NON-DEFECTIVITY OF GRASSMANNIAN BUNDLES OVER A CURVE

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Abstract. Let $\mathrm{Gr}(2, E)$ be the Grassmann bundle of two-planes associated to a general bundle E over a curve X. We prove that an embedding of $\mathrm{Gr}(2, E)$ by a certain twist of the relative Plücker map is not secant defective. This yields a new and more geometric proof of the Hirschowitz-type bound on the Lagrangian Segre invariant for orthogonal bundles over X, analogous to those given for vector bundles and symplectic bundles in [2, 3]. From the non-defectivity we also deduce an interesting feature of a general orthogonal bundle over X, contrasting with the classical and symplectic cases: Any maximal Lagrangian subbundle intersects at least one other maximal Lagrangian subbundle in positive rank.

1. Introduction

Let X be a smooth complex projective curve of genus $g \geq 2$. In the 1980s, Hirschowitz [6] found that there do not exist vector bundles of a fixed rank and degree over X with maximal subbundles of arbitrarily small degree. Precisely; let $V \to X$ be any vector bundle of rank $n \geq 2$. For $1 \leq r \leq n-1$, the *Segre invariant* $s_r(V)$ is defined by

$$s_r(V) \ := \ \min\{r \cdot \deg V - n \cdot \deg E : \ E \text{ a rank } r \text{ subbundle of } V\}.$$

Hirschowitz [6, Théorème 4.4] showed that one always has $s_r(V) \leq r(n-r)(g-1)+\delta$ for a certain $\delta \in \{0, \ldots, n-1\}$, with equality if V is general. A geometric proof of this result was given in [2, §5], exploiting the secant non-defectivity of a certain embedded Segre fibration proven in [2, Theorem 5.1]. This proof can be regarded as a generalization of Lange and Narasimhan's proof [9, §3] of Nagata's bound for rank two bundles, which exploited the non-defectivity of certain curves in projective space.

Suppose now that V admits an orthogonal or symplectic structure. We recall that a subbundle $E \subset V$ is called *Lagrangian* if E is isotropic and has the largest possible rank $\left\lfloor \frac{1}{2}\mathrm{rk}\, V \right\rfloor$. The *Lagrangian Segre invariant* is defined as

$$t(V) \ := \ \min\{-2 \cdot \deg E : \ E \subset V \text{ a Lagrangian subbundle}\}.$$

A Lagrangian subbundle will be called *maximal* if it has maximal degree among all Lagrangian subbundles. In [3, §3], with arguments analogous to those in [2], the non-defectivity of certain embedded Veronese fibrations was proven and used to compute the sharp upper bound on $t(V)$ in the symplectic case. (Note that if V is symplectic then $\mathrm{rk}\, V$ is even.)

In [4], a sharp upper bound on $t(V)$ was given for orthogonal bundles of rank $2n$, by a different method (for comparison, this is briefly sketched in Remark 4.4). However, compared with the treatment of vector bundles and symplectic bundles in [2, 3], there is a missing geometric picture in the orthogonal case, namely, the non-defectivity of the object corresponding to the aforementioned Segre and Veronese fibrations. By [4, §2], this turns out to be an embedding of the Grassmannian bundle $\mathrm{Gr}(2, E)$ whose fiber at a point x is the Grassmannian of planes $\mathrm{Gr}(2, E_x)$ for a generic vector bundle $E \to X$ of rank n.

The first goal of the present note is to complete the picture for orthogonal bundles by showing the non-defectivity of these Grassmannian bundles. It is relevant to point out that the Grassmannian parameterizing projective lines in $\mathbb{P}^N$ is secant defective in most cases; see Catalisano–Geramita–Gimigliano [1].

Here is an overview of the paper. In §2 we recall some results on the geometry of orthogonal extensions. In §3 we prove the desired non-defectivity statement (Theorem 3.1) for $\mathrm{Gr}(2, E)$. As in the classical and symplectic cases, the strategy is to describe the embedded tangent spaces of $\mathrm{Gr}(2, E)$ and apply Terracini's Lemma.

In §4, we use Theorem 3.1 to give a proof of the Hirschowitz-type bound on the Lagrangian Segre invariant of orthogonal bundles, analogous to those mentioned above in [2, §5] and [3, Theorem 1.4].

Furthermore, in §5 we use Theorem 3.1 to answer a question which we were unable to solve with the methods in [4]: We show that any maximal Lagrangian subbundle of a general orthogonal bundle meets another maximal Lagrangian subbundle in a sheaf of positive rank. In this way orthogonal bundles behave differently from general vector bundles and symplectic bundles. More information and precise statements are given in Theorem 5.3.

Regarding future investigations: If $Q \to X$ is a principal G-bundle, the notion of a subbundle or isotropic subbundle generalizes to that of a reduction of structure group to a maximal parabolic subgroup $P \subset G$; equivalently, a section $\sigma\colon X \to Q/P$. The Segre invariant $s_r(V)$ or $t(V)$ is replaced by the number

$$s_P(Q) \ := \ \min\left\{\deg \sigma^* T^{\mathrm{vert}}_{(Q/P)/X} \ : \ \sigma \text{ a reduction of structure group to } P\right\}$$

where $T^{\mathrm{vert}}_{(Q/P)/X}$ is the tangent bundle along fibers of $Q/P \to X$. Holla and Narasimhan [8] computed an upper bound on s_P, which is not always sharp. The strategy of exploiting secant non-defectivity has given sharp upper bounds on certain s_P if G is $\mathrm{GL}_r\mathbb{C}$, $\mathrm{Sp}_{2n}\mathbb{C}$ or $\mathrm{SO}_{2n}\mathbb{C}$. It would be interesting to investigate whether these ideas can be used to give a sharp upper bound on s_P in general.

