Design of RC Sections with Single Reinforcement according to EC2-1-1 and the Rectangular Stress Distribution

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Abstract. Nowadays, the design of concrete structures in Europe is governed by the application of Eurocode 2 (EC2). In particular, EC2 – Part 1-1 deals with the general rules and the rules for concrete buildings. An important aspect of the design is specifying the necessary tensile (and compressive, if needed) steel reinforcement required for a Reinforced Concrete (RC) section. In this study we take into account the equivalent rectangular stress distribution for concrete and the bilinear stress-strain relation with a horizontal top branch for steel. This chapter presents three detailed methodologies for the design of rectangular cross sections with tensile reinforcement, covering all concrete classes, from C12/15 up to C90/105. The purpose of the design is to calculate the necessary tensile steel reinforcement. The first methodology provides analytic formulas and an algorithmic procedure that can be easily implemented in any programming language. The second methodology is based on design tables that are provided in Appendix A, requiring less calculations. The third methodology provides again analytic formulas that can replace the use of tables and even be used to reproduce the design tables. Apart from the direct problem, the inverse problem is also addressed, where the steel reinforcement is given and the purpose is to find the maximum bending moment that the section can withstand, given also the value and position of the axial force. For each case analytic relations are extracted in detail with a step-by-step procedure, the relevant assumptions are highlighted and results for four different cross section design examples are presented.

Keywords: Reinforced Concrete, RC, Section, Design, Eurocode, Steel, Reinforcement.

1 LITERATURE REVIEW AND INTRODUCTION

During the last decades, well-established procedures have been used for the design of reinforced concrete cross-sections against bending and/or axial loads (Rüsch 1960). Three Model Codes have been published in the past (Comité Euro-International du Béton 1978; Comité Euro-International du Béton 1993; Fédération Internationale du Béton 2012a; Fédération Internationale du Béton 2012b), which are guiding documents for future codes, making recommendations for the design of reinforced and prestressed concrete structures. In the first two, improved models were developed for a more accurate representation of the structural behaviour of reinforced concrete structures. In Model Code 1990 (Comité Euro-International du Béton 1993) constitutive equations for the proper description of concrete material properties were introduced (concrete strengths up to C80 were considered), in view of the possibility of nonlinear finite element analysis of structures. Model Code 1990 (Comité Euro-International du Béton 1993) became the most important reference document for the future development of EC2-1-1 (European Committee for Standardisation 2004). A detailed presentation of the Model Code 2010 (Fédération Internationale du Béton 2012a; Fédération Internationale du Béton 2012b) is given in (Walraven and Bigaj-van Vliet 2011).

It is common knowledge that all relevant national standards of European countries regarding the design and construction of reinforced concrete structures will eventually be replaced by the Eurocode 2 (EC2), which will be valid throughout the whole Europe and not only. EC2-Part 1-1 (European Committee for Standardisation 2004) specifies the strength and deformation characteristics of 14 classes of concrete, classified according to their strength. For all of these, stress–strain relationships are defined for: (a) structural analysis, (b) design of cross-section and (c) confinement of concrete. In the second case, three stress-strain relationships are defined for concrete as follows: (a) parabolic-rectangular stress distribution (b) bi-linear stress distribution, (c) rectangular stress distribution.

In the past research has been conducted regarding the degree of simplification, conservative design, safety and equivalence in between the three above cases of stress-strain distributions, as well as their application for modern types of reinforcement (e.g. Fibre Reinforced Polymer, FRP). In (Roşca and Petru 2009) the design of a reinforced concrete section subjected to bending using two stress–strain relationships mentioned in EC2, namely the parabola-rectangle stress distribution and the rectangular distribution, is studied and the differences are underlined. Two dimensionless quantities are used to convert the parabola-rectangle stress distribution to an equivalent concentrated force for the concrete in compression. Also analytic relations which determine the limit between single reinforcement (only tensile) and double reinforcement (tensile and compressive) are provided. The results drawn from the use of these two stress distributions, namely, parabola–rectangle and rectangle, showed that the differences between the amounts of reinforcement are less than 1% for singly reinforced sections and less than 2% for doubly reinforced sections.

Due to the different characteristics of higher strength concrete (higher strain before reaching yield, and much reduced stress plateau after yield) some design procedures traditionally used in normal strength concrete structures had to be revised. In (Jenkins 2011), Jenkins compared the results of the revised rectangular stress block specified in the Australian Standard Concrete Structures Code AS 3600-2009 (Standards Australia 2009) regarding concrete strengths higher than 50 MPa, with those in the main international codes (e.g. ACI 318-2005 (ACI Committee 318 2005), EC2-1-1 (European Committee for Standardisation 2004)), and with stress-strain distributions closer to the actual behaviour of high strength concrete. It was found that the equivalent rectangular stress block derived from the parabolic-rectangular stress block of EC2 (assuming the same positions of centroids and the same resultant compressive force) gave almost identical results to the parabolic-rectangular curve of EC2 for all concrete strengths when used on a rectangular section.

In (Shehata et al. 2000) the influence of the assumed stress-strain curve for concrete on the prediction of the strength of conventional and high strength concrete columns under eccentric axial load is investigated. It was concluded that the traditional parabola-rectangle stress-strain relationship of the CEB-FIP Model Code 90 (for f_{ck}<50 MPa) leads to unsafe results when used for high strength concrete.

A general methodology for determining the moment capacity of FRP RC sections by using the general parabola-rectangle diagram for concrete in compression, according to the model of EC2 is proposed in (Torres et al. 2012). Non-dimensional equations are derived independently of the characteristics of concrete and FRP reinforcement, and a simplified closed-form equation is also proposed for the case of failure due to FRP rupture. These equations can be used to obtain universal design charts and tables, which can facilitate the design process. A comparative study is also presented between the predictions of the proposed methodology and experimental results from 98 tests available in the literature.

Although the above studies deal with the application of the most suitable stress-strain diagram for concrete for the “optimal” design of cross sections using different approaches, to the authors’ knowledge, there is no study in which explicit closed formulas, design charts and design tables are provided to achieve the design of RC sections according to EC2-1-1. In the present study, the case of the rectangular stress distribution of EC2-1-1 for concrete is thoroughly studied and three different but equivalent methodologies are provided for the design of RC sections with single tensile reinforcement. The first and the third of the methodologies provide analytic formulas and step-by-step instructions for the design, while the second is based on easy-to-use design tables that are provided in the Appendix. In addition, the inverse problem is also investigated, again using the three methodologies, where given the steel reinforcement the aim is to find the maximum bending moment that the RC section can withstand, given also the axial force acting on the section.

2 CONCRETE

2.1 Concrete properties

According to EC2-1-1 the compressive strength of concrete is denoted by concrete strength classes which relate to the characteristic (5%) cylinder strength f_{ck}, or cube strength $f_{ck,cube}$, in accordance with EN 206-1. Higher strength concretes, up to the class C90/105 are covered by Eurocode 2. The strength classes for concrete are presented in Table 1 where f_{ck} is the characteristic compressive cylinder strength of concrete at 28 days and $f_{ck,cube}$, is the corresponding cube strength.

Table 1. Strength classes for concrete according to EC2-1-1.

f_{ck} (MPa)	12	16	20	25	30	35	40	45	50	55	60	70	80	90
$f_{ck,cube}$ (MPa)	15	20	25	30	37	45	50	55	60	67	75	85	95	105

The design compressive strength is defined as

$$f_{cd} = a_{cc} \frac{f_{ck}}{\gamma_c} \quad (1)$$

where:

- γ_c is the partial safety factor for concrete at the Ultimate Limit State, which is given in Table 2.1N of EC2-1-1. For persistent and transient design situations, $\boldsymbol{\gamma_c}$**=1.5**
- a_{cc} is the coefficient taking account of long term effects on the compressive strength and of unfavourable effects resulting from the way the load is applied. The value of a_{cc} for use in a country should lie between 0.8 and 1.0 and may be found in its National Annex. **The recommended value is 1**, although various countries have adopted lower values, leading to more conservative designs.

It should be noted that higher concrete strength shows more brittle behaviour, reflected by shorter horizontal branch, as will be shown in the stress-strain relations and diagrams, later.

2.2 Concrete stress – strain relations for the design of cross sections

Eurocode 2 Part 1-1 suggests the use of three approaches for the stress-strain relations of concrete for the design of cross sections:

1. Parabola-rectangle diagram (more detailed) – EC2-1-1 Par. 3.1.7(1)
2. Bi-linear stress-strain relation (less detailed) – EC2-1-1 Par. 3.1.7(2)
3. Rectangular stress distribution (simplest approach) – EC2-1-1 Par. 3.1.7(3)

The three different approaches are described in detail in the following sections. In the present study, only the 3rd approach has been used for the design of RC sections.

2.2.1. Rectangular stress distribution

According to Paragraph 3.1.7(3) of EC2-1-1, a rectangular stress distribution may be assumed for concrete, as shown in Figure 1 (Figure 3.5 of EC2-1-1).

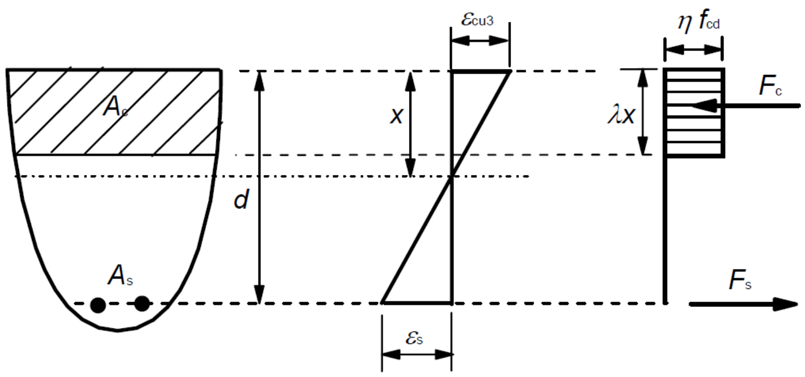


Figure 1. Rectangular stress distribution.

In the figure, d is the effective depth of the cross-section, x is the neutral axis depth, A_s is the cross sectional area of the tensile steel reinforcement, ε_s is the tensile strain at the position of the steel reinforcement, F_c is the concrete force (compressive, positive, as in the figure), F_s is the steel reinforcement force (tensile, positive, as in the figure). The factor λ defining the effective height of the compression zone and the factor η defining the effective strength, are calculated from:

$$\lambda = \begin{cases} 0.8 & \text{for} \quad f_{ck} \leq 50MPa \\ 0.8 - \dfrac{f_{ck} - 50}{400} & \text{for} \quad 50 < f_{ck} \leq 90MPa \end{cases} \tag{2}$$

$$\eta = \begin{cases} 1.0 & \text{for} \quad f_{ck} \leq 50MPa \\ 1.0 - \dfrac{f_{ck} - 50}{200} & \text{for} \quad 50 < f_{ck} \leq 90MPa \end{cases} \tag{3}$$

According to EC2-1-1, Table 3.1 the value of ε_{cu3} is given by

$$\varepsilon_{cu3}(‰) = \begin{cases} 3.5 & \text{for} \quad f_{ck} \leq 50MPa \\ 2.6 + 35\left(\dfrac{90 - f_{ck}}{100}\right)^4 & \text{for} \quad 50 < f_{ck} \leq 90MPa \end{cases} \tag{4}$$

Table 2 and Figure 2 show the values of the parameters λ, η and ε_{cu3} for each concrete class.

Table 2. The parameters λ, η and ε_{cu3} for each concrete class.

Concrete Class	**λ**	**η**	**ε_{cu3} (‰)**
C12/15–C50/60	0.80	1.00	3.50
C55/67	0.79	0.98	3.13
C60/75	0.78	0.95	2.88
C70/85	0.75	0.90	2.66
C80/95	0.73	0.85	2.60
C90/105	0.70	0.80	2.60

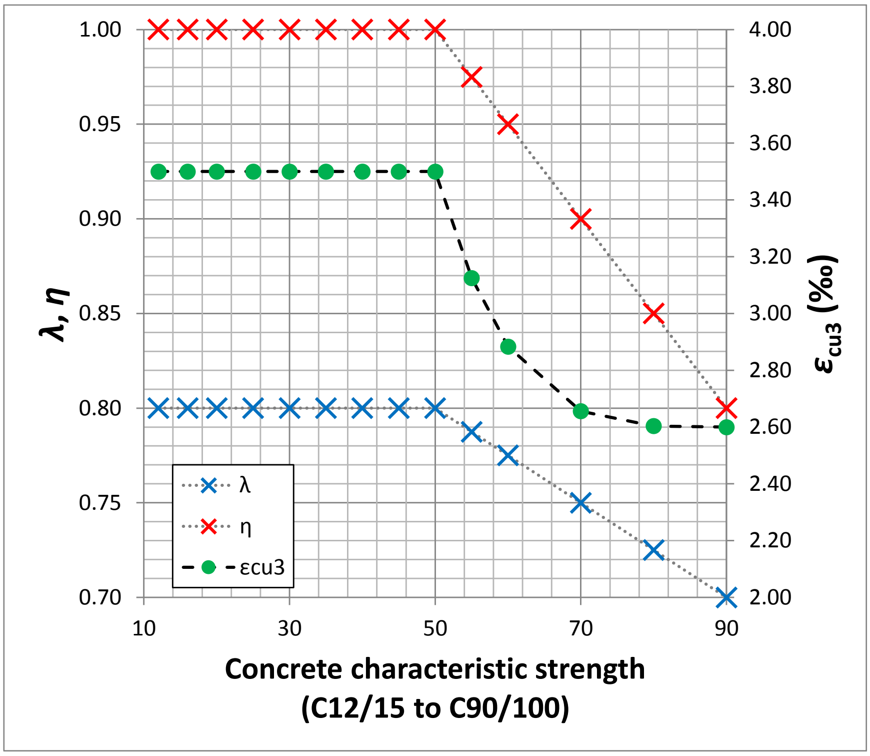


Figure 2. The parameters λ, η and ε_{cu3} for each concrete class.

