Celebrating the centenary of the Schwarzschild solutions

Øyvind Grøn
Faculty of Technology, Art and Design, Oslo and Akershus University College of Applied Sciences, P.O. Box 4 St. Olavs Plass, NO-0130 Oslo, Norway

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This article is a celebration of the centenary of Schwarzschild's presentations of his external and internal solutions describing spacetime outside and inside an incompressible, spherically symmetric body. I give a review of these solutions and how they have been interpreted physically.

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I. A BRIEF HISTORY OF THE SCHWARZSCHILD SOLUTIONS

Karl Schwarzschild was born on October 9, 1873. He was very gifted and already at sixteen he had published two papers on the dynamics of double stars. When the First World War broke out in August of 1914, Schwarzschild offered his service for Germany in spite of being more than 40 years old. The papers he is now mostly remembered for were written at the Russian front during the First World War (Fig. 1).

At a meeting at the Prussian Academy of Sciences on November 18, 1915, Einstein had presented a solution to an old problem concerning Mercury's perihelion precession. It had turned out that this motion could not be explained using Newton's theory of gravity, even when including the effects of the other planets. Using his gravitational field equations for empty space, Einstein found a solution in the weak-field approximation and used this solution to calculate the relativistic contribution to the perihelion precession of Mercury. The results of the calculations was in agreement with observations.

Karl Schwarzschild was present at this meeting, being on leave from his military duties at the Russian front. Einstein's calculations made a deep impression on him, and within a month he had found a solution of the exact field equations of empty space. In a letter dated December 22, 1915, Schwarzschild offered the new solution to Einstein and wrote: "It is a wonderful thing that from such an abstract idea the Mercury anomaly emerges so stringently."[1]

Schwarzschild's paper with this solution was first presented to the Prussian Academy of Sciences by Einstein on behalf of Schwarzschild, and then submitted to the editor of Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften who received it on January 13, 1916.[2] Einstein sent a letter to Schwarzschild, writing: "I had not expected that the exact solution to the problem could be formulated. Your analytic treatment of the problem appears to me splendid."[3]

Fig. 1. Karl Schwarzschild, October 9, 1873–May 11, 1916.

As written by Thorne,[4] in less than two weeks after seeing Einstein's field equations, Schwarzschild had calculated the exact result for the curvature of spacetime outside *any* spherical star. His calculation was elegant and beautiful, and the curved spacetime geometry that it predicted, the *Schwarzschild geometry* as it soon came to be known, was destined to have enormous impacts on our understanding of gravity and the universe.

Karl Schwarzschild then went on to calculate the solution of Einstein's field equations for spacetime inside a spherically symmetric static and incompressible fluid.[5] The paper with this solution was received by the journal on February 24, 1916. Schwarzschild was transported home in March because of a serious disease, and he died at the age of 42 on May 11, 1916.

It is remarkable that Schwarzschild managed to arrive at these solutions under such difficult circumstances. It was not an easy task. Einstein himself had only found a solution of the approximate equations for empty space, which was sufficient to calculate the perihelion precession of Mercury. One reason for Schwarzschild's success was that he took advantage of the spherical symmetry of the problem and introduced spherical coordinates in his calculations.

II. PHYSICAL INTERPRETATION OF THE EXTERIOR SCHWARZSCHILD SOLUTION

The title of Schwarzschild's first relativity paper is: "On the gravitational field of a mass point according to Einstein's theory." In this paper, he deduces the solution of Einstein's field equations for empty space outside a point particle. Schwarzschild explicitly made the following assumptions (as formulated with modern notation):

(1) All the components of the metric tensor are independent of the time.
(2) Only the diagonal components of the metric tensor are non-vanishing.
(3) The solution is spherically symmetric.
(4) There is flat Minkowski spacetime at infinity.

He then made the following comment: "Mr. Einstein showed that this problem, in first approximation, leads to

Newton's law and that the second approximation correctly reproduces the known anomaly in the motion of the perihelion of Mercury. The following calculation yields the exact solution of the problem. It is always pleasant to avail of exact solutions of simple form. More importantly, the calculation proves also the uniqueness of the solution, about which Mr. Einstein's treatment still left doubt, and which could have been proved only with great difficulty, in the way shown below, through such an approximation method. The following lines therefore let Mr. Einstein's result shine with increased clearness."

