based applications, etc. [10]. In [6], Granmo and Oommen presented an approach for solving resources allocation problems under noisy Environment using a learning machine that is basically an SPL. The basic SPL version is used to determine the probability of polling a resource among two possible resources at each time instant. The scheme was also generalized to handle the case of more than one material using an hierarchical structure. The paradigm has been applied to determining the optimal polling frequencies of a web-page and to solving sampling estimation problems with constraints [5].

In [30], it is proposed to apply the SPL paradigm to solve the stochastic root finding problem which is a well-known stochastic optimization problem. The classical solution to solve this problem is based on stochastic approximation. Yazidi and Oommen show that it is possible to model the problem as variant of the SPL with adaptive d-ary search.

Recently, Yazidi et al. [29] showed that quantiles can be estimated using an SPL type search. The scheme has computational advantages as it uses discretized memory and it is able to adapt to dynamic environments. Another recent application of the SPL [23] is estimating the optimal parameters of Distance Estimation Functions (DEF). Distance Estimation (DE) [9] is a classical problem where the aim is to estimate an accurate value for the real (road) distance between two points which is typically tackled by utilizing parametric functions called Distance Estimation Functions. The authors use the Adaptive Tertiary Search strategy [23], to calculate the best parameters for the DEF. The proposed method uses the current estimate of the distances, the feedback from the Environment, and the set of known distances, to determine the unknown parameters of the DEF. It is suggested that SPL is a better way to determine DEF parameters rather than the traditional Goodness of-Fit (GoF) based paradigm [9].

Furthermore, SPL can also be used to find the appropriate dose in clinical practices and experiments [15].

A possible interesting application, which we focus on in our ongoing research, is to determine the difficulty level of a cognitive training method by SPL. One of the key challenges, faced by many learning methods, is to find the cognitive level of a participant in order of designing suitable level of training. To the best of our knowledge, in most legacy methods, alternating between different training levels and scenarios is simply done by increasing the difficulty if the task is managed, once or over a set of repeated iterations, or by decreasing/fixing the difficulty level if the task is not managed. This problem could be modeled by SPL with certain conditions, such as non-stationary point location, since the manageable difficulty level will change as time goes for trained participant, and unknown certainty/probability of the results. Because there are many factors that might affect the response to a training test that are not related to the real ability of the participant. For instance, Titrated delayed matching-to-sample (TDMTS) method, which is used by behaviour analysts, could easily be modeled as a SPL problem. TDMTS can be used to study important variables for analyzing short-term memory problems [1].

Spaced Retrieval Training (SRT) [16] is also a method of learning and retaining target information by recalling that information over increasingly longer intervals; a method which is especially used for people with dementia [2]. For progressive diseases like dementia, it is so important to estimate the ability level, i.e. point location in SPL, as quickly as possible, since the ability will rapidly change during time, affected by training, disease, and patient's condition.

This paper is partially based on our previous work published in [18]. In [18], we show that the SPL problem can be solved by introducing two key multinomially distributed random variables and tracking them using the Stochastic Learning Weak Estimator (SLWE) method. SLWE [25] figures among the most prominent estimators for non-stationary distributions. We proposed to integrate the SLWE as the inherent part of a more sophisticated and accurate solution for the SPL. The recursive updated form of the SLWE makes it a viable strategy in our problem since the tracked distribution in the case of SLWE is updated incrementally. Therefore, our strategy for estimation of point location revolves around tracking the distribution at each time step and estimating the point based upon it. We applied different statistical operators: maximum, expectation, and median on the estimated probability vectors to obtain our estimates. The results indicate that, the estimates obtained from these methods are smoother than those obtained from legacy SPL solutions and can track the changes more efficiently. The results, also, confirm that using the concept of mutual probability flux between states, according to which transitions are considered as the events of multinomially distributed random variable, is a superior alternative to [20]. We name the contribution as Flux-based Estimation Solution (FES). In the simulation part of initial work reported [18], Environment effectiveness fixed to $p = 0.7$ and the resolution fixed to $N = 16$. It was shown there that the estimated error reduced up to 75%. In the current paper, we do not fix the resolution and consider the case where we can tune the resolution. A new contribution in this paper is to introduce the Last Transition-based Estimation Solution (LTES). This estimator is much simpler than FES and in the case that we have no constraint on the resolution, LTES could estimate the point location equally well with FES.

The Environment effectiveness, i.e. probability of correct answer, is unknown and might vary over time. As the second contribution of this paper, we estimate the probability in tandem with the unknown location estimation.

The remainder of this paper is organized as follows. In Section 2, the SPL problem is defined formally. Section 3 is devoted to presenting our solution for both estimating the point location as well as the Environment effectiveness probability. In this perspective, Section 3.1 introduces the concept of mutual probability flux which is formally proved to be a stronger method compared with the last visited state of the Markov Chain. In Section 3.2, we introduce our estimation approach reckoned as Flux-based Estimation Solution (FES) that is based on a subtle usage of the concept of flux probability. We show that the LTES is a special case of the FES method, and a comparison between the LTES with the FES method is provided at the end of this part. Section 3.3 deals with the related fundamental problem of estimation of the Environment effectiveness. To evaluate the behavior of estimators, extensive simulation results based on synthetic data are presented and discussed in Section 4. Experiments based on real-life data related to online tracking of topics are presented in Section 5. Finally, we drew final conclusions in Section 6.

## 2 Stochastic point location problem in a dynamic setting

This problem considers that the learning mechanism (LM) moves within $[0, 1]$ interval and attempts to locate a point ($0 \leq \lambda^*(n) \leq 1$) that may change over time $n$. The Environment $E$ is considered to be informative; LM receives the right direction to the point location with probability $p^*(n) > 0.5$. This probability of receiving a correct response, which reflects the "effectiveness" of the Environment, is unknown by LM and assumed to be varying.

As aforementioned, we intend to track $\lambda^*(n)$ in an efficient manner. We follow the model presented in [20] and discretize the interval and perform a controlled random walk on it, characterized by $\lambda(n)$. More precisely, we subdivide the unit interval into $N + 1$ discrete points

$$\{0, 1/N, 2/N, \cdots, (N-1)/N, 1\},$$

where $N$ is called the resolution of the learning scheme. Let $\lambda(n)$ be the current location at time step $n$:

- If $E$ suggests increasing $\lambda(n)$:
  $\lambda(n+1) = \min(\lambda(n) + 1/N, 1)$
- If $E$ suggests decreasing $\lambda(n)$:
  $\lambda(n+1) = \max(0, \lambda(n) - 1/N)$

Hereafter, the binary function $E(n, i)$ stands for the Environment answer at step $n$ and location $\lambda(n) = i/N$, where $E(n,i)=1$ refers to the Environment suggestion to increase $\lambda(n)$ and $E(n,i)=0$ refers to the Environment suggestion to decrease $\lambda(n)$. Let $Z$ be an integer value between 0 and $N-1$, based on above rules, if $Z/N \le \lambda^*(n) < (Z+1)/N$ at time $n$ we have:

$$\begin{aligned}
Pr(E(n,i)=1) &= p^*(n) \ if \ 0 \le i \le Z \\
&= q^*(n) \ if \ Z < i \le N \\
Pr(E(n,i)=0) &= q^*(n) \ if \ 0 \le i \le Z \\
&= p^*(n) \ if \ Z < i \le N
\end{aligned} \tag{1}$$

Where $q^*(n)=1-p^*(n)$.

Based on the results presented in [20], in the stationary case in which $\lambda^*(n)=\lambda^*$, this random walk will converge into a value arbitrarily close to $\lambda^*$, when $N\to\infty$ & $n\to\infty$. However, the above asymptotic results are not valid for the non-stationary SPL. In practice, we might experience some constraints, both on time $n\le T$ and on the resolution $N\le R$. Throughout the rest of this paper, we pursue better estimates for $\lambda^*(n)$ than $\lambda(n)$.

## 3 Estimation strategies

In this section, we first show the superiority of the Last Transition-based Estimation Solution (LTES) over the last location estimate. Then, a multinomially distributed random variable is considered. We track its probability distribution with SLWE method [25] and estimate the $\lambda^*(n)$ from the estimated distributions. Then, we explain how we can estimate the probability $p^*(n)$ using the estimation of $\lambda^*(n)$.

In [18], we showed that tracking probability distribution for different state transitions, instead of the point locations, yields a better performance. The reason is that the estimation by Markov chain will have many transitions around the true and unknown $\lambda^*(n)$. In the following, we prove that using the concept of mutual probability flux is a stronger tool for solving the SPL problem than using the current point location. In the proof, we consider the static case, i.e. $\lambda^*(n)=\lambda^*$.

### 3.1 Superior accuracy with the concept of mutual probability flux

For simplicity, let $x_i = i/N$ for $i = 0,1,\ldots,N$. So, the Markov chain states will be the possible value of $x_i$ for $0 \le i \le N$ which belongs to the set of values $\{0, 1/N, 2/N, \cdots, 1\}$. Let $\pi_i$ be the stationary (or equilibrium) probability of the chain being in state $x_i$. Then, the equilibrium probability distribution vector will be $\Pi = [\pi_0, \pi_1, \cdots, \pi_N]^T$.

We know that the Markov chain is an instantiation of the birth-death process.[1] It is also known that, such a process is a time reversible Markov chain, i.e. satisfies the detailed balance equation:

$$\pi_i M_{i,j} = \pi_j M_{j,i} \quad \text{for all } i \neq j$$

where $M_{i,j}$'s are transition probabilities. For a complete overview about time reversibility, we refer the reader to an excellent book by Kelly [14]. The following simple proof shows time reversibility of our Markov chain.

If $|i-j|>1$ for $0\le i,j\le N$, i.e. $x_i$ and $x_j$ are not adjacent, then the detailed balance equation is obviously true. For a given $i$, we can divide the states into two parts, $L=\{x_k|k\le i\}$ and $R=\{x_k|k>i\}$. Since the Markov chain is a birth-death chain, the only passage between the two parts is the transition $x_i$ to $x_{i+1}$ or $x_{i+1}$ to $x_i$. The flow from $L$ to $R$ is $\pi_i M_{i,i+1}$ and from $R$ to $L$ is $\pi_{i+1}M_{i+1,i}$. Since $\Pi$ is stationary, the total flow must be 0, which concludes what is desired:

$$\pi_i M_{i,i+1} = \pi_{i+1} M_{i+1,i}. \tag{2}$$

Let $x_i^+$ denotes the event according to which the Markov chain makes a transition from $x_i$ to $x_{i+1}$ or from $x_{i+1}$ to $x_i$. The informed reader would observe the latter event can be related to the concept of flux probability [19, Chapter 8.4]. In fact, in the literature, the flux probability between two neighboring states $x_i$ and $x_{i+1}$ is given by $M_{i,i+1}\pi_i$ which represents the absolute probability of observing a transition from $x_i$ to $x_{i+1}$. We can see that the probability of $x_i^+$ can be described as the sum of two flux probabilities; namely the flux probability corresponding to transiting from $x_i$ to $x_{i+1}$, and the flux probability of transiting in the opposite direction from state $x_{i+1}$ to $x_i$. In other words, the probability of the event $x_i^+$, which is shown by $\pi_i^+$, equals to the following sum

$$\pi_i^+ = M_{i,i+1}\pi_i + M_{i+1,i}\pi_{i+1}. \tag{3}$$

We call this quantity as *mutual probability flux* between states $x_i$ and $x_{i+1}$. In the light of this explanation, we call $\Pi^+ = [\pi_0^+, \pi_1^+, \cdots, \pi_{N-1}^+]^T$ the *mutual flux probability vector* between two neighboring states.

Now we intend to investigate the relation between $\Pi$ and $\Pi^+$. Let $x_Z \le \lambda^* < x_{Z+1}$ and $e = \frac{p}{q} > 1$.[2] As a result

[1] Since the only possible transitions are moving one state to the left or right.

[2] Suppose the Environment is stationary; $\lambda^*(n)=\lambda^*$, $p^*(n)=p>0.5$, and $q^*(n)=q=1-p$.

of (2) and referring to relations in (1), the following balance equations hold.

$$\pi_i = e.\pi_{i-1} \quad \text{whenever} \quad i \leq Z \tag{4}$$

In the case $i \leq Z$ we have $M_{i,i+1} = p$ and $M_{i+1,i} = q$.

$$\pi_i = \frac{\pi_{i-1}}{e} \quad \text{whenever} \quad i > Z+1 \tag{5}$$

In the case $i > Z+1$ we have $M_{i,i+1} = q$ and $M_{i+1,i} = p$. Finally, since we have $M_{Z,Z+1} = p$ and $M_{Z+1,Z} = p$:

$$\pi_{Z+1} = \pi_Z \tag{6}$$

These relations show that values are increasing from $\pi_0$ to $\pi_Z$ and decreasing from $\pi_{Z+1}$ to $\pi_N$; and therefore, $\pi_Z$ and $\pi_{Z+1}$ take the maximum value.

Let $\lambda_{\text{tr}}(n)$ be the mean of last two states, i.e. $\lambda_{\text{tr}}(n) = \frac{\lambda(n-1)+\lambda(n)}{2}$. In this case, $\pi_i^+$ would be the stationary probability of $\lambda_{\text{tr}}(n)$ chain being in transition $x_i^+$. We can easily see that the probabilities of $\Pi^+ = [\pi_0^+, \pi_1^+, \cdots, \pi_{N-1}^+]^T$ are larger for indexes around the $\lambda^*$.

Whenever $i < Z$, using (3), we have

$$\pi_i^+ = p\pi_i + (1-p)\pi_{i+1} = p\frac{q}{p}\pi_{i+1} + q\pi_{i+1}.$$

and therefore

$$\pi_i^+ = 2q\pi_{i+1} \quad \text{whenever} \quad i < Z. \tag{7}$$

In the case $i = Z$ we have:

$$\pi_Z^+ = p\pi_Z + p\pi_{Z+1} = 2p\pi_Z, \tag{8}$$

and finally whenever $i > Z$

$$\pi_i^+ = q\pi_i + p\pi_{i+1} = q\pi_i + p\frac{q}{p}\pi_i,$$

$$\pi_i^+ = 2q\pi_i \quad \text{whenever} \quad i > Z. \tag{9}$$

Up to this point, we have showed the relation between $\Pi$ and $\Pi^+$. Now, to show the convergence of $\Pi^+$, we just need to prove $\pi_Z^+$ is greater than $\pi_i^+$ for $i \neq Z$ (i.e. $i < Z$ and $i > Z$).

**Case 1: $i < Z$** Based on (4), (5), and (6), we know that $\pi_Z > \pi_i$ for $i < Z$. We also know that $2q < 1$, when $p > 1/2$; and therefore, $2q\pi_i < \pi_Z$. However, we showed that $2q\pi_i = \pi_i^+$, and as a result:

$$\pi_i^+ = 2q\pi_{i+1} < \pi_Z < 2p\pi_Z = \pi_Z^+$$

**Case 2: $i > Z$** Again, we observe that $\pi_Z > \pi_i$ for $i > Z$. So, we have

$$\pi_i^+ = 2q\pi_i < \pi_i < \pi_Z < 2p\pi_Z = \pi_Z^+.$$

Thus, we have proved that $\pi_i^+ < \pi_Z^+$ for $i \neq Z$, which means that, the transition has higher probabilities at $\pi_Z^+$ and lower values at other locations.

Since $\pi_Z^+$ is greater than $\pi_Z$, we expect that the $\lambda_{\text{tr}}(n)$ estimator, or LTES, performs better than $\lambda(n)$. This can be investigated by comparing the expected estimation error of SPL and LTES. Let $E_{\text{SPL}}$ be the expected estimation error for SPL and let $E_{\text{LTES}}$ be the expected estimation error for the LTES. For the sake of simplicity, we suppose that $\lambda^*$ is in the middle of the interval $[Z/N, (Z+1)/N]$ which means $\lambda^* = \frac{Z+(Z+1)}{2N}$.

$$\begin{aligned}
\mathrm{E_{SPL}} &= \sum_i \pi_i \mid \lambda^* - x_i \mid \\
&= \sum_i \pi_i \mid \frac{Z+(Z+1)}{2N} - \frac{i}{N} \mid \\
&= \sum_{i \neq Z} \pi_i \mid \frac{Z+(Z+1)}{2N} - \frac{i}{N} \mid \\
&\quad + \pi_Z \mid \frac{Z+(Z+1)}{2N} - \frac{Z}{N} \mid
\end{aligned} \tag{10}$$

On the other hand for the LTES we have

$$\begin{aligned}
\mathrm{E_{LTES}} &= \sum_i \pi_i^+ \mid \lambda^* - x_i^+ \mid \\
&= \sum_i \pi_i^+ \mid \frac{Z+(Z+1)}{2N} - \frac{i+(i+1)}{2N} \mid \\
&= \sum_{i \neq Z} \pi_i^+ \mid \frac{Z+(Z+1)}{2N} - \frac{i+(i+1)}{N} \mid \\
&\quad + \pi_Z^+ \mid \frac{Z+(Z+1)}{2N} - \frac{Z+(Z+1)}{2N} \mid \\
&= \sum_{i \neq Z} 2q\pi_i \mid \frac{Z+(Z+1)}{2N} - \frac{i+(i+1)}{2N} \mid
\end{aligned} \tag{11}$$

As for large $N$, $\frac{2i+1}{2N} \approx \frac{i}{N}$, we can write

$$\mathrm{E_{LTES}} = \sum_{i \neq Z} 2q\pi_i \mid \frac{Z+(Z+1)}{2N} - \frac{i}{N} \mid .$$

From the above equations we get:

$$\begin{aligned}
\mathrm{E_{SPL}} &> \sum_{i \neq Z} \pi_i \mid \frac{Z+(Z+1)}{2N} - \frac{i}{N} \mid \\
&> 2q \left( \sum_{i \neq Z} \pi_i \mid \frac{Z+(Z+1)}{2N} - \frac{i}{N} \mid \right) \\
&= \mathrm{E_{LTES}}
\end{aligned} \tag{12}$$

The last inequality is due to the fact that $2q < 1$.

Therefore we conclude that the expected estimation error for LTES is smaller than SPL for large enough $N$. The results in Section 4 confirm the discussion above.

### 3.2 Flux-based estimation solution (FES)

Let $X^+(n)$ denote a multinomially distributed variable over the possible transitions $x_i^+, i = 0, \ldots, N-1$; where the concrete realization of $X^+(n)$ at time step $n$ is $\lambda_{tr}(n)$. Please note that the distribution of $X^+(n)$ can be explained using the mutual flux probability vector $\Pi^+(n)$. The portion of transitions defined as $P(X^+(n) = x_i^+) = \pi_i^+(n), i = 0, \ldots, N-1$.

The SLWE method estimates the probabilities

$$\Pi^+(n) = [\pi_0^+(n), \pi_1^+(n), \ldots, \pi_{N-1}^+(n)]^T$$

by maintaining a running estimate $S(n) = [s_0(n), s_1(n), \cdots, s_{N-1}(n)]^T$ of $\Pi^+(n)$ where $s_i(n)$ is the estimate of $\pi_i^+(n)$ at time $n$. The updating rule is (the rules for other values of $s_j(n)$, $j \neq i$, are similar):

$$\begin{aligned} s_i(n+1) &\leftarrow \alpha s_i(n) + (1-\alpha) \text{ when } \lambda_{tr}(n) = x_i^+ \\ &\leftarrow \alpha s_i(n) \text{ when } \lambda_{tr}(n) \neq x_i^+ \end{aligned} \tag{13}$$

$0 < \alpha < 1$ is a user-defined parameter for updating the probability distribution. The intuition behind the updating rule is that if $\lambda_{tr}(n) \neq x_i^+$ we should decrease our estimate $s_i(n)$ which is given by the second part of the updating rule. Similarly, if $\lambda_{tr}(n) = x_i^+$ we should increase our estimate which is given by the first part of the updating rule.