Note: According to EC2-1-1, if the width of the compression zone decreases in the direction of the extreme compression fibre, the value $\eta \cdot f_{cd}$ should be reduced by 10%. This case is not examined in the present study, as the cross section is assumed to be rectangular and the width of the compression zone does not decrease. In any case, if needed, this correction can be very easily implemented in the calculations.

3 STEEL

3.1 Steel properties

The design strength for steel is given by

$$f_{yd} = \frac{f_{yk}}{\gamma_s} \tag{5}$$

where γ_s is the partial safety factor for steel at the Ultimate Limit State, which is given in Table 2.1N of EC2-1-1 (for persistent and transient design situations, $\boldsymbol{\gamma_s}$**=1.15**) and f_{yk} is the characteristic yield strength of steel reinforcement.

Table 3 (derived from Table C.1 of Annex C of EC2-1-1) gives the properties of reinforcement suitable for use with the Eurocode. The properties are valid for temperatures between -40ºC and 100ºC for the reinforcement in the finished structure. Any bending and welding of reinforcement carried out on site should be further restricted to the temperature range as permitted by EN 13670.

Table 3. Properties of steel reinforcement according to EC2-1-1.

Product form	**Bars and de-coiled rods**			**Requirement or quantile value (%)**
Class	A	B	C	-
Characteristic yield strength f_{yk} or $f_{0,2k}$ (MPa)	400 to 600			5.0
Minimum value of $k=(f_t/f_y)_k$	≥1.05	≥1.08	≥1.15 <1.35	10.0
Characteristic strain at maximum force, ε_{uk} (%)	≥2.5	≥5.0	≥7.5	10.0

The application rules for design and detailing in Eurocode 2 are valid for a specified yield strength range, f_{yk} from 400 to 600 MPa. The yield strength f_{yk} is defined as the characteristic value of the yield load divided by the nominal cross sectional area. The reinforcement should have adequate ductility as defined by the ratio of tensile strength to the yield stress, $(f_t/f_y)_k$ and the characteristic strain at maximum force, ε_{uk}. Typical values of f_{yk} used in the design practice nowadays are 400 MPa and 500 MPa.

3.2 Steel stress-strain relations for the design of cross-sections

According to Paragraph 3.2.7(2) of EC2-1-1, for normal design, either of the following assumptions may be made for the stress-strain relation for steel, as shown in Figure 3 (Figure 3.8 of EC2-1-1):

1. An inclined top branch with a strain limit of ε_{ud} and a maximum stress of $k \cdot f_{yk}/\gamma_s$ at ε_{uk}, where $k=(f_t/f_y)_k$.
2. A horizontal top branch <u>without the need to check the strain limit</u>.

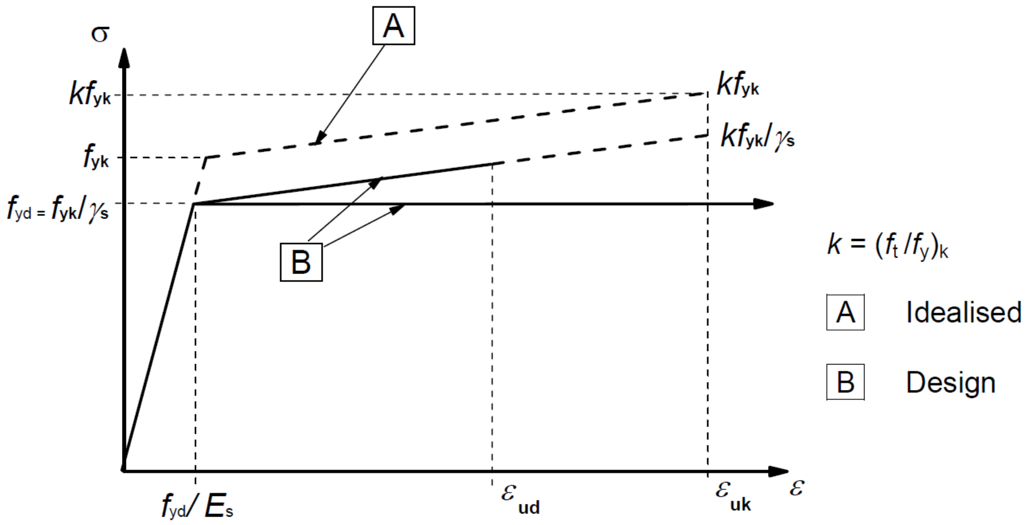


Figure 3. Idealised and design stress-strain diagrams for reinforcing steel (for tension and compression)

The parameter k defines the inclination of the top branch. The special case k=1 corresponds to a horizontal top branch (no inclination).

In the present study we use the second of the above approaches, i.e. a horizontal top branch for steel (k=1). According to this approach, there is no need to check the strain limit of steel and as a result in the design of RC cross sections, the concrete is always assumed to be the critical material. In this case, the steel design stress is given by

$$\sigma_s = \begin{cases} f_{yd} \cdot \dfrac{\varepsilon_s}{\varepsilon_{ys}} = E_s \cdot \varepsilon_s & \text{if } 0 < \varepsilon_s < \varepsilon_{ys} \\ f_{yd} & \text{if } \varepsilon_s \geq \varepsilon_{ys} \end{cases} \tag{6}$$

where f_{yd} is the design steel strength given by Eq. (5) and ε_{ys} is the design yield strain given by

$$\varepsilon_{ys} = \frac{f_{yd}}{E_s} \tag{7}$$

The design value of the steel modulus of elasticity E_s may be assumed to be 200 GPa according to EC2-1-1.

Table **4** shows the parameters f_{yk}, f_{yd} and ε_{ys} for each steel class (B400, B500, B600), with the assumptions E_s=200 GPa and γ_s=1.15, in accordance with EC2-1-1.

Table 4. The parameters f_{yk}, f_{yd} and ε_{ys} for each steel class, assuming E_s=200 GPa and γ_s=1.15.

Steel Class	f_{yk} (MPa)	f_{yd} (MPa)	ε_{ys} (‰)
B400	400	347.83	1.74
B500	500	434.78	2.17
B600	600	521.74	2.61

4 DESIGN ASSUMPTIONS

The following design assumptions are made in this study, in accordance with Eurocode 2 - Part 1-1:

1. The design is based on characteristic concrete cylinder strengths, not cube strengths.
2. Plane sections remain plane.
3. Strain in the bonded reinforcement, whether in tension or compression, is the same as that of the surrounding concrete.
4. The tensile strength of concrete is completely ignored.
5. The concrete stress is considered according to the simplified rectangular distribution shown in Figure 1. This gives the opportunity to obtain elegant closed-form solutions for the design process.
6. Stress in steel reinforcement is considered according to the stress-strain relation of EC 2-1-1 for steel (Figure 3), with a horizontal top branch without the need to check the

strain limit. As a result, concrete is assumed to always be the critical material, reaching its maximum strain at ULS.

5 RECTANGULAR STRESS DISTRIBUTION CASE DEFINITIONS

Figure 4 shows a typical rectangular cross section and the distribution of strains, stresses and corresponding forces.

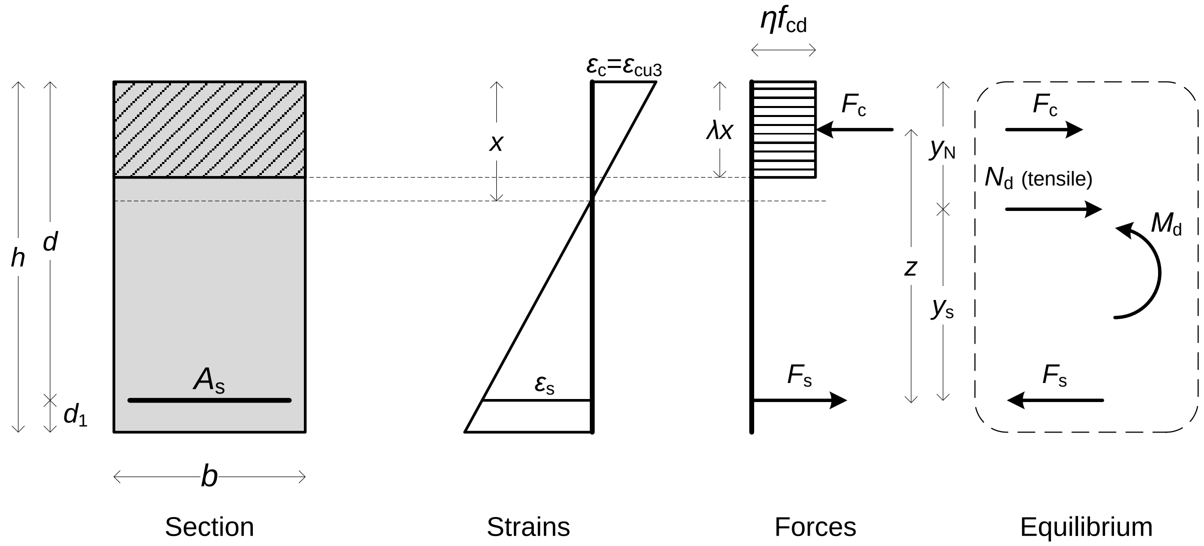


Figure 4. Cross section, strain, stresses and forces distribution and section equilibrium, assuming $\varepsilon_c = \varepsilon_{cu3}$ (concrete at limit strain).

Since the horizontal top branch for the steel stress-strain relationship is adopted in this study (Figure 3), there is no need to check the strain limit of steel and at the Ultimate Limit State (ULS) the concrete is the critical material ($\varepsilon_c = \varepsilon_{cu3}$) as shown in Figure 4, where:

- h and b are the height and width of the rectangular section, respectively
- d_1 is the distance from the lower edge of the section to the centre of the tensile reinforcement
- d is the effective depth of the rectangular section
- x is the neutral axis depth
- ε_s is the tensile strain in the steel reinforcement
- $\varepsilon_c = \varepsilon_{cu3}$ is the compressive strain in the concrete upper edge
- λ is a factor defining the effective height of the compression zone, given by Eq. (2)
- η is a factor defining the effective strength of the compression zone, given by Eq. (3)
- M_d is the applied external bending moment (if positive, it puts the lower edge of the section in tension)
- N_d is the applied external axial force (tensile for the section if positive), applied at a position y_N measured from the top of the section towards the lower edge of it. Note: If the axial force is central, acting at the middle of the section height, then $y_N = h/2$

- y_s is the distance from the tensile steel reinforcement to the position of the external applied axial force
- z is the distance of the resultant concrete force F_c from the steel reinforcement
- F_c is the concrete (compressive) force
- F_s is the steel (tensile) force
- A_s is the required steel reinforcement

5.1 Definition of the direct and the inverse problem

In the direct problem, the loading conditions (bending moment M_d, axial force N_d,) are given and the purpose is to calculate the required tensile reinforcement (steel area) A_s. In the inverse problem, A_s and N_d (applied at y_N) are given and the purpose is to calculate the maximum bending moment M_d that the cross section can withstand.

6 INVESTIGATION OF THE DIRECT PROBLEM

6.1 Analytical calculation of the required tensile reinforcement area A_s

In the direct problem, the loading conditions are given and the purpose is to calculate the required tensile reinforcement (steel area) A_s. In order to calculate A_s, the unknown quantities x and z for the given loading conditions have to be calculated first. After moving the external force N_d to the position of the steel reinforcement and imposing force and moment equilibrium for the cross-section, the situation is depicted in Figure 5.

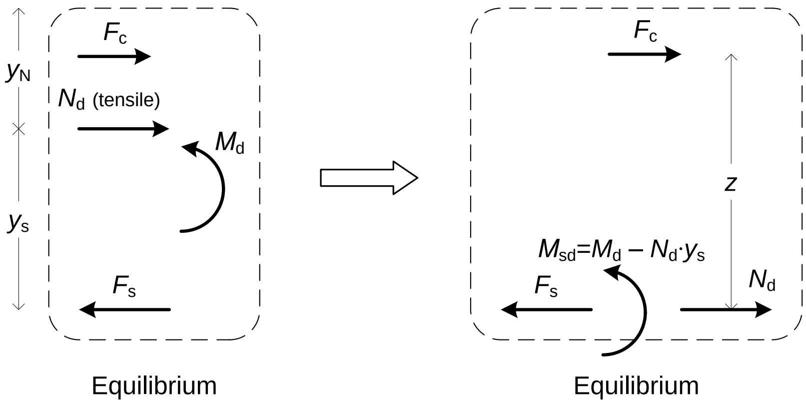


Figure 5. Equilibrium after moving the external force N_d to the position of the steel reinforcement.