In Schwarzschild's notation (and using units so that $c = 1$) the solution is given by the line element

$$ds^2 = -(1-\alpha/R)dt^2 + \frac{dR^2}{1-\alpha/R} + R^2 d\theta^2 + R^2 \sin^2\theta\, d\phi^2, \quad R = \left(r^3+\rho\right)^{1/3}, \tag{1}$$

where α and ρ are integration constants, and r is a radial polar coordinate with range $0 \le r < \infty$. Schwarzschild noted that there is what he called a "discontinuity" (we would say a coordinate singularity today) at $R = \alpha$. He then required that this shall "coincide with the origin," $r = 0$, where $R = \rho^{1/3}$. This requires that $\rho = \alpha^3$ and hence reduces the number of independent constants to one. Furthermore, he wrote that the value of this constant "depends on the value of the mass at the origin."

Usually, the value of α is determined by requiring that Einstein's theory shall be in agreement with Newton's theory in the weak field limit. However, in this paper, Schwarzschild determines α in a different way. He uses his solution to calculate the perihelion precession of Mercury and then compares his result with that of Einstein. In this way, he finds that α has a small positive value, which, however, is not specified. Instead Schwarzschild writes that for Mercury the quantity $(1+\alpha^3/r^3)^{1/3}$ differs from 1 only by quantities of the order 10^{-12}. In his article where the interior Schwarzschild solution was found, Schwarzschild deduced that

$$\alpha = R_S = 2GM/c^2, \tag{2}$$

which is now called the Schwarzschild radius.

It is clear from the title of Schwarzschild's paper and his presentation that he considered a physical situation with a mass point at the center of a polar coordinate system with radial coordinate $r > 0$ extending down to the center. But the radial coordinate R appearing in the line element only had values $R \ge \alpha$. Therefore, it was not necessary for him to discuss the physical nature of the singularity at $R = 0$. It was outside the allowable range of the R-coordinate.

Schwarzschild interpreted r to be a radial coordinate with origin at the symmetry center. With this interpretation, there is no interior region, and the symmetry center has no physical singularity, but there is a coordinate singularity there. In the line element (1) $R = \alpha$ represents the symmetry center according to Schwarzschild's interpretation. However, it was soon shown that there were certain difficulties with this interpretation.

The first observation, which was made by Droste,[6] was that according to the third term in the line element (1), a circle with radius r about the origin does not have the expected length $2\pi r$, but instead $2\pi R$. Taking the limit $r \to 0$ the length of the circle approaches $2\pi\alpha$ and not zero as one would expect for a point. Similarly, a spherical surface with radius r about the origin does not have the expected area $4\pi r^2$, but instead $4\pi R^2$. Taking the limit $r \to 0$ the area of the surface approaches $4\pi\alpha^2$ and not zero, as one again would expect for a point. Hence, it seems clear that $r = 0$ represents a surface with radius α, not a point. This issue was further discussed by Corda.[7] (Droste only considered the region $R > \alpha$.)

Historically, the "Schwarzschild solution" is the solution represented by the line element (1) where $r = 0$ is interpreted as the symmetry center, and the "Droste solution" is the solution represented by the same line element where $R = 0$ is interpreted as the symmetry center. However, at the present time, the term "the Schwarzschild solution" is used for the Droste solution, and I think this cannot be changed even if it would have been historically correct to use the term the "Schwarzschild-Droste solution." In the present article, I will use the term "Schwarzschild solution" in the historically correct way with symmetry center at $r = 0$ and the "Schwarzschild-Droste solution" for the solution with symmetry center at $R = 0$.

The time t in the line element (1) is measured by coordinate clocks that everywhere run at the same rate as a standard clock at rest infinitely far from the symmetry center. Droste studied free particles in this region. The travelling time of a particle falling vertically from a state of rest at R_0 to R is

$$t = \sqrt{\frac{R_0}{\alpha} - 1}\int_R^{R_0} \frac{x^{3/2}dx}{(x-\alpha)\sqrt{R_0 - x}}. \tag{3}$$

In the limit that $R \to \alpha$ this integral diverges. Hence, Droste concluded that as measured on clocks measuring t in the line element (1), it takes an infinitely long time for a freely falling particle to arrive at the surface $R = \alpha$. However, Droste did not calculate the corresponding travel time as measured by a clock carried by the falling particle, given by[8]

$$\tau = R_0\sqrt{\frac{R_0}{\alpha}}\left[\arccos\left(\sqrt{\frac{R}{R_0}}\right) + \sqrt{\frac{R}{R_0}}\sqrt{1-\frac{R}{R_0}}\right]. \tag{4}$$