It is worth mentioning that in [25], $X(n) = X$, i.e. it is not modeled as a function of time and as a result $\Pi(n) = [\pi_0, \pi_1, \cdots, \pi_N]^T$ is time-invariant. The theorems and results are also proven in the asymptotic case when $n \to \infty$ which is in contradiction with the non-stationary assumption for Environment. It is discussed that in practice the convergence takes place after a relatively small value of $n$. For instance, if the Environment switches its multinomial probability vector after 50 steps, the SLWE could track this change. However, we prefer to use the notation in a way that the point location, and thereafter, the multinomially probability vector is clearly shown to be non-stationary. SLWE converges weakly, independently of $\alpha$ value, however the rate of convergence is a function of $\alpha$. Based on previous section where we showed $\pi_i < \pi_Z$, $i \neq Z$, and as $S(n)$ converges to $\Pi^+(n)$, we are able to estimate the point location, $\lambda^*(n)$, by finding the maximum probability, i.e.

$$\begin{aligned} &z = \arg\max_i(s_i(n)) \\ &\lambda_{\max}(n) = x_z^+ \end{aligned} \tag{14}$$

Note that the maximum value refers to a pair that the LM transits to the most. For non-unique $z$, the last visited pair with the max probability value is chosen (see Algorithm 1).

As $n \to \infty$, and for appropriate choices of $\alpha \to 1$; $S(n) \to \Pi^+(n)$. Thus, (14) reduces to $z = Z$, as we know that $\pi_Z^+$ is the largest component in the vector $\Pi^+(n)$. Then, the error will be $\approx 0$ as time goes to infinity.

As a side remark, if $\alpha \leq 0.5$ and $\lambda_{tr}(n) = x_i^+$, then $s_i(n) \geq 0.5$ for event $x_i^+$. In other words, $\lambda_{\max}(n) = \lambda_{tr}(n)$ if $\alpha \leq 0.5$. Because of this, we set $\alpha > 0.5$ in our simulations to avoid repeating the same estimation.

**Algorithm 1** Estimation of $\lambda^*(n)$ by FES

```
input : N, T, i = ⌊N/2⌋, E(n,i), X(n), α
initialization
λ(0) = x_i, S(0) = [s_0(0), s_1(0), ..., s_{N−1}(0)]^T =
[1/N, 1/N, ..., 1/N]^T
begin
    for n = 1 to T do
        j = i − (−1)^{E(n,i)}
        if j ≤ 0 or j ≥ N then
            j = i
        i = j
        λ(n) = x_i
        x_i^+ = (λ(n)+λ(n−1))/2
        S(n) = αS(n − 1) + (1 − α)I_i
        /* I_i = [0, 0, ··· , 1, ··· , 0]^T ; a vector of
        size N with 1 at ith position and 0
        elsewhere.                              */
        λ_max(n) = x_z^+ where z = arg max_i(s_i(n))
output: λ(n), λ_max(n)
```

#### 3.2.1 LTES as a special case of FES

The informed reader would remark that the FES scheme needs to keep track of the maximum component of the mutual flux probability vector. For each component, the middle point of the corresponding pair of states is used as an estimate of the point location. A special case of the FES method is to operate without memory, and in this case, the maximum component of the mutual flux probability vector will simply correspond to the middle point of the last visited pair of states. This is also true regarding Algorithm 1, where we see that if we replace $\alpha$ by 0, then FES reduces to the LTES algorithm.

A potential strength of LTES ($\lambda_{tr}$) is that we only need to tune the parameter, namely $N$, while the FES estimator ($\lambda_{max}$) contains two parameters $N$ and $\alpha$. However, both parameters are related to how rapidly the estimator adjusts to changes in the Environment. This suggests that if we are able to tune over $N$, the LTES approach and Oommen's method could perform equally well as the more sophisticated algorithm with weak estimation.

In the following, we show how to estimate the $p^*(n)$ using $\lambda_{tr}$ and weak estimators. The Oommen estimate i.e. $\lambda(n)$ can not be a basis for estimation of $p^*(n)$. The reason is that we increase or decrease the probability by comparing the estimation of point location $\hat{\lambda}(n)$ and Environment suggestion $E(n, i)$ at point $\lambda(n)$. In Oommen's method since $\hat{\lambda}(n) = \lambda(n)$ the probability estimation always would be 0.5.

### 3.3 Estimation of environment effectiveness probability

To estimate $p^*(n)$ based on the estimation of $\lambda^*(n)$, we use simple binomial weak estimator. Let $\hat{\lambda}(n)$ be the estimation of $\lambda^*(n)$. We adjust over $\gamma$ which is the parameter for binomial weak estimator (see Algorithm 2). Since the probability assumed to change over [0.5, 1], the initial guess of the probability is set to $\hat{p}(0) = 0.75$.

– If ($\lambda(n) < \hat{\lambda}(n)$ and $E(n, i) = 1$) OR ($\lambda(n) > \hat{\lambda}(n)$ and $E(n, i) = 0$):

$$\hat{p}(n) = 1 - \gamma(1 - \hat{p}(n-1))$$

– Else if ($\lambda(n) < \hat{\lambda}(n)$ and $E(n, i) = 0$) or ($\lambda(n) > \hat{\lambda}(n)$ and $E(n, i) = 1$):

$$\hat{p}(n) = max(0.5, \gamma\hat{p}(n-1)) \tag{15}$$

– Else if ($\lambda(n) = \hat{\lambda}(n)$):

$$\hat{p}(n) = \hat{p}(n-1)$$

Basically, the probability $\hat{p}(n)$ increases by a multiplicative parameter $\gamma$ if the Environment direction $E(n, i)$ agrees with the estimation of point location, $\hat{\lambda}(n)$, and vice versa; the opposite probability $(1 - \hat{p}(n))$ increases by a multiplicative factor $\gamma$ if they disagree. Since we know that $p^*(n)$ change over [0.5, 1], we restrict our estimations to this domain by setting the lower bound 0.5 in (15).

**Algorithm 2** Estimation of $p^*(n)$

**input** : $N,\ T,\ E(n,i),\ \lambda(n),\ \hat{\lambda}(n),\ \gamma$
initialization
$\hat{p}(0) = 0.75$
**begin**
  **for** $n = 1$ *to* $T$ **do**
    **if** *(*$\lambda(n) < \hat{\lambda}(n)$ *AND* $E(n,i) = 1$*) OR*
    *(*$\lambda(n) > \hat{\lambda}(n)$ *AND* $E(n,i) = 0$*)* **then**
      $\hat{p}(n) = 1 - \gamma(1 - \hat{p}(n-1))$
    **else if** *(*$\lambda(n) < \hat{\lambda}(n)$ *AND* $E(n,i) = 0$*) OR*
    *(*$\lambda(n) > \hat{\lambda}(n)$ *AND* $E(n,i) = 1$*)* **then**
      $\hat{p}(n) = \max(0.5, \gamma\hat{p}(n-1))$
    **else**
      $\hat{p}(n) = \hat{p}(n-1)$
**output**: $\hat{p}(n)$

## 4 Experimental results

In this section, we resort to simulation experiments to evaluate the performance of the estimators suggested in this paper. As mentioned before, both $\lambda^*(n)$ and $p^*(n)$, which are not known by LM, could be either constant or dynamic. In this regard, there are many possibilities to define the Environment in which two general types of Environments are considered. Those Environments can show the characteristics of estimators in the best manner.

– Both $\lambda^*(n)$ and $p^*(n)$ change after a fixed amount of time. So their values are fixed for a while until a sharp change happens. We use the sample abbreviation SWITCH-1000-10000 for this type, which means $\lambda^*(n)$ changes after 1000 steps and $p^*(n)$ changes after 10000 steps. The next value of $\lambda^*(n)$ is randomly chosen from [0, 1], and for $p^*(n)$ the random value is chosen from [0.5, 1].
– Both $\lambda^*(n)$ and $p^*(n)$ vary gradually as continuous functions of time. We consider the changes as sine functions. A sample abbreviation for this type would be SINE-1080-10080, which means that $\lambda^*(n)$ has a period of 1080 and $p^*(n)$ has a period of 10080. More precisely, $\lambda^*(n) = 0.5 + 0.5\sin((n/540)\pi)$ where the sine argument changes by $\pi/180$ radians every 3 steps. Therefore, period equals $3 \cdot 360 = 1080$. Moreover, $p^*(n) = 0.75 + 0.25\sin((n/5040)\pi)$ where the sine argument changes by $\pi/180$ radians every 28 steps; so the period equals $28 \cdot 360 = 10080$.

The key aspects of presented estimators can be discussed through eight cases that highlight the salient features of our

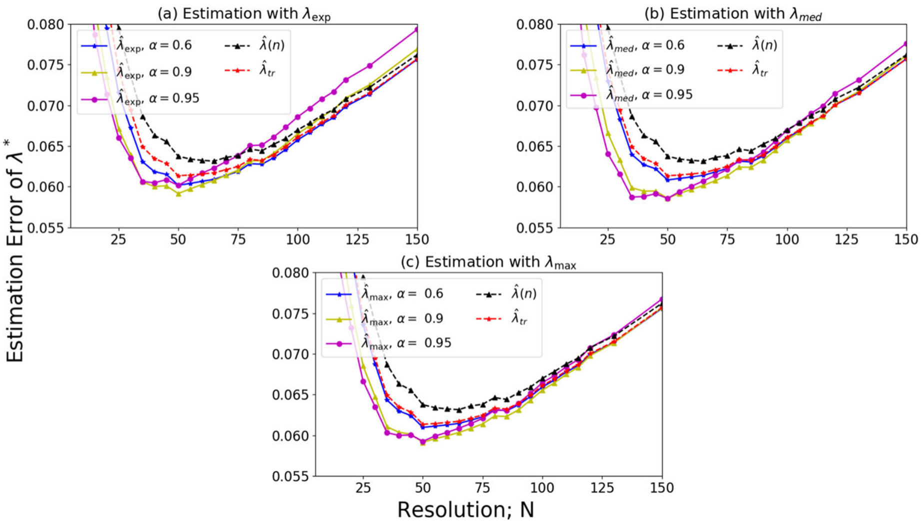



**Fig. 1** **SWITCH-1000-1000.** In each of the three sub-figures, one of the max, med, and exp along with the Oommen's method $\lambda(n)$ and the transition $\lambda_{\text{tr}}$ are depicted

scheme. For the sake of clarity, these cases are classified into seven headings which are introduced briefly in the following.

The first Section 4.1 presents the initial settings when both $\lambda^*(n)$ and $p^*(n)$ change moderately. Cases SWITCH-1000-1000 and SINE-1080-1080 are presented in this part in Figs. 1 and 2 respectively. Next, in Section 4.2 the effect of faster changes in $\lambda^*(n)$ and $p^*(n)$ are addressed through cases SWITCH-100-100 (Fig. 3) and SINE-360-360 (Fig. 4). In the third Section 4.3 the effect of changing rate of $p^*(n)$ on estimating $\lambda^*(n)$ in SWITCH dynamic is examined. To do so, the changes of $\lambda^*(n)$ are fixed on 1000, and two alternative cases SWITCH-1000-100 (Fig. 5) and SWITCH-1000-10000 (Fig. 6) are compared with SWITCH-1000-1000 (Fig.1). Additionally, in Fig. 7 a trace plot for tracking $\lambda^*(n)$ via LTES ($\lambda_{\text{tr}}$) is presented and the behavior of the estimator is discussed through three cases SWITCH-1000-100, SWITCH-1000-1000, and SWITCH-1000-10000. Table 1 summarizes the choices of tuning parameters resulting into the minimum error for $\lambda_{\text{tr}}$

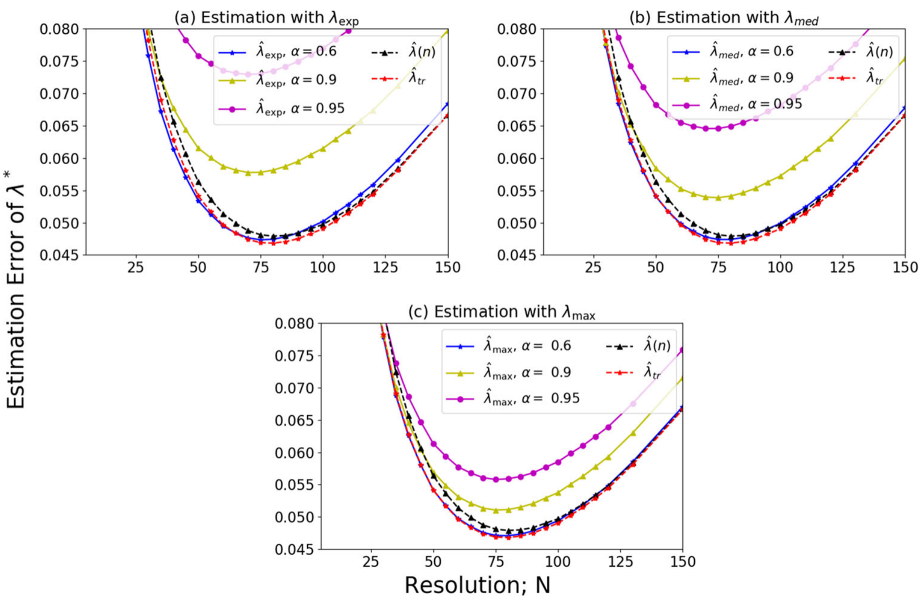



**Fig. 2** **SINE-1080-1080.** In each of the three sub-figures, one of the max, med, and exp along with the Oommen's method $\lambda(n)$ and the transition $\lambda_{\text{tr}}$ are depicted

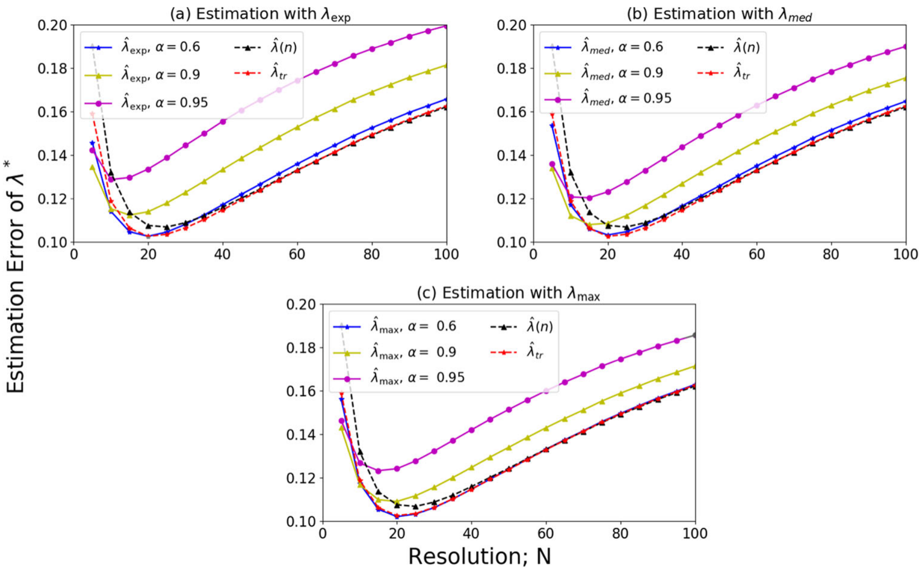



**Fig. 3** **SWITCH-100-100.** In each of the three sub-figures, one of the max, med, and exp along with the Oommen's method $\lambda(n)$ and the transition $\lambda_{tr}$ are depicted

and $\lambda_{\max}$ to the SWITCH cases. The fourth Section 4.4 focuses on the SINE dynamic and presents the results for the effect of changing rate of $p^*(n)$ on estimating $\lambda^*(n)$. Similarly, the periods of sine function at $\lambda^*(n)$ are fixed on 1080, and two alternative cases SINE-1080-360 (Fig. 8) and SINE-1080-10080 (Fig. 9) are compared with SINE-1080-1080 (Fig. 2). Table 2 summarizes the same data as Table 1 for SINE cases. Fifth Section 4.5 is devoted to study the effect of relation between $\lambda^*(n)$ and $p^*(n)$ dynamics on the estimators. Figure 10 depicts tracking $\lambda^*(n)$ throughout the two scenarios SINE-1080-1080 and SINE-1080-1080-Shift where the second scenario has a shift in the phase of $\lambda^*(n)$. The differences in the tracking performance are discussed in detail. Table 3 is assisting the discussion in this section. Estimation of Environment effectiveness is addressed in the last two sections. In Section 4.6, Figs. 11

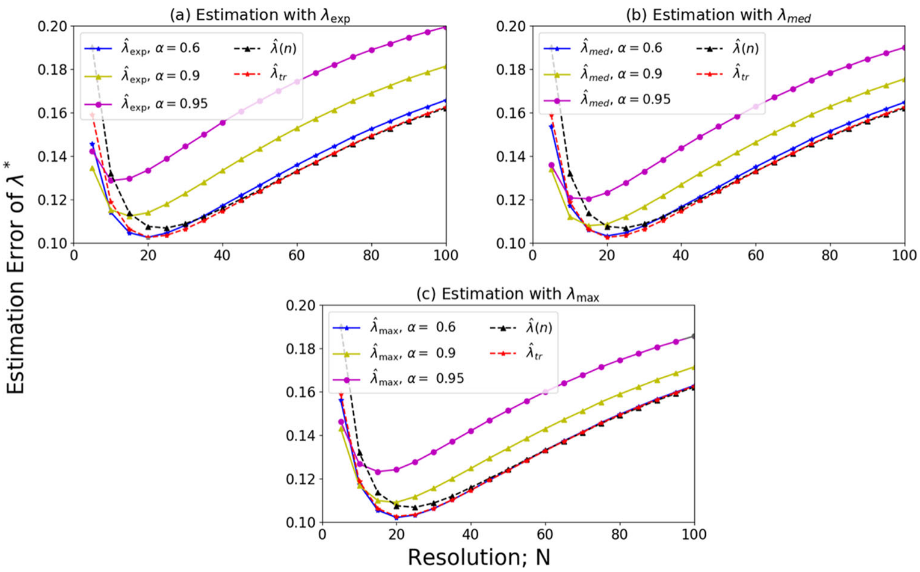



**Fig. 4** **SINE-360-360.** In each of the three sub-figures, one of the max, med, and exp along with the Oommen's method $\lambda(n)$ and the transition $\lambda_{tr}$ are depicted

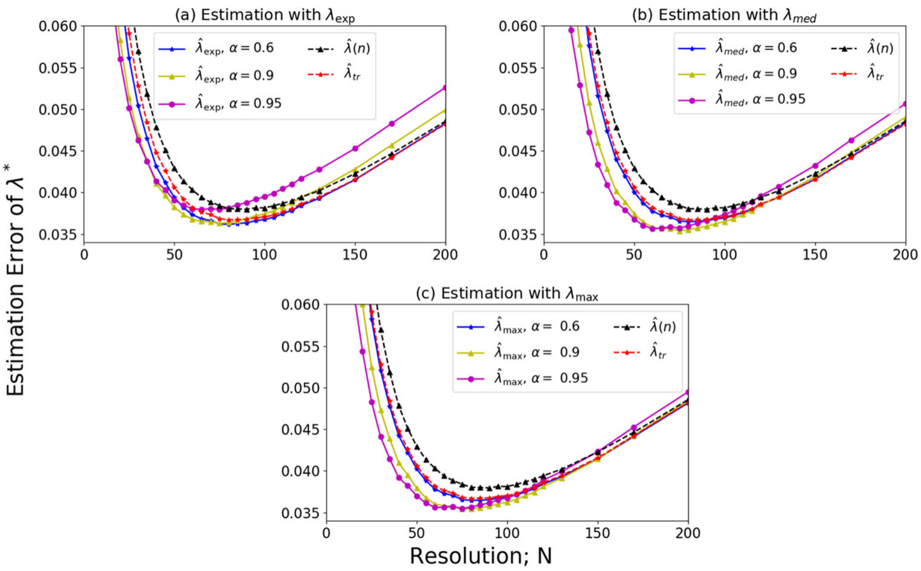



**Fig. 5** **SWITCH-1000-100.** In each of the three sub-figures, one of the max, med, and exp along with the Oommen's method $\lambda(n)$ and the transition $\lambda_{tr}$ are depicted

and 12 present the estimation error for various SWITCH and SINE cases respectively. Moreover, Table 4 summarizes the choices of tuning parameters resulting into the minimum error while Environment effectiveness is estimated. In order to compare the effect of tuning parameters, in Table 4, the estimation error for $p^*(n)$ with $N = 5$ and $\alpha = 0.9$ is reported as well. Finally, Section 4.7 analyses the results of estimation of Environment effectiveness for the tracking process depicted in Figs. 13 and 14.