From the equilibrium of the section in the x-direction, we have:

$$\Sigma F_x = 0 \Rightarrow F_c + N_d - F_s = 0 \Rightarrow F_s = F_c + N_d \tag{8}$$

We have also:

$$d_1 + d = h \Rightarrow d = h - d_1 \tag{9}$$

$$y_s + y_N = d \Rightarrow y_s = d - y_N \tag{10}$$

The effective bending moment applied at the location of the steel reinforcement is:

$$M_{sd} = M_d - N_d \cdot y_s \tag{11}$$

From the geometry of the section (Figure 4), we have:

$$d = z + \frac{\lambda x}{2} \Rightarrow z = d - \frac{\lambda x}{2} \tag{12}$$

The concrete force, assuming a rectangular distribution of stresses, is given by:

$$F_c = \lambda x \eta b f_{cd} \tag{13}$$

From the equilibrium of moments at the position of the steel reinforcement (Figure 5) we have (clockwise moment taken as positive):

$$\Sigma M_{steel} = 0 \Rightarrow F_c \cdot z - M_{sd} = 0 \Rightarrow M_{sd} = F_c \cdot z \tag{14}$$

By substituting Eq. (13) into Eq. (14), we obtain:

$$M_{sd} = \lambda x \eta b z f_{cd} \tag{15}$$

By substituting Eq. (12) into Eq.(15), we have:

$$M_{sd} = \lambda x \eta b f_{cd} \cdot \left(d - \frac{\lambda x}{2} \right) = x \cdot \left(\lambda \eta b d f_{cd} \right) - x^2 \cdot \left(nb \frac{\lambda^2}{2} f_{cd} \right) \Rightarrow \tag{16}$$

$$\left(\frac{\eta b \lambda^2 f_{cd}}{2} \right) \cdot x^2 - \left(\lambda \eta b d f_{cd} \right) \cdot x + M_{sd} = 0 \tag{17}$$

The above quadratic equation needs to be solved for the neutral axis depth *x*. All quantities except for *x* are known and the solution of the quadratic equation can be easily obtained as

$$x_{1,2} = \frac{d}{\lambda} \pm \frac{\sqrt{\Delta_1}}{2A_1} \tag{18}$$

where
$$A_1 = \frac{\eta b \lambda^2 f_{cd}}{2} \tag{19}$$

and Δ_1 is the discriminant of the quadratic equation:

$$\Delta_1 = \lambda^2 \eta b f_{cd} \left(\eta b d^2 f_{cd} - 2M_{sd} \right) \tag{20}$$

According to Eq. (2), it is always $\lambda<1$, as $\lambda=0.80$ for $f_{ck} \leq 50$ MPa and $\lambda<0.80$ for $50<f_{ck} \leq 90$ MPa and as a result $d/\lambda>d$ which leads to $x_2>d$ which is not acceptable, since the requirement is that $0 \leq x \leq d$ for sections under bending. Therefore the only acceptable solution is $x=x_1$ and thus:

$$x = x_1 = \frac{d}{\lambda} - \frac{\sqrt{\Delta_1}}{2A_1} \tag{21}$$

After calculating *x* from Eq. (21), it is easy to calculate also *z*, F_c and F_s from Eqs. (12), (13), (8). The required tensile reinforcement is then calculated by

$$A_s = \frac{F_s}{\sigma_s} \tag{22}$$

where σ_s is the steel stress at the Ultimate Limit State (ULS) of the section, calculated by Eq. (6). In our case, at the ULS the concrete zone is always at the critical strain, $\varepsilon_c=\varepsilon_{cu3}$ while the steel strain ε_s can be calculated considering the geometry of Figure 4 as follows:

$$\frac{\varepsilon_{cu3}}{x}=\frac{\varepsilon_{cu3}+\varepsilon_s}{d}\Rightarrow\varepsilon_s=\left(\frac{d}{x}-1\right)\varepsilon_{cu3} \tag{23}$$

If the steel does not work in full stress ($\sigma_s<f_{yd}$), although the required reinforcement area A_s can be calculated, the design with a single tensile reinforcement is not economic. Either compressive reinforcement should be also added, or an increase in the dimensions of the cross section, in particular its effective depth *d*.

6.1.1. Maximum effective moment $M_{sd,max}$ that the section can withstand

The maximum effective bending moment that the section can withstand (either economically, with steel working at full strength or not) can be calculated by setting $x=d$, so that the concrete compressive zone obtains its maximum value. In order to find the corresponding maximum effective bending moment $M_{sd,max}$, we set $x=d$ in Eq. (16) and we obtain:

$$M_{sd,\max}=\lambda\left(1-\frac{\lambda}{2}\right)\eta bd^2 f_{cd} \tag{24}$$

It should be noted that the maximum effective bending moment $M_{sd,max}$ is the upper limit of the effective moment, but the design for $M_{sd,max}$ is in fact impossible, as for $x=d$, it is $\varepsilon_s=0$, $\sigma_s=0$ and as a result an infinite amount of steel reinforcement would be needed according to Eq. (22).

The effective bending moment M_{sd} can be also expressed in general in a normalized (dimensionless) form as follows

$$\mu_{sd}=\frac{M_{sd}}{bd^2 f_{cd}} \tag{25}$$

where μ_{sd} is called the normalized effective bending moment. For the maximum normalized effective bending moment, we have

$$\mu_{sd,\max}=\frac{M_{sd,\max}}{bd^2 f_{cd}}=\lambda\left(1-\frac{\lambda}{2}\right)\eta \tag{26}$$

It can be seen that $\mu_{sd,max}$ depends only on the concrete class, as λ and η are both direct functions of the concrete strength only (Eqs (2) and (3)).

6.1.2. Critical effective moment $M_{sd,lim}$ that the section can withstand economically

Theoretically, the steel area can be calculated for any $M_{sd}<M_{sd,max}$ (or equivalently $\mu_{sd}<\mu_{sd,max}$) but as mentioned earlier, for the cases $M_{sd,lim}<M_{sd}<M_{sd,max}$ (or $\mu_{sd,lim}<\mu_{sd}<\mu_{sd,max}$) the design is not economic as steel works below its yield point. In order for the design to be economic, the steel reinforcement has to work above the yield limit, at full strength ($\varepsilon_s\geq\varepsilon_{ys}$ and $\sigma_s=f_{yd}$). At the limit of this condition, we set $\varepsilon_s=\varepsilon_{ys}$ in Eq. (23), and solving for *x*, we have the corresponding limit value x_{lim} of *x*:

$$x_{\lim} = \frac{\varepsilon_{cu3}}{\varepsilon_{cu3} + \varepsilon_{ys}} d \tag{27}$$

In order to find the corresponding effective moment $M_{\text{sd,lim}}$, we set $x=x_{\text{lim}}$ in Eq. (16)

$$M_{sd,\lim} = x_{\lim} \cdot \left(\lambda \eta b d f_{cd}\right) - x_{\lim}^2 \cdot \left(\eta b \frac{\lambda^2}{2} f_{cd}\right) \tag{28}$$

By substituting x_{lim} from Eq. (27) into Eq. (28), we finally obtain:

$$M_{sd,\lim} = \frac{\varepsilon_{cu3}\left(1 - \frac{\lambda}{2}\right) + \varepsilon_{ys}}{\left(\varepsilon_{cu3} + \varepsilon_{ys}\right)^2} \varepsilon_{cu3} \cdot \eta \lambda b d^2 f_{cd} \tag{29}$$

The corresponding dimensionless limit value $\mu_{\text{sd,lim}}$ is then

$$\mu_{sd,\lim} = \frac{M_{sd,\lim}}{b d^2 f_{cd}} = \frac{\varepsilon_{cu3}\left(1 - \lambda/2\right) + \varepsilon_{ys}}{\left(\varepsilon_{cu3} + \varepsilon_{ys}\right)^2} \eta \lambda \varepsilon_{cu3} \tag{30}$$

If for a given design problem $M_{\text{sd}} \leq M_{\text{sd,lim}}$ (or equivalently $\mu_{\text{sd}} \leq \mu_{\text{sd,lim}}$) then an economic design *can* be achieved using single steel reinforcement only. On the other hand, if $M_{\text{sd}} > M_{\text{sd,lim}}$ (or $\mu_{\text{sd}} > \mu_{\text{sd,lim}}$) then an economic design cannot be achieved using only single steel reinforcement. Either double reinforcement (tensile and also compressive) is needed, or an increase in the dimensions of the cross section (especially d, but also b). As shown in Eq. (30), the value of $\mu_{\text{sd,lim}}$ depends on the concrete strength class and the steel yield strain ε_{ys} which is dependent on the steel strength, as shown in Eq. (7) and Table 4.

6.1.3. Summary of the analytical methodology for the design of cross sections

The full methodology for the calculation of the needed steel reinforcement A_{s} is summarized below:

Known quantities for the design:

- **Materials properties**: f_{ck}, f_{yk}, E_{s} (EC2-1-1 value is 200 GPa)
- **Safety factors**: γ_{c} (EC2-1-1 value is 1.5), γ_{s} (EC2-1-1 value is 1.15), a_{cc} (EC2-1-1 recommended value is 1, National Annexes can enforce values between 0.8 and 1.0)
- **Section geometry**: b, h, d_1
- **Loading conditions**: M_{d}, N_{d} applied at y_{N} position

Quantities to be calculated and corresponding equation to use:

- λ: Eq. (2), η: Eq. (3)
- f_{cd},: Eq. (1), ε_{cu3}: Eq. (4), f_{yd}: Eq. (5), ε_{ys}: Eq. (7)
- d: Eq. (9), y_{s}: Eq. (10), M_{sd}: Eq. (11)

- $M_{sd,max}$: Eq. (24). If $M_{sd}<M_{sd,max}$ then proceed with the next calculations, otherwise stop, the design *cannot* be achieved
- $M_{sd,lim}$: Eq. (29). If $M_{sd}<M_{sd,lim}$ then the design using single steel reinforcement is economic (steel working at full strength), otherwise the design using single steel reinforcement can be achieved, but it is not economic (steel working below full strength)
- A_1: Eq. (19), Δ_1: Eq. (20), x: Eq. (21), z: Eq. (12)
- F_c: Eq. (13), F_s: Eq. (8)
- ε_s: Eq. (23), σ_s: Eq. (6), A_s: Eq. (22)

The above procedure is straightforward and can be easily implemented in any programming language. A simple spreadsheet program, such as Microsoft Excel, can be also used in order to make the necessary calculations, without even the need for any complicated programming macros.

6.2 Design of cross sections using design tables

In this section, we explain how the steel reinforcement area can be calculated using the design tables that are provided in Appendix A. We define the dimensionless value ω as

$$\omega = \frac{F_c}{bdf_{cd}} \tag{31}$$

From Eq. (8) we have

$$F_c = F_s - N_d \tag{32}$$

Substituting F_s from Eq. (22) in Eq. (32) and then substituting F_c from Eq. (32) into Eq. (31) we obtain

$$\omega = \frac{A_s\sigma_s - N_d}{bdf_{cd}} \tag{33}$$

By solving Eq. (33) for A_s, we obtain

$$A_s = \frac{1}{\sigma_s}\left(\omega bdf_{cd} + N_d\right) \tag{34}$$

It is obvious that if ω and σ_s are both known, then it is easy to calculate the needed steel reinforcement area A_s from Eq. (34). In Appendix A there are six tables which provide the values of ω and σ_s for given values of the normalized effective bending moment μ_{sd}, for each concrete class. In section 6.3 we will explain how the values of the tables can be calculated. Each table gives the value of ω for a given value of μ_{sd}, together with the values of $\xi=x/d$, $\zeta=z/d$, ε_s (‰) and also σ_s for three different steel classes (B400, B500, B600). Of these parameters, only σ_s is affected by the steel quality and that's why it is given in three columns.

It should be noted that the first nine concrete classes (C12/15, C16/20, C20/25, C25/30, C30/37, C35/45, C40/50, C45/55, C50/60) share the same table (Table 10) while for the other five concrete classes (C55/67, C60/75, C70/85, C80/95 and C90/105) there are separate tables for each case.

The tables are independent of the values of the concrete parameters a_{cc} and γ_c. Of course these parameters affect the final design, but they are taken into account through the calculation of f_{cd} in Eq. (34) which affects the calculation of A_s. The first 5 columns, μ_{sd}, ω, ξ, ζ, ε_s are also independent of the steel parameters γ_s and E_s. Only the steel stress at the ultimate state (last three columns of the tables) depends on the steel parameters γ_s and E_s and these three columns have been calculated with the assumption E_s=200 GPa and γ_s=1.15 (in accordance with EC2-1-1). This is also the case for the limit values $\mu_{sd,lim}$ and ω_{lim} which depend also on E_s and γ_s.

6.2.1. Linear interpolation for the ω-μ_{sd} tables

In most cases, the value of μ_{sd} is not an exact value of the table, but rather lies between two neighbouring values μ_{sd1} and μ_{sd2} ($\mu_{sd1}<\mu_{sd}<\mu_{sd2}$). In this case linear interpolation is needed in order to obtain the value of ω that corresponds to the given μ_{sd}. This is of course an easy-to-solve problem, but nevertheless we will provide an explicit analytic solution here.

If ω_1 corresponds to μ_{sd1} and ω_2 corresponds to μ_{sd2} then we have the linear interpolation problem that is depicted in Table 5.

Table 5. The linear interpolation problem of the μ_{sd}-ω tables.

μ_{sd} values from Table	ω values from Table
μ_{sd1}	ω_1
Our μ_{sd} ($\mu_{sd1}<\mu_{sd}<\mu_{sd2}$)	Our ω = ?
μ_{sd2}	ω_2

The solution is given below

$$\frac{\mu_{sd2}-\mu_{sd1}}{\omega_2-\omega_1}=\frac{\mu_{sd}-\mu_{sd1}}{\omega-\omega_1}\Rightarrow \tag{35}$$

$$\omega=\omega_1+\frac{\mu_{sd}-\mu_{sd1}}{\mu_{sd2}-\mu_{sd1}}\left(\omega_2-\omega_1\right) \tag{36}$$

6.3 Analytic formulas and investigation of the design parameters ω, ξ, ζ, ε_s

In this section, we will investigate the parameters ω, ξ, ζ, ε_s and we will end up to closed formulas for their calculation. Using these formulas, one can easily generate the design tables of the Appendix.