This expression gives a finite proper time to fall down to $R = \alpha$. In addition, Droste calculated the physical distance from this surface to a point with radius R and found

$$\hat{R} = \int_\alpha^R \frac{dR}{\sqrt{1-\alpha/R}} = R\sqrt{1-\frac{\alpha}{R}} + \alpha\,\mathrm{arcosh}(R/\alpha). \tag{5}$$

With $\rho = \alpha^3$ the transformation between r and R is given in Eq. (1) as $R = (r^3+\alpha^3)^{1/3}$. One may argue that since the Schwarzschild- and Schwarzschild-Droste solutions are related by a coordinate transformation, the Schwarzschild-Droste solution must be equivalent to the solution found by Schwarzschild. This is clearly so in the external region $R > \alpha$. However, the global spacetime encompassing the internal region that cannot be described by a coordinate system comoving with a static reference frame is different in these solutions according to Schwarzschild's and the present interpretation. According to Schwarzschild, there is a point particle at the symmetry center of his solution, and according to the present interpretation, the Schwarzschild-Droste solution contains a black hole inside $R = \alpha$. Droste himself refrained from saying anything about the internal region.

Crothers[9] has argued that the quantity R in the line-element (1) represents the inverse square root of the Gaussian curvature K of the spherically symmetric surface in the 3-space $t = \text{constant}$, since the Gaussian curvature of the spherical surface defined by putting $t = \text{constant}$, $r = \text{constant}$ in the line-element (1) gives $K = 1/R^2$. Hence the curvature is finite at $r = 0$, again indicating a surface and not a point.

Hilbert considered the same problem[10] and his article was later discussed by Abrams[11] and Antoci.[12] Hilbert pointed out that the solution contains two singularities, one at $R = \alpha$ and another one at $R = 0$. However, as pointed out by Antoci, he wrote in a footnote that he considered Schwarzschild's interpretation, that $R = \alpha$ represents the symmetry center, as "not advisable," and pointed out $R = 0$ as the symmetry center. This has later become the "canonical" conception of the term "the Schwarzschild solution" due to the form of the angular part of the line element $R^2(d\theta^2 + \sin^2\theta\, d\phi^2)$.

Originally, there was not any clear understanding of the physical significance of the singularity at $R = \alpha$. Schwarzschild considered it to represent the position of a point particle at the symmetry center, but later physicists noted that it represents a spherical surface. In this connection, Antoci and Liebscher[13] have discussed the question of the removability of this singularity. They considered the acceleration scalar a of an observer O permanently at rest in the coordinate system, meaning that the observer has fixed spatial coordinates. The acceleration scalar is a physical quantity representing the proper acceleration of O, i.e., O's acceleration relatively to a free particle instantaneously at rest in the coordinate system, as measured by standard measuring rods and clocks carried by the observer. Calculating this acceleration scalar one finds[14]

$$a = \left(1 - \frac{\alpha}{R}\right)^{-1/2} \frac{\alpha}{2R^2}, \tag{6}$$

which diverges at $R = \alpha$. As stated by Antoci and Liebscher, "the value of a scalar cannot be altered by any transformation."[13] They further write that a physical argument for this singular behavior should be given.

There exists such an argument. From the line element (1), it follows that a standard clock at rest in the Schwarzschild coordinate system does not run at $R = \alpha$. That is the physical reason for the divergence of the acceleration scalar at this position.

Senovilla[15] has succinctly summarized some salient points concerning the Schwarzschild exterior solution:

- It was the first exact solution of Einstein's field equations that was found.
- It is unique for the exterior (empty space) spacetime outside any spherically gravitating source.
- It is the basis for most of the experimental tests of General Relativity.
- It gives a very good first approximation for spacetime outside a body that is slowly rotating or slightly deformed from a spherical shape.
- It describes to a good approximation the gravitational field outside compact bodies such as white dwarf stars, neutron stars, and black holes.
- It provides the theoretical basis for describing gravitational lensing.