It is worth mentioning that there are two main other approaches to solve the SPL problem which we do not compare with here. The first approach was pioneered by Yazidi et al. [28] and is based on arranging the search space into a tree structure. The second main approach is the CPL-ATS strategy [23, 24] and is based on diving the search interval into d sub-intervals and then recursively eliminating at least one sub-interval, thus shrinking the search space. We did not compare with these methods because in contrast to our solution and to Oommen's original SPL solution [20], much more queries are required per iteration. In fact, when it comes to the hierarchical solution [28], three queries are required in the case of a binary tree structure while the CPL-ATS strategy requires as many queries as the number d of sub-interval. Therefore, it would be inappropriate to compare against our method and Oommen's original SPL which use only one query per iteration. Furthermore, the CPL-ATS strategy suffers from the fact that is not suitable for dynamic Environment as it eliminates irreversibly parts of the search space at each epoch. Before proceeding to the experimental results, it is necessary to clarify some

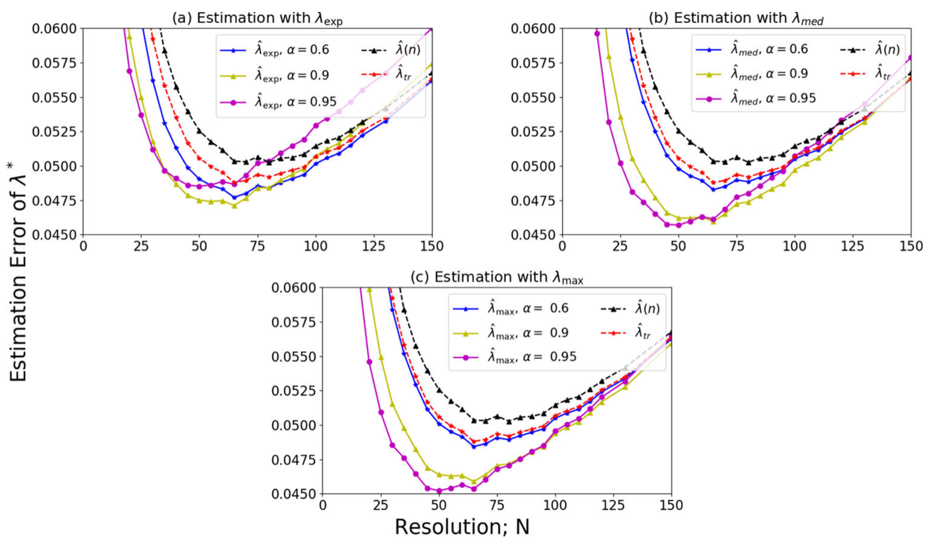



**Fig. 6** **SWITCH-1000-10000.** In each of the three sub-figures, one of the max, med, and exp along with the Oommen's method $\lambda(n)$ and the transition $\lambda_{tr}$ are depicted

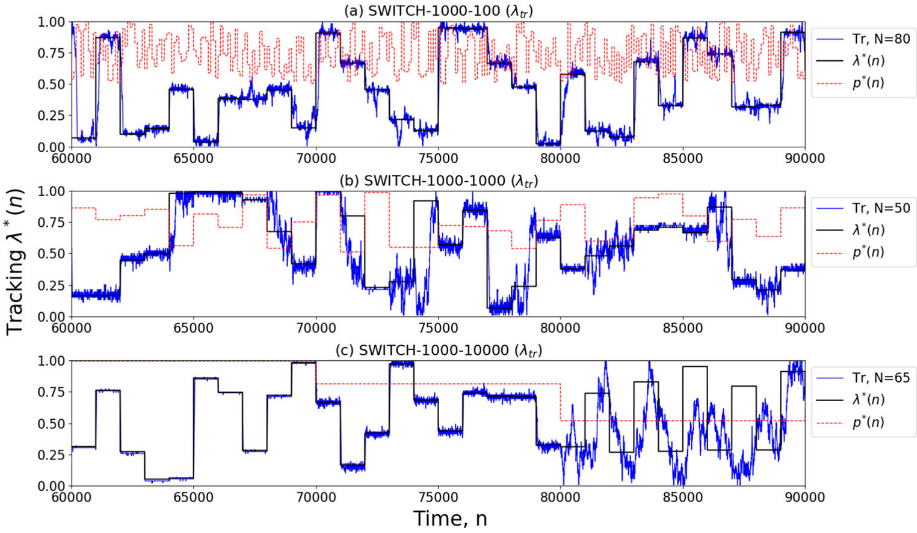



**Fig. 7** **a** shows how $\lambda_{tr}$ tracks $\lambda^*(n)$ in case SWITCH-1000-100. **b** and **c** show the same for SWITCH-1000-1000 and SWITCH-1000-10000 respectively. In all cases, a slice of Environment from $n = 60000$ to $n = 90000$ are represented

general issues regarding the reported data and figures. First, some figures shows the estimation error for a variety of tuning parameters $N$ and $\alpha$. Below, we refer to this as "error plots". In all experiments we have considered $\alpha \in [0.6, 0.7, 0.8, 0.9, 0.95, 0.99]$, however, in the sake of clarity, we only depict $\alpha \in [0.6, 0.9, 0.95]$ cases in the error plots.

Along with $\lambda_{max}(n)$ estimation, $\lambda_{med}(n)$ and $\lambda_{exp}(n)$ estimations are presented in [18] respectively as the median and expectation of probability vector. Formally, $\lambda_{med}(n)$ and $\lambda_{exp}(n)$ are defined by

– the expected value of the $X^+(n)$ at step $n$

$$\lambda_{exp}(n) = \sum_{i=0}^{N-1} x_i^+ s_i(n), \tag{16}$$

– the median of the $X^+(n)$ at step $n$:

$$\lambda_{med}(n) = x_z^+ \text{ where } z \text{ is the index satisfying:}$$
$$\sum_{i=0}^{z} s_i(n) \geq 0.5 \text{ and } \sum_{i=z}^{N-1} s_i(n) \geq 0.5. \tag{17}$$

Intuitively, it makes sense to estimate $\lambda^*(n)$ by the most visited transition which is given by $\lambda_{max}(n)$. However, if the system varies rapidly, the probability vector estimate $S(n)$ will be quite poor. In such a case, taking the expectation might be a more robust alternative, as given by $\lambda_{exp}(n)$.

Although the main proposal of this paper is $\lambda_{tr}(n)$ and $\lambda_{max}(n)$, in order of comparison, we include error plots for $\lambda(n)$, $\lambda_{med}(n)$ and $\lambda_{exp}(n)$.

The presented plots in Section 4.5, show estimation error of $p^*(n)$ as a function of tuning parameter $\gamma$. Since the main objective of this paper is to track $\lambda^*(n)$, and there are many

**Table 1** Summary of the choices of tuning parameters resulting into minimum error for $\lambda_{tr}$ and $\lambda_{max}$ in SWITCH experiments. The smallest error value in each experiment is represented in bold font

| Estimator \ SWITCH- | | 1000–1000 | | 100–100 | | 1000–100 | | 1000–10000 | |
|---|---|---|---|---|---|---|---|---|---|
| | | N | Error | N | Error | N | Error | N | Error |
| Oommen($\lambda(n)$) | | 65 | 0.06315 | 25 | 0.1069 | 90 | 0.03797 | 80 | 0.05027 |
| LTES($\lambda_{tr}$) | | 50 | 0.06135 | 20 | 0.10258 | 80 | 0.03671 | 65 | 0.04879 |
| FES($\lambda_{max}$) | $\alpha = 0.6$ | 50 | 0.06098 | 20 | **0.1021** | 80 | 0.03646 | 65 | 0.04843 |
| | $\alpha = 0.7$ | 50 | 0.06069 | 20 | 0.10253 | 80 | 0.03628 | 65 | 0.04806 |
| | $\alpha = 0.8$ | 50 | 0.06 | 20 | 0.10349 | 80 | 0.03586 | 65 | 0.047312 |
| | $\alpha = 0.9$ | 50 | **0.05914** | 20 | 0.10916 | 75 | **0.03539** | 65 | 0.04589 |
| | $\alpha = 0.95$ | 50 | 0.05928 | 15 | 0.1233 | 75 | 0.0355 | 50 | **0.04521** |
| | $\alpha = 0.99$ | 35 | 0.07094 | 10 | 0.20845 | 40 | 0.0406 | 45 | 0.06035 |

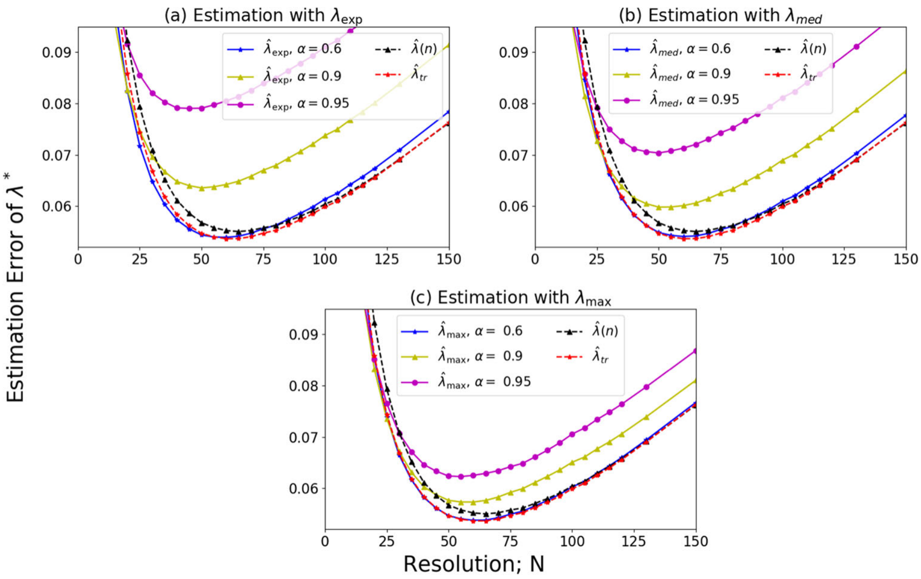



**Fig. 8** **SINE-1080-360.** In each of the three sub-figures, one of the max, med, and exp along with the Oommen's method $\lambda(n)$ and the transition $\lambda_{tr}$ are depicted

parameters in estimation of $p^*(n)$, we restrict the plots to the best choices of $N$ in $\lambda_{tr}$, and ($N$ and $\alpha$) in $\lambda_{max}$. However, we added minimum estimation error for $p^*(n)$ when $N = 5$ and $\alpha = 0.9$ to discuss the effect of resolution on estimation of $p^*(n)$.

To measure the estimation error in the estimation of $\lambda^*(n)$ and $p^*(n)$, the Mean Absolute Error (MAE) will be used. For $\lambda^*(n)$ this becomes

$$\text{MAE}_\lambda = \frac{1}{T}\sum_{n=1}^{T} \mid \hat{\lambda}(n) - \lambda^*(n) \mid \tag{18}$$

where $T$ is the total number of time steps and $\hat{\lambda}(n)$ is the estimate at time step $n$. Similarly, for $p^*(n)$ this becomes

$$\text{MAE}_p = \frac{1}{T}\sum_{n=1}^{T} \mid \hat{p}(n) - p^*(n) \mid \tag{19}$$

where $\hat{p}(n)$ is the estimate at time step $n$.

Finally, to remove any Monte Carlo error in the results, we ran a total of 100 experiments of length $T = 10^5$ for all cases.

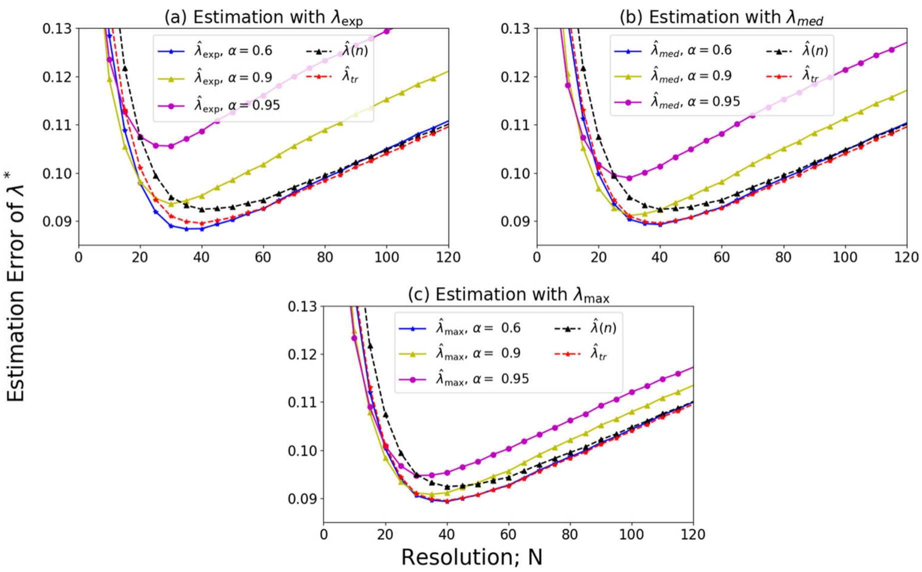



**Fig. 9** **SINE-1080-10080.** In each of the three sub-figures, one of the max, med, and exp along with the Oommen's method $\lambda(n)$ and the transition $\lambda_{tr}$ are depicted

**Table 2** Summary of the choices of tuning parameters resulting in minimum error for $\lambda_{tr}$ and $\lambda_{max}$ in SINE Experiments. The smallest error value in each experiment is represented in bold font

| Estimator \ SINE– | | 1080–1080 | | 360–360 | | 1080–360 | | 1080–10080 | |
|---|---|---|---|---|---|---|---|---|---|
| | | N | Error | N | Error | N | Error | N | Error |
| Oommen($\lambda(n)$) | | 80 | 0.04791 | 35 | 0.08634 | 65 | 0.05502 | 40 | 0.09244 |
| LTES($\lambda_{tr}$) | | 80 | **0.04682** | 35 | **0.08425** | 60 | **0.05363** | 40 | 0.08952 |
| FES($\lambda_{max}$) | $\alpha = 0.6$ | 80 | 0.0471 | 35 | 0.08533 | 60 | 0.05377 | 40 | 0.08937 |
| | $\alpha = 0.7$ | 80 | 0.04749 | 35 | 0.08659 | 60 | 0.05398 | 40 | **0.08935** |
| | $\alpha = 0.8$ | 75 | 0.0485 | 35 | 0.08965 | 60 | 0.0548 | 35 | 0.08959 |
| | $\alpha = 0.9$ | 75 | 0.05106 | 35 | 0.09706 | 60 | 0.05733 | 35 | 0.09085 |
| | $\alpha = 0.95$ | 75 | 0.05578 | 35 | 0.11246 | 55 | 0.06228 | 30 | 0.09474 |
| | $\alpha = 0.99$ | 65 | 0.111 | 20 | 0.31157 | 30 | 0.12002 | 25 | 0.14673 |

### 4.1 Moderate changes of both $\lambda^*(n)$ and $p^*(n)$

In this section both $\lambda^*(n)$ and $p^*(n)$ change moderately. Figure 1 shows the estimation error as a function of resolution for some choices of $\alpha$. At any resolution, $\lambda_{tr}$ has lower estimation error than $\lambda(n)$ and indeed, all the cases $\lambda_{max}$, $\lambda_{med}$, and $\lambda_{exp}$ perform more efficiently than $\lambda(n)$ for, at least, a specific choice of $\alpha$.

We also note that, the higher resolution will not result in a smaller error in all the cases. For instance, for $\lambda(n)$, the estimation error increases after resolution $N = 65$ in which there is a minimum of errors. As it is represented in Table 1, the minimum error for $\lambda(n)$ equals $e = 0.063149$ when $N = 65$. We reach error $e = 0.061353$ for $\lambda_{tr}$ at resolution $N = 50$. The best error for $\lambda_{max}$ is $e = 0.059138$ when $N = 50$ and $\alpha = 0.9$. The minimum error over all scenarios is $e = 0.058559$ which is achieved by $\lambda_{med}$ estimator when $N = 50$ and $\alpha = 0.95$.

Figure 2 shows the estimation error as a function of resolution for some choices of $\alpha$ for SINE-1080-1080. All the curves have an optimum resolution point in which any higher resolution cause higher estimation error. In the SINE-1080-1080 case, $\lambda_{max}$ and $\lambda_{tr}$ are best satisfying estimators. $\lambda_{tr}$ with minimum error $e = 0.04682$ at $N = 80$, slightly outperforms $\lambda_{max}$ with minimum error $e = 0.047095$ at ($N = 80$, and $\alpha = 0.6$).

### 4.2 Fast changes of both $\lambda^*(n)$ and $p^*(n)$

Here the effect of faster changes in $\lambda^*(n)$ and $p^*(n)$ are addressed through cases SWITCH-100-100 and SINE-360-360.

Figure 3 is devoted to SWITCH-100-100 that both $\lambda^*(n)$ and $p^*(n)$ randomly switch to a new value in their domain. As expected, comparing the error with the SWITCH-1000-1000 case, the estimation errors are higher. From Table 1 we see that the minimum error for the estimators $\lambda(n)$, $\lambda_{tr}$, and $\lambda_{max}$ are $e = 0.1069$, $e = 0.10258$, and $e = 0.1021$ respectively. The minimum error in case SWITCH-100-100 equals to $e = 0.1021$ and is achieved by $\lambda_{max}$ when $N = 20$ and $\alpha = 0.6$

As expected, we see that faster changing Environment could be tracked more accurately with smaller values of resolution and $\alpha$. For instance, compare resolution $N = 20$ in this case, for $\lambda_{max}$, to $N = 50$ in case SWITCH-1000-1000. The same comparison between $\alpha = 0.9$ and 0.6 shows that to track faster changing Environment we must rely on less on memory.

In Fig. 3, we observe that the best choice of $\alpha$ is dependent on the resolution; for example, if $N = 5$, the $\lambda_{max}$, $\lambda_{med}$, and $\lambda_{exp}$ with $\alpha = 0.95$ and $\alpha = 0.9$ are superior to the choices with $\alpha = 0.6$. However $N = 15$, $\alpha = 0.6$ would be a more desirable option.

Regarding fast changes, Fig. 4 is devoted to SINE-360-360 in which both $\lambda^*(n)$ and $p^*(n)$ change continuously as a sine function with period 360 degree. The minimum error in this case equals $e = 0.08425$ that is achieved by $\lambda_{tr}$ at $N = 35$. Again, the simpler estimator $\lambda_{tr}$ outperforms $\lambda_{max}$ with minimum error $e = 0.085329$ when ($N = 35$ and $\alpha = 0.6$). Note that, the $\lambda_{max}$ estimator is more efficient than $\lambda(n)$ that has minimum error $e = 0.08634$ when $N = 35$. In comparison with SINE-1080-1080, the estimated error is higher and the best resolution is much smaller in case SINE-360-360. Compare the best resolution $N = 35$ to the case SINE-1080-1080 which equals to $N = 80$. Notice that $\alpha$ values closer to 1, produce weaker estimations.

### 4.3 Effect of changing rate of $p^*(n)$ on estimation of $\lambda^*(n)$ in SWITCH cases

In this section the effect of changing rate of $p^*(n)$ on estimating $\lambda^*(n)$ in SWITCH dynamic is examined. Two alternative cases SWITCH-1000-100 (Fig. 5) and

SWITCH-1000-10000 (Fig. 6) are compared to SWITCH-1000-1000 (Fig.1). In Fig. 7 a trace plot for tracking $\lambda^*(n)$ through LTES ($\lambda_{tr}$) is presented and the behavior of the estimator is discussed. Additionally, Table 1 summarizes the choices of tuning parameters resulting into the minimum error for $\lambda_{tr}$ and $\lambda_{max}$ to the SWITCH cases.