Although the present note can be read independently of [2, 3, 4, 5], we use several results from these articles. In particular, access to [4, §2 and §5] may be helpful for the reader.

2. Grassmannian bundles inside the extension spaces

Here we recall some notions from [3, 4]. Let X be a projective curve over $\mathbb{C}$ which is smooth and irreducible of genus $g \geq 2$. Let W be a vector bundle over X.

Via Serre duality and the projection formula, there are identifications

$$H^1(X,W) \;\cong\; H^0(X,K_X\otimes W^*)^* \;\cong\; H^0\left(\mathbb{P}W,\pi^*K_X\otimes\mathcal{O}_{\mathbb{P}W}(1)\right)^*.$$

Thus we obtain naturally a rational map $\phi\colon \mathbb{P}W \dashrightarrow \mathbb{P}H^1(X,W)$.

Suppose now that $W = \wedge^2 E$ for a vector bundle E of rank $n \geq 2$. Consider the fiber bundle $\mathrm{Gr}(2,E)$ over X whose fiber at $x \in X$ is the Grassmannian of 2-dimensional subspaces of E_x. Then we get a rational map

$$\psi\colon \mathrm{Gr}(2,E) \dashrightarrow \mathbb{P}H^1(X,\wedge^2 E)$$

by composing ϕ with the fiberwise Plücker embedding. In fact there is a diagram

$$\begin{tikzcd} & \mathbb{P}(E\otimes E) \arrow[r, dashed, "\widetilde{\phi}"] & \mathbb{P}H^1(X,E\otimes E) \\ \mathrm{Gr}(2,E) \arrow[r, hook] & \mathbb{P}(\wedge^2 E) \arrow[u, hook] \arrow[r, dashed, "\phi"] & \mathbb{P}H^1(X,\wedge^2 E). \arrow[u, hook] \end{tikzcd} \tag{2.1}$$

By the above discussion, it is easy to see that the line bundle on $\mathrm{Gr}(2,E)$ inducing ψ is $\pi^*K_X\otimes\det\mathcal{U}^*$, where $\mathcal{U}^*$ is the relative universal bundle on $\mathrm{Gr}(2,E)$.

Recall that the *slope* of a bundle E is defined as the ratio $\mu(E) := \deg(E)/\mathrm{rk}(E)$. The following is a consequence of [4, Lemma 2.2]:

Lemma 2.1. *Let $E \to X$ be a stable bundle with $\mu(E) < -1$. Then*

$$\widetilde{\phi}\colon \mathbb{P}(E\otimes E) \dashrightarrow \mathbb{P}H^1(X,E\otimes E) \quad \textit{and} \quad \psi\colon \mathrm{Gr}(2,E) \dashrightarrow \mathbb{P}H^1(X,\wedge^2 E)$$

are embeddings. □

Now the space $H^1(X,E\otimes E)$ is a parameter space for extensions

$$0 \to E \to V \to E^* \to 0.$$

By [7, Criterion 2.1], the subspace $H^1(X,\wedge^2 E)$ parameterizes extensions V with an orthogonal structure with respect to which E is Lagrangian. As discussed in [4], there is a relationship between the Segre stratification on the moduli space of orthogonal bundles and the stratification given by the higher secant variety of $\mathrm{Gr}(2,E)$ inside $\mathbb{P}H^1(X,\wedge^2 E)$. This motivates the work in the next section, and will be discussed in more detail in §4.

3. Non-defectivity of Grassmannian bundles

In this section, we assume that E is a general stable bundle of rank n and slope $\mu(E) < -1$, and consider the embedding $\psi\colon \mathrm{Gr}(2,E) \hookrightarrow \mathbb{P}H^1(X,\wedge^2 E)$. For each positive integer k, the k-th secant variety $\mathrm{Sec}^k\mathrm{Gr}(2,E)$ is the Zariski closure of the union of all the linear spans of k general points of $\mathrm{Gr}(2,E)$. We say that $\mathrm{Gr}(2,E)$ is *non-defective* if for all $k \geq 1$, we have

$$\dim \mathrm{Sec}^k\mathrm{Gr}(2,E) \;=\; \min\{k\cdot\dim\mathrm{Gr}(2,E) + (k-1),\; \dim\mathbb{P}H^1(X,\wedge^2 E)\}.$$

Theorem 3.1. *For a general stable bundle E of rank n and degree $d < -n$, the Grassmannian bundle $\mathrm{Gr}(2,E) \subset \mathbb{P}H^1(X,\wedge^2 E)$ is non-defective.*

We will prove this theorem by applying Terracini's Lemma. To do this, we must first describe the embedded tangent spaces of $\mathrm{Gr}(2, E)$. Now a point of $\mathrm{Gr}(2, E)$ corresponds to a two-dimensional subspace $P \subseteq E|_x$ for some $x \in X$. Let $\hat{E}$ be the elementary transformation of E along this subspace:

$$0 \to E \to \hat{E} \to \mathbb{C}_x \oplus \mathbb{C}_x \to 0.$$

We may regard $\hat{E}$ as the sheaf of sections of E which are regular apart from at most simple poles at x in directions corresponding to P. This induces a sequence $0 \to \wedge^2 E \to \wedge^2 \hat{E} \to \tau_P \to 0$, where τ_P is torsion of degree $2(n-1)$. The associated cohomology sequence is

$$\cdots \to \Gamma(\tau_P) \to H^1(X, \wedge^2 E) \longrightarrow H^1(X, \wedge^2 \hat{E}) \longrightarrow 0. \tag{3.1}$$

Let us describe τ_P more explicitly. Choose a local coordinate z centered at x, and a local frame $e_1, e_2, \ldots, e_n$ of E near x, where $e_1(x)$ and $e_2(x)$ span P. Then $\Gamma(\tau_P)$ has a basis consisting of the following principal parts:

$$\frac{e_1 \wedge e_i}{z} : \ i = 2, \ldots, n, \quad \frac{e_j \wedge e_2}{z} : \ j = 3, \ldots, n, \quad \text{and} \quad \frac{e_1 \wedge e_2}{z^2}. \tag{3.2}$$

Note that $\frac{e_i \wedge e_j}{z}$ depends only on the values of the sections e_i and e_j at x, but $\frac{e_1 \wedge e_2}{z^2}$ also depends on the 1-jets of e_1 and e_2. However, the image of $\Gamma(\tau_P)$ in $H^1(X, \wedge^2 E)$ depends only on the subspace P.