6.3.1. Parameter ω

Although the values of the parameter ω can be taken from the design tables using the design approach described before, it is very interesting to investigate also ω analytically, using closed formulas. From Eq. (8) we have

$$F_s=F_c+N_d \tag{37}$$

By substituting the concrete force from Eq. (13) and the steel force from Eq. (22) into Eq. (37), we have

$$A_s\cdot\sigma_s=\lambda x\cdot\eta f_{cd}\cdot b+N_d \tag{38}$$

By substituting A_s from Eq. (34) into Eq. (38), we have

$$\frac{1}{\sigma_s}\left(\omega bdf_{cd}+N_d\right)\cdot\sigma_s=\lambda x\cdot\eta f_{cd}\cdot b+N_d\Rightarrow \tag{39}$$

$$\frac{\lambda x\eta}{d}=\omega \tag{40}$$

By definition it is

$$\mu_{sd}=\frac{M_{sd}}{bd^2 f_{cd}} \tag{41}$$

By substituting M_{sd} from Eq. (15) into Eq. (41) we have

$$\mu_{sd}=\frac{\lambda x\cdot\eta f_{cd}\cdot b\cdot z}{bd^2 f_{cd}}=\frac{\lambda\eta xz}{d^2} \tag{42}$$

By substituting z from Eq. (12) into Eq. (42) we obtain

$$\mu_{sd}=\frac{\lambda\eta x\left(d-\frac{\lambda x}{2}\right)}{d^2}=\frac{\lambda x\eta}{d}-\frac{1}{2\eta}\left(\frac{\lambda x\eta}{d}\right)^2 \tag{43}$$

By substituting $\lambda x\eta/d$ from Eq. (40) into Eq. (43) we finally get

$$\mu_{sd}=\omega-\frac{1}{2\eta}\omega^2 \tag{44}$$

The above is a simple analytic formula for the calculation of μ_{sd} when ω is known. This is very useful in the inverse problem which will be investigated later. Now we will try to solve Eq. (44) for ω. It can be written in the following form:

$$\frac{1}{2\eta}\omega^2-\omega+\mu_{sd}=0 \tag{45}$$

The solution of the quadratic equation is:

$$\omega_{1,2}=\eta\left(1\pm\sqrt{1-\frac{2\mu_{sd}}{\eta}}\right) \tag{46}$$

From the above two solutions, only the one with the negative sign (ω_1) is acceptable (proof will follow) and as a result:

$$\omega=\eta\left(1-\sqrt{1-\frac{2\mu_{sd}}{\eta}}\right) \tag{47}$$

Proof that ω_2 (with the positive sign) is not an acceptable solution of Eq. (45)

Assuming that ω_2 is an acceptable solution, then from Eq. (40) we have

$$\omega_2 = \eta \cdot \frac{\lambda x_2}{d} \tag{48}$$

Since

$$\omega_2 = \eta \left(1 + \sqrt{1 - \frac{2\mu_{sd}}{\eta}} \right) \tag{49}$$

Then it should be

$$\eta \cdot \frac{\lambda x_2}{d} = \eta \left(1 + \sqrt{1 - \frac{2\mu_{sd}}{\eta}} \right) \Rightarrow \tag{50}$$

$$\lambda \frac{x_2}{d} = 1 + \sqrt{1 - \frac{2\mu_{sd}}{\eta}} \Rightarrow \tag{51}$$

$$x_2 = \frac{1 + \sqrt{1 - \frac{2\mu_{sd}}{\eta}}}{\lambda} \cdot d \tag{52}$$

Since the numerator is greater than 1 and the denominator λ is less than 1, then $x_2 > d$ which is not acceptable. As a result, ω_2 is *not* an acceptable solution.

Figure 6 depicts Eq. (47) showing ω as a function of the dimensionless effective bending moment μ_{sd}, for every concrete class.

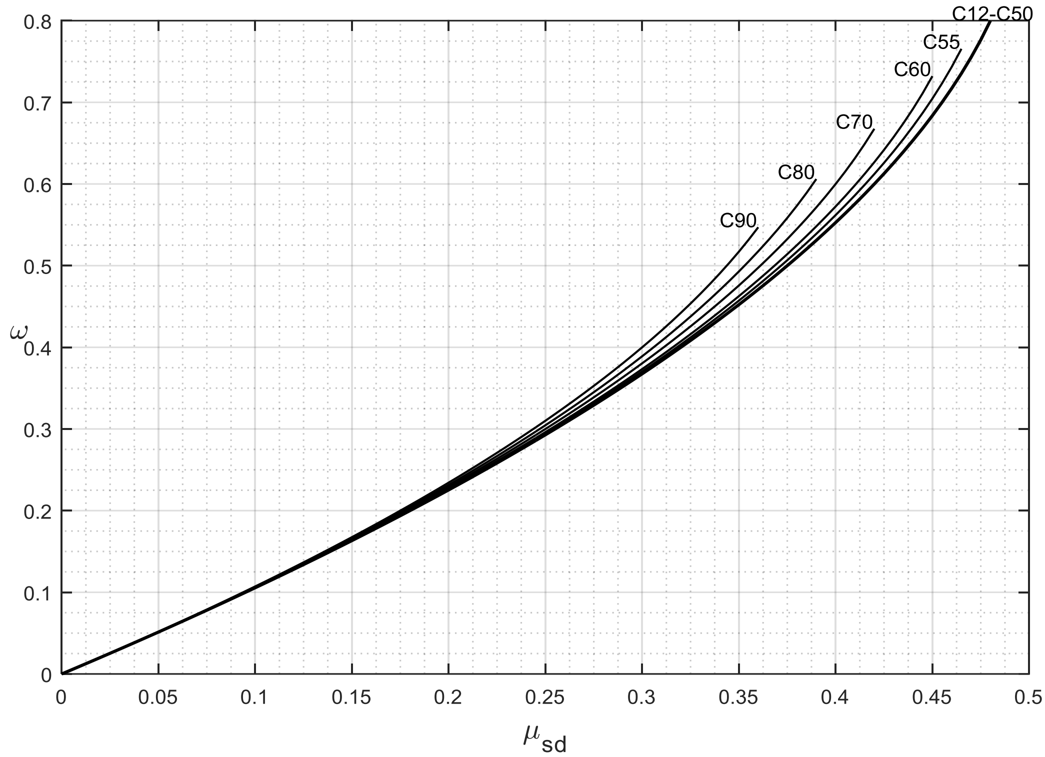


Figure 6. ω as a function of μ_{sd} for every concrete class.

Using Eq. (47) for ω and setting as μ_{sd} the values of $\mu_{sd,max}$ (Eq. (26)) and $\mu_{sd,lim}$ (Eq. (30)) for each steel class, it is easy to calculate the corresponding values ω_{max} and ω_{lim} for every steel class, and obtain the closed formulas as follows:

$$\omega_{\max} = \eta\lambda \tag{53}$$

$$\omega_{\lim} = \eta\left(1-\sqrt{1-\frac{\varepsilon_{cu3}(2-\lambda)+2\varepsilon_{ys}}{\left(\varepsilon_{cu3}+\varepsilon_{ys}\right)^2}\lambda\varepsilon_{cu3}}\right) \tag{54}$$

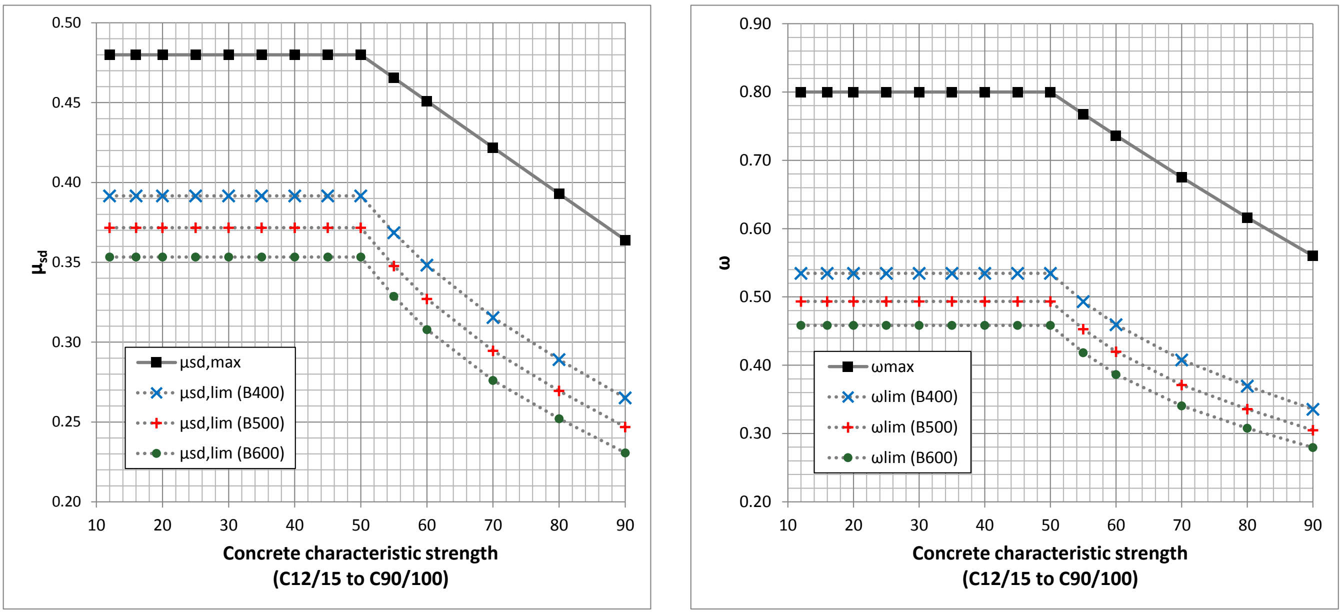
Maximum and limit values for μ_{sd} and ω

Table 6 shows the values of the parameters $\mu_{sd,max}$, ω_{max} (the same for all steel classes) and $\mu_{sd,lim}$, ω_{lim} (for steel B400, B500 and B600), for every concrete class. It should be noted that the values of the limit parameters (lim) of the table have been calculated for E_s=200 GPa and γ_s=1.15, in accordance with EC2-1-1.

Table 6. The values of the parameters $\mu_{sd,max}$, ω_{max}, $\mu_{sd,lim}$, ω_{lim}.

Concrete Class	max. (any steel)		lim (B400)		lim (B500)		lim (B600)	
	$\mu_{sd,max}$	ω_{max}	$\mu_{sd,lim}$	ω_{lim}	$\mu_{sd,lim}$	ω_{lim}	$\mu_{sd,lim}$	ω_{lim}
C12/15–C50/60	0.4800	0.8000	0.3916	0.5344	0.3717	0.4935	0.3533	0.4584
C55/67	0.4655	0.7678	0.3685	0.4933	0.3477	0.4528	0.3287	0.4185
C60/75	0.4510	0.7363	0.3482	0.4593	0.3270	0.4198	0.3079	0.3865
C70/85	0.4219	0.6750	0.3155	0.4079	0.2946	0.3712	0.2761	0.3405
C80/95	0.3929	0.6163	0.2892	0.3695	0.2695	0.3358	0.2521	0.3078
C90/105	0.3640	0.5600	0.2652	0.3356	0.2469	0.3050	0.2307	0.2795

Figure 7 shows the corresponding $\mu_{sd,max}$, ω_{max} and $\mu_{sd,lim}$, ω_{lim}, as functions of the concrete strength. Figure 8 depicts ω_{max} and ω_{lim} vs $\mu_{sd,max}$ and $\mu_{sd,lim}$ for each concrete and steel class.



(a) (b)

Figure 7. For each concrete class and steel class: (a) $\mu_{sd,max}$ and $\mu_{sd,lim}$, (b) ω_{max} and ω_{lim}.

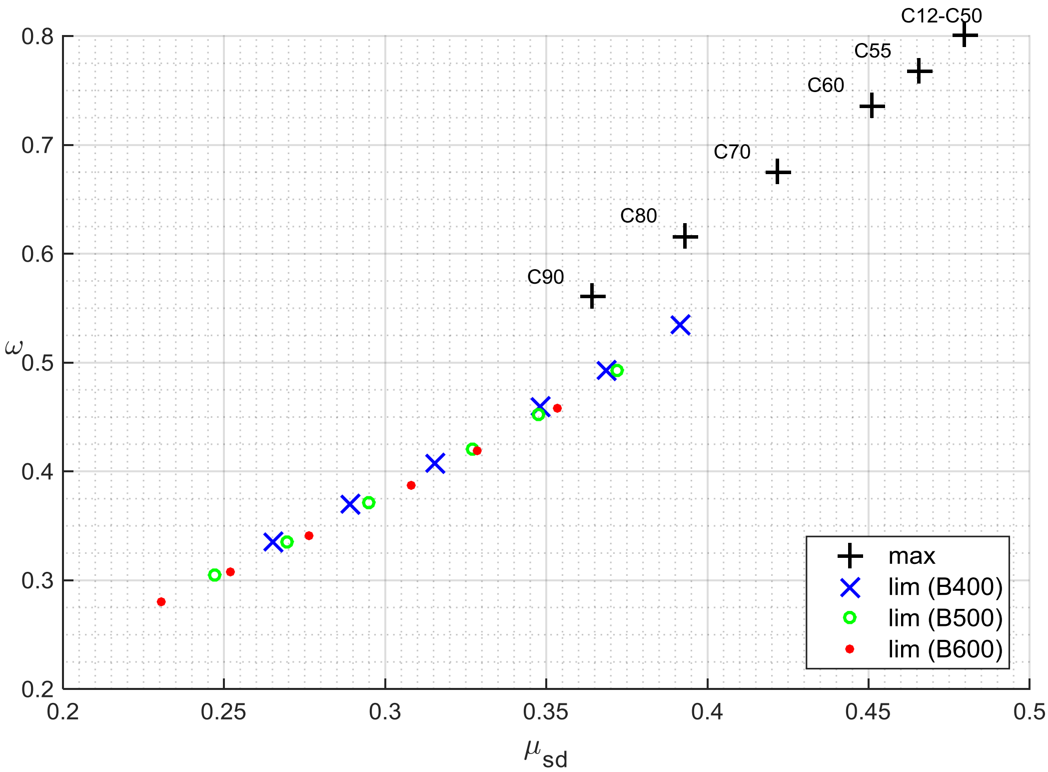


Figure 8. ω_{max} and ω_{lim} vs $\mu_{sd,max}$ and $\mu_{sd,lim}$ for each concrete and steel class.