From Fig. 5 and Table 1, we observe that estimators perform better in SWITCH-1000-100 in comparison with SWITCH-1000-1000. For instance, compare the minimum error of estimator $\lambda_{tr}$ in case SWITCH-1000-100 which is $e = 0.03671$ for $N = 80$ with $e = 0.06135$ for $N = 50$ in case SWITCH-1000-1000.

The minimum error in various settings is $e = 0.03538$ which is achieved by $\lambda_{med}$ estimator when $N = 75$ and $\alpha = 0.9$.

Figure 6 presents the case SWITCH-1000-10000 where $p^*(n)$ changes ten times slower than SWITCH-1000-1000. It is observable that estimators show a better performance in case SWITCH-1000-10000 compared with SWITCH-1000-1000. As presented in Table 1 we see that the minimum error for the estimators $\lambda(n)$, $\lambda_{tr}$, and $\lambda_{max}$ are $e = 0.05027$, $e = 0.048795$, and $e = 0.04521$ respectively. The best estimator is $\lambda_{max}$ when $\alpha = 0.95$ and $N = 50$.

In summary, the results, as shown in Figs. 5, 6, and Table 1, indicate that when the Environment effectiveness changes fast, the minimum estimation error will be smaller. Compare minimum errors $e = 0.03539$ to $e = 0.05914$ and $e = 0.04521$ for cases SWITCH-1000-100, SWITCH-1000-1000, and SWITCH-1000-10000 respectively. However, the error in SWITCH-1000-10000 when $p^*(n)$ changes very slow is better than moderate changes in SWITCH-1000-1000. This result is somewhat counterintuitive. In order to understand it, we compare the trace plots of SWITCH-1000-100, SWITCH-1000-1000, and SWITCH-1000-10000 together in Fig. 7.

Figure 7 shows tracking $\lambda^*(n)$ under optimal choices of parameters for $\lambda_{tr}$ in order to study the impact of Environment effectiveness on estimation of $\lambda^*(n)$. For the sake of simplicity, suppose there is a same chain $\lambda^*(n)$ in all the cases. Consider $\lambda^*(n)$ along with three Environment effectiveness chains $p^*_f(n)$, $p^*_m(n)$, and $p^*_s(n)$, for SWITCH-1000-100 (fast changes), SWITCH-1000-1000 (moderate changes), and SWITCH-1000-10000 (slow changes) respectively, in which their average value are approximately the same i.e.

$$\frac{1}{T}\sum_{t=1}^{T} p^*_f(n) \approx \frac{1}{T}\sum_{t=1}^{T} p^*_m(n) \approx \frac{1}{T}\sum_{t=1}^{T} p^*_s(n).$$

Consider SWITCH-1000-10000 with $p^*_s(n)$ and let the estimation of $\lambda^*(n)$ be $\hat{\lambda}(n)$; suppose the following three scenarios:

1. The Environment effectiveness is close to 1, see $n = 60000$ to $n = 70000$ in Fig. 7c. $\lambda^*(n)$ is easily tracked in this segment and the estimation $\hat{\lambda}(n)$ is satisfactory.
2. The Environment effectiveness is slightly distant from 1, but it is informative, see $n = 70000$ to $n = 80000$ in Fig. 7c where $p^*_s(n)$ value is close to 0.8. Because the information from Environment is somewhat faulty, tracking the point location in this segment is more difficult, but still satisfactory.
3. The Environment effectiveness has a value close to 0.5, see $n = 80000$ to $n = 90000$ in Fig. 7c. The estimation $\hat{\lambda}(n)$ is unsatisfactory and it is almost a random chain with a lot of fluctuations. The reason is that estimator does not receive new information from Environment and after a short time $\hat{\lambda}(n)$ will deviate from $\lambda^*(n)$.

In summary, we keep the well estimation in the first segment, the estimation performance is reduced in the second segment, but still satisfactory. Within the third segment, possibility of error is rather high, and $\hat{\lambda}(n)$ fluctuates at a distant point from $\lambda^*(n)$. For $p^*_s(n)$ segments like the last one is discouraging, since we remain in an unsatisfactory situation for a long period of time.

Alternatively, consider the Environment effectiveness $p^*_f(n)$, Fig. 7a. It is possible to detect segments like the above three segments but with a much shorter length. So, the behavior of $\hat{\lambda}(n)$ in each of them is not long lasting. Faster changes make the behavior of estimators more like the second segment, with fluctuations around $\lambda^*(n)$.

In Fig. 7b; i.e. SWITCH-1000-1000 case, the error is the highest among the three cases. In this case, $p^*_m(n)$ and $\lambda^*(n)$ are changing at the same time. So, the changes of $p^*_m(n)$ has no positive effect on estimation of $\lambda^*(n)$. The best resolution in this case equals $N = 50$, which suggests more changes than SWITCH-1000-10000 with $N = 65$ and SWITCH-1000-100 with $N = 80$. This smaller resolution, produces a higher error. To investigate the negative effect of the simultaneous changes more, we run the SWITCH-1000-1000 case where there is a 500 steps delay between $\lambda^*(n)$ and $p^*_m(n)$ changes. That reduces the minimum error to $e = 0.05144$ for $N = 65$, and approves the negative effect of simultaneous changes in SWITCH cases.

The main observations in Fig. 7 are:

– Tracking $\lambda^*(n)$ is heavily affected by Environment effectiveness. In Fig. 7c, there are no fluctuations when $p^*_s(n) \approx 1$, however, when $p^*_s(n) \approx 0.8$, the estimator fluctuates more around the optimal $\lambda^*(n)$, and then when $p^*_s(n)$ is slightly larger than 0.5 the fluctuations are much more bigger.
– Faster changes in Environment effectiveness leads to better estimations of $\lambda^*(n)$. Note that, the rate of changes must be regulated in a way that $\hat{\lambda}(n)$ can

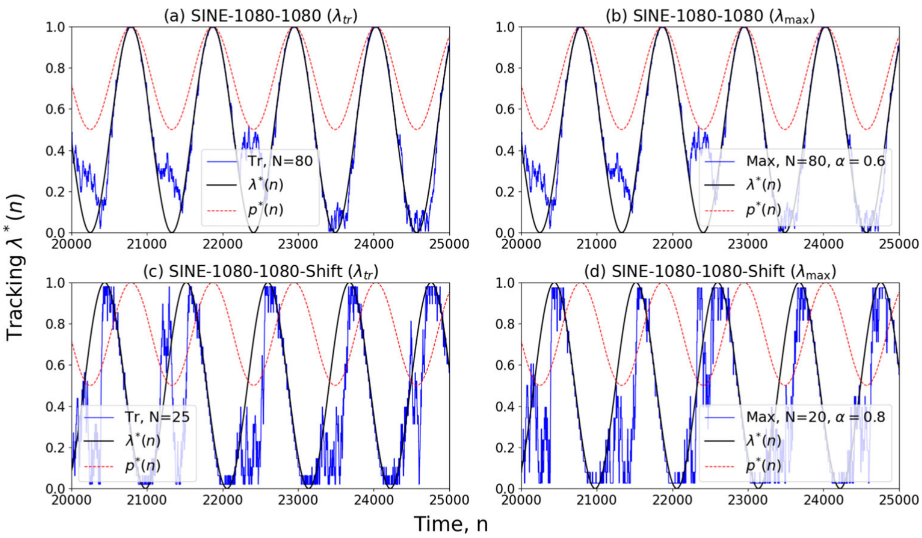



**Fig. 10** **SINE-1080-1080.** **a** and **b** subfigures show how the estimation tracks $\lambda^*(n)$ when the argument of sine function is the same by $\lambda_{tr}$ and $\lambda_{max}$, respectively. The **c** and **d** sub-figures show the same when the argument of sine function differ by $\pi/2$

converge to $\lambda^*(n)$ when Environment effectiveness is close to 1.

- When $\lambda^*(n)$ and $p^*(n)$ changes together, it is much harder to track $\lambda^*(n)$.
- Since the average values of Environment effectiveness in three cases are supposed to be the same, and the most estimation error is produced in the third segment, there is a better performance in case $p^*_f(n)$ in total.

From Table 1, we observe that the best estimations belong to the case SWITCH-1000-100 and estimator $\lambda_{max}$.

### 4.4 Effect of changing rate of $p^*(n)$ on estimation of $\lambda^*(n)$ in SINE cases

This section focuses on the SINE dynamic and presents the results for the effect of changing rate of $p^*(n)$ on estimating $\lambda^*(n)$. Two alternative cases SINE-1080-360 (Fig. 8) and SINE-1080-10080 (Fig. 9) are compared with SINE-1080-1080 (Fig. 2), and Table 2 summarizes the same data as Table 1 for SINE cases.

As reported in Table 2, the best estimation error for case SINE-1080-360 (Fig. 8) is achieved through $\lambda_{tr}$ at $N = 60$ which equals to $e = 0.05363$; compare to the best estimation error for SINE-1080-1080 that equals $e = 0.04682$. In contrast to the SWITCH case, we see that faster changes of probability does not result in smaller estimation errors. We later explain that along with the rate of changes, another factor which plays a role is the phase of changes. In SINE-1080-1080 both $\lambda^*(n)$ and $p^*(n)$ are in phase but in SINE-1080-360 they have different periods and can not be in phase. The effect of this will be addressed in Section 4.5 in details. The final case, SINE-1080-10080 in Fig. 9, provides a more clear insight.

In SINE-1080-10080, the changes are asymmetric and the Environment effectiveness varies slower. The minimum estimation error equals $e = 0.08935$ and occurs for $\lambda_{max}$ at $(N = 40, \alpha = 0.7)$. In this case, we observe that $\lambda_{max}$ estimator slightly outperforms $\lambda_{tr}$. The minimum error for $\lambda_{tr}$ is $e = 0.08952$ at $N = 40$. Moreover, $\lambda_{exp}$ results the best minimum error, $e = 0.08835$ when $N = 35$ and $\alpha = 0.6$

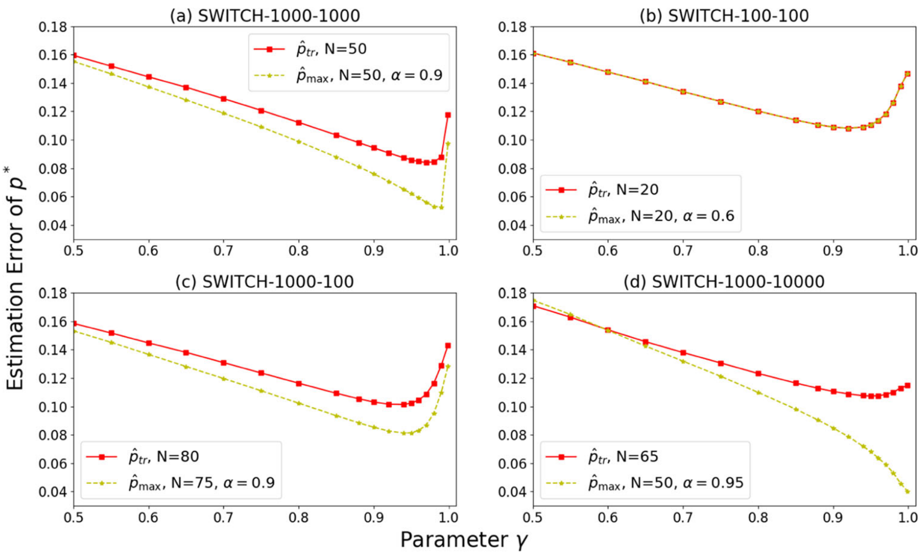
By comparing SINE-1080-10080 with SINE-1080-1080 and SINE-1080-360, we observe that its estimation error is the weakest.

If only SINE-1080-360 and SINE-1080-10080 are compared with together, we detect a better estimation at

**Table 3** Summary of SINE-1080-1080 alternatives

| $n$ range | $\theta$ range | $\lambda^*(n)$ range | $p^*_1(n)$ range | $p^*_2(n)$ range |
|---|---|---|---|---|
| $n_1 - n_2$ | $(2k\pi + \pi/4) - (2k\pi + 3\pi/4)$ | [0.85, 1] | [0.93, 1] | [0.57, 0.93] |
| $n_2 - n_3$ | $(2k\pi + 3\pi/4) - (2k\pi + 5\pi/4)$ | [0.15, 0.85] | [0.57, 0.93] | [0.93, 1] |
| $n_3 - n_4$ | $(2k\pi + 5\pi/4) - (2k\pi + 7\pi/4)$ | [0, 0.15] | [0.5, 0.57] | [0.57, 0.93] |
| $n_4 - n_5$ | $(2k\pi + 7\pi/4) - (2(k + 1)\pi + \pi/4)$ | [0.15, 0.85] | [0.57, 0.93] | [0.5, 0.57] |

Fig. 11 Estimation of $p^*(n)$ with binomial weak estimation via two alternative choices of $\lambda^*(n)$ estimations ($\lambda_{\max}$ and $\lambda_{\mathrm{tr}}$) for SWITCH-1000-1000 **a**, SWITCH-100-100 **b**, SWITCH-1000-100 **c**, and SWITCH-1000-10000 **d**




faster changing Environment effectiveness. While SINE-1080-1080 is not following this hypothesis. In contrast to SWITCH cases, where moderate changes of $p^*(n)$ in SWITCH-1000-1000 show the weakest performance, moderate changes of $p^*(n)$ in SINE-1080-1080 show the best results. This suggests that another factor affects the estimation. Later in Section 4.5, Fig. 10, we explain it through the assessment of two different trace plots for SINE-1080-1080 case.

We have collected the best parameter values and resulted minimum errors of $\lambda(n)$, $\lambda_{\mathrm{tr}}$, and $\lambda_{max}$ in Table 2. The best estimations belong to the case SINE-1080-1080. Here, LTES estimator ($\lambda_{\mathrm{tr}}$) is the best estimator.

### 4.5 The relation between $\lambda^*(n)$ and $p^*(n)$ changes and the estimation performance

To study the effect of the relation between $\lambda^*(n)$ and $p^*(n)$ dynamics on the estimators, we consider the case SINE-1080-1080 that both $\lambda^*(n)$ and $p^*(n)$ are changing according a sine curve. So, we re-run the SINE-1080-1080 case when the arguments of sine functions for $\lambda^*(n)$ and $p^*(n)$ differ by $\pi/2$. More formally, what we have reported on Fig. 2 and on the top of Fig. 10 is $\lambda^*(n) = 0.5 + 0.5\sin(\frac{\pi}{3\cdot 180})$ and $p^*(n) = 0.75 + 0.25\sin(\frac{\pi}{3\cdot 180})$. In the second run, $\lambda^*(n)$ argument is added by $\pi/2$, so $\lambda^*(n) = 0.5 + 0.5\sin(\frac{\pi}{3\cdot 180} + \pi/2)$ (Fig. 10c and d).

Fig. 12 Estimation of $p^*(n)$ with binomial weak estimation via two alternative choices of $\lambda^*(n)$ estimations ($\lambda_{\max}$ and $\lambda_{\mathrm{tr}}$) for SINE-1080-1080 **a**, SINE-360-360 **b**, SINE-1080-360 **c**, and SINE-1080-10080 **d** Environments

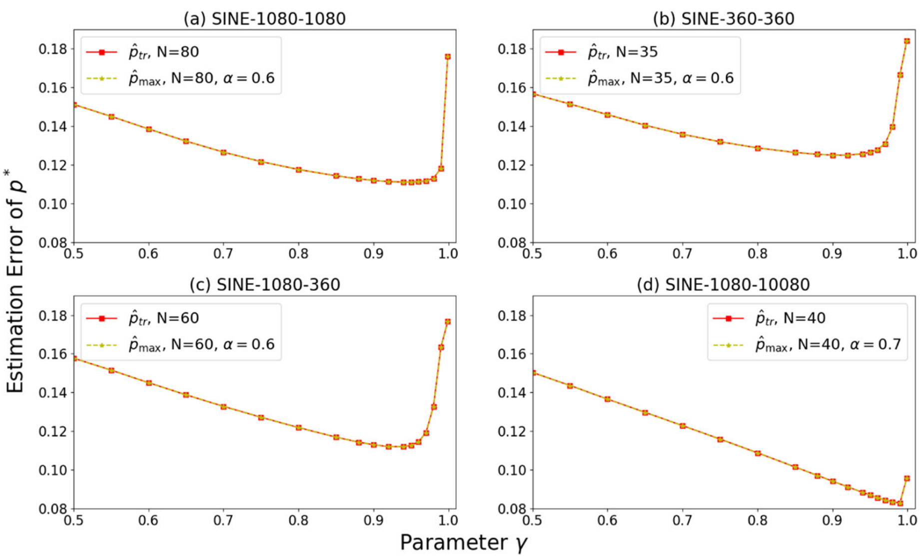


**Table 4** Summary of tuning parameters resulting into minimum error, along with the parameters $N = 5$ and $\alpha = 0.9$ for $p_{tr}$ and $p_{max}$. The lowest error value among $p_{tr}$ and $p_{max}$ for each case is represented in bold font

| Case \ Estimator | $\lambda_{tr}$ | | | $\lambda_{max}$ | | | |
|---|---|---|---|---|---|---|---|
| | N | $\gamma$ | Error | N | $\alpha$ | $\gamma$ | Error |
| SWITCH-1000-1000 | 50 | 0.97 | 0.08406 | 50 | 0.9 | 0.99 | **0.0524** |
| SWITCH-1000-1000 | 5 | 0.98 | 0.06812 | 5 | 0.9 | 0.99 | **0.04199** |
| SWITCH-100-100 | 20 | 0.92 | **0.10819** | 20 | 0.6 | 0.92 | **0.10819** |
| SWITCH-100-100 | 5 | 0.94 | 0.08967 | 5 | 0.9 | 0.95 | **0.07445** |
| SWITCH-1000-100 | 80 | 0.94 | 0.10135 | 75 | 0.9 | 0.94 | **0.08126** |
| SWITCH-1000-100 | 5 | 0.94 | 0.08697 | 5 | 0.9 | 0.95 | **0.07086** |
| SWITCH-1000-10000 | 65 | 0.95 | 0.10736 | 50 | 0.95 | 0.999 | **0.04036** |
| SWITCH-1000-10000 | 5 | 0.999 | 0.08444 | 5 | 0.9 | 0.999 | **0.0399** |
| SINE-1080-1080 | 80 | 0.94 | **0.11103** | 80 | 0.6 | 0.94 | **0.11103** |
| SINE-1080-1080 | 5 | 0.97 | 0.06308 | 5 | 0.9 | 0.97 | **0.05056** |
| SINE-360-360 | 35 | 0.92 | **0.125** | 35 | 0.6 | 0.92 | **0.125** |
| SINE-360-360 | 5 | 0.94 | 0.077 | 5 | 0.9 | 0.94 | **0.07433** |
| SINE-1080-360 | 60 | 0.94 | **0.11201** | 60 | 0.6 | 0.94 | **0.11201** |
| SINE-1080-360 | 5 | 0.94 | 0.07282 | 5 | 0.9 | 0.94 | **0.06873** |
| SINE-1080-10080 | 40 | 0.99 | **0.08299** | 40 | 0.7 | 0.99 | **0.08299** |
| SINE-1080-10080 | 5 | 0.99 | 0.04871 | 5 | 0.9 | 0.99 | **0.0345** |
| SINE-1080-1080-Shift | 25 | 0.97 | 0.07681 | 20 | 0.8 | 0.97 | **0.762** |
| SINE-1080-1080-Shift | 5 | 0.97 | 0.0586 | 5 | 0.9 | 0.97 | **0.05437** |

Hereafter, we call it SINE-1080-1080-Shift. We observe significant differences between tracking $\lambda^*(n)$ throughout the two scenarios SINE-1080-1080 (Fig. 10a and b) and SINE-1080-1080-Shift (Fig. 10c and d). The estimators $\lambda_{tr}$ and $\lambda_{max}$ track $\lambda^*(n)$ more accurately in SINE-1080-1080 than SINE-1080-1080-Shift. Moreover, the minimum estimation error for $\lambda_{tr}$ in SINE-1080-1080-Shift is $e = 0.10504$, while it equals to $e = 0.04682$ for $\lambda_{tr}$ in SINE-1080-1080. Similarly, the minimum estimation error for $\lambda_{max}$ in SINE-1080-1080-Shift is $e = 0.10382$. Compare it to $e = 0.0471$ for $\lambda_{max}$ in SINE-1080-1080. We explain it by analyzing Fig. 10. In general, when $\lambda^*(n)$ value is close to 0 or 1, the effect of wrong guidance from Environment is reduced. The reason is that $\lambda(n)$ cannot pass the boundaries.