Lemma 3.2. *For $P \in \mathrm{Gr}(2, E)$, the embedded tangent space $\mathbb{T}_P \mathrm{Gr}(2, E)$ to $\mathrm{Gr}(2, E)$ at P coincides with*

$$\mathbb{P}\mathrm{Ker}\left[H^1(X, \wedge^2 E) \longrightarrow H^1\left(X, \wedge^2 \hat{E}\right)\right].$$

Proof. Let z and $e_1, e_2, \ldots, e_n$ be as above. For $1 \leq i \leq 2$, let E_i be the elementary transformation of E satisfying

$$\mathrm{Ker}\,(E|_x \to E_i|_x) = \mathbb{C} \cdot e_i(x).$$

Recall that the *decomposable locus* Δ of $\mathbb{P}(E \otimes E)$ is defined by

$$\Delta \;=\; \bigcup_{x \in X} \mathbb{P}\{e \otimes f : e, f \text{ nonzero in } E|_x\} \;\cong\; \mathbb{P}E \times_X \mathbb{P}E.$$

Now by hypothesis and by Lemma 2.1, we also have an embedding

$$\tilde{\phi} \colon \mathbb{P}(E \otimes E) \hookrightarrow \mathbb{P}H^1(X, E \otimes E).$$

By [2, Lemma 5.3], the embedded tangent space $\mathbb{T}_{e_1 \otimes e_2} \Delta$ is given by

$$\mathbb{P}\mathrm{Ker}\left[H^1(X, E \otimes E) \longrightarrow H^1(X, E_1 \otimes E_2)\right].$$

Therefore, by the cohomology sequence of

$$0 \to E \otimes E \to E_1 \otimes E_2 \to \frac{E_1 \otimes E_2}{E \otimes E} \to 0,$$

we see that $\mathbb{T}_{e_1 \otimes e_2} \Delta$ is (freely) spanned by the cohomology classes of the principal parts

$$\frac{e_1 \otimes e_i}{z} : i = 1, \ldots, n, \quad \frac{e_j \otimes e_2}{z} : j = 2, \ldots, n, \quad \text{and} \quad \frac{e_1 \otimes e_2}{z^2}. \tag{3.3}$$

Now $\mathrm{Gr}(2, E)$ is precisely the image of Δ under the projection $E \otimes E \to \wedge^2 E$. Thus the embedded tangent space to $\mathrm{Gr}(2, E)$ at P is exactly the image of $\mathbb{T}_{e_1 \otimes e_2}\Delta$ under the projection $\mathbb{P}H^1(X, E \otimes E) \dashrightarrow \mathbb{P}H^1(X, \wedge^2 E)$. Hence $\mathbb{T}_P \mathrm{Gr}(2, E)$ is spanned by the cohomology classes of the antisymmetrizations of (3.3):

$$\frac{e_1 \wedge e_i}{z}, \ i = 2, \dots, n, \quad \frac{e_j \wedge e_2}{z}, \ j = 3, \dots, n, \quad \text{and} \quad \frac{e_1 \wedge e_2}{z^2}.$$

But this is exactly the basis (3.2). The lemma follows by (3.1). □

Recall now that Hirschowitz' lemma [6, §4.6] states that the tensor product of two general bundles is non-special. We require also the following variant:

Lemma 3.3. *Suppose $F \to X$ is a general stable bundle of rank n and degree e. Then $\wedge^2 F$ is non-special; that is,*

$$\dim H^0(X, \wedge^2 F) \ = \ \begin{cases} (n-1)e - \frac{1}{2}n(n-1)(g-1) & \text{if } \ e > \frac{1}{2}n(g-1), \\ 0 & \text{if } \ e \leq \frac{1}{2}n(g-1). \end{cases}$$

Proof. If $e \leq \frac{1}{2}n(g-1)$, then by [3, Lemma A.1] we have $\dim H^0(X, F \otimes F) = 0$, and hence also $\dim H^0(X, \wedge^2 F) = 0$. The other case follows by an argument practically identical to that in [3, Corollary A.3]. □

Proof of Theorem 3.1. To ease notation, write $G = \mathrm{Gr}(2, E)$. By the Terracini lemma, the dimension of the higher secant variety $\mathrm{Sec}^k G$ coincides with that of the linear span of k general embedded tangent spaces:

$$\dim(\mathrm{Sec}^k G) \ = \ \dim \langle \mathbb{T}_{P_1} G, \mathbb{T}_{P_2} G, \dots, \mathbb{T}_{P_k} G \rangle ,$$

where $P_1, P_2, \dots, P_k$ are k general points of G supported at $x_1, x_2, \dots, x_k$ respectively. For $1 \leq i \leq k$, let $0 \to E \to F_i \to \mathbb{C}^2_{x_i} \to 0$ be the elementary transformation of E at the plane P_i. Then by Lemma 3.2, we have

$$\mathbb{T}_{P_i} G \ = \ \mathbb{P}\mathrm{Ker} \left[H^1(X, \wedge^2 E) \longrightarrow H^1(X, \wedge^2 F_i) \right] .$$