6.3.2. Parameter ξ

The parameter ξ is the normalized neutral axis depth. The neutral axis depth is normalized with respect to the effective height d of the section and is defined as

$$\xi = \frac{x}{d} \tag{55}$$

Using Eq. (40) and also substituting ω from Eq. (47) we have

$$\xi = \frac{x}{d} = \frac{\omega}{\lambda\eta} = \frac{1}{\lambda}\left(1 - \sqrt{1 - \frac{2\mu_{sd}}{\eta}}\right) \tag{56}$$

The corresponding values ξ_{max} and ξ_{lim} are

$$\xi_{\text{max}} = 1 \tag{57}$$

$$\xi_{\text{lim}} = \frac{1}{\lambda}\left(1 - \sqrt{1 - \frac{\varepsilon_{cu3}(2-\lambda) + 2\varepsilon_{ys}}{\left(\varepsilon_{cu3} + \varepsilon_{ys}\right)^2}\lambda\varepsilon_{cu3}}\right) \tag{58}$$

In Figure 9 ξ is shown as a function of the normalized moment μ_{sd} for various concrete strength classes. It is apparent that for higher concrete classes, the normalized neutral axis depth is higher, for the same value of μ_{sd}. All curves increase with increasing normalized moment, until ξ gets equal to one (x=d).

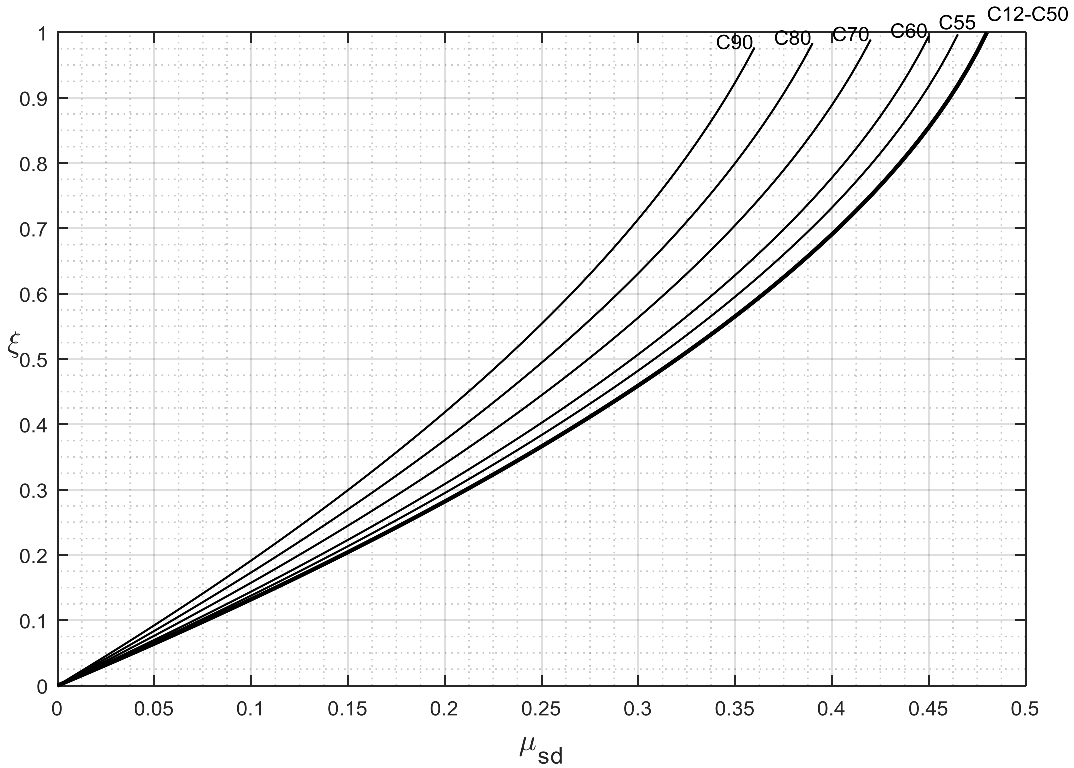


Figure 9. ξ as a function of μ_{sd} for every concrete class.

6.3.3. Parameter ζ

The parameter ζ is the normalized distance of the resultant concrete force from the tensile reinforcement z with respect to the effective section height d and is defined as

$$\zeta = \frac{z}{d} \tag{59}$$

Using Eq. (12) and also Eq. (56) we have

$$\zeta = \frac{z}{d} = \frac{\left(d - \frac{\lambda x}{2}\right)}{d} = 1 - \frac{\lambda}{2} \cdot \frac{x}{d} = 1 - \frac{\lambda}{2} \cdot \xi = 1 - \frac{\omega}{2\eta} \tag{60}$$

Substituting ξ from Eq. (56) we obtain also

$$\zeta = 1 - \frac{\lambda}{2} \cdot \frac{1}{\lambda}\left(1 - \sqrt{1 - \frac{2\mu_{sd}}{\eta}}\right) = 0.5\left(1 + \sqrt{1 - \frac{2\mu_{sd}}{\eta}}\right) \tag{61}$$

The corresponding values $\zeta_{\min}$ (corresponding to $\omega_{\max}$ and $\mu_{sd,max}$) and $\zeta_{\lim}$ are

$$\zeta_{\min} = 1 - \frac{\lambda}{2} \tag{62}$$

$$\zeta_{\lim} = 0.5\left(1 + \sqrt{1 - \frac{\varepsilon_{cu3}\left(2-\lambda\right) + 2\varepsilon_{ys}}{\left(\varepsilon_{cu3} + \varepsilon_{ys}\right)^2} \lambda \varepsilon_{cu3}}\right) \tag{63}$$

In Figure 10 ζ is plotted against the dimensionless design bending moment μ_{sd} for various concrete strength classes. It is observed that ζ decreases with increasing concrete strength class for the same value of μ_{sd} and it decreases generally with increasing μ_{sd}.

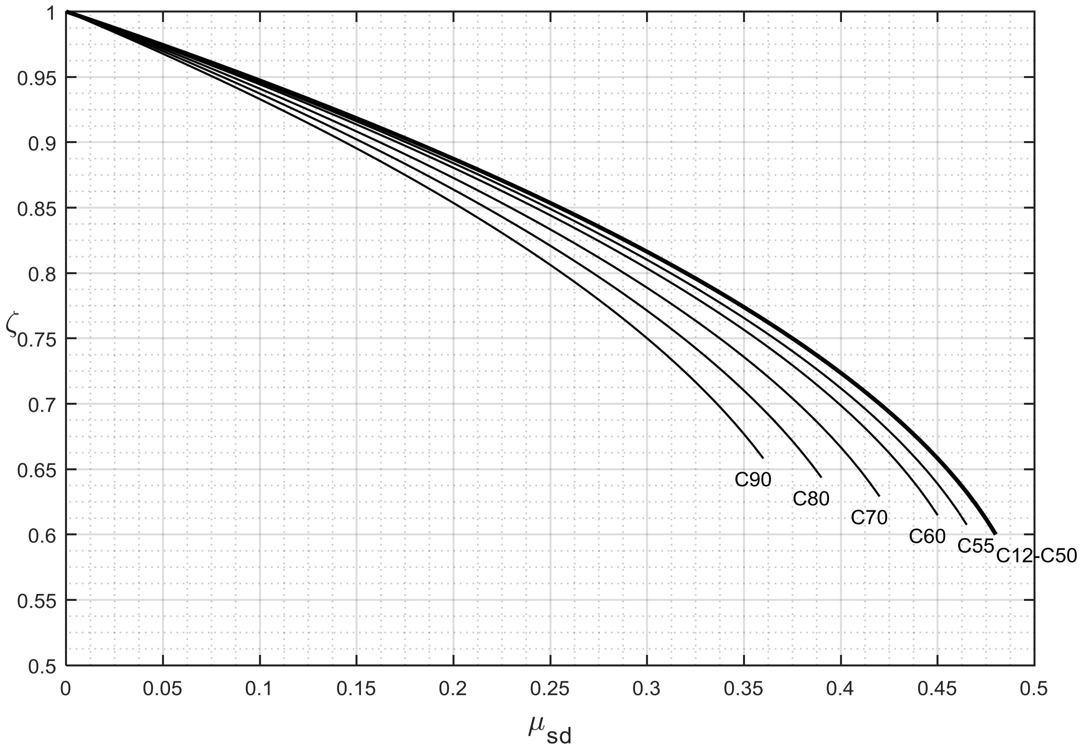


Figure 10. ζ as a function of μ_{sd} for every concrete class.

Table 7 shows the values of the parameters ξ_{max}, ζ_{min} (the same for all steel classes) and ξ_{lim}, ζ_{lim} (for steel B400, B500 and B600), for every concrete class. It should be noted that the values of the limit parameters (lim) of the table have been calculated for E_s=200 GPa and γ_s=1.15, in accordance with EC2-1-1.

Table 7. The values of the parameters ξ_{max}, ζ_{min}, ξ_{lim}, ζ_{lim}.

Concrete Class	max./min. (any steel class)		lim (B400)		lim (B500)		lim (B600)	
	ξ_{max}	ζ_{min}	ξ_{lim}	ζ_{lim}	ξ_{lim}	ζ_{lim}	ξ_{lim}	ζ_{lim}
C12/15–C50/60	1	0.6000	0.6680	0.7328	0.6169	0.7533	0.5730	0.7708
C55/67	1	0.6063	0.6425	0.7470	0.5898	0.7678	0.5450	0.7854
C60/75	1	0.6125	0.6238	0.7583	0.5702	0.7791	0.5250	0.7966
C70/85	1	0.6250	0.6043	0.7734	0.5499	0.7938	0.5045	0.8108
C80/95	1	0.6375	0.5995	0.7827	0.5450	0.8025	0.4995	0.8189
C90/105	1	0.6500	0.5992	0.7903	0.5446	0.8094	0.4992	0.8253

6.3.4. Steel strain ε_s

From the definition of ξ, it is

$$\frac{d}{x} = \frac{1}{\xi} \tag{64}$$

Substituting Eq. (64) into Eq. (23) and also using Eq. (56) we have

$$\varepsilon_s = \left(\frac{1}{\xi} - 1\right)\varepsilon_{cu3} = \left(\frac{1}{\frac{\omega}{\lambda\eta}} - 1\right)\varepsilon_{cu3} = \left(\frac{\lambda\eta}{\omega} - 1\right)\varepsilon_{cu3} \tag{65}$$

or in terms of μ_{sd}

$$\varepsilon_s = \left(\frac{1}{\xi} - 1\right)\varepsilon_{cu3} = \left(\frac{1}{\frac{1}{\lambda}\left(1 - \sqrt{1 - \frac{2\mu_{sd}}{\eta}}\right)} - 1\right)\varepsilon_{cu3} = \left(\frac{\lambda\eta\left(1 + \sqrt{1 - \frac{2\mu_{sd}}{\eta}}\right)}{2\mu_{sd}} - 1\right)\varepsilon_{cu3} \tag{66}$$

The strain of the reinforcement ε_s is shown in Figure 11, as a function of the normalized design bending moment μ_{sd} (for $\mu_{sd} \geq 0.01$), for various concrete strength classes, where the x-axis (μ_{sd}) is in logarithmic scale for better clarity. In general, it is shown that the steel strain decreases for increasing normalized bending moment μ_{sd}. If the horizontal top branch of the steel stress-strain diagram is considered (as in this study), the steel strain is not supposed to have a maximum and in theory it can extend to infinity. Therefore, for very small values of the dimensionless bending moment μ_{sd} the ε_s curves tend asymptotically towards infinity. Furthermore, for higher values of μ_{sd}, the steel strain decreases and for $\mu_{sd,max}$ it becomes zero, as shown in the figure.

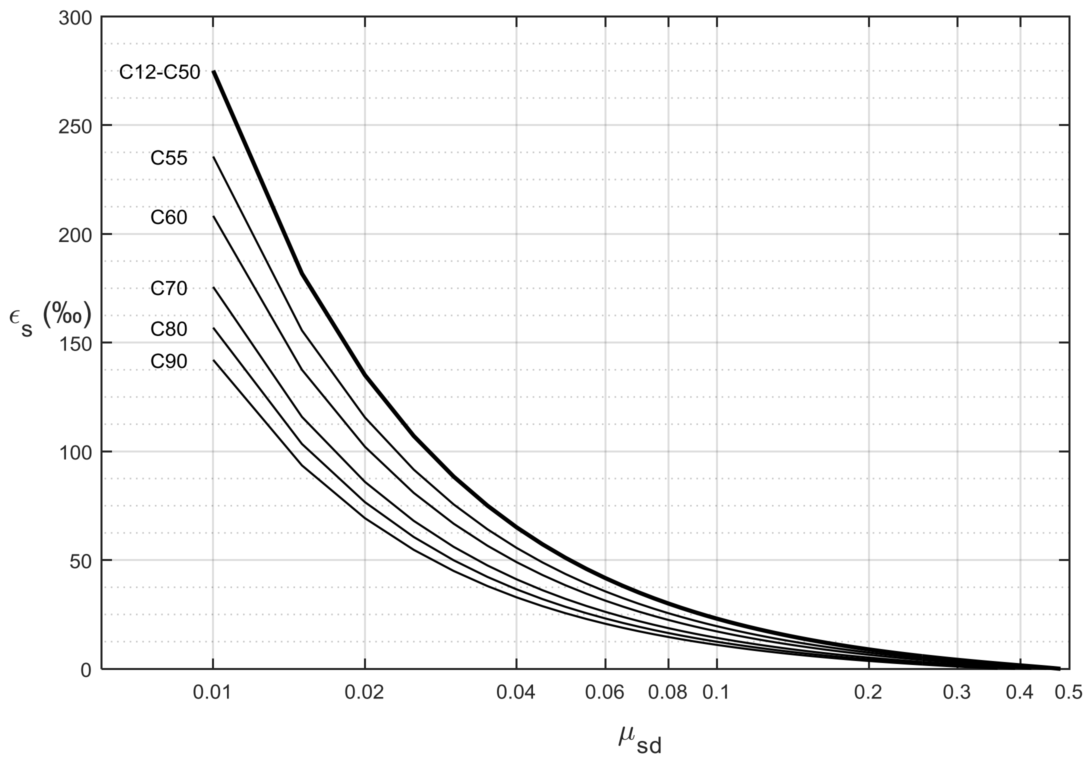


Figure 11. ε_s as a function of μ_{sd} for every concrete class (μ_{sd} in logarithmic scale).