Let $\lambda^*(n) = 0.5 + 0.5\sin(\theta)$, so we have $p_1^*(n) = 0.75+0.25\sin(\theta)$ for SINE-1080-1080 and $p_2^*(n) = 0.75 + 0.25\sin(\theta - \pi/2)$ for SINE-1080-1080-Shift.

Let us take the case SINE-1080-1080 where $p^*(n)$ and $\lambda^*(n)$ are in phase, i.e. $p^*(n) = p_1^*(n)$ . Interestingly, the valley of $p_1^*(n)$ corresponds to the valley of $\lambda^*(n)$. Since $p_1^*(n)$ has a valley around 0.5, then the tracking of $\lambda^*(n)$ will be handicapped during that period but this will not affect much the accuracy as $\lambda^*(n)$ is also experiencing a valley and the changes are slow over that valley. However, in case SINE-1080-1080-Shift where $p_2^*(n)$ and $\lambda^*(n)$ are out of phase, a valley of $p_2^*(n)$ coincides with a change of $\lambda^*(n)$ from its lowest value to its biggest value. Then, during that valley of $p_2^*(n)$, $\lambda^*(n)$ tracking gets handicapped and the error is big due to the scheme not being able to track the true underlying $\lambda^*(n)$ that changes dramatically from its min to its max. To be more precise, let

$\theta_1 = 2k\pi + \pi/4$, $\theta_2 = 2k\pi + 3\pi/4$, $\theta_3 = 2k\pi + 5\pi/4$, $\theta_4 = 2k\pi + 7\pi/4$, and $\theta_5 = 2(k+1)\pi + \pi/4$ where $k$ is a positive integer. This way we divide a period of 1080 steps to four equal parts each with 270 steps. For the above values, $\lambda^*(n_1)$ up to $\lambda^*(n_2)$ is situated in the range [0.85, 1]; i.e. in 270 successive steps the point location is placed within this range. However, for the next 270 steps, i.e. from $\lambda^*(n_2)$ to $\lambda^*(n_3)$, the values locate in range [0.15, 0.85]. We observe that the rate of changes is not uniform. Similarly, $\lambda^*(n_3)$ to $\lambda^*(n_4)$ is placed in the range [0, 0.15] and $\lambda^*(n_4)$ to $\lambda^*(n_5)$ is situated in the range [0.15, 0.85].

Similar to the discussions we had about SWITCH cases, we have

$$\frac{1}{T}\sum_{t=1}^{T} p_1^*(n) \approx \frac{1}{T}\sum_{t=1}^{T} p_2^*(n).$$

This time $p_1^*(n)$ and $p_2^*(n)$ are exactly the same, but their relation to $\lambda^*(n)$ makes them different.

In range $n_1$ to $n_2$, where $\lambda^*(n)$ is located in [0.85, 1], $p_1^*(n_1)$ to $p_1^*(n_2)$ is situated in the range [0.93, 1]. In range $n_2$ to $n_3$, where $\lambda^*(n)$ is located in [0.15, 0.85], $p_1^*(n_2)$ to $p_1^*(n_3)$ is situated in the range [0.57, 0.93]. Moreover, in the range $n_3$ to $n_4$, where $\lambda^*(n)$ is located in [0, 0.15], $p_1^*(n_3)$ to $p_1^*(n_4)$ is placed in the range [0.5, 0.57]. Finally, for range $n_4$ to $n_5$, where $\lambda^*(n)$ is located in [0.15, 0.85], $p_1^*(n_4)$ to $p_1^*(n_5)$ is situated in the range [0.57, 0.93].

The intervals for values of $p_2^*(n)$ are achieved through shifting $p_1^*(n)$ values. When $\lambda^*(n)$ is in range [0.85, 1] it takes values in [0.57, 0.93], and when $\lambda^*(n)$ is in range [0.15, 0.85] it takes values in [0.93, 1]. When $\lambda^*(n)$ is placed in range [0, 0.15] it takes values in [0.57, 0.93], and when $\lambda^*(n)$ is placed in range [0.15, 0.85] it takes values in [0.5, 0.57]. See Table 3 for a summary:

A comparison of the two Environments reveals why estimation of SINE-1080-1080 ($p_1^*(n)$) outperforms SINE-1080-1080-Shift ($p_2^*(n)$):

- In range $n_1 - n_2$, since $p_1^*(n)$ values are higher, we will have more promising estimations. Note that $p_2^*(n)$ is in range [0.57, 0.93], $\lambda^*(n)$ is close to 1, when it reaches its peak, hence its value changes slowly. That is to say in this period, estimations in SINE-1080-1080-Shift are satisfactory. See, for instance, around $n = 24500$ in Fig. 10c and d.
- In range $n_2 - n_3$, the changes in $\lambda^*(n)$ are fast. Estimation in SINE-1080-1080 case is more difficult than SINE-1080-1080-Shift, because $p_1^*(n)$ is in range [0.57, 0.93] and $p_2^*(n)$ is in range [0.93, 1].
- In range $n_3 - n_4$, the changes in $\lambda^*(n)$ are not fast and the value is close to the boundary. $p_1^*(n)$ is in range [0.5, 0.57] while the information from Environment is almost random. However, since $\lambda^*(n)$ is in a peak, its value is close to boundary and does not change fast, as we explained before, the most fluctuations will be nearby the true $\lambda^*(n)$; see around $n = 21000$ in Fig. 10a and b. Tracking $\lambda^*(n)$ changes in SINE-1080-1080-Shift case is more accurate than in SINE-1080-1080 case.
- In range $n_4 - n_5$, the changes in $\lambda^*(n)$ are fast. Tracking $\lambda^*(n)$ in SINE-1080-1080 Environment, similar to the range $n_2 - n_3$, is acceptable to some extent. However, tracking the point location in SINE-1080-1080-Shift Environment is almost impossible. As can be seen in Fig. 10c and d around $n = 22500$, the combination of fast changes of $\lambda^*(n)$ and distance from boundaries, cause huge deviation. Such periods result in higher estimation error in SINE-1080-1080-Shift Environment than SINE-1080-1080.

Therefore, both the rate of changes in Environment effectiveness and its relationship to the point location might affect the estimations.

### 4.6 Estimation of environment effectiveness: $p^*(n)$

In Figs. 11 and 12 the estimation error for various SWITCH and SINE cases are presented respectively. Moreover, Table 4 summarizes the choices of tuning parameters resulting into the minimum error while Environment effectiveness is estimated.

In order to depict the estimation performance of $p^*(n)$, we restrict the results to estimation based on two cases $\lambda_{\max}$ and $\lambda_{\text{tr}}$, for these are the main contribution in this paper which perform the best. Moreover, we consider the best parameters of these two estimators based on results of previous error plots. The tuning parameters resulted to the best minimum error reported in Table 4. We will consider and discuss the results for $N = 5$ and $\alpha = 0.9$ in Table 4 later. In the following we will compare the results from best $\lambda^*(n)$ estimations. The best minimum error in case SWITCH-1000-1000 is achieved for $\hat{p}_{\max}$ when ($N = 50, \alpha = 0.9$ and $\gamma = 0.99$) equals to $e = 0.0524$. The error for alternative method, i.e. $\hat{p}_{\text{tr}}$ equals to $e = 0.08406$ when ($N = 50$, and $\gamma = 0.97$), see Table 4.

The best minimum error in case SWITCH-100-100 is simultaneously achieved for $\hat{p}_{\max}$ and $\hat{p}_{\text{tr}}$ at value $e = 0.10819$ with parameters ($N = 20, \alpha = 0.6$, and $\gamma = 0.92$). In comparison with SWITCH-1000-1000, it is weaker than the previous case in which Environment changes more slowly.

In case SWITCH-1000-100, the best minimum error is obtained through $\hat{p}_{\max}$ when ($N = 75, \alpha = 0.9$ and $\gamma = 0.94$), that equals to $e = 0.08126$. Error for $\hat{p}_{\text{tr}}$ equals to $e = 0.10135$ in the case ($N = 80$, and $\gamma = 0.94$).

Finally, the case SWITCH-1000-10000 where the minimum error for $\hat{p}_{\max}$ when ($N = 50, \alpha = 0.95$ and $\gamma = 0.999$) equals to $e = 0.04036$. The error for alternative method $\hat{p}_{\text{tr}}$ equals to $e = 0.10736$ in case that ($N = 65$, and $\gamma = 0.95$).

Overall, it seems like $\lambda_{\max}$ performs a little better than $\lambda_{\text{tr}}$. However, a significant disadvantage of $\lambda_{\max}$ compared with $\lambda_{\text{tr}}$ is that the tracking of $\lambda^*(n)$ requires tuning of two parameters compared to only one for $\lambda_{\text{tr}}$. For dynamically changing environments it is usually hard enough to tune one parameter.

In case SINE-1080-1080 illustrated in Fig. 12a, we choose $N = 80$ for $\lambda_{\text{tr}}$ and ($N = 80, \alpha = 0.6$) for $\lambda_{\max}$ as the best parameters. As reported in Table 4, the best estimation error for both $\hat{p}_{\max}$ and $\hat{p}_{\text{tr}}$ occurs at $\gamma = 0.94$ and equals $e = 0.11103$.

Similarly, we observe that both estimators are equally well for cases SINE-360-360, SINE-1080-360, and SINE-1080-10080; where the best estimation error equals to $e = 0.125$, $e = 0.11201$, and $e = 0.08299$ respectively, see Table 4 for more details.

Even though the estimation error for point location in case SINE-1080-10080 is weaker than all the cases SINE-1080-1080, SINE-360-360, and SINE-1080-360, its estimated probability is preferred; because slower changes can be tracked more easily.

In SINE cases, apart from the case SINE-1080-10080 and SINE-1080-1080-Shift, the estimation error is poor, i.e. optimal error is greater than 0.1 in both $\hat{p}_{\text{tr}}$ and $\hat{p}_{\max}$. In Figs. 11, 12 and Table 4, we see that in some cases $\hat{p}_{\text{tr}}$ and $\hat{p}_{\max}$ perform equally well, even though the estimators $\lambda_{\text{tr}}$ and $\lambda_{\max}$ results are different. The reason is in estimation of Environment effectiveness, the important data is whether the suggested direction by Environment agrees with the estimation or not. In other words the distance between estimation and the point location is not important, and the crucial issue is that both point location and its estimation, are at the same side- left or right- of the query location. So we can have exactly the same results even if the estimated point is not the same in two estimators.

A natural question that might arise is that whether the best parameters for estimation of $\lambda^*(n)$ are the best for estimating $p^*(n)$ or not. A simple simulation, where we set $N = 5$ and $\alpha = 0.9$, provides a negative answer to this question. These parameters result into a smaller estimation error for all the cases compare to the best parameters. Moreover, the $\hat{p}_{\max}$ estimations are all better than $\hat{p}_{\text{tr}}$. For instance, in case SWITCH-1000-1000 the estimation error for $\hat{p}_{\text{tr}}$ drops from $e = 0.08406$ for $N = 50$, to $e = 0.06812$ for $N = 5$. For $\hat{p}_{\max}$, error drops from $e = 0.0524$ to $e = 0.04199$ for the same resolutions. Similarly, for SINE-1080-1080, please compare the error $e = 0.11103$ for $N = 80$ and $\alpha = 0.6$ to $e = 0.06308$ and $e = 0.05056$ for $\hat{p}_{\text{tr}}$ and $\hat{p}_{\max}$ respectively, when $N = 5$ and $\alpha = 0.9$; see Table 4. We try to justify the reason behind this in the following.

Recall (4), (5), and (6) where we have:

$$
\begin{aligned}
&\pi_i = e.\pi_{i-1} \text{ whenever } i \le Z,\\
&\pi_{Z+1} = \pi_Z, \text{ and}\\
&\pi_i = \frac{\pi_{i-1}}{e} \text{ whenever } i > Z+1,
\end{aligned}
$$

where $e = \frac{p}{q}$. To find a relation between resolution and $\pi_Z^+$ we have:

$$
\begin{aligned}
1 &= \sum_{i=0}^{N-1} \pi_i^+\\
&= \sum_{i=0}^{Z-1} \pi_i^+ + \sum_{i=Z+1}^{N-1} \pi_i^+ + \pi_Z^+
\end{aligned}
\tag{20}
$$

By substituting the relations (7), (8), and (9):

$$
\begin{aligned}
&= 2q\left(\sum_{i=1}^{Z-1} \pi_i + \sum_{i=Z+1}^{N-1} \pi_i\right) + 2p(\pi_Z)\\
&= 2q\pi_Z\left(\sum_{i=1}^{Z-1}\left(\frac{1}{e}\right)^i + \sum_{i=Z+1}^{N-1}\left(\frac{1}{e}\right)^{i-Z}\right) + 2p(\pi_Z)
\end{aligned}
\tag{21}
$$

By removing $q$ and simplification:

$$
= 2p\pi_Z\left[1 + \frac{1}{e^2}\left(\sum_{i=0}^{Z-2}\left(\frac{1}{e}\right)^i + \sum_{i=Z}^{N-2}\left(\frac{1}{e}\right)^{i-Z}\right)\right]. \text{ So}
$$

$$
\pi_Z = \frac{1}{2p\left[1 + \frac{1}{e^2}\left(\sum_{i=0}^{Z-2}\left(\frac{1}{e}\right)^i + \sum_{i=Z}^{N-2}\left(\frac{1}{e}\right)^{i-Z}\right)\right]} \tag{22}
$$

The above equation implies that for a static environment the larger $N$, the smaller $\pi_Z$. Since for the estimation of $p^*(n)$ the accuracy of point location is not important, a smaller resolution will increase the probability to be at the correct pair, i.e. $Z/N \le \lambda^*(n) < (Z+1)/N$. Based on this argument, we can formally prove that a smaller resolution gives a better estimation of $p^*(n)$ while a larger resolution yields a better estimation of $\lambda^*(n)$. Based on the above theoretical result that is in accordance with our experimental results, we therefore suggest to run the SPL in parallel using two different resolutions: a smaller resolution for better estimation of $p^*(n)$ and a larger resolution for better estimation of $\lambda^*(n)$.

## 4.7 Environment effectiveness tracking

The results of estimation of Environment effectiveness through tracking curves, which are depicted in Figs. 13 and 14, are analyzed in this section.

Figure 13 compares tracking $p^*(n)$ for two cases SWITCH-1000-100 (a-b) and SWITCH-1000-10000 (c-d) based on estimations $\lambda_{\max}$ and $\lambda_{\text{tr}}$. We observe that $\hat{p}_{\text{tr}}$ fluctuation is higher than $\hat{p}_{\max}$. Indeed, $\hat{p}_{\max}$ documents a little better peak performance (as the trace plots show), but at the price of requiring tuning of an additional parameter in $\lambda_{\max}$.

Moreover, as we see more clear at SWITCH-1000-10000 case, whenever probability is closer to 1, any change in $\lambda^*(n)$ intensely affects the $p^*(n)$ estimators. So we detect sharper changes in $p^*(n)$ estimators within the range of $n = 60000$ to $n = 70000$. Then, there is a middle range probability around 0.8 from $n = 70000$ to $n = 80000$. In this range estimators fluctuate more but change less shapely. Interestingly, within the range of $n = 80000$ to $n = 90000$ there are fewer fluctuations. The reason is that

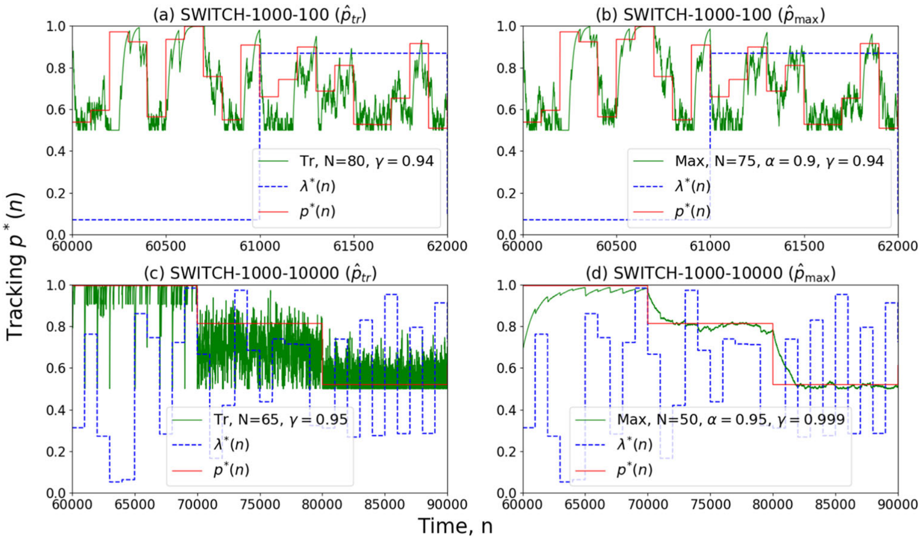



**Fig. 13** Comparison of how the estimators track $p^*(n)$ with different dynamics. The sub-figures **a** and **b** show how $\lambda^*(n)$ is tracked by $\lambda_{tr}$ and $\lambda_{max}$ respectively in case SWITCH-1000-100. Sub-figures **c** and **d** track $\lambda^*(n)$ by $\lambda_{tr}$ and $\lambda_{max}$ respectively, in case SWITCH-1000-10000. In SWITCH-1000-100 cases a slice of Environment from $n = 60000$ to $n = 62000$ are represented. The represented slice for SWITCH-1000-100 is $n = 60000$ to $n = 90000$

the Environment provides almost random directions and the changes of $\lambda^*(n)$ are not followed by the estimators efficiently. Since $\lambda^*(n)$ estimation is the basis for $p^*(n)$ estimation, the changes in $\lambda^*(n)$ could not affect $p^*(n)$ estimations. Therefore, there are no sharp changes when $p^*(n)$ is close to 0.5.

Even though the estimation error for point location in case SWITCH-1000-10000 is weaker than SWITCH-1000-100, its estimated probability is preferred; because slower changes can be tracked more easily.

Figure 14 compares tracking $p_1^*(n)$ with $p_2^*(n)$ for SINE-1080-1080 and SINE-1080-1080-Shift, based on two estimations $\lambda_{max}$ and $\lambda_{tr}$. An interesting observation regarding the Fig. 14 is the different behavior of estimations near value 1. The estimation for $p_1^*(n)$ is more accurate comparing to $p_2^*(n)$, which can be explained due to the value of $\lambda^*(n)$. The tracking is promoted by the fact that in SINE-1080-1080 (Fig. 14a and b), $\lambda^*(n)$ is both close to 1 and changes more slowly. However, the tracking is weakened in SINE-1080-1080-Shift due to $\lambda^*(n)$ changes faster in the middle ranges. Through comparing the two results, it can be seen that although the estimation of $\lambda^*(n)$ is weaker in SINE-1080-1080-Shift, the proposed estimators for $p^*(n)$ in SINE-1080-1080-Shift are more precise. Compare the minimum error $e = 0.07681$ in SINE-1080-1080-Shift to $e = 0.11103$ in SINE-1080-1080.