Write F for the elementary transformation of E determined by $P_1, \dots, P_k$. Then F_i is contained in F for each i, and the linear span $\langle \mathbb{T}_{P_i} G : 1 \leq i \leq k \rangle$ is given by

$$\mathbb{P}\mathrm{Ker} \left[H^1(X, \wedge^2 E) \longrightarrow H^1(X, \wedge^2 F) \right] .$$

Thus, to prove the theorem, we must show that

$$\dim \mathrm{Ker} \left[H^1(X, \wedge^2 E) \longrightarrow H^1(X, \wedge^2 F) \right] \ = \ \min\{ k \cdot \dim G + k, \ \dim H^1(X, \wedge^2 E) \}.$$

Note that

$$k \cdot \dim G + k \ = \ k(2n-3) + k \ = \ 2k(n-1) \ = \ \deg(\wedge^2 F) - \deg(\wedge^2 E). \tag{3.4}$$

Firstly, assume that $k \cdot \dim G + k \ < \ \dim H^1(X, \wedge^2 E)$, which is equivalent to

$$\deg F \ = \ d + 2k \ < \ \frac{n(g-1)}{2}.$$

Claim: For general E and general $P_1, \dots, P_k$ in $\mathrm{Gr}(2, E)$, the bundle $\wedge^2 F$ is non-special in the sense of Lemma 3.3.

By the claim, $h^0(X, \wedge^2 F) = 0$, and so

$$\begin{aligned}\dim \operatorname{Ker} \left[H^1(X, \wedge^2 E) \longrightarrow H^1(X, \wedge^2 F) \right] &= \dim H^1(X, \wedge^2 F) - \dim H^1(X, \wedge^2 E) \\ &= \deg(\wedge^2 F) - \deg(\wedge^2 E) \\ &= k \cdot \dim G + k \text{ by (3.4).}\end{aligned}$$

On the other hand, suppose $k \cdot \dim G + k \geq \dim H^1(X, \wedge^2 E)$, so

$$\deg F \ \geq \ \frac{1}{2} n(g-1).$$

By the above claim and by Lemma 3.3, then,

$$\dim H^0(X, \wedge^2 F) \ = \ (n-1)(d+2k) - \frac{1}{2} n(n-1)(g-1).$$

Therefore,

$$\begin{aligned}&\dim \operatorname{Ker} \left[H^1(X, \wedge^2 E) \longrightarrow H^1(X, \wedge^2 F) \right] \\ &\quad = \dim H^1(X, \wedge^2 F) - \dim H^1(X, \wedge^2 E) - \dim H^0(X, \wedge^2 F) \\ &\quad = \deg(\wedge^2 F) - \deg(\wedge^2 E) - (n-1)(d+2k) + \frac{1}{2} n(n-1)(g-1) \\ &\quad = -(n-1)d + \frac{1}{2} n(n-1)(g-1) \\ &\quad = \dim H^1(X, \wedge^2 E).\end{aligned}$$

Thus we are done once we have proven the claim. Note that the condition in Lemma 3.3 can be restated as

$$h^0(X, \wedge^2 F) \cdot h^1(X, \wedge^2 F) = 0. \tag{3.5}$$

It suffices to show that there exists a stable E such that some elementary transformation F of E of the stated form satisfies (3.5). By Lemma 3.3, we may choose an F_0 satisfying (3.5). Let E_0 be some elementary transformation of F_0 fitting into a sequence

$$0 \to E_0 \to F_0 \to \bigoplus_{i=1}^{k} \mathbb{C}^2_{x_i} \to 0. \tag{3.6}$$

Since (3.5) is an open condition on families, it holds for a general deformation

$$0 \to E_t \to F_t \to \tau_t \to 0$$

over a small disk T, where $\tau_t = \bigoplus_{i=1}^{k} \mathbb{C}^2_{x_i(t)}$ for $t \in T$. Thus for general $t \in T$, the deformation F_t satisfies (3.5). Since a general deformation of E_0 is a general stable bundle, we are done. $\square$

Theorem 3.1 asserts the non-defectivity of the image of $\mathrm{Gr}(2, E)$ in the projective space $|\pi^* K_X \otimes \det \mathcal{U}^*|^*$, where $\mathcal{U}$ is the relative universal bundle over $\mathrm{Gr}(2, E)$. It is not difficult to generalize this to certain other line bundles over $\mathrm{Gr}(2, E)$ restricting to $\det \mathcal{U}^*$ on each fiber:

Corollary 3.4. *Let $E \to X$ be a general stable bundle, and suppose $L \to X$ is a general line bundle satisfying* $\deg L > 1 + \mu(E)$. *Then the map*

$$\mathrm{Gr}(2, E) \dashrightarrow |\pi^*(K_X L^2) \otimes \det \mathcal{U}^*|^*$$

is an embedding, and the image is secant non-defective.