In Figure 12 we zoom in the area of higher values of μ_{sd}, $0.2 \leq \mu_{sd} \leq 0.48$. The yield (limit) values for ε_s (ε_{ys}, shown in Table 4) have been plotted in this diagram also, as horizontal lines, for each steel class.

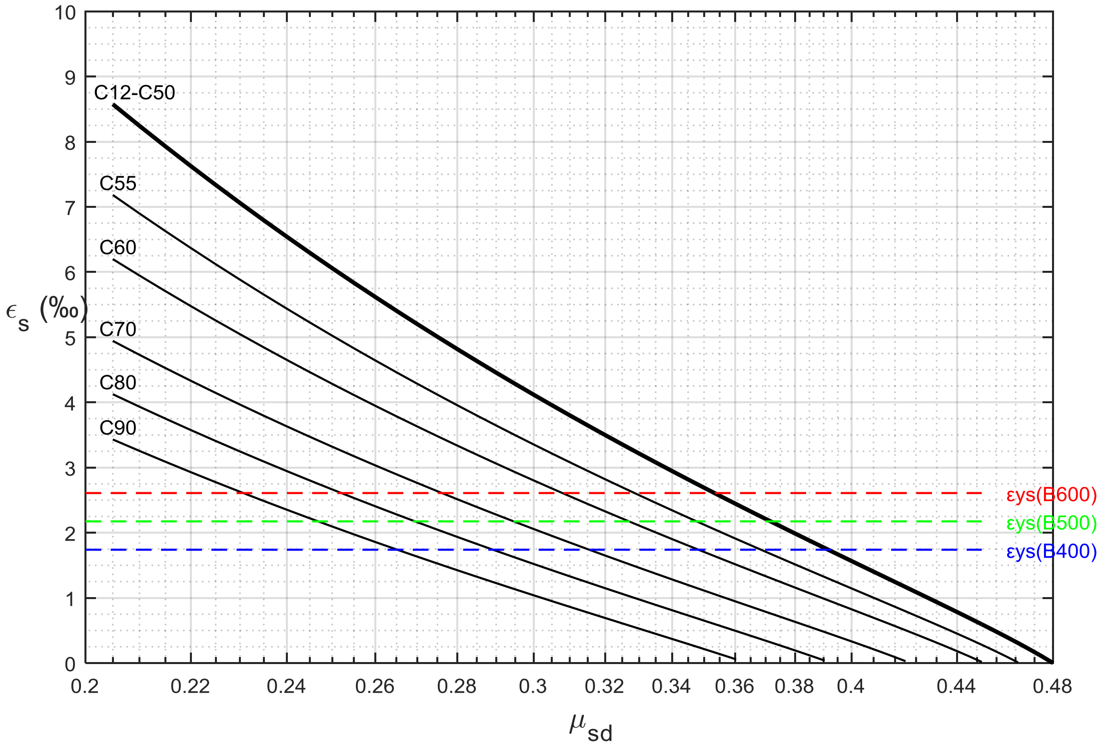


Figure 12. ε_s as a function of μ_{sd} for every concrete class ($\mu_{sd} \geq 0.2$, μ_{sd} in logarithmic scale).

6.3.5. Analytic formulas of μ_{sd}, ω, ξ, ζ, ε_s for concrete classes up to C50/60

For the special case of concrete classes up to C50/60, calculations are much simpler. For this case, it is η=1 and λ=0.8 and as a result we obtain the following simplified formulas.

For μ_{sd}:

$$\mu_{sd} = \omega - 0.5 \cdot \omega^2 \tag{67}$$

$$\mu_{sd,\max} = 0.48 \tag{68}$$

$$\mu_{sd,\lim} = \frac{0.48\varepsilon_{cu3} + 0.8\varepsilon_{ys}}{\left(\varepsilon_{cu3} + \varepsilon_{ys}\right)^2}\varepsilon_{cu3} \tag{69}$$

For ω:

$$\omega = 1 - \sqrt{1 - 2\mu_{sd}} \tag{70}$$

$$\omega_{\max} = 0.8 \tag{71}$$

$$\omega_{\lim} = 1 - \sqrt{1 - \frac{0.96\varepsilon_{cu3} + 1.6\varepsilon_{ys}}{\left(\varepsilon_{cu3} + \varepsilon_{ys}\right)^2}\varepsilon_{cu3}} \tag{72}$$

For ξ:

$$\xi = 1.25\omega = 1.25\left(1-\sqrt{1-2\mu_{sd}}\right) \tag{73}$$

$$\xi_{\max} = 1 \tag{74}$$

$$\xi_{\lim} = 1.25 - 1.25\sqrt{1-\frac{0.96\varepsilon_{cu3}+1.6\varepsilon_{ys}}{\left(\varepsilon_{cu3}+\varepsilon_{ys}\right)^2}\varepsilon_{cu3}} \tag{75}$$

For ζ:

$$\zeta = 1-0.5\omega = 0.5\left(1+\sqrt{1-2\mu_{sd}}\right) \tag{76}$$

$$\zeta_{\min} = 0.6 \tag{77}$$

$$\zeta_{\lim} = 0.5 + 0.5\sqrt{1-\frac{0.96\varepsilon_{cu3}+1.6\varepsilon_{ys}}{\left(\varepsilon_{cu3}+\varepsilon_{ys}\right)^2}\varepsilon_{cu3}} \tag{78}$$

For ε_s:

$$\varepsilon_s = \left(\frac{0.8}{\omega}-1\right)\varepsilon_{cu3} = \left[\frac{0.4}{\mu_{sd}}\left(1+\sqrt{1-2\mu_{sd}}\right)-1\right]\varepsilon_{cu3} \tag{79}$$

Figure 13 shows the parameters ω, ξ, ζ and ε_s as functions of the normalized bending moment μ_{sd} for concrete classes C12/15 up to C50/60.

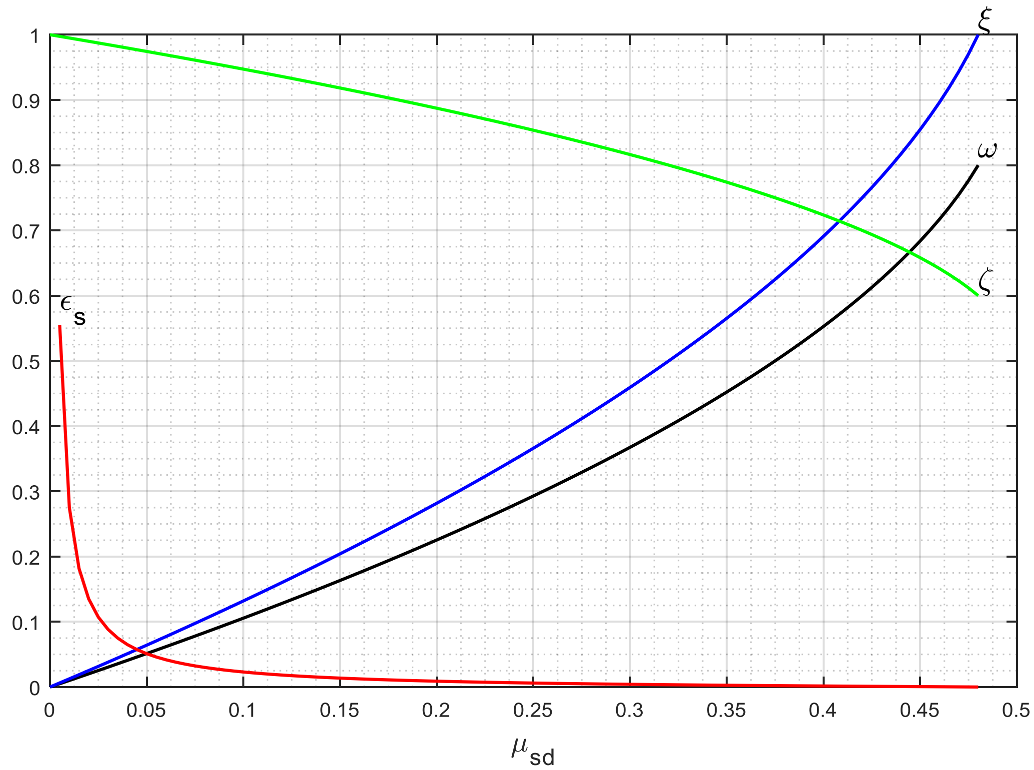


Figure 13. ω, ξ, ζ, ε_s as functions of μ_{sd} for concrete classes C12/15 up to C50/60.

7 INVESTIGATION OF THE INVERSE PROBLEM

In the inverse problem, the tensile reinforcement (steel area) A_s and the axial force N_d (which is applied at y_N) are given and the purpose is to calculate the maximum bending moment M_d that the cross section can withstand.

7.1 Analytical calculation of the maximum bending moment M*d*

In this problem there are generally again two cases:

- Steel working at full strength ($\varepsilon_s \geq \varepsilon_{ys}$, $\sigma_s = f_{yd}$)
- Steel working below full strength ($\varepsilon_s < \varepsilon_{ys}$, $\sigma_s < f_{yd}$)

Case A: We assume that steel works at full strength

If steel works at full strength, then $\varepsilon_s \geq \varepsilon_{ys}$ and $\sigma_s = f_{yd}$ and we have:

$$A_s = \frac{F_s}{f_{yd}} \Rightarrow F_s = A_s \cdot f_{yd} \tag{80}$$

$$\Sigma F_x = 0 \Rightarrow F_c + N_d - F_s = 0 \Rightarrow F_c = F_s - N_d \tag{81}$$

$$F_c = \lambda x n b f_{cd} \Rightarrow x = \frac{F_c}{\lambda n b f_{cd}} \tag{82}$$

$$\frac{\varepsilon_{cu3}}{x} = \frac{\varepsilon_{cu3} + \varepsilon_s}{d} \Rightarrow \varepsilon_s = \left(\frac{d}{x} - 1\right)\varepsilon_{cu3} \tag{83}$$

Using Eq. (83) we can now check our principal assumption. If $\varepsilon_s \geq \varepsilon_{ys}$ then the assumption was right and we can continue, otherwise the assumption was not right and we have to move to Case B. By substituting Eqs (80), (81), (82) into Eq. (83) and doing some calculations, the criterion for Case A becomes as follows:

$$\text{if} \left[\left(\frac{d\lambda n b f_{cd}}{A_s f_{yd} - N_d} - 1\right)\frac{\varepsilon_{cu3}}{\varepsilon_{ys}}\right] \geq 1 \quad \text{then} \quad \varepsilon_s \geq \varepsilon_{ys} \quad \text{otherwise} \quad \varepsilon_s < \varepsilon_{ys} \tag{84}$$

If the criterion of Eq. (84) is satisfied, then $\varepsilon_s \geq \varepsilon_{ys}$. If this is the case, then we calculate x from Eq. (82) and we continue with the Final step below, otherwise we move to Case B where $\varepsilon_s < \varepsilon_{ys}$.