III. THE INTERIOR SCHWARZSCHILD SOLUTION

We now come to Schwarzschild's interior solution—the solution of Einstein's field equations inside a static, spherically symmetric body with radius R_E and mass M consisting of an incompressible fluid. Such a situation is of course not physically realistic, because the sound velocity is infinitely great in such matter. Nevertheless, Schwarzschild's incompressible star is now the standard first example when teaching the Tolman-Oppenheimer-Volkoff equation, the relativistic equation of hydrostatic equilibrium, and it demonstrates important relativistic effects.

I will first give Schwarzschild's solution using modern notation.[16] Then the line element takes the form

$$ds^2 = -\left(\frac{3}{2}\sqrt{1 - \frac{R_S}{R_E}} - \frac{1}{2}\sqrt{1 - \frac{R_S}{R_E^3}R^2}\right)^2 dt^2 + \frac{dR^2}{1 - \left(R_S/R_E^3\right)R^2} + R^2 d\Omega^2, \quad R \le R_E, \tag{7}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ and $R_S = 2GM$ is the Schwarzschild radius of the body. Schwarzschild required that the metric is continuous at the surface of the sphere $R = R_E$. Comparing the line elements in Eqs. (1) and (7) then leads to $\alpha = R_S$. Hence, Newton's gravitational constant in the external solution does not come by taking the Newtonian limit in Schwarzschild's deduction, but more elegantly, without any reference to Newton's theory, from Einstein's field equations and the requirement of a continuous matching of the external and internal solution at the surface of the body.

Schwarzschild then made some interesting observations. Consider the 3-space $t = \text{constant}$ in the line element (7) and compare with the 3-space of the Friedmann-Lemaitre-Robertson-Walker universe models at the present time, i.e., setting the scale factor of the cosmic expansion equal to 1.[8] (The mathematical representation of a curved, spherically symmetric 3-space was known many years before it was applied in the FLRW universe models.) We have

$$dl^2 = \frac{dR^2}{1 - \left(k/R_0^2\right)R^2} + R^2 d\Omega^2, \tag{8}$$

where R_0 is the curvature radius of the 3-space. There are three possibilities: $k = -1$ for hyperbolic space with negative curvature, $k = 0$ for Euclidean space, and $k = 1$ for spherical space with positive curvature.

As noted by Schwarzschild, the 3-space is spherical inside the body with a 3-space described by the last terms of the line element (7). The curvature radius is $R_0 = \sqrt{R_E^3/R_S}$. Note that if the radius of the body is $R_E = R_S$ then $R_0 = R_E$. Hence, the constant curvature radius of the 3-space inside a body compressed so that its radius is equal to its Schwarzschild radius, is equal to its radius.

Furthermore, $R_S = 2GM = (8\pi G/3)\rho R_E^3 = (\kappa/3)\rho R_E^3$, where ρ is the density of the body and $\kappa = 8\pi G$ is Einstein's gravitational constant. In this way, Schwarzschild demonstrated that the curvature radius can be expressed by the density of the body as $R_0 = \sqrt{3/\kappa\rho}$.

In cosmology, one often employs a standard radial coordinate $\hat{\chi}$ so that the line element of a spherical 3-space takes the form[8]

$$dl^2 = d\hat{\chi}^2 + R_0^2 \sin^2(\hat{\chi}/R_0)\, d\Omega^2. \tag{9}$$

We see that in such a space the standard radial coordinate represents physical radial distance. Schwarzschild further introduced a dimensionless radial coordinate $\chi = \hat{\chi}/R_0$, i.e., an angle representing radial distance. The coordinate transformation between R and χ is

$$R = R_0 \sin\chi, \quad 0 < \chi < \chi_E, \tag{10}$$

where χ_E is given by $R_E = R_0 \sin\chi_E$. I will argue below that χ_E must obey $\chi_E < \arccos(1/3) = 1.23$ in order that the body shall not collapse under its own gravity.

In terms of the dimensionless standard radial coordinate the spacetime line element of the internal Schwarzschild solution takes the form given by Schwarzschild

$$\begin{aligned} ds^2 = &-(1/4)(3\cos\chi_E - \cos\chi)^2 dt^2 \\ &+ R_0^2(d\chi^2 + \sin^2\chi d\Omega^2). \end{aligned} \tag{11}$$

The non-Euclidean spatial geometry here appears in the expression for the area of a sphere with radius $\hat{\chi}$, namely, $4\pi R_0^2 \sin^2\chi$, which is smaller than the corresponding Euclidean area $4\pi\hat{\chi}^2$.