Proof. Write π_1 for the projection $\mathbb{P}(\wedge^2(E \otimes L)) \to X$, and $\mathcal{U}_1$ for the relative universal bundle over $\mathrm{Gr}(2, E \otimes L)$. A straightforward calculation shows that

$$H^0(\mathrm{Gr}(2, E), \pi^*(K_X L^2) \otimes \det \mathcal{U}^*)^* \cong H^0(\mathrm{Gr}(2, E \otimes L^{-1}), \pi_1^* K_X \otimes \det \mathcal{U}_1^*)^*.$$

By hypothesis, the bundle $E \otimes L^{-1}$ is general and satisfies $\mu(E \otimes L^{-1}) < -1$. Therefore, $\mathrm{Gr}(2, E \otimes L^{-1}) \cong \mathrm{Gr}(2, E) \dashrightarrow \mathbb{P}H^1(X, \wedge^2(E \otimes L^{-1}))$ is an embedding by Lemma 2.1. By Theorem 3.1, the image is secant non-defective. Since $\mathrm{Gr}(2, E)$ is canonically isomorphic to $\mathrm{Gr}(2, E \otimes L^{-1})$, the corollary follows. □

Remark 3.5. The above definitions of the secant variety $\mathrm{Sec}^k \mathrm{Gr}(2, E)$ and non-defectivity still make sense when $\psi \colon \mathrm{Gr}(2, E) \dashrightarrow \mathbb{P}H^1(X, \wedge^2 E)$ is only a generically finite rational map. If we assume E is such that both ψ and $\tilde{\phi} \colon \Delta \dashrightarrow \mathbb{P}H^1(X, E \otimes E)$ are generically finite and rational, then the proof of Theorem 3.1 is valid with a few minor technical modifications.

4. Application to Lagrangian Segre invariants

We return to the study of orthogonal bundles V of rank $2n$. In [4, Theorem 1.3 (1)], a sharp upper bound on the value of $t(V)$ was given, based on the computation of the dimensions of certain Quot schemes. In this section we use Theorem 3.1 together with a lifting criterion from [4] to give a more geometric proof of this upper bound.

Recall that the second Stiefel–Whitney class $w_2(V) \in H^2(X, \mathbb{Z}/2) = \mathbb{Z}/2$ is the obstruction to lifting the $\mathrm{SO}_{2n}\mathbb{C}$ structure on V to a spin structure (see Serman [12] for details). We recall another characterisation of $w_2(V)$ from [4, Theorem 1.2 (2)]:

Theorem 4.1. *Let V be an orthogonal bundle of rank $2n$. Then $w_2(V)$ is trivial (resp., nontrivial) if and only if all Lagrangian subbundles of V have even degree (resp., odd degree).* □

The link between the situation of Theorem 3.1 and the invariant $t(V)$ is given by the following:

Proposition 4.2.
1. *The subspace $H^1(X, \wedge^2 E)$ of $H^1(X, E \otimes E)$ parameterizes extensions $0 \to E \to V \to E^* \to 0$ admitting an orthogonal structure with respect to which E is Lagrangian.*
2. *Let $0 \to E \to V \to E^* \to 0$ be an orthogonal extension with class $[V] \in H^1(X, \wedge^2 E)$. Then some elementary transformation F of E^* satisfying $\deg(E^*/F) \leq 2k$ lifts to a Lagrangian subbundle of V if and only if $[V] \in \mathrm{Sec}^k \mathrm{Gr}(2, E)$. In this case,* $\deg E \equiv \deg F \mod 2$.
3. *If $[V] \in \mathrm{Sec}^k \mathrm{Gr}(2, E)$, then $t(V) \leq 2(2k + \deg E)$.*

Proof. (1) follows from [7, Criterion 2.1]. Statement (2) is [4, Criterion 2.2 (2)], and (3) is immediate from (2) and the definition of $t(V)$. $\square$

Now we can derive the upper bound on $t(V)$:

Theorem 4.3. *Let V be an orthogonal bundle of rank $2n$. If $w_2(V)$ is trivial (resp., nontrivial), then $t(V) \le n(g-1)+\varepsilon$, where $\varepsilon \in \{0,1,2,3\}$ is such that $n(g-1)+\varepsilon \equiv 0 \mod 4$ (resp., $n(g-1)+\varepsilon \equiv 2 \mod 4$).*

Proof. Suppose V is a general orthogonal bundle of rank $2n$ with $w_2(V)$ trivial. Firstly, we show that V has a Lagrangian subbundle F which is general as a vector bundle. We adapt the argument for symplectic bundles in [3, Lemma 3.2]: Choose an open set $U \subset X$ over which V is trivial. By linear algebra, we may choose a Lagrangian subbundle $\tilde{F}$ of $V|_U$. Since X is of dimension one, we may extend $\tilde{F}$ uniquely to a Lagrangian subbundle $F \subset V$. Deforming if necessary, we may assume F is general as a vector bundle. By Proposition 4.2 (1), the bundle V is represented in the extension space $H^1(X, \wedge^2 F)$. Moreover, $\deg F$ is even by Theorem 4.1.

We now compute the smallest value of k for which $\mathrm{Sec}^k \mathrm{Gr}(2,F)$ sweeps out the whole of $\mathbb{P}H^1(X,\wedge^2 F)$, and hence must contain $[V]$. By the non-defectivity of $\mathrm{Sec}^k\mathrm{Gr}(2,F)$ proven in Theorem 3.1, we have

$$\dim\left(\mathrm{Sec}^k\mathrm{Gr}(2,F)\right) \;=\; \min\left\{k(\dim \mathrm{Gr}(2,F)+1)-1, h^1(X,\wedge^2 F)-1\right\}.$$

Therefore, the number k we require is the smallest integral solution to the inequality

$$k(\dim \mathrm{Gr}(2,F)+1)-1 \;\ge\; h^1(X,\wedge^2 F)-1.$$

Computing, we obtain $2k \;\ge\; -\deg F + \frac{1}{2}n(g-1)$. Since $\deg F$ is even, we have

$$2k + \deg F \;=\; \frac{1}{2}\left(n(g-1)+\varepsilon\right)$$

where $\varepsilon \in \{0,1,2,3\}$ is such that $n(g-1)+\varepsilon \equiv 0 \mod 4$. By Proposition 4.2 (3), we obtain $t(V) \le n(g-1)+\varepsilon$ as required. Since V was chosen to be general, the bound is valid for all V by semicontinuity.