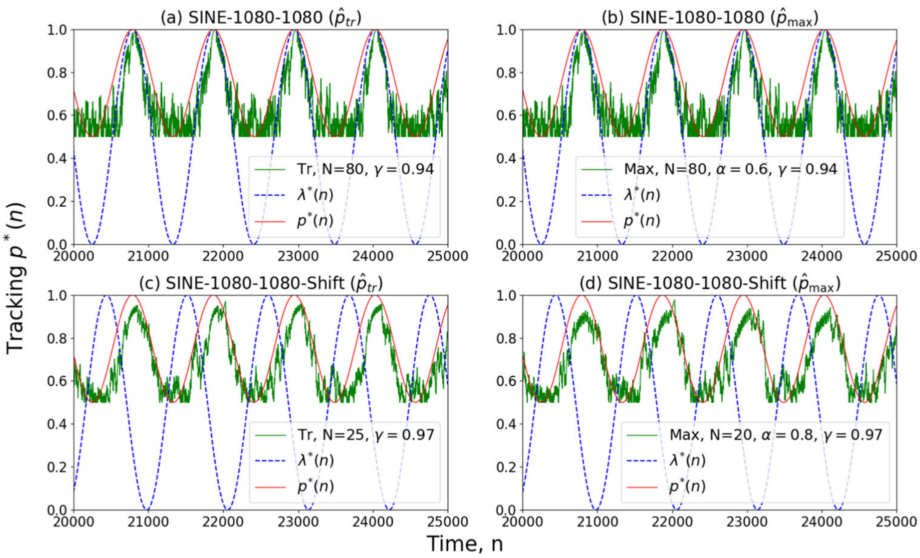
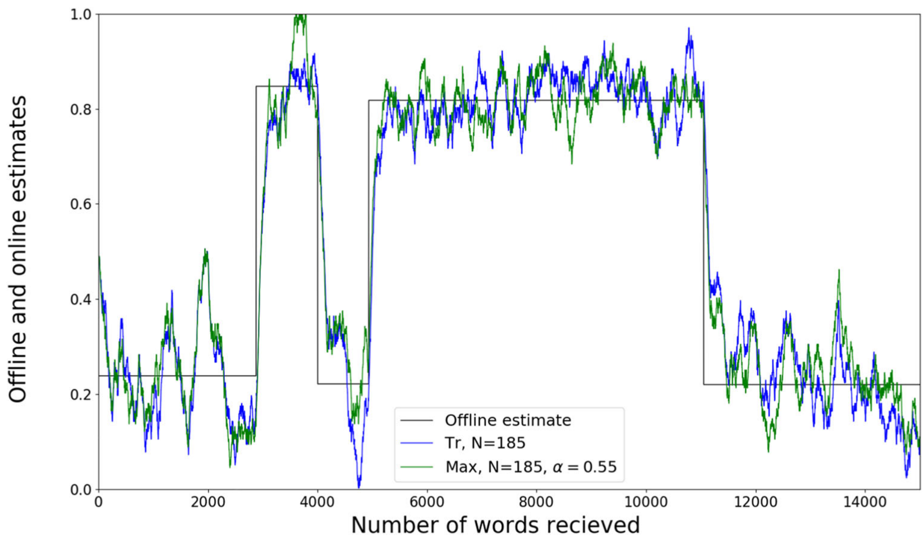



**Fig. 14** **SINE-1080-1080.** Comparison of how the estimations track $p^*(n)$, when $\lambda^*(n)$ and $p^*(n)$ are either in-phase **a**–**b**, or out of phase **c**–**d**

**Fig. 15** Trace plot for FES (Max) and LTES (Tr) for tracking the probability of the current topic be News in topic tracking experiment with keyword list approach. The black curve show the offline estimate




## 5 Real-Life experiment

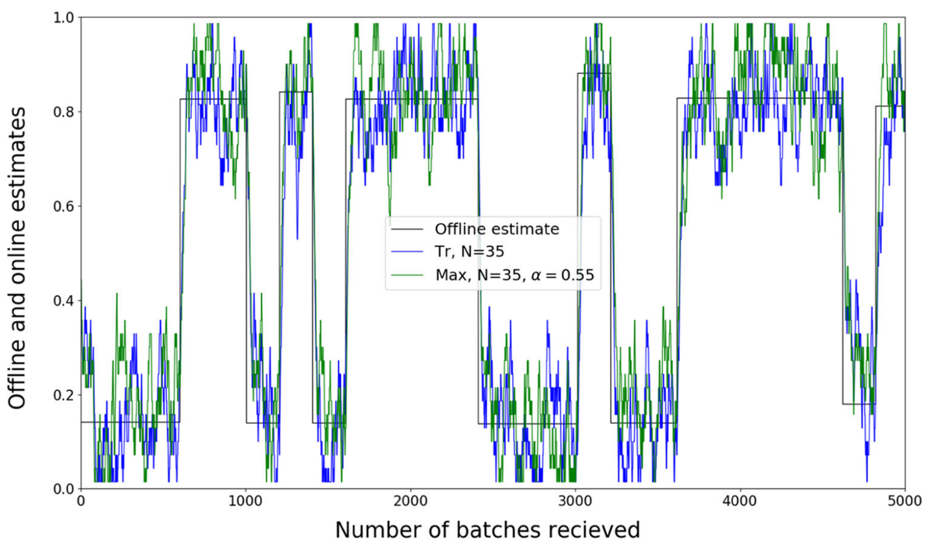
In this Section, we show how our proposed algorithms can be used for topic tracking in a stream of text by enhancing an existing estimator proposed in the literature [32]. Online tracking of topics in a stream of text, such as news/social media feeds, has been addressed in several research [3, 8, 25].

Consider *News* and *Entertainment* (including sports) as the two topics of interest. The aim is to model this problem such that the point location would be the probability of current topic being News. This quantity has the characteristics of point location ($\lambda^*(n)$). Additionally, we need some guidance from Environment to be able to run the proposed algorithms. As we will explain in the following, we can consider $x(n) \in \{0, 1\}$ to be a stream of zero and ones, where zero stands for Entertainment and one stands for News. So, $x(n)$ is a Binomial variable and $\lambda^*(n)$- i.e. probability that the current topic is News- is the Bernoulli parameter for each trial.

In [32], the *Stochastic Search on the Line-based Discretized weak Estimator (SSLDE)* is used to estimate the parameters of a distribution, when these parameters change with time. Note that in the distribution parameter estimation problem, the Environment is rather artificial and is constructed to suggest whether to increase or decrease the current estimate. We follow the same method as the SSLDE for the online tracking problem and create an artificial Environment that guides us to the point location.

Recall that for resolution $N$, we have $\lambda(n) \in \{0, 1/N, 2/N, \cdots, i/N, \cdots, (N-1)/N, 1\}$. The estimator

**Fig. 16** Evaluation of FES (Max) and LTES (Tr) for tracking the News in feed experiment in machine learning approach. The black curve show the offline estimate



is assigned initially the value $\lambda(0) = \frac{\lfloor N/2 \rfloor}{N}$. The updating rules for SSLDE [32] depends on whether the current estimate is greater or less than $N/2$. Suppose that $\lambda(n) = \frac{i}{N}$ and, as mentioned above, let $x(n)$ be the Binomial variable that takes zero or one at time $n$.

1. Case 1: $[i \geq (N/2)]$:
   - If $x(n) = 1$ and $rand() \leq \frac{N}{2 \cdot i}$:
   $$E(n,i) = 1 \;\rightarrow\; \lambda(n+1) = \frac{\min((i+1), N)}{N}$$
   - Else:
   $$E(n,i) = 0 \;\rightarrow\; \lambda(n+1) = \frac{i-1}{N}$$
2. Case 2: $[i < (N/2)]$:
   - If $x(n) = 0$ and $rand() \leq \frac{N}{2(N-i)}$:
   $$E(n,i) = 0 \;\rightarrow\; \lambda(n+1) = \frac{\max((i-1), 0)}{N}$$
   - Else:
   $$E(n,i) = 1 \;\rightarrow\; \lambda(n+1) = \frac{i+1}{N}$$

where $0 \leq rand() \leq 1$ is a uniform random number generator. Now, we are able to track the probability of the current topic be News by using the above suggestions by Environment.

## 5.1 Tracking problem

As mentioned above, News and Entertainment are the two topics we consider in this experiment. To generate the text feed, a large set of related articles are collected from the popular Norwegian newspaper site *vg.no*. The articles are shuffled randomly with the assumption that the algorithm is unaware of when transitions between News and Entertainment take place. In the same line as in [8], based on the stream of text, two methods are used for generating binary observations namely the keyword-list approach and the Machine learning approach.

- **Keyword lists.** A keyword list is a set of words for each topic, here News and Entertainment. For generation of the keyword lists, the popular Pointwise Mutual Information criterion [17] is used. We assume that one word at time is received from the News feed and the task is to track the probability of the current topic of the text stream is News. The best possible estimate based on the keyword list approach is to compute the portion of keywords in each article that are News keywords. This approach is called offline approach. The performance of our algorithm can be compared to this offline approach and see how close our online estimates are to the optimal offline approach.

  Figure 15 shows the tracking of the probability that the current topic is News for FES (Max) and LTES (Tr) for the first 15000 words. The total number of keywords in the experiment was 800400, while there was not a fixed period for changing between topics. We see that our algorithms are able to track changes in the News stream well. For instance, look at period $n = 5000$ to $n = 11000$. A difference between these real data from the simulations in Section 4 is that here the rate of changes is not fixed. As we see in Fig. 15 the offline estimate changes rapidly in some periods (for $n = 2000$ to $n = 5000$) and does not change for a long period ($n = 5000$ to $n = 11000$). Since in average the data has long fixed periods, the best achieved resolution is $N = 185$, which is better for the long periods.

  It is worth mentioning that this tracking data can be used as a classifier. Consider what we really want to understand from the data is that if the current feed belongs to the News or Entertainment. Indeed, the required answer is if the probability of the current topic be News is greater than 0.5 or not. Interestingly, for classification application, the best resolution is much smaller, i.e $N = 45$. The reason is that for classification, the flexibility is much more important compared to accuracy.
- **Machine Learning.** The most used approach to automatically classify text into different classes like topics or sentiment is to train a machine learner. The process starts by dividing the training text stream in batches of 20 words, each within one of the News or Entertainment topics. In the machine learning approach, the documents (batches) were represented by word frequencies in a bag of word matrix. These batches are used to train a machine learning model. For this experiment multinomial ridge regression [4] is used through the glmnet package in R [27]. For the testing part, the single words of the text stream were collected into batches of 20 words. Each batch in this phase were classified into one of the News or Entertainment topics using the trained multinomial regression model. The probabilities of the current topic were updated in the same manner as for the keyword list approach.

Figure 16 shows the tracking of the probabilities for the different topics for the machine learning approach. The total number of batches was 189141 which we depict the tracking in the first 5000 batches. Data changes faster in the ML approach and therefore the best resolution is much smaller; $N = 35$. We see that the fluctuations are greater than Fig. 15 because of this smaller resolution, but in turn, it is more adaptable with fast changes. Similar to the keyword list, if we use the algorithms for classification, the resolution will be even smaller; i.e. $N = 11$. So, being aware of that there might be different best parameters in the estimation for different applications is an important point.

## 6 Conclusion

A wide range of real-life problems can be modeled as a SPL problem, especially when the Environment is considered to be non-stationary. The random walk method, that Oommen presented for solving the SPL problem, is known to converge into a value arbitrarily close to the point location, when both resolution and time tend to infinity. Oommen's method simply discretizes the interval and performs a controlled random walk on it. This paper is an extention of the preliminary work presented [18] where we propose a new method to estimate the point location in the SPL problem domain. In the current paper, we have introduced the mutual probability flux concept and have proved that Flux-based Estimation Solution (FES) and Last Transition-based Estimation Solution (LTES), as a special case, always outperform Oommen's method. Moreover, we present a method to estimate Environment effectiveness, $p^*(n)$. This simple method could track the probability of receiving correct response from the Environment in tandem with the unknown location estimation.

Apart from theoretical proofs several experiments are presented in order to understand the characteristics of each method. We argued that $\lambda_{tr}$, proposed in this paper, is equally simple but with better estimation performance than Oommen's method. $\lambda_{max}$, $\lambda_{exp}$ and $\lambda_{med}$ show better estimation performance than $\lambda_{tr}$ in low resolutions, but this comes at the price of tuning one additional parameter. This suggests that if we have no constraint on $N$, i.e. $\lambda^*(n)$ represents a continuous quantity, we can tune just with $N$ and estimate with LTES. But in the case where freely tuning over $N$ is not possible, tuning with $\alpha$ and using one of $\lambda_{max}$, $\lambda_{exp}$, and $\lambda_{med}$ could provide more accurate estimations.

As experiments show, the tracking of $\lambda^*(n)$ performs better when $p^*(n)$ value is close to 1. The estimation performance of $\lambda^*(n)$ drops drastically when $p^*(n)$ is close to 0.5. This is as expected, since in case $p^*(n) = 1$, our estimation procedure will be correct, i.e. $\hat{\lambda}(n)$ switches back and forth around the true $\lambda^*(n)$. In contradiction, in the case $p^*(n)$ is close to 0.5, we have more faulty feedback, and so an unsatisfactory estimation of $\lambda^*(n)$.

Based on the results, we have also discussed when $\lambda^*(n)$ value is close to 0 or 1, the effect of faulty guidance from Environment will be reduced to some extent and the estimation is slightly better. Moreover, if $p^*(n)$ takes the same value in average, faster changes of $p^*(n)$ are preferable; with the condition that changes in $p^*(n)$ are slow enough that estimator could converge into $\lambda^*(n)$ when $p^*(n)$ is reaching to 1. In this case, faster changes interrupt a long lasting weak estimation and bring the estimator back into a more accurate value. However, if $p^*(n)$ and $\lambda^*(n)$ changes simultaneously, the positive effect of faster changes of $p^*(n)$ is lost. We have also discussed that, not only the rate of changes, but also the relation between $\lambda^*(n)$ and $p^*(n)$ affects the estimation error where $p^*(n)$ represents reliability of the feedback from Environment. A satisfactory estimation of $p^*(n)$ informs us to what extent we can trust the feedback and subsequently the estimations we have built upon that.

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## Study IV

# Balanced Difficulty Task Finder: An Adaptive Recommendation Method for Learning Tasks Based on the Concept of State of Flow

**Anis Yazidi, Asieh Abolpour Mofrad, Morten Goodwin, Hugo L. Hammer and Erik Arntzen**

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RESEARCH ARTICLE

# Balanced difficulty task finder: an adaptive recommendation method for learning tasks based on the concept of state of flow

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## Abstract

An adaptive task difficulty assignment method which we reckon as balanced difficulty task finder (BDTF) is proposed in this paper. The aim is to recommend tasks to a learner using a trade-off between skills of the learner and difficulty of the tasks such that the learner experiences a state of *flow* during the learning. Flow is a mental state that psychologists refer to when someone is completely immersed in an activity. Flow state is a multidisciplinary field of research and has been studied not only in psychology, but also neuroscience, education, sport, and games. The idea behind this paper is to try to achieve a flow state in a similar way as Elo's chess skill rating (Glickman in Am Chess J 3:59–102) and TrueSkill (Herbrich et al. in Advances in neural information processing systems, 2006) for matching game players, where "matched players" should possess similar capabilities and skills in order to maintain the level of motivation and involvement in the game. The BDTF draws analogy between choosing an appropriate opponent or appropriate game level and automatically choosing an appropriate difficulty level of a learning task. This method, as an intelligent tutoring system, could be used in a wide range of applications from online learning environments and e-learning, to learning and remembering techniques in traditional methods such as adjusting delayed matching to sample and spaced retrieval training that can be used for people with memory problems such as people with dementia.

**Keywords** Adaptive task difficulty · State of flow · Intelligent tutoring system · Game ranking systems · Online learning · Adjusting delayed matching-to-sample · Computerized adaptive testing · Stochastic point location

## Introduction

Attempts to achieve computer tutoring systems that are as effective as human tutors can be traced back to the earliest days of computers (Smith and Sherwood 1976). *Online learning* is becoming a significant driving force in today's educational systems. The lack of faculty members is a common trend in today's universities which makes personalized *one to one* teaching challenging, or practically impossible. Students may struggle to fulfill their full potential because the assigned tasks are generic and not tailored to their specific needs and skill level. Several studies show that personalized learning is the key to increased fulfillment of potential (see, e.g., Miliband 2004). A possible solution to the latter problem is resorting to the advances in *AI* in order to personalize the teaching process. AI could be defined as: "*The automation of activities that we associate with human thinking, activities such as*

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*decision-making, problem solving and learning"* (Bellman 1978).

Some of early studies that allude to the term *Intelligent Tutoring System (ITS)* dates back to 1982, where D. Sleeman and J.S Brown pioneered the idea of a system designed to help students reach their full potential in a limited amount of time (see Sleeman and Brown 1982). A few years later, a study is published demonstrating that individual tutoring is twice as effective as group teaching (Bloom 1984). Later, online e-learning platforms such as *Kahn Academy*[1] and *Udemy*,[2] massive open online course (MOOC) such as *MIT OpenCourseWare*,[3] digital hand in tools like *Fronter*, plagiarism controls like *Ephorus (Fronter)*, and autograding assignment tools such as *Bakpax*[4] have emerged. True ITS also exists with open tools like *Codeacademy*[5] and other e-learning platforms.

ITSs can raise student performance beyond the level of traditional classes and even beyond the level of students who learn from human tutors (see Kulik and Fletcher 2016, for a survey). A recent study by Chirikov et al. (2020) shows that online education platforms could scale high-quality science, technology, engineering, and mathematics (STEM) education through national online education platforms at universities. Such instruction can produce similar learning outcomes for students as traditional, in-person classes with a much lower cost (see also VanLehn 2011, for a review of relative effectiveness of human tutoring, intelligent tutoring systems, and other tutoring systems or no tutoring).

An ITS is supposed to *"provide immediate and customized instruction or feedback to learners"* (Psotka et al. 1988). In this paper, we provide algorithms that aspire to fulfill the latter statement for the purpose of task selection.

Many ITSs are based on *Computerized Adaptive Testing (CAT)* which is a form of computer-based test in which the correctness of the student's responses shapes the difficulty level of upcoming tasks (see, e.g. Hatzilygeroudis et al. 2006; Kozierkiewicz-Hetmańska and Nguyen 2010; Jansen et al. 2016, for instance). The aims of testing and practicing through tutoring differ; testing should efficiently estimate the student's ability (Birnbaum 1968; Eggen and Verschoor 2006), while training and practicing need to consider motivation and involvement of students in line with the length of the test (Jansen et al. 2016). A probability of success of 0.5 could minimize the test length, but this level of challenge could be frustrating for some students. For instance, in *Math Garden*, which is a web-based application for monitoring and practicing math skills based on CAT principles (Klinkenberg et al. 2011), a success rate of 75% is considered on average.

There is a substantial body of work on *Learning Automata (LA)* and ITSs (see, e.g. Oommen and Hashem 2013). In simple terms, LA is a stochastic machine attempting to find the optimal strategy from a set of actions in a random environment. LA, as a fundamental problem in AI, is particularly important in decision making under uncertainty (see Narendra and Thathachar 2012, for an introduction to LA). The term tutorial-like systems refers to study tutorial systems while no entity needs to be a real-life individual. Research in this field tries to model components of the system with appropriate learning models, such as LA (Oommen and Hashem 2013).

In a tutorial-like system, the teacher also might be stochastic and learns through the process of training (Hashem 2007). The design and analysis of a tutorial-like system model could involve modeling of a student (Oommen and Hashem 2009b), modeling of a classroom of students where artificial students can interact and learn from each other as well as the teacher (Oommen and Hashem 2009a), modeling of a (stochastic) teacher (Hashem and Oommen 2007), modeling the domain knowledge (Oommen and Hashem 2010), and modeling how teaching abilities of a teacher can be improved (Oommen and Hashem 2013).

ITSs can also be applied in some traditional learning methods in behavior analysis such as titrated delayed Matching-to-Sample (MTS) method, also referred as adjusting delayed MTS (Cumming and Berryman 1965; Sidman 2013).[6] Titrated delayed MTS has been used to study remembering in a variety of settings, including to study important variables in analyzing short-term memory problems (Arntzen and Steingrimsdottir 2014). Similar applications of ITSs in MTS and titrated delayed MTS procedures, can proposed to the computational models of these experimental methods which are usually introduced in the sake of research (see, e.g. Mofrad et al. 2020, for a recent computational model that simulates MTS procedure). ITSs can be used as a tool in the simulation part of training phase of MTS or titrated delayed MTS procedures to study the effect of adaptive training in a simulator model.

[1] www.khanacademy.com.