The pressure inside the body is[8]

$$p(R) = \frac{\sqrt{1 - \frac{R_S}{R_E^3}R^2} - \sqrt{1 - \frac{R_S}{R_E}}}{3\sqrt{1 - \frac{R_S}{R_E}} - \sqrt{1 - \frac{R_S}{R_E^3}R^2}}\rho. \tag{12}$$

In terms of the dimensionless standard radial coordinate, the pressure distribution is

$$p(\chi) = \frac{\cos\chi - \cos\chi_E}{3\cos\chi_E - \cos\chi}\rho. \tag{13}$$

To first order in R_S/R_E Eq. (12) gives the Newtonian expression for the pressure inside an incompressible spherical body[16]

$$p_N(R) = \frac{2\pi G}{3}\left(R_E^2 - R^2\right)\rho^2. \tag{14}$$

Using the fact that $\cos\chi \approx 1 - (1/2)\ \sin^2\chi \approx 1 - (1/2)\chi^2 = 1 - (1/2)\hat{\chi}^2/R_0^2 \approx 1 - (4\pi G/3)\rho R^2$, Eq. (13) gives the same expression.

Schwarzschild noted that the general theory of relativity does not permit an arbitrarily compressed mass distribution. The pressure at the center is

$$p(0) = \frac{1 - \cos\chi_E}{3\cos\chi_E - 1}\rho. \tag{15}$$

This pressure must be positive in order that the body shall not collapse, which requires $\cos\chi_E > 1/3$, leading to $R_E > (9/8)R_S$. Having noted this, Schwarzschild concluded his article by writing that for an outside observer a sphere of gravitational mass given by R_S cannot have a radius smaller than $(9/8)R_S$. (For the Sun R_S is equal to 3 km, for a mass of 1 gram R_S is equal to $1.5 \cdot 10^{-28}$ cm.) Hence, according to the theory of relativity, if a star becomes too compressed, it will be unstable and collapse. There is no corresponding Newtonian result.

IV. CONCLUSION

Within three months after Einstein had presented the correct field equations on November 24, 1915, Karl Schwarzschild had published two papers, one with the solution of the field equations for empty space outside a spherical mass distribution, and one with the solution of the field equations with an energy-momentum tensor describing spacetime inside an incompressible spherical body. He worked out the solutions while at the Russian front during the First World War, having a painful disease from which he died on May 11, 1916. It was a victory of his dedication, great talent, and mathematical experience that he managed to do so.

In the first article, Schwarzschild interpreted his exterior solution as representing spacetime outside a point mass, but in the second as describing spacetime outside a spherical body with finite radius. The Schwarzschild radius containing Newton's gravitational constant appeared in the exterior solution not by taking the weak field approximation and comparing with Newton's theory, but from Einstein's field equations by demanding continuity of the solutions at the boundary between the interior and exterior solutions.

Schwarzschild argued from the requirement of a finite positive pressure at the center of the mass distribution that the general theory of relativity restricts how much a body can be compressed before it collapses. According to the present interpretation of the exterior Schwarzschild solution, it predicts the existence of black holes, i.e., regions where the central body is compressed so much that a horizon appears outside the body. The horizon acts as a one way membrane where it is possible to move through it in the inwards direction, but not in the outwards direction. This means that nothing can come out of a black hole (disregarding quantum effects).

This phenomenon is not due to a very large curvature of spacetime or 3-space at the horizon. For a supermassive black hole, the curvature is not large at the horizon. The reason is that the "river of space" flows inwards with the velocity of light with respect to imaginary stationary observers at the horizon and with superluminal velocity inside it.[17,18]

This seems a little strange, though. We know that the special theory of relativity is valid locally. The second postulate of this theory says that the velocity of light is independent of the velocity of the light source. Hence even for a light source flowing inwards together with the river of space one would expect that the light moves outwards with the usual velocity. However, this is only valid locally according to the general theory of relativity. As measured by the clocks and measuring rods of an external observer far away from the central body in the asymptotic Minkowski spacetime, the slowing down of time in a gravitational field also makes light move slower. At the horizon it does not move at all.