The case where $w_2(V)$ is nontrivial is proven similarly. $\square$

Remark 4.4. The above result was proven by a different method in [4, §5], which we outline here for comparison. Consider firstly bundles with trivial w_2. For each even number $e \ge 0$, one constructs a family of extension spaces of the form $\mathbb{P}H^1(X,\wedge^2 E)$ with $\deg E = -e$, admitting a classifying map to the moduli space $\mathcal{M}O^+_{2n}$ of semistable orthogonal bundles of rank $2n$ over X with trivial w_2. The fiber over a stable $V \in \mathcal{M}O^+_{2n}$ is identified with a Quot-type scheme of degree $-e$ Lagrangian subbundles of V. Computing the dimension of this Quot scheme, one sees that for $\varepsilon \in \{0,1,2,3\}$ such that $n(g-1)+\varepsilon \equiv 0 \mod 4$, the classifying map corresponding to $e = \frac{1}{2}(n(g-1)+\varepsilon)$ dominates $\mathcal{M}O^+_{2n}$. Thus $t(V) \le n(g-1)+\varepsilon$ for a general V with $w_2(V)$ trivial, and hence for all V by semicontinuity. A similar method works for bundles with nontrivial w_2, taking e to be odd instead of even.

Remark 4.5. In Theorem 4.3, the secant geometry of $\mathrm{Gr}(2,E) \subset \mathbb{P}H^1(X,\wedge^2 E)$ is applied in the "opposite" sense to that in [4, §5]. In Theorem 4.3, the density of

$\mathrm{Sec}^k \mathrm{Gr}(2, F)$ in $\mathbb{P}H^1(X, \wedge^2 F)$ is used to produce a *lower* bound on the degrees of maximal Lagrangian subbundles of a general orthogonal extension $0 \to E \to V \to E^* \to 0$. On the other hand, in [4, Theorem 5.2], the fact that certain $\mathrm{Sec}^k \mathrm{Gr}(2, E)$ are *not* dense in their respective $\mathbb{P}H^1(X, \wedge^2 E)$ is used to give *upper* bounds on the degrees of maximal Lagrangian subbundles of the corresponding extensions.

5. Intersection of maximal Lagrangian subbundles

Lange and Newstead showed in [10, Proposition 2.4] that if W is a generic vector bundle of rank r and if $k \le r/2$, then two maximal subbundles of rank k in W intersect generically in rank zero. The analogous statement for maximal Lagrangian subbundles of a generic symplectic bundle was proven in [4, Theorem 4.1 (3)]. However, as noted in [4, Remark 5.1], the corresponding approach in the orthogonal case does not exclude the possibility that two maximal Lagrangian subbundles intersect in a line bundle. In this section, we use the non-defectivity statement of Theorem 3.1 to show that this (somewhat unexpected) situation in fact arises for a general orthogonal bundle of even rank.

Firstly, we make precise the statement of [4, Remark 5.1].

Proposition 5.1. *Suppose X has genus $g \ge 5$, and $n \ge 2$. Let $V \to X$ be a general orthogonal bundle of rank $2n$. Then the generic rank of the intersection of any two maximal Lagrangian subbundles of V is at most 1.*

Proof. Let E be a general bundle of degree $-e := -\frac{1}{2}(n(g-1) + \varepsilon)$, where $\varepsilon \in \{0, 1, 2, 3\}$ is determined as in Theorem 4.3 by n and g together with a choice of Stiefel–Whitney class w_2. We consider orthogonal extensions $0 \to E \to V \to E^* \to 0$, which by Lemma 4.2 (1) are parameterized by $H^1(X, \wedge^2 E)$.

Suppose $H \subset E$ is a subbundle of rank r and degree $-h$, and write $q \colon E \to E/H$ for the quotient map. Then by the proof of [4, Criterion 2.3 (2)] the bundle $H^\perp/H$ is an orthogonal extension

$$0 \to \frac{E}{H} \to \frac{H^\perp}{H} \to \left(\frac{E}{H}\right)^* \to 0$$

with class $q_* {}^t q^* [V] \in H^1(X, \wedge^2(E/H))$. Furthermore, V admits a Lagrangian subbundle F of degree $-f \ge -e$ fitting into a diagram

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] & V \arrow[r] & E^* \arrow[r] & 0 \\
0 \arrow[r] & H \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & F/H \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

if and only if

$$e - h \ \ge \ 0 \tag{5.1}$$

and $\left[H^\perp/H\right] = q_* {}^t q^* [V]$ belongs to the secant variety

$$\mathrm{Sec}^{e-h} \mathrm{Gr}(2, E/H) \ \subseteq \ \mathbb{P}H^1\left(X, \wedge^2(E/H)\right).$$

By Step 1 of the proof of [4, Theorem 5.1], the locus of those V in $H^1(X, \wedge^2 E)$ admitting some such configuration of isotropic subbundles F and H has dimension at most

$$e(n-r) - (e-h)(r+1) - \frac{1}{2}(n-r)(n+r-1)(g-1) + h^1(X, \wedge^2 E). \tag{5.2}$$

Suppose firstly that $r \leq n-2$, so that $H^1(X, \wedge^2(E/H))$ is nonzero. Since $e - h \geq 0$, the above dimension is strictly smaller than $h^1(X, \wedge^2 E)$ if

$$\varepsilon \ < \ (r-1)(g-1).$$

Since $0 \leq \varepsilon \leq 3$, this is satisfied for all $g \geq 5$ if $r \geq 2$.