Case B: Steel working below the yield limit (with less than full strength)

If the criterion of Eq. (84) is not satisfied then steel works below the yield point, $\varepsilon_s < \varepsilon_{ys}$ and $\sigma_s < f_{yd}$ and we have:

$$\sigma_s = f_{yd} \cdot \frac{\varepsilon_s}{\varepsilon_{ys}} = E_s \cdot \varepsilon_s \tag{85}$$

$$A_s = \frac{F_s}{\sigma_s} \Rightarrow F_s = A_s \cdot \sigma_s \tag{86}$$

$$\frac{\varepsilon_{cu3}}{x} = \frac{\varepsilon_{cu3} + \varepsilon_s}{d} \Rightarrow \varepsilon_s = \left(\frac{d}{x} - 1\right)\varepsilon_{cu3} \tag{87}$$

By substituting ε_s from Eq. (87) into Eq. (85) and then σ_s from Eq. (85) into Eq. (86) we obtain:

$$F_s = A_s E_s \cdot \left(\frac{d}{x} - 1\right) \varepsilon_{cu3} \tag{88}$$

We have also:

$$F_c = \lambda x n b f_{cd} \tag{89}$$

$$F_s = F_c + N_d \tag{90}$$

By substituting F_s from Eq. (88) and F_c from Eq. (89) into Eq. (90) we have:

$$A_s E_s \cdot \left(\frac{d}{x} - 1\right) \varepsilon_{cu3} = \lambda x n b f_{cd} + N_d \Rightarrow \tag{91}$$

$$\left(\lambda n b f_{cd}\right) \cdot x^2 + \left(N_d + A_s E_s \varepsilon_{cu3}\right) \cdot x - A_s E_s d \varepsilon_{cu3} = 0 \Rightarrow \tag{92}$$

The above quadratic equation needs to be solved for the neutral axis depth x. It can be written as:

$$A_2 x^2 + B_2 x + C_2 = 0 \tag{93}$$

where $$A_2 = \lambda n b f_{cd}, \quad B_2 = N_d + A_s E_s \varepsilon_{cu3}, \quad C_2 = -A_s E_s d \varepsilon_{cu3} \tag{94}$$

The quantities A_2, B_2 and C_2 are all known, so by solving the quadratic Eq. (93) we can determine the quantity x. The discriminant Δ_2 of the quadratic equation is given by:

$$\Delta_2 = B_2^{\,2} - 4A_2 \cdot C_2 = \left(N_d + A_s \cdot E_s \cdot \varepsilon_{cu3}\right)^2 + 4\lambda \cdot n \cdot f_{cd} \cdot b \cdot A_s \cdot E_s \cdot d \cdot \varepsilon_{cu3} \tag{95}$$

The solution of the quadratic equation is:

$$x_{1,2} = \frac{-B_2 \pm \sqrt{\Delta_2}}{2A_2} \Rightarrow \begin{cases} x_1 = \dfrac{-B_2 - \sqrt{\Delta_2}}{2A_2} \\ x_2 = \dfrac{-B_2 + \sqrt{\Delta_2}}{2A_2} \end{cases} \tag{96}$$

Given that $-B_2 - \sqrt{\Delta_2} < 0$ and according to the requirement $0 \le x \le d$, the only acceptable solution is $x = x_2$ and thus:

$$x = x_2 = \frac{-B_2 + \sqrt{\Delta_2}}{2A_2} \tag{97}$$

After calculating x from Eq. (97), it is easy to calculate also ε_s from Eq. (87). We can now check again the validity of the principal assumption. It should certainly be $\varepsilon_s < \varepsilon_{ys}$ otherwise the assumption for Case B was not right and there must be a problem in the calculations. If indeed $\varepsilon_s < \varepsilon_{ys}$ then we continue with the Final step below, with the value of x calculated with Eq. (97).

Final step:

Having obtained the value of x, either from Case A or Case B, we continue with the following calculations:

$$M_{sd} = \lambda x \eta b f_{cd}\left(d - \frac{\lambda x}{2}\right) \tag{98}$$

$$M_{sd} = M_d - N_d \cdot y_s \Rightarrow M_d = M_{sd} + N_d \cdot y_s \tag{99}$$

7.1.1. Summary of the analytical methodology for the calculation of the maximum bending moment M_d

The full methodology for the calculation of the maximum bending moment M_d that the section can withstand given the existing steel reinforcement A_s and the axial force N_d (which is applied at y_N) is summarized below:

Known quantities for the calculation of the strength: The known quantities for the calculation of the cross section strength are the same as the ones of the direct problem, with the exception of the applied external bending moment M_d which is now not known (and needs to be calculated). Instead, the existing steel reinforcement A_s is now known.

Quantities to be calculated and corresponding equation to use:

- λ: Eq. (2), η: Eq. (3), f_{cd}: Eq. (1), ε_{cu3}: Eq. (4), f_{yd}: Eq. (5), ε_{ys}: Eq. (7), d: Eq. (9), y_s: Eq. (10)
- If the Criterion of Eq. (84) is satisfied, then proceed with Case A, otherwise proceed with Case B
- **Case A**
  - F_s: Eq. (80), F_c: Eq. (81), x: Eq. (82), ε_s: Eq. (83) (should be $\geq\varepsilon_{ys}$), $\sigma_s=f_{yd}$
- **Case B**
  - A_2, B_2, C_2: Eq. (94), Δ_2: Eq. (95), x: Eq. (97), ε_s: Eq. (87) (should be $<\varepsilon_{ys}$), σ_s: steel stress, Eq. (85)
- M_{sd}: Eq. (98), M_d: Eq. (99).

The above is again a straightforward procedure that can be very easily implemented in any programming language.

7.2 Solution of the inverse problem using design tables

The inverse problem can be solved using the design tables provided in Appendix A, without any complicated analytic calculations in the usual case of economic design (steel working at full strength). In the case where the steel does not work at full strength, then it is not very easy to use the design tables, as the unknowns in this case are two (ω and σ_s) and an iterated process is needed in order to calculate the real value of ω, as described in detail in the following sections.

Case A: We assume that steel works at full strength

Setting $\sigma_s=f_{yd}$ in Eq. (33) we obtain

$$\omega = \frac{A_s f_{yd} - N_d}{bdf_{cd}} \tag{100}$$

Now we must calculate ω with Eq. (100) and then read the design table and ensure that for the given value of ω, steel works indeed above the yield limit, at full strength (σ_s=f_{yd}) so our assumption was right. For this we can also simply read the ω_{lim} value for the given steel class and check if the calculated ω is below ω_{lim} ($\omega \leq \omega_{lim}$). Otherwise, if $\omega > \omega_{lim}$ then the assumption was not right and we have to move to Case B. If indeed steel works at full strength, then for the given value of ω, we use the design table to take the corresponding value of μ_{sd} (linear interpolation may be needed) Then we calculate M_{sd} with the following formula which is derived by solving Eq. (25) for M_{sd}:

$$M_{sd} = \mu_{sd} \cdot bd^2 f_{cd} \tag{101}$$

Then, as previously, M_d can be easily calculated using Eq. (99)

Case B: Steel working below the yield limit (with less than full strength)

If using Eq. (100) for the given A_s and N_d, we obtain a value of ω equal to $\omega_{calc,in}$ for which it is $\omega_{calc,in} > \omega_{lim}$, then the assumption that steel works at full strength was wrong. In this case for the real value of ω, it is $\omega < \omega_{calc,in}$ because in fact $\sigma_s < f_{yd}$. We must start an iterative process in order to calculate the real value of ω from the values of the table. We continue with the first pair of ω_{table} and $\sigma_{s,table}$ values from the table which correspond to an uneconomic design (first $\sigma_{s,table}$ for which it is $\sigma_{s,table} < f_{yd}$). From each $\sigma_{s,table}$ we calculate ω_{calc} as follows:

$$\omega_{calc} = \frac{A_s \sigma_{s,table} - N_d}{bdf_{cd}} \tag{102}$$

and we move on with the next pairs (ω_{table}, $\sigma_{s,table}$) until we find a value of ω_{calc} for which $\omega_{calc} < \omega_{table}$. Then we stop and the real value of ω should be between the last two values from the table, as shown in Table 8.

Table 8. Schematic representation of how to use design tables when Steel works below the yield limit.

ω (from table)	**σ_s (from table)**	**ω (calculated from σ_s with Eq. (102))**
ω_{lim}	σ_s=f_{yd}	$\omega_{calc,in}$ (from σ_s=f_{yd}) > ω_{lim}
ω_{table}	$\sigma_{s,table}$	ω_{calc} (from $\sigma_{s,table}$) > ω_{table}
…	…	…
$\omega_{1,table}$	$\sigma_{s1,table}$	$\omega_{a,calc}$ (from $\sigma_{s1,table}$) **> $\omega_{1,table}$**
$\omega_{2,table}$	$\sigma_{s2,table}$	$\omega_{b,calc}$ (from $\sigma_{s2,table}$) **< $\omega_{2,table}$**

In Table 8, the real value of ω should be between the two values ω_1 and ω_2 (the word “table” has been omitted) of the table. In order to find ω we have to find the intersection of two lines in the 2D space of (σ_s, ω), namely the line passing through points (σ_{s1}, ω_1) and (σ_{s2}, ω_2) and the line passing through points (σ_{s1}, ω_a) and (σ_{s2}, ω_b). The intersection point can be easily calculated as follows:

$$\sigma_s = \frac{\sigma_{s1}\left(\omega_2 - \omega_b\right) + \sigma_{s2}\left(\omega_a - \omega_1\right)}{\omega_2 - \omega_b + \omega_a - \omega_1} \tag{103}$$

$$\omega=\frac{\omega_2\omega_a-\omega_1\omega_b}{\omega_2-\omega_b+\omega_a-\omega_1} \tag{104}$$

Having calculated ω, we read μ_{sd} from the table (linear interpolation may be needed). Then as previously, we can calculate M_{sd} and M_d, by using Eq. (101) and Eq. (99), respectivelly.

7.2.1. Linear interpolation for the ω-μ_{sd} tables

Generally, the value of ω is not an exact value of the table, but rather lies between two neighboring values ω_1 and ω_2 ($\omega_1<\omega<\omega_2$), corresponding to μ_{sd} values μ_{sd1} and μ_{sd2}. In this case linear interpolation is needed again. Solving Eq. (35) for μ_{sd} we obtain

$$\mu_{sd}=\mu_{sd1}+\frac{\omega-\omega_1}{\omega_2-\omega_1}\left(\mu_{sd2}-\mu_{sd1}\right) \tag{105}$$

7.3 Analytic formulas of ω, ξ, ζ, ε_s for the solution of the inverse problem

Again we have two cases: Steel working at full strength and steel working below full strength.

Case A: We assume that steel works at full strength ($\mu_{sd}\leq\mu_{sd,lim}$)

Setting $\sigma_s=f_{yd}$ in Eq. (33) we obtain

$$\omega=\frac{A_s f_{yd}-N_d}{bdf_{cd}} \tag{106}$$

Substituting ω from Eq. (47) we have:

$$\eta\left(1-\sqrt{1-\frac{2\mu_{sd}}{\eta}}\right)=\frac{A_s f_{yd}-N_d}{bdf_{cd}}\Rightarrow \tag{107}$$

$$\mu_{sd}=\frac{\eta}{2}\left[1-\left(1-\frac{A_s f_{yd}-N_d}{\eta bdf_{cd}}\right)^2\right] \tag{108}$$

Now we check if the μ_{sd} calculated from Eq. (108) is indeed less than $\mu_{sd,lim}$ (see Table 6). If indeed $\mu_{sd}\leq\mu_{sd,lim}$ then the assumption was right, otherwise we move to case B. If the assumption was right, then we can calculate M_{sd} and M_d as previously, by using Eq. (101) and Eq. (99).

Case B: Steel working below the yield limit (less than full strength, $\mu_{sd}>\mu_{sd,lim}$)

If using Eq. (108) for the given A_s and N_d, the obtained value μ_{sd} is greater than $\mu_{sd,lim}$, then steel works below yield strain and the design is not economic. In this case, we have $\varepsilon_s<\varepsilon_{ys}$ and from Eq. (6) we have

$$\sigma_s=E_s\cdot\varepsilon_s \tag{109}$$

Substituting σ_s from Eq. (109) into Eq. (33) we have

$$\omega=\frac{A_s E_s\varepsilon_s-N_d}{bdf_{cd}} \tag{110}$$

Substituting ε_s from Eq. (65) into Eq. (110) we have

$$\omega = \frac{A_s E_s \left(\frac{\lambda\eta}{\omega} - 1 \right) \varepsilon_{cu3} - N_d}{bdf_{cd}} \Rightarrow \tag{111}$$

$$\left(bdf_{cd}\right) \cdot \omega^2 + \left(A_s E_s \varepsilon_{cu3} + N_d\right) \cdot \omega - \lambda\eta A_s E_s \varepsilon_{cu3} = 0 \tag{112}$$

The above quadratic equation needs to be solved for ω. It can be written in the form:

$$A_3 \omega^2 + B_3 \omega + C_3 = 0 \tag{113}$$

Where

$$A_3 = bdf_{cd} \tag{114}$$

$$B_3 = A_s E_s \varepsilon_{cu3} + N_d \tag{115}$$

$$C_3 = -\lambda\eta A_s E_s \varepsilon_{cu3} \tag{116}$$

The quantities A_3, B_3 and C_3 are all known. The discriminant Δ_3 of the quadratic equation is given by:

$$\Delta_3 = B_3^2 - 4A_3 C_3 = \left(A_s E_s \varepsilon_{cu3} + N_d\right)^2 + 4\lambda\eta bdf_{cd} A_s E_s \varepsilon_{cu3} \tag{117}$$

The solution of the quadratic equation is:

$$\omega_{1,2} = \frac{-B_3 \pm \sqrt{\Delta_3}}{2A_3} \Rightarrow \begin{cases} \omega_1 = \dfrac{-B_3 - \sqrt{\Delta_3}}{2A_3} \\ \omega_2 = \dfrac{-B_3 + \sqrt{\Delta_3}}{2A_3} \end{cases} \tag{118}$$

Of the above solutions, only the second is acceptable, as the first leads to negative values for ω. So we have

$$\omega = \frac{-B_3 + \sqrt{\Delta_3}}{2A_3} \tag{119}$$

Having calculated ω, we calculate μ_{sd} with Eq. (44) and then as previously, we can calculate M_{sd} and M_d, by using Eq. (101) and Eq. (99).

8 NUMERICAL EXAMPLES

Four concrete sections will be examined in total. For each section, the direct and the inverse problem are solved using three methodologies:

1. Analytical calculations
2. Using the design tables provided in Appendix A
3. Using ω analytic formulas without the use of tables

Below are the common properties for all numerical examples:

- γ_c=1.50, a_{cc}=1
- Steel class B500 (f_{yk}=500 MPa)

- E_s=200 GPa, γ_s=1.15

The main different characteristics of the four test examples are summarized below:

1. Concrete Class C20/25, **no axial force** (steel working at full strength).
2. Concrete Class C30/37, **with tensile axial force** (steel working at full strength).
3. **Higher Concrete Class (C70/85)**, with tensile axial force (steel working at full strength).
4. Concrete Class C30/37, with compressive axial force (**steel working below the yield limit**, with less than full strength).