This is a physical reality for the observer far away, but is nevertheless considered to be a coordinate effect because one can introduce coordinate clocks that are synchronized and adjusted so that light moving inwards has the usual velocity c independent of the position. With such so-called ingoing Eddington-Finkelstein coordinates,[14] the light cones are seen to point inwards (see Fig. 8.2 in Ref. 16). At the horizon light emitted in the outwards direction does not move at all, and inside the horizon it moves inwards. This may be understood as a consequence of the superluminal inwards velocity of the river of space in this region, which does not permit anything to move outwards. Also in this region there does not exist any Lorentz transformation to an observer permanently at rest in the Schwarzschild coordinate system.

Schwarzschild's deduction of the solutions of Einstein's field equations bearing his name was a great work performed under extremely difficult circumstances. It is a demonstration of what may be possible for truly gifted and dedicated people. These solutions have become the foundations for testing several predictions of the general theory of relativity. With good approximation, they describe the spacetime in which we live.

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APPENDIX: A POINT PARTICLE SOURCE OF THE EXTERIOR SCHWARZSCHILD SOLUTION

A point particle interpretation of the Schwarzschild-Droste solution has been considered by Narlikar and Padmanabhan[19] even though the coordinates of the line element (1) are not defined inside $R = \alpha$. However, just out of curiosity one may calculate the properties such a point particle must have in order to fulfill Einstein's field equations. They assumed that there is a point with mass M at the origin of the coordinate system, so that the time-time component of the energy-momentum tensor is $T_{\hat{t}\hat{t}} = M\delta(R)$, where $\delta(R)$ is Dirac's delta-function. From Einstein's field equations, we then have that the time-time component of the Einstein tensor is $E_{\hat{t}\hat{t}} = \kappa M\delta(R)$, where $\kappa = 8\pi G$ is Einstein's gravitational constant. For the Schwarzschild solution, the $\hat{r}\hat{r}$-component of the Einstein tensor then is $E_{\hat{r}\hat{r}} = E_{\hat{t}\hat{t}} = \kappa M\delta(R)$ and hence $T_{\hat{r}\hat{r}} = M\delta(R)$.

In order to calculate $E_{\hat{\theta}\hat{\theta}} = E_{\hat{\phi}\hat{\phi}}$, which was not given in Ref. 17, we need the expressions for the components of the Einstein tensor for the line element

$$ds^2 = -e^{2\alpha(R)}dt^2 + e^{2\beta(R)}dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta\, d\phi^2. \tag{A1}$$

For the Schwarzschild solution $\beta(R) = -\alpha(R)$ and so

$$E_{\hat{t}\hat{t}} = -E_{\hat{R}\hat{R}} = \frac{1}{R^2}\left(1 - e^{2\alpha}\right) - \frac{2}{R}e^{2\alpha}\alpha',$$
$$E_{\hat{\theta}\hat{\theta}} = E_{\hat{\phi}\hat{\phi}} = e^{2\alpha}\left(\alpha'' + 2\alpha'^2 + \frac{2\alpha'}{R}\right). \tag{A2}$$

It follows from these equations that

$$E_{\hat{\theta}\hat{\theta}} = E_{\hat{\phi}\hat{\phi}} = -\frac{1}{2R}\left(R^2 E_{\hat{t}\hat{t}}\right)'. \tag{A3}$$

Inserting $T_{\hat{t}\hat{t}} = M\delta(R)$ and using the fact that the derivative of Dirac's delta-function is $\delta'(R) = -\delta(R)/R$, gives $E_{\hat{\theta}\hat{\theta}} = E_{\hat{\phi}\hat{\phi}} = -M\delta(R)$. Hence, the "point particle" at the center is characterized by the energy-momentum tensor

$$E_{\hat{\mu}\hat{\nu}} = M\delta(R)diag(1, 1, -1, -1). \tag{A4}$$

This means that in order to be a solution of Einstein's field equations the "point particle" at the center must have a structure giving it a radial pressure and a tangential stress. It is like a Zel'dovich medium in the radial direction—as rigid as allowed in order that the sound velocity in the radial direction shall not be greater than the velocity of light—and like a domain wall in the tangential direction. Hence the material of the particle is extremely anisotropic.

However, it would be more in the spirit of Schwarzschild to assume a mass distribution $T_{\hat{t}\hat{t}} = M\delta(R - \alpha) = M\delta(r)$ and interpret this as a point particle at the symmetry center $r = 0$. According to the modern interpretation, however, the symmetry center is at $R = 0$, and the mass distribution represents a spherical shell.

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