[2] www.udemy.com.

[3] https://ocw.mit.edu.

[4] www.bakpax.com.

[5] www.codecademy.com.

[6] Matching-to-sample procedures, have been frequently used to study complex human behavior (see for instance Cumming and Berryman 1965; Sidman 1994). Arntzen (2012) provides an overview of MTS experiments and several variables that can be manipulated when designing an experiment through MTS procedures. In adjusting delayed MTS, the length of the delay changes as a function of the participants' responses, which makes it similar to the adaptive task assignment problem.

Spaced retrieval training (SRT) (Camp et al. 1989) is another method of learning and retaining a piece of information by recalling that piece of information over increasingly longer intervals. The underlying problem in SRT is also similar to the adaptive difficulty task assignment which is addressed here. The SRT method is especially used for people with dementia (Camp et al. 1996).

Note that defining or measuring *task difficulty* can be addressed in many ways. A definition based on whether or not a task is performed, has applications in developmental research. In this context, easier tasks can be performed at earlier stages of development (see, e.g. Gilbert et al. 2012). For healthy adults, a difficult task can be defined as a quantitative measure, say percentage of task compliance in a series of trials. Response time is another measure of task difficulty, where a longer response time in average is equivalent to a more difficult task. Accuracy and response time, however, trade against each other (Fitts 1966; Wickelgren 1977) and both must be considered in a well-defined and standard task difficulty measure. Difficult tasks in this respect, can be defined as those with long response time and/or high frequency error (see, e.g. Gilbert et al. 2012, for other accounts in defining task difficulty).

In this paper, we present a formal theory by which an ITS can select the difficulty of task in a similar manner to selecting an opponent of similar capabilities in *balanced difficulty game* (Herbrich et al. 2006), which is called Balanced Difficulty Task Finder (BDTF). As suggested by systems such as Elo's chess skill rating (Glickman 1995) and TrueSkill (Herbrich et al. 2006) for matching game players, matched players should have similar capabilities and skills in order to achieve a balance between skills and challenges to experience the state of flow. We draw analogy between choosing an appropriate opponent or appropriate game level and automatically choosing an appropriate level of a learning task. It is noteworthy that by way of analogy, we can model the student as the player and the chosen task by the system as the opponent.

### Paper organization

The remainder of this paper is organized as follows. "State of art" section reviews the state of the art and various approaches to ITS modeling. "Modeling task selection as balanced game using balanced difficulty task finder" section models task selection as balanced difficulty game by resorting to our devised BDTF. "The concept of flow" section addresses the concept of flow from psychological point of view. In "Related work on games" section, related works from research on games are reported. "Neural basis of adaptive learning and state of flow experience" section addresses some literature on neural basis of adaptive learning and state of flow. Furthermore, theoretical formulation of BDTF is provided in "Formulating learning as a balanced difficulty game" section. Experimental results in "Experimental results" section catalogues the convergence properties of the BDTF discussed in the theory part. Finally, concluding remarks and future works are addressed in "Conclusions and future work" section.

## State of art

In this section, relevant studies and papers are discussed to give the reader an overview over the current state of the art. Although several papers on this topic exist dating back several years, the literature reviewed in this section is limited to content published (preferably) after 2005.

There are several approaches to create an ITS. In the most recent papers, we are presented with a mix of different artificial intelligence approaches to solve the problem. Common for most of the papers reviewed is the need for a model of student including different properties like learning-rate, previous experience and knowledge, and other variables. An approach for such a model (from now referred to as the *student model*) is represented in numerous studies (see for instance Brusilovsky and Millán 2007; Clement et al. 2014, 2015; Millán et al. 2010).

The use of the student model in recent papers suggests that this approach is fairly common in the field of ITS. Even though the model itself is fairly common, the implementation varies significantly. As an example, Clement et al. (2015) resort to a combination of a student model and a cognitive model to create a tutoring model. With this approach, the authors try to eliminate the need for a strongly typed student model. The goal is to adjust the learning tasks to individual students with as little information as possible. The use of a Learning Automata (LA) algorithm enables the system to find the optimal learning sequence for a specific student subject to some constraints; such as requiring certain activities to happen before others. A disadvantage of the latter approach is particularly the assumption that some tasks should be carried out in an order. The authors (Clement et al. 2015) assume that after task A1, either A2 or B1 need to follow. If students move to B1, they can not move back to any task in A category. This is in most cases a simplification of the learning process, since students should be able to work on several categories and practice by repeating previous categories.

Clement et al. (2015) use *partial-observable Markov decision process (POMDP)* for optimization of task selection, which is inspired by Rafferty et al. (2011) who used the students acquisition level to propose activities. This method requires the system to assume all students learn in the same way. It is also stated that this approach can be optimal, but requires sophisticated student and cognitive models. In most cases these methods are based on *knowledge tracing-methods (KTM)* which attempt to estimate student knowledge in a parametric manner. Usually, the lack of data causes this form of modeling to be inaccurate. POMDPs also has been suggested to be used for modeling a population of students, instead of individuals. This approach has been proven to be suboptimal in an ITS setting (Clement et al. 2015; Lee and Brunskill 2012).

On the other hand, several improved versions of the KTM have been proposed in the literature. A Representative example is the *Bayesian knowledge tracing (BKT)* with skill-specific parameters for each student. There are strong indicators that BKT models accounting for the student variance is superior to the Bayesian knowledge model (Pardos and Heffernan 2010; Yudelson et al. 2013). This partially nuances the criticism proposed by Clement et al. (2015).

A significant number of studies indicate that intrinsically motivated students perform better. Thus, this requires a good ITS keeps motivating the student throughout the whole learning experience. Lumsden (1994) investigated the optimal strategy for motivating the student, and found that one of the main keystones for a motivational experience is task mastery. This is backed up by Clement et al. (2015) who proposes a solution where the student is presented with tasks that are neither too easy nor too hard, but slightly beyond their current abilities. Psychologists refer to this experience as state of flow (see, e.g. Csikzentmihalyi 1996).

In this article, we propose a solution where each student starts with a predefined *optimal-difficulty* (Clement et al. 2015) which will be adjusted over time based on the student answers. Some students may be more prone to be motivated with challenging tasks, and therefore the overall learning outcome may be more effective for these students. On the other hand, we might find students struggling with the default or optimal-difficulty. In such cases, the learning-rate should be decreased, allowing these students to participate at a slower pace.

There are several possible alternatives to design an ITS. We have looked at several candidates in this study, including *multi-armed bandits* (Clement et al. 2015), *Bayesian-networks* (Millán et al. 2010) and *neural-networks* (Zatarain Cabada et al. 2015), each with its own advantages. As mentioned earlier the student model is an important part of this ITS. In the latter reviewed papers, the neural network and Bayesian-network both relied on comprehensive student models, with a solid core of data in order to be able to draw accurate assumptions and decisions. These systems are shown to be reliable and effective, but comprehensive data models are required in order to achieve optimal operation (Clement et al. 2015). With the use of LA it is possible to eliminate the need for prior-knowledge about the students. The LA is efficient, and it requires a weaker link between student and the cognitive model. Clement et al. (2015) propose an LA for seven to eight years old school-children learning to decompose numbers while manipulating money. Even though a generic solution is presented by Clement et al. (2015) relying on multi-armed bandit, there is no evidence that a similar approach is viable for use for adults and contexts addressed in online learning (see also Hashem and Oommen 2007; Hashem 2007; Oommen and Hashem 2009a, b, 2010, 2013, for LA based models for a generalized framework of tutoring system, called tutoring-like systems).

A limited number of studies describe the use of ITS in programming courses. As representative studies, we identified *Java Sensei* (Zatarain Cabada et al. 2015) and *ASK-ELLE* (Jeuring et al. 2012), each of the latter studies use a different machine learning approach. Java Sensei resorts to a combination of neural-network strategies and emotion sensors to register information and to make decisions based on input. ASK-ELLE ITS utilizes a domain reasoner using a Haskel Compiler called *Helios*. This compiler was developed to give feedback on wrong syntax. The system requires each student to complete a given task, but helps the student to accomplish the tasks by giving hints and examples relevant to found error(s).

Before moving to the model and contribution of this paper, we refer to the Stochastic Point Location (SPL) problem which has some similarities to the current work. A considerable amount of literature has been published on SPL since the Oommen work (Oommen 1997) (see for instance Yazidi et al. 2014; Mofrad et al. 2019). In SPL, an LA search for a point location in a line through the guidance of an external environment which might give faulty advice. Many scientific and real-life problems can be modeled as the instances of SPL problem, including adaptive task assignment problem. For instance, in Mofrad et al. (2019), some authors of this paper discuss that the point location can represent the difficulty level of a task that a participant can handle, and tries to find that point as fast and accurate as possible. The participant performance in Mofrad et al. (2019) is modeled using a stair function with two levels: a high performance for difficulties under the optimal *manageable* difficulty level and a low performance for difficulties just above the same level, i.e., the *manageable* optimal difficulty level. However, if we rather use a more realistic performance function according to which the performance is continuous and monotonically decreases as a function of the difficulty level, the approach proposed in Mofrad et al. (2019) will basically converge to difficulty level for which the participant performance is at 50% under some mild conditions. In other words the model finds a manageable difficulty level and can be used in titrated delayed MTS, SRT and online environments. Such remark motivated the current study in which we resort to the latter realistic performance model, for efficiently finding a higher rates of performance that are motivating enough for the learner, and provides a balance between challenge and skills, usually above 50% such as 70%. In comparison with Mofrad et al. (2019), where the adjustment technique is symmetric, in the current work the effect of correct and incorrect responses are not the same, i.e. the adjustment is asymmetric.

## Modeling task selection as balanced game using balanced difficulty task finder

In this section, we present BDTF as the main contribution in this article which is a theory that aspires to learn the appropriate difficulty of a task rather than exploring different types of tasks as in the case of work in Andersen et al. (2016). Although both approaches can be combined, we clearly distinguish between them as the second case can be seen as a novel theory for determining the adequate difficulty level of an assignment for the purpose of keeping the learning activity *motivating* and not *exploring* (as in Andersen et al. (2016), which is more concerned about exploring the different tasks in a similar manner to bandit problem).

Difficulty is a subjective concept, or more precisely, it is more *individual* and personal (see, e.g. Gilbert et al. 2012). We argue that difficulty should be tailored to the ability of the student. In fact, as in video games, or chess, the player is motivated by an appropriate level of challenge or equivalently difficulty. For example, the purpose of Xbox TrueSkill system (Herbrich et al. 2006) is to match players that have similar capabilities so that the outcome of the game is unpredictable (optimally equi-chance of winning and losing). Elo tries to find a global ranking among players and TrueSkill is similar to the Elo rating system for matching chess players. We advocate that, in a similar manner to TrueSkill and Elo, a student needs to find an enough challenging assignment that matches his capabilities.

After a brief introduction on psychological concept of flow experience ("The concept of flow" section), reviewing related works on games ("Related work on games" section), and related works addressing neural basis of adaptive task difficulty and the state of flow ("Neural basis of adaptive learning and state of flow experience" section), we provide a sound mathematical formulation ("Formulating learning as a balanced difficulty game" section) that emanates from the field of stochastic approximation (Kushner and Yin 2003).

## The concept of flow

The history of optimal human functioning in humanistic and health psychology can be tracked back to the work of Maslow (1959) who refereed to these moments of self-actualization *peak experiences*. These experiences are described as instances of happiness, fulfillment, and achievement with a feeling of awareness to one's human potential. Csikzentmihalyi (1996) has described such an experience as a state of flow since it is characterized by *"an almost automatic, effortless, yet highly focused state of consciousness"* (p. 110).

Any mental or physical activity, according to Csikzentmihalyi (1996), can generate flow if: it is a challenging enough task that requires intense concentration and commitment, involves clear goals, provides immediate feedback, and is perfectly balanced to the skill level of the person.

Delle Fave and Massimini (1988) discuss that balancing challenges and skills is not enough for optimizing the quality of experience and the notion of *skill stretching* inherent in the flow concept. They redefined flow as the balance of challenges and skills at the time both are above average levels for the person. Moreover, the quality of experience intensifies in a channel by moving away from a person's average levels in the challenge/skills space. Figure 1 depicts a classification of experiences based on the level of challenge and skill in eight categories. The rings depict increasing intensity of experience in each channel or

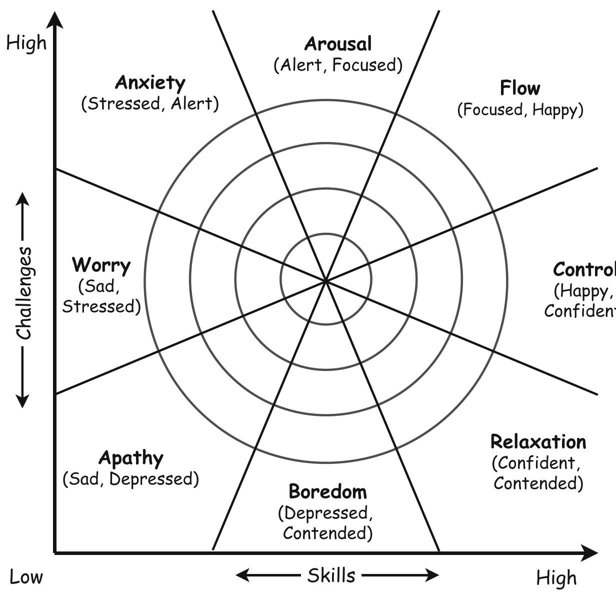



**Fig. 1** Model of the flow state adapted from Csikszentmihalyi (2020). Perceived challenges and skills must be above the person average level in order to experience a state of a flow. The apathy is the case when both are below the average and the experience intensity is increased by distancing from average, shown by rings

quadrant (see Nakamura and Csikszentmihalyi 2014, for a detailed overview of the concept of flow).

## Related work on games

A representative study that sheds light on the relationship between three inter-related concepts: difficulty, motivation and learning is presented by Chen (2007) that introduces the flow Channel to the filed of games. According to Schell (2014) and Chen (2007), when the difficulty exceeds the learner's skill, the learner experience a feeling of anxiety at the thought of his learning skills are insufficient, and as a result gets demotivated. Consequently, the learner tends to abandon the activity after short time. On the other hand, boredom takes place in the other extreme case where the student level is much higher than the assignment's difficulty. In this sense, the student perceives the assignment as a waste of time. The ideal case according to Schell (2014) and Chen (2007) takes place when the aptitude of the learner and the difficulty level are in *state of balance*. In this case, similar to the psychological definition of flow, the learner is said to achieve a state of flow. Chen (2007) defines the flow as: "the feeling of complete and energized focus in an activity, with a high level of enjoyment and fulfillment".

As reported by Gallego-Durán et al. (2016), the notion of difficulty in games does not seem to have attracted much attention in the field of education in general. In this perspective, the proposed BDTF tries to bridge the gap between two seemingly disjoint fields of research, namely, ITSs and game ranking/matching systems.

The most pertinent work to our approach emanates from the realm of computer games and chess where it was remarked that when the level of the game is either too difficult or too easy, the players abandon playing (Chen 2007; Schell 2014). Extensive literature has been centred on the design of adaptive method to adjust the difficulty of the game so that to match the level of the players, but in the interest of brevity, we skip them (see, e.g. Hunicke 2005).

## Neural basis of adaptive learning and state of flow experience

There are many studies on the neural basis of state of flow that we briefly review some of them. Due to the complexity of concept of flow, it must be measured through its components. Dietrich (2004) analyses the flexibility/efficiency trade-off in the flow state and concludes that a prerequisite to the experience of flow period is "a state of transient hypofrontality that enables the temporary suppression of the analytical and meta-conscious capacities of the explicit system". Klasen et al. (2012) use brain imaging to study neural basis of flow and showed an influence of flow on midbrain reward structures as well as complex network of sensorimotor, cognitive and emotional brain circuits. Some of the components of flow that identified in this study are focus, direct feedback, balance between skill and difficulty, clear goals and having control over the activity. Flow association with prefrontal functions such as emotion and reward processing was suggested by Yoshida et al. (2014) where brain activity in the prefrontal cortex during a flow state is examined using functional near-infrared spectroscopy (fNIRS). Cheron (2016) addresses some possible ways to measure the psychological flow from a neuroscience perspective. The neuroscience studies on games are not limited to the flow state, but we leave it since it is out of the scope of this article (see Palaus et al. 2017, for a systematic review on neural basis of video gaming).

To achieve and keep the state of flow, we use adaptive task difficulty methods. The neural basis of adaptive task difficulty has been studied by researches of the field (see, e.g. Flegal et al. 2019). An important issue is to see if the cognitive training effect could transfer to untrained tasks and neural plasticity. Kalbfleisch et al. (2007) study the influences of task difficulty and response correctness during fluid reasoning on neural systems using functional magnetic resonance imaging (fMRI). Von Bastian and Eschen (2016) compared conditions in which the difficulty of working memory training tasks was adaptive, self-selected, or randomly varied, in a behavioral study. The reported results indicate that all three procedures produced equivalent improvement on trained tasks, in comparison with an active control group. However, no significant difference between the training groups and the active control group, was reported for the transfer effects on untrained working memory tasks and far transfer (reasoning) tasks. So the transfer effects could not link to adaptivity or variability of task difficulty. McKendrick et al. (2014) examined mechanisms of training-induced plasticity by comparing a group that received adaptive working memory training with an active control group where task difficulty was matched to the performance of participants in the adaptive group, i.e. training was variable but not individually adaptive. The method was continuous monitoring of working memory training with near infrared spectroscopy (NIRS) during a dual verbal–spatial working memory task. The results suggested refuting the hypothesis that the effectiveness of adaptive task difficulty and variable task difficulty are alike. Flegal et al. (2019) study the effect of adaptive task difficulty on transfer of training and neural plasticity by measuring behavioral and neural plasticity in fMRI sessions before and after 10 sessions of *working memory updating (WMU)* training. The tasks difficulty was either fixed or adaptively increased in response to performance. The results show the transfer to an untrained episodic memory task activation decreases in striatum and hippocampus on a trained WMU task in adaptive training. Flegal et al. (2019) support the use of adaptive training as the best practice and suggest that cognitive training programs need to incorporate adaptive task difficulty to extend the transfer of training gains and optimize the efficiency of task-related brain activity (see also Gaume et al. 2019; Mora-Sánchez et al. 2020, for brain-computer interfaces which are able to monitor the working memory load and cognitive load in real-time based on biomarkers derived from EEG).

## Formulating learning as a balanced difficulty game

Without loss of generality, we suppose that the difficulty of any given task can be characterized by a real number from [0, 1], where 0 denotes the lowest possible difficulty and 1 denotes the highest possible difficulty.

The main intuition behind BDTF is the fact that the chance of a student for succeeding in a given task decreases monotonically as the difficulty level increases. We suppose that a student possesses a characterizing skill-curve that describes the relationship between the difficulty of the task and the student chance for succeeding in solving the task. We assume that the tasks are ranked on scale from 0 to 1 by an expert such as teacher where 0 denotes the lowest level of difficulty and 1 denotes the highest level of difficulty.

We suppose that $s^*$ is the optimal success probability that we want a learner (student) to experience. It is up to the designer of the intelligent tutoring system to fix the desired target chance of the succeeding in a task for a student. Thus, our approach will try to adjust the difficulty of the given tasks in an online manner that drives the system towards a state of flow (Chen 2007). Inspired by Elo system, one can choose $s^* = 0.5$ which basically means that the designer desires that the student finds the tasks challenging enough by fixing the target success probability to 50%.

Please note that this reflects the most uncertain case since the outcome of the task in terms of success or failure is unpredictable. However, deciding on $s^*$ value requires more in depth study that takes into account many factors including psychological factors. In this paper, and in all the experiments presented in the rest of the article, we will fix $s^* = 0.7$ which basically reflects the fact that we desire the student to succeed most of the time in solving the given task while failing just 30% of the time.