If $2 \leq r = n-1$, then E/H is a line bundle, so $h^1(X, \wedge^2(E/H)) = 0$. Thus

$$\frac{H^\perp}{H} \ \cong \ \frac{E}{H} \oplus \left(\frac{E}{H}\right)^*$$

and the inverse image F of $(E/H)^*$ is a Lagrangian subbundle of V, of degree

$$\deg H + \deg(E/H)^* \ = \ e - 2h.$$

Since V is general, $\deg F = e - 2h \leq -e$, so $e - h \leq 0$. In view of (5.1), therefore, $e = h$. We claim that a general bundle E of rank n and degree $-e$ has no rank $n-1$ subbundle of degree $\geq -e$. By Hirschowitz [6, Théorème 4.4], we have

$$(n-1)\deg E - n \deg H \ = \ e \ = \ \frac{1}{2}\left(n(g-1) + \varepsilon\right) \ \geq \ (n-1)(g-1)$$

since E is general. Thus we can have $\deg H = -e$ only if $\varepsilon \geq (n-2)(g-1)$. But this is excluded for $g \geq 5$ since $\varepsilon \leq 3$ and $n = r+1 \geq 3$ by hypothesis. □

Next, we will need the following result from Reid [11, §1] on the even orthogonal Grassmannian $\mathrm{OG}(n, 2n)$ parameterizing Lagrangian subspaces of $\mathbb{C}^{2n}$:

Proposition 5.2. *The space* $\mathrm{OG}(n, 2n)$ *consists of two disjoint, irreducible and mutually isomorphic components. Two Lagrangian subspaces* F_1 *and* F_2 *belong to the same component if and only if* $\dim(\mathsf{F}_1 \cap \mathsf{F}_2) \equiv n \mod 2$. □

In the same way, if V is an orthogonal bundle of rank $2n$, then there is a Lagrangian Grassmannian bundle $\mathrm{OG}(n, V) \subset \mathrm{Gr}(n, V)$, also with two connected components. A Lagrangian subbundle of V corresponds to a section $X \to \mathrm{OG}(n, V)$. We write $\mathrm{rk}\,(E_1 \cap E_2)$ for the dimension of the intersection of E_1 and E_2 at a general point of X. The subbundles E_1 and E_2 belong to the same component of $\mathrm{OG}(n, V)$ if and only if $\mathrm{rk}\,(E_1 \cap E_2) \equiv n \mod 2$.

Now we come to the main result of this section. Fix $w_2 \in H^2(X, \mathbb{Z}/2)$. By Theorem 4.3, a general orthogonal bundle V of rank $2n$ with $w_2(V) = w_2$ satisfies $t(V) = n(g-1) + \varepsilon$ where $0 \leq \varepsilon \leq 3$ is determined as above by n, g and w_2.

Theorem 5.3. *Assume that* $g \geq 5$ *and* $n \geq 2$. *Let* V *be a general orthogonal bundle as above. Let* E *be a maximal Lagrangian subbundle which is a general vector bundle of rank* n *and degree* $-e := -\frac{1}{2}(n(g-1) + \varepsilon)$.

(1) *There exist maximal Lagrangian subbundles* $\widetilde{E}$ *and* $\widetilde{F}$ *of* V *which intersect* E *in rank* 0 *and* 1 *respectively.*

(2) *The locally free part H of $E \cap \widetilde{F}$ satisfies*

$$-e \ \leq \ \deg H \ \leq \ -e + \frac{\varepsilon(n-1)}{4}.$$

(3) *If $n(g-1)$ is divisible by 4, then $E \cap \widetilde{F}$ is a line subbundle of degree $-e$.*

Proof. (1) By Proposition 4.2 (1), the bundle V is an extension $0 \to E \to V \to E^* \to 0$, with class $[V] \in H^1(X, \wedge^2 E)$. By the non-defectivity statement Theorem 3.1, the secant variety $\mathrm{Sec}^e \mathrm{Gr}(2, E)$ fills up the whole extension space $\mathbb{P}H^1(X, \wedge^2 E)$. Thus by the geometric lifting criterion Proposition 4.2 (2), the extension V contains a Lagrangian subbundle $\widetilde{E}$ lifting from E^* such that $\deg \widetilde{E} \geq -e$. Since E is maximal, in fact $\deg \widetilde{E} = -e$, so $\widetilde{E}$ is also maximal. Clearly $E \cap \widetilde{E}$ is trivial at a general point.

As for $\widetilde{F}$: Let I be any rank $n-1$ subbundle of E, which is necessarily isotropic in V. Over an open set $U \subset X$ upon which V is trivial, complete $I|_U$ to a Lagrangian subbundle of $V|_U$ intersecting $E|_U$ in $I|_U$. This gives a section of $\mathrm{OG}(n, V)$ over U. Since X is of dimension one, we can extend this uniquely to a global section over X, and get a Lagrangian subbundle F intersecting E in rank $n-1$. Write $\deg F = -f$. Since E is maximal Lagrangian, we have $f \geq e$.

Consider the extension space $H^1(X, \wedge^2 F)$, in which V is represented again. We claim that $h^0(X, \wedge^2 F) = 0$. For; if $\gamma \colon F^* \to F$ were a nonzero map, then the composition $V \to F^* \to F \to V$ would be a nonzero nilpotent endomorphism of V, contradicting stability. Thus no such γ exists, so

$$h^0(X, \wedge^2 F) \ \leq \ h^0(X, \mathrm{Hom}(F^*, F)) \ = \ 0.$$

Deforming V, E, I and $F|_U$ if necessary, we may assume F is a general stable bundle. By the non-defectivity proven in Theorem 3.1, the secant variety $\mathrm{Sec}^{\frac{1}{2}(f+e)} \mathrm{Gr}(2, F)$ has the expected dimension

$$\min \left\{ (e+f)(n-1) - 1, h^1(X, \wedge^2 F) - 1 \right\}.$$

Since $f \geq e = \frac{1}{2}(n(g-1) + \varepsilon)$ and $h^0(X, \wedge^2 F) = 0$, one checks easily that this dimension is $h^1(X, \wedge^2 F) - 1$. Therefore $\mathrm{Sec}^{\frac{1}{2}(f+e)} \mathrm{Gr}(2, F)$ fills the extension space $\mathbb{P}H^1(X, \wedge^2 F)$. Hence by Proposition 4.2 (2), the extension V contains a Lagrangian subbundle $\widetilde{F}$ lifting from F^*, satisfying $\deg \widetilde{F} \geq -e$. Since E is maximal, again $\deg \widetilde{F} = -e$, so $\widetilde{F}$ is a maximal Lagrangian subbundle.

Now $\mathrm{rk}\,(E \cap F) = n-1 \not\equiv n \mod 2$. Hence by Proposition 5.2 and the discussion following it, E and F belong to opposite components of $\mathrm{OG}(n, V)$. Then the fact that $\mathrm{rk}\,(F \cap \widetilde{F}) = 0$ implies that F and $\widetilde{F}$ belong to the same component if n is even, and to opposite components if n is odd. In both cases, the rank of $E \cap \widetilde{F}$ is odd, so in particular non-zero. By Proposition 5.1 and by generality, $\mathrm{rk}\,(E \cap \widetilde{F}) = 1$.

(2) Let us calculate the minimum value of h in order for a general extension $0 \to E \to V \to E^* \to 0$ to admit a Lagrangian subbundle of degree $-e$ whose intersection with E contains a line bundle H of degree $-h$. As before, the dimension of the locus of such extensions is in general bounded above by the expression (5.2).

The required inequality in h is therefore

$$e(n-r) - (e-h)(r+1) - \frac{1}{2}(n-r)(n+r-1)(g-1) \ \geq \ 0$$

where $r = 1$ and $e = \frac{1}{2}(n(g-1)+\varepsilon)$. Computing, we obtain the inequality

$$h \ \geq \ e - \frac{\varepsilon(n-1)}{4}.$$

Moreover, by (5.1), we have $h \leq e$. The desired inequality follows.

(3) If $n(g-1) \equiv 0 \mod 4$, then $\varepsilon = 0$ by Theorem 4.3. By part (2), we have $h = e$. By the proof of Proposition 5.1, we have $[V] \in \mathrm{Ker}\left(q_* {}^t q^*\right)$ for some such H (here we adhere to the convention that the empty set $\mathrm{Sec}^0\mathrm{Gr}(2, E/H)$ has dimension -1). Thus $H^\perp/H$ splits as $E/H \oplus (E/H)^*$, and the bundle F is the inverse image of $(E/H)^*$ in $H^\perp \subset V$. Thus $E \cap F$ is exactly the line bundle H. □

Remark 5.4. If $e = \frac{1}{2}n(g-1)$, we can determine the line bundle H up to a point of order two in Pic^0X. Since $[V]$ belongs to $H^1(X, \wedge^2 E)$, the extension structures $0 \to E \to V \to E^* \to 0$ and $0 \to E \to V^* \to E^* \to 0$ are isomorphic, and similarly for $0 \to F \to V \to F^* \to 0$. Thus we can identify the diagrams

$$\begin{array}{ccccc} H & \longrightarrow & F & \longrightarrow & F/H \\ \downarrow & & \downarrow & & \downarrow \\ E & \longrightarrow & V & \longrightarrow & E^* \\ \downarrow & & \downarrow & & \downarrow \\ E/H & \longrightarrow & F^* & \longrightarrow & \frac{E^*}{F/H} \end{array} \quad \text{and} \quad \begin{array}{ccccc} \left(\frac{E^*}{F/H}\right)^* & \longrightarrow & F & \longrightarrow & (E/H)^* \\ \downarrow & & \downarrow & & \downarrow \\ E & \longrightarrow & V^* & \longrightarrow & E^* \\ \downarrow & & \downarrow & & \downarrow \\ (F/H)^* & \longrightarrow & F^* & \longrightarrow & H^{-1} \end{array}$$

Hence $F/H \cong (E/H)^*$. Taking determinants, we obtain $H^2 \ \cong \ \det E \cdot \det F$. □

We conclude by describing the intersection of two maximal Lagrangian subbundles of a general orthogonal bundle of odd rank. This is an easy consequence of [5, Proposition 5.5]:

Proposition 5.5. *Suppose X has genus $g \geq 5$, and let $V \to X$ be a general orthogonal bundle of rank $2n+1$ where $n \geq 1$. Then any pair of maximal Lagrangian subbundles of V intersect trivially in a generic fiber of V.*

Proof. We may assume $n \geq 2$, the statement being obvious for $n = 1$. Let E be a general bundle of rank n and degree $-e = -\frac{1}{2}((n+1)(g-1)+\varepsilon)$ as above. Let $0 \to E \to F \to \mathcal{O}_X \to 0$ be a general extension. We consider orthogonal extensions $0 \to E \to V \to F^* \to 0$ in the sense of [5, §3], where $F = E^\perp$. By the proof of [5, Proposition 5.5], for $0 \leq r \leq n-2$, a general such V admits no maximal Lagrangian subbundle intersecting E in rank r if

$$e \ = \ \frac{1}{2}\left((n+1)(g-1)+\varepsilon\right) \leq \ \frac{1}{2}(n+r+1)(g-1).$$

This is true for any $r \geq 1$ if $\varepsilon < g-1$, which holds since $\varepsilon \leq 3$ and $g \geq 5$. □

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