In addition, we suppose that we are operating in a discrete time space and $t$ referring to the current time instant. The difficulty of the next assignment at time $t+1$ depends on the difficulty of the solved assignment at time $t$ as well as the previous achievement (success or failure).

$$d(t+1) = \begin{cases} \min(1, d(t) + \lambda(1 - s^*)): & \text{if } x(t) = 1 \\ \max(0, d(t) - \lambda s^*): & \text{if } x(t) = 0 \end{cases} \tag{1}$$

where $d(t)$ denotes the difficulty of the task at time $t$,[7] $\lambda$ is an update parameter that is in the interval ]0, 1[, and $x(t)$ denotes the binary variable that records the result of solving the task given at time instant $t$. $x(t) = 0$ in case of failure and $x(t) = 1$ in case of success.

Equation (1) describes a recursive update of the difficulty of the tasks depending on the performance of the student, $x(t)$. According to Eq. (1), the difficulty gets increased upon success and decreased upon failure in an asymmetric manner. We suppose that at time $t = 0$, the BDTF starts by suggesting a task with difficulty $d(0) = 0.5$, i.e, we start with tasks with *medium level*. We suppose that for student $i$, there is a function $S_i(d)$ that describes the success probability given the difficulty of the task. Whenever there is no ambiguity, we drop the index $i$. As explained previously, the latter function is monotonically decreasing. Please note that $x(t) = 1$ with probability $S(d(t))$ and $x(t) = 0$ with probability $1 - S(d(t))$. We will later provide theoretical results that demonstrate that if there exists a point $d^*$ such that $S(d^*) = s^*$ then the update equation converges to it. Since $d$ is defined over [0, 1] and $S(d)$ is decreasing over [0, 1] and admits values in [0, 1], then for any function $S_i$ such point $d^*$ is unique (if it exists). A simple and sufficient condition for the existence as well as uniqueness of $d^*$ is that $S_i(0) = 1$ and $S_i(1) = 0$. This has an intuitive interpretation: the success probability for the min difficulty is one and for the max difficulty is zero. However, in general, $S(0)$ might be different from one and $S(1)$ might be different from zero. The following theorem catalogues the convergence of our scheme for an arbitrary monotonically decreasing function $S$ such that $S$ is mapping from [0, 1] to [0, 1].[8]

It is noteworthy that the proof of the coming theorem is based on the results of the stochastic approximation theory (Kushner and Yin 2003). The informed reader would observe that our algorithm is very similar to the seminal algorithm of Robbins and Monro (1951) who pioneered the field of stochastic approximation. The main differences are the following:

- They use a time dependent update parameter $\lambda$.
- In Robbins and Monro (1951), the response function is increasing, while in our case it is decreasing.

[7] When relation to time is not important, we simply use $d$ to refer to difficulty.

[8] The function $S(.)$ has values within [0, 1] since it denotes the probability of success.

Those differences can be tackled easily in the proof as within the field of stochastic approximation, there are two types of algorithms: algorithms with fixed step size and algorithms with time varying step size, usually decreasing. We are working in this paper with a fixed step size algorithm. The second difference concerns the response function. The monotonicity of the function gives uniqueness of the equilibrium. If our function was increasing, we would simply change $\lambda$ by $-\lambda$. This form of update is similar to gradient descent where the direction of movement is determined according to whether we are facing a minimization or maximization problem.

**Theorem 1** *The stochastic process $d(t)$ converges to one of the three following cases as the learning parameter $\lambda$ tends to zero:*

*Case 1 If* $\min S(d) \le s^* \le \max S(d)$, *then* $\lim_{t\to\infty} \lim_{\lambda\to 0} d(t) = S^{-1}(s^*) = d^*$.
*Case 2 If* $\max S(d) < s^*$, *then* $\lim_{t\to\infty} \lim_{\lambda\to 0} d(t) = 0$.
*Case 3 If* $\min S(d) > s^*$, *then* $\lim_{t\to\infty} \lim_{\lambda\to 0} d(t) = 1$.

***Proof*** Similar to Altman et al. (2009), we can re-write the update equations as per:

$$d(t+1) = \Pi_H(d(t) + \lambda(x(t) - s^*)) \tag{2}$$

where $\Pi_H$ denote the following projection

$$\Pi_H(d) = \begin{cases} d, & \text{if } 1 < d < 0, \\ 1, & \text{if } d \ge 1, \\ 0, & \text{if } d \le 0. \end{cases}$$

The usage of projection is common with the field of stochastic approximation to force the iteration to stay with a bounded set $H = [0, 1]$, and they are projected back to the set whenever they go outside it. Without loss of generality, the boundary set we are using here, consisting of zero and one, is a well-behaved one as described by Borkar (2009, Chapter 5.4). We can show that process converges to some limit set of the following Ordinary Differential Equation (ODE):

$$\dot{d} = E[x(t)|d] - s^*. \tag{3}$$

We know that $E[x(t)|d] = S(d)$, therefore the ODE is

$$\dot{d} = S(d) - s^*. \tag{4}$$

The decreasing nature of $S(d)$ provides the uniqueness of the fixed point $s^*$ whenever $\min S(d) \le s^* \le \max S(d)$. Whenever $s^*$ lies outside $H = [0, 1]$, we will converge towards the boundary point, zero and one, according to whether $\max S(d) < s^*$ or $\min S(d) > s^*$ respectively. □

## Experimental results

In this section, we provide some experimental results which confirm the theoretical results presented in Theorem 1.

In order to describe the relationship between difficulty and success, we define $S(d) = a - b/(1 + \exp(-20 * (d - 0.5)))$, where $0 < b \le a \le 1$, ensuring that $S$ is decreasing. In the reported results for three cases of the theorem, $\lambda = 0.01$, and the target success probability is $s^* = 0.7$. Please note that the aim of the section is to rather confirm the theoretical properties of our scheme so any decreasing function suffices.

Figure 2 depicts the time evolution of $d$ and the corresponding success probability $S(d)$ where $S(d) = 1 - 1/(1 + \exp(-20 * (d - 0.5)))$ for an update parameter $\lambda = 0.01$. Please note that since $\min S(d) = 0 \le s^* = 0.7 \le \max S(d) = 1$, then according to Theorem 1, $d(t)$ converges to $d^* = S^{-1}(s^*) = 0.458$.

Figure 3 depicts the time evolution of $d$ and the corresponding success probability $S(d)$ where $S(d) = 0.6 - 0.5/(1 + \exp(-20 * (d - 0.5)))$ for an update parameter $\lambda = 0.01$. Please note that since $\max S(d) = 0.6 < s^* = 0.7$, then $d(t)$ converges to $d^* = 0$.

Finally, Fig. 4 depicts the time evolution of $d$ and the corresponding success probability $S(d)$ where $S(d) = 1 - 0.2/(1 + \exp(-20 * (d - 0.5)))$ for an update parameter $\lambda = 0.01$. Since $\min S(d) = 0.8 > s^* = 0.7$, then $d(t)$ converges to $d^* = 1$.

Please note that the convergence time is a function of both starting point distance to optimal difficulty and value of $\lambda$. In Fig. 2, the optimal difficulty is $d^* = 0.458$ which means it is about 0.14 far from the starting point. After around 100 iterations, the optimal difficulty is reached. In Figs. 3 and 4 the optimal difficulty is about 0.5 far from the starting point, and in both cases after about 600 steps, the optimal difficulty is reached. In all the three cases, $\lambda = 0.01$. To study the role of $\lambda$ in the convergence time, we fix the success probability function to $S(d) = 1 - 1/(1 + \exp(-20 * (d - 0.5)))$, which is depicted in Fig. 2 and test it for three different values of $\lambda = 0.1$, $\lambda = 0.01$, and $\lambda = 0.001$. As we see in Fig. 5, smaller values of $\lambda$ result into a slower, but smoother convergence. In Fig. 5a, with $\lambda = 0.1$, the convergence is just about 10 steps, in Fig. 5b, with $\lambda = 0.01$, the convergence happens after about 100 steps, and finally in Fig. 5c, with $\lambda = 0.001$, the convergence happens after about 1000 steps. Hence, the value of $\lambda$ can be chosen in a way to find a compromise between convergence speed and convergence accuracy.

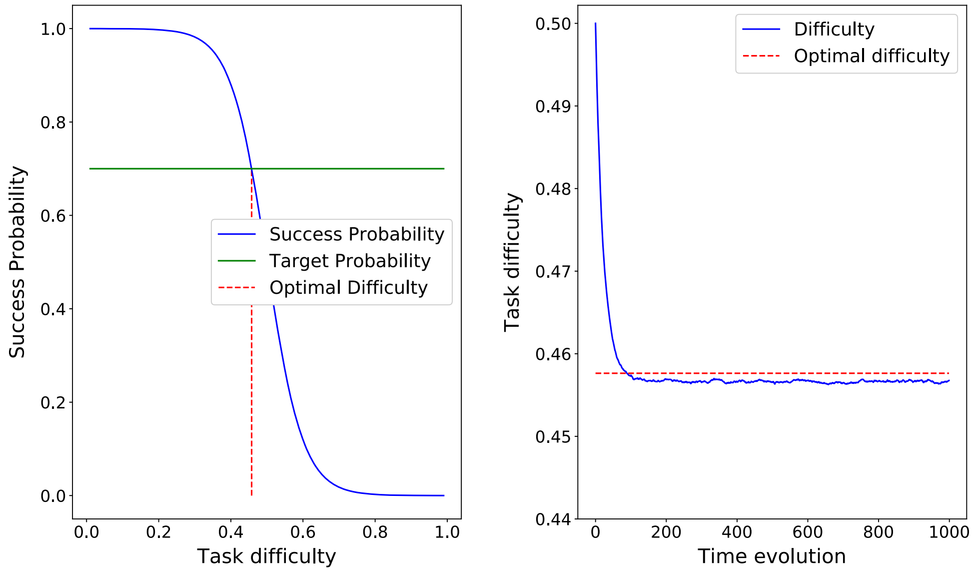



**Fig. 2** Case 1 in the theorem. $S(d) = 1 - 1/(1 + \exp(-20 * (d - 0.5)))$ so $d(t)$ converges to $d^* = S^{-1}(s^*) = 0.458$

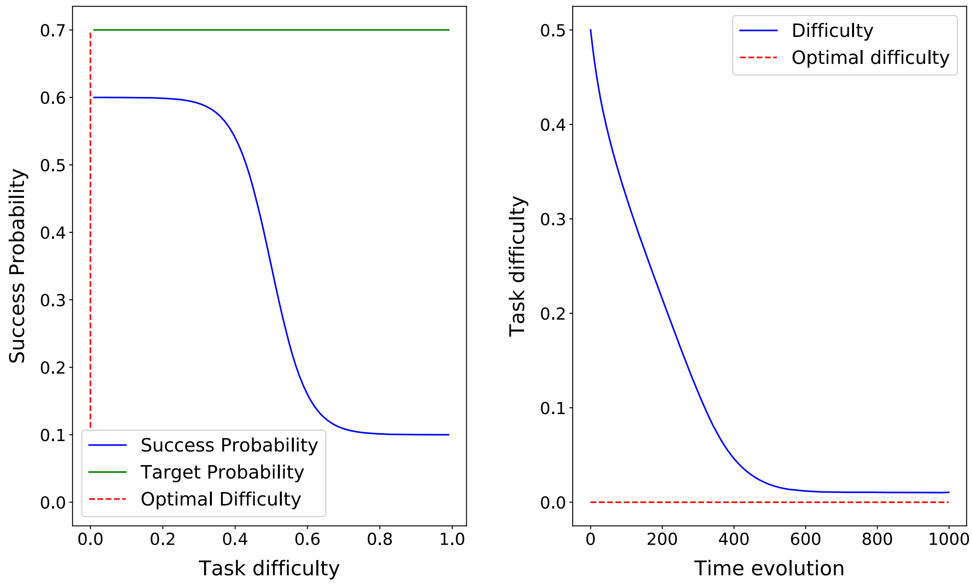


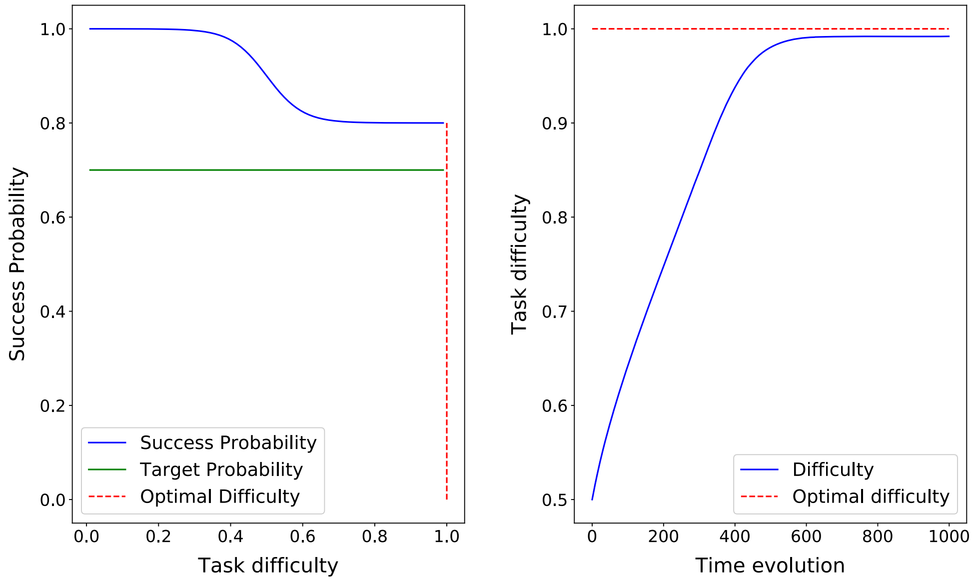

**Fig. 3** Case 2 in the theorem. $S(d) = 0.6 - 0.5/(1 + \exp(-20 * (d - 0.5)))$ so $d(t)$ converges to $d^* = 0$

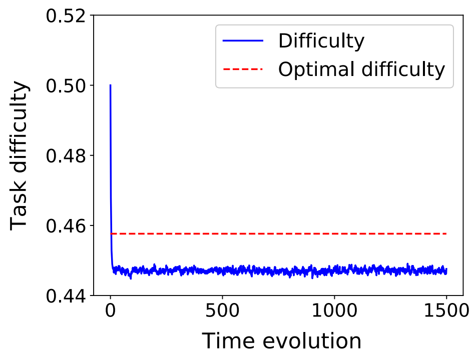
The aim of the last experiment is to demonstrate the ability to track the changes in optimal difficulty. This is analogous to the cases where instructor or teacher decides to give easier or harder tasks based on the feedback from learner. In Fig. 6 the optimal success probability is set to $s^* = 0.7$ at the beginning where the learner achieves this success rate when the optimal difficulty is $d^* = 0.458$. Then at time instance $t = 1500$, the teacher see that this is still challenging for the student and decided to provide student with tasks that 90% of the time handled by student. Figure 6a shows the case that $\lambda = 0.01$ and therefore the change rate it higher. Figure 6b is when changes are slower, $\lambda = 0.001$. The optimal difficulty for $s^* = 0.9$ equals $d^* = 0.39$.

## Conclusions and future work

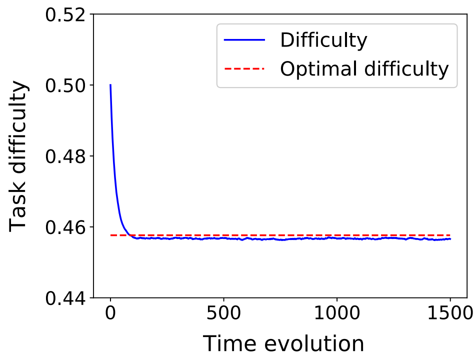
In this paper, we tackled the problem of personalized task assignment in online learning environment as well as training methods for retaining information. We present the BDTF which is a formal theory by which an ITS can fine tune the difficulty of a task to a level that matches the student level. The underlying assumption of the BDTF is that the ITSs can fine tune the difficulty of the task to a continuous level. The BDTF application to the learning methods that focus on memory and retaining information such as adjusting delayed MTS and spaced retrieval training methods is discussed. These methods are looking for the best delay time between two consecutive tasks and can be used for memory training.

**Fig. 4** Case 3 in the theorem. $S(d) = 1 - 0.2/(1 + \exp(-20 * (d - 0.5)))$ so $d(t)$ converges to $d^* = 1$

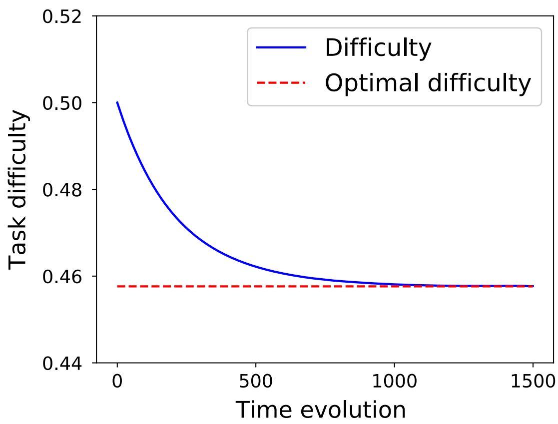



**(a)** $\lambda = 0.1$



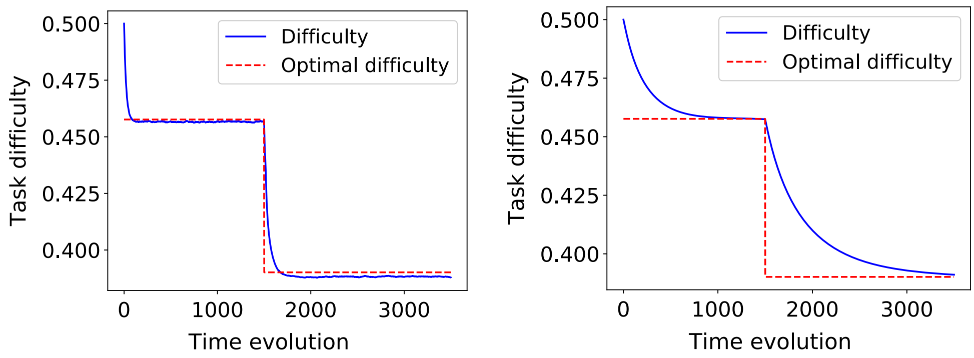

**(b)** $\lambda = 0.01$




**(c)** $\lambda = 0.001$

**Fig. 5** The success probability function based on difficulty is $S(d) = 1 - 1/(1 + \exp(-20 * (d - 0.5)))$, which is depicted in Fig. 2. The optimal task difficulty for success probability $s^* = 0.7$ is $d^* = 0.458$ and shown by dashed red line

The BDTF approach deals only with binary feedback. It is possible to extend our work so that to accommodate non-binary feedback in the form of a continuous or discrete score reflecting the achievement of the student in solving a given task. Furthermore, as a future work, we intend to explore the effect of learning on the progress of the student. Intuitively, the success probability $S(d)$ shall also be frequency dependent, i.e, the more assignments the student




**(a)** $\lambda = 0.01$ **(b)** $\lambda = 0.001$

**Fig. 6** The success probability function based on difficulty is $S(d) = 1 - 1/(1 + \exp(-20 * (d - 0.5)))$, which is depicted in Fig. 2. The optimal task difficulty for success probability $s^* = 0.7$ is $d^* = 0.458$ in the first 1500 time instances, then the target success probability changes to $s^* = 0.9$ which means the optimal task difficulty is $d^* = 0.39$. The optimal task difficulty is shown by dashed red line

tries, the higher the chance of success in future tasks. This is also described as the learning effect that results from repetitive trials. The latter effect can be easily accommodated in our model by rendering $S(d)$ a function of the number of trials, meaning the dynamics of $S(d)$ shall include a frequency dependent term. An interesting avenue for research is the possibility of introducing the recency and spacing in time between the different student trials as an extra parameter in $S(d)$. BDTF approach could be extended to the tutorial-like systems similar to the LA applications for a generalized concept of ITS proposed by Hashem (2007). Since we are using LA, we can integrate the idea of having an stochastic teacher (Hashem and Oommen 2007), modeling a classroom of students where artificial students can interact and learn from each other as well as the teacher (see Oommen and Hashem 2009a, for such a model), and propose an adaptive learning model of teacher and how teaching abilities of a teacher can be improved during the process (inspired by Oommen and Hashem (2013)).

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