8.1 Numerical example 1

The section of the first numerical example has the following properties:

- Concrete class C20/25, Height h=50 cm, Width b=25 cm, d_1=5 cm
- For the direct problem, we have: M_d=60 kNm, N_d=0 (no axial force), y_N: Not applicable

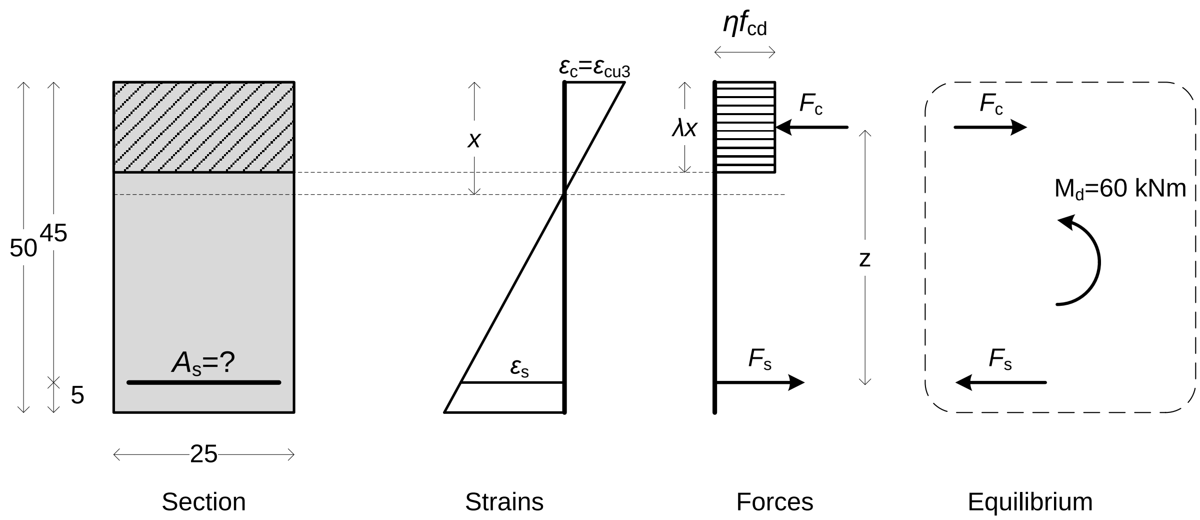


Figure 14. The direct problem of the 1st numerical example (dimensions in cm).

8.1.1. Direct problem

In the direct problem, the external forces are known and we need to find the required steel reinforcement area A_s.

A. Analytical calculations

1. λ=0.8
2. η=1
3. f_{cd} =13333.33 kPa
4. ε_{cu3}=3.5‰
5. f_{yd}=434782.61 kPa

12. A_1=1066.67
13. M_d=60 kNm
14. N_d=0
15. Δ=1184000
16. x=0.052 m

6. ε_{ys}=2.17‰
7. d=0.45 m
8. y_s= Not applicable
9. M_{sd}=60 kNm
10. $M_{sd,max}$=324 kNm, M_{sd}<$M_{sd,max}$ so proceed with the next calculations
11. $M_{sd,lim}$=250.91 kNm, M_{sd}<$M_{sd,lim}$ so the design using single steel reinforcement is economic (steel working at full strength)
17. z=0.429 m
18. F_c=139.85 kN
19. F_s=139.85 kN
20. ε_s=26.53‰
21. σ_s=434782.61 kPa
22. **A_s=3.22 cm^2**

B. Using design tables

After calculating M_{sd} as above, we calculate μ_{sd} from Eq. (25). Then using linear interpolation we obtain the corresponding value of ω from the values of μ_{sd1}, μ_{sd2}, ω_1, ω_2 of Table 10. Finally, we read the corresponding value of σ_s from the table (linear interpolation is not needed for σ_s, unless we are in the area of μ_{sd}>$\mu_{sd,lim}$ of uneconomic design) and we calculate the value of A_s using Eq. (34), as follows

1. M_{sd}=60 kNm
2. μ_{sd}=0.0889
3. For μ_{sd1}=0.08, ω_1=0.0835 (Table 10)
4. For μ_{sd2}=0.09, ω_2=0.0945 (Table 10)
5. ω=0.0933 (obtained with linear interpolation)
6. σ_s=434.78 MPa
7. **A_s=3.22 cm^2**

C. Using ω analytic formulas without the use of tables

Again, after calculating M_{sd}, we calculate μ_{sd} from Eq. (25). Then, instead of using the design tables in order to obtain ω and σ_s, we calculate the value of ω using Eq. (47), the value of ε_s using Eq. (66) and the value of σ_s using Eq. (6). Finally, we obtain the value of A_s again using Eq. (34), as follows

1. M_{sd}=60 kNm
2. μ_{sd}=0.0889
3. ω=0.0932
4. ε_s=26.53‰ > ε_{ys}
5. σ_s=434.78 MPa
6. **A_s=3.22 cm^2**

8.1.2. Inverse problem

In the inverse problem, the tensile reinforcement (steel area) A_s and the axial force N_d (which is applied at y_N) are given and the purpose is to calculate the maximum bending moment M_d that the cross section can withstand. We assume that we have the same problem as previously, therefore:

- N_d=0, y_N: Not applicable
- A_s=3.22 cm^2

A. Analytical calculations

1. λ = 0.8
2. η = 1
3. f_{cd} =13333.33 kPa
4. ε_{cu3} = 3.5‰
5. f_{yd} = 434782.61 kPa
6. ε_{ys} = 2.17‰
7. d = 0.45 m
8. y_s = Not applicable
9. Criterion of Eq. (84) = 12.19≥1, thus we have Case A, steel working at full strength
10. F_s = 140.00 kN
11. F_c = 140.00 kN
12. x = 0.053 m
13. ε_s = 26.50‰ $\geq \varepsilon_{ys}$
14. M_{sd} = 60.06 kNm
15. **M_d = 60.06 kNm**

We see that we get a value of M_d equal to 60.06 kNm, instead of 60.00 kNm of the direct problem. This is because of the fact that in the inverse problem we set A_s=3.22 cm^2 while in the direct problem, the exact value of the needed A_s had more decimal digits (3.21662 cm^2), but it was rounded to two decimal digits for the definition of the inverse problem.

B. Using design tables

We assume that steel works at full strength. We calculate ω using Eq. (100)

- ω=0.0933

We read σ_s from the table (Table 10) and we confirm that steel works at full strength (σ_s=434.78 MPa), so we proceed with Case A. We take the value of μ_{sd} from the table (linear interpolation is needed):

- For ω_1=0.0835, μ_{sd1}=0.08 (Table 10)
- For ω_2=0.0945, μ_{sd2}=0.09 (Table 10)
- With linear interpolation: μ_{sd}=0.0889<$\mu_{sd,lim}$=0.3717

Then we calculate M_{sd} from Eq. (101) and M_d from Eq. (99) as follows:

- M_{sd}=60.03 kNm
- **M_d=60.03 kNm**

C. Using ω analytic formulas without the use of tables

We assume that steel works at full strength. We calculate μ_{sd} using Eq. (108)

- μ_{sd}=0.0890

It is $\mu_{sd} \leq \mu_{sd,lim}$=0.3713, so indeed steel works at full strength and the assumption was right. We then calculate M_{sd} from Eq. (101) and M_d from Eq. (99) as follows:

- M_{sd}=60.06 kNm
- **M_d=60.06 kNm**

8.2 Numerical example 2

The section of the second numerical example has the following properties:

- Concrete class C30/37, Height h=60 cm, Width b=30 cm, d_1=5 cm
- For the direct problem, we have: M_d=100 kNm, N_d=50 kN, y_N=h/2=30 cm

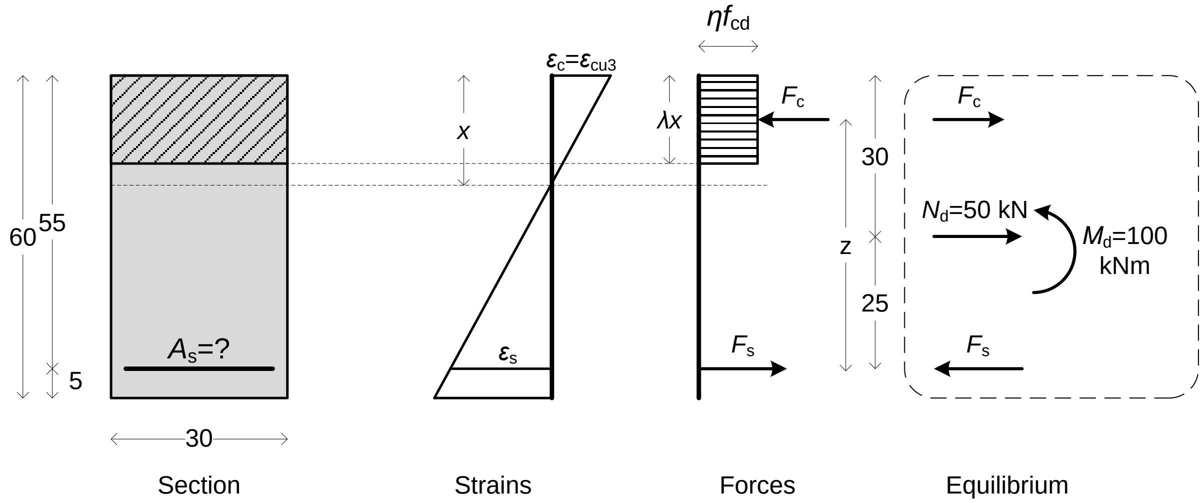


Figure 15. The direct problem of the 2nd numerical example (dimensions in cm).

8.2.1. Direct problem

A. Analytical calculations

1. λ=0.8
2. η=1
3. f_{cd} =20000 kPa
4. ε_{cu3}=3.5‰
5. f_{yd}=434782.61 kPa
6. ε_{ys}=2.17‰
7. d=0.55 m
8. y_s= 0.25 m
11. $M_{sd,lim}$=674.68 kNm, M_{sd}<$M_{sd,lim}$ so the design using single steel reinforcement is economic (steel working at full strength)
12. x=0.034 m
13. z=0.536 m
14. F_c=163.12 kN
15. F_s=213.12 kN
16. ε_s=53.14‰

9. M_{sd}=87.50 kNm
10. $M_{sd,max}$=871.20 kNm, M_{sd}<$M_{sd,max}$ so proceed with the next calculations
17. σ_s=434782.61 kPa
18. **A_s=4.90 cm^2**

B. Using design tables

Using the same methodology as in the first example, we have:

1. M_{sd}=87.5 kNm
2. μ_{sd}=0.0482
3. For μ_{sd1}=0.04, ω_1=0.0408 (Table 10)
4. For μ_{sd2}=0.05, ω_2=0.0513 (Table 10)
5. ω=0.0494 (linear interpolation)
6. σ_s=434.78 MPa
7. **A_s=4.90 cm^2**

C. Using ω analytic formulas without the use of tables

Using the same methodology as in the first example, we have:

1. M_{sd}=87.50 kNm
2. μ_{sd}=0.0482
3. ω=0.0494
4. ε_s=53.14‰ > ε_{ys}
5. σ_s=434.78 MPa
6. **A_s=4.90 cm^2**

8.2.2. Inverse problem

We assume that we have the same problem as previously, therefore:

- N_d=50 kN, y_N=h/2=30 cm
- A_s=4.90 cm^2

A. Analytical calculations

1. λ = 0.8
2. η = 1
3. f_{cd} =20000 kPa
4. ε_{cu3}=3.5‰
5. f_{yd} = 434782.61 kPa
9. Criterion of Eq. (84) = 24.46≥1, thus we have Case A, steel working at full strength
10. F_s = 213.04 kN
11. F_c = 163.04 kN

6. $\varepsilon_{ys} = 2.17‰$
7. $d = 0.55$ m
8. $y_s = 0.25$ m
12. $x = 0.034$ m
13. $\varepsilon_s = 53.17‰ \geq \varepsilon_{ys}$
14. $M_{sd} = 87.46$ kNm
15. **M_d = 99.96 kNm**

Again, there is a small errors due to rounding A_s to two decimal digits.

B. Using design tables

We assume that steel works at full strength. We calculate ω using Eq. (100)

- ω=0.0494

We read σ_s from the table and we confirm that steel works at full strength (σ_s=434.78 MPa), so we proceed with Case A. We take the value of μ_{sd} from the table (linear interpolation is needed):

- For ω_1=0.0408, μ_{sd1}=0.04 (Table 10)
- For ω_2=0.0513, μ_{sd2}=0.05 (Table 10)
- With linear interpolation: μ_{sd}=0.0482<$\mu_{sd,lim}$=0.3717

Then we calculate M_{sd} from Eq. (101) and M_d from Eq. (99) as follows:

- M_{sd}=87.48 kNm
- **M_d=99.98 kNm**

C. Using ω analytic formulas without the use of tables

We assume that steel works at full strength. We calculate μ_{sd} using Eq. (108)

- μ_{sd}=0.0482

It is $\mu_{sd} \leq \mu_{sd,lim}$=0.3713, so indeed steel works at full strength and the assumption was right. We then calculate M_{sd} from Eq. (101) and M_d from Eq. (99):

- M_{sd}=87.46 kNm
- **M_d=99.96 kNm**

8.3 Numerical example 3

The section of the third numerical example has the following properties:

- Concrete class C70/85, Height h=70 cm, Width b=30 cm, d_1=5 cm
- For the direct problem, we have: M_d=150 kNm, N_d=100 kN, y_N=h/2=35 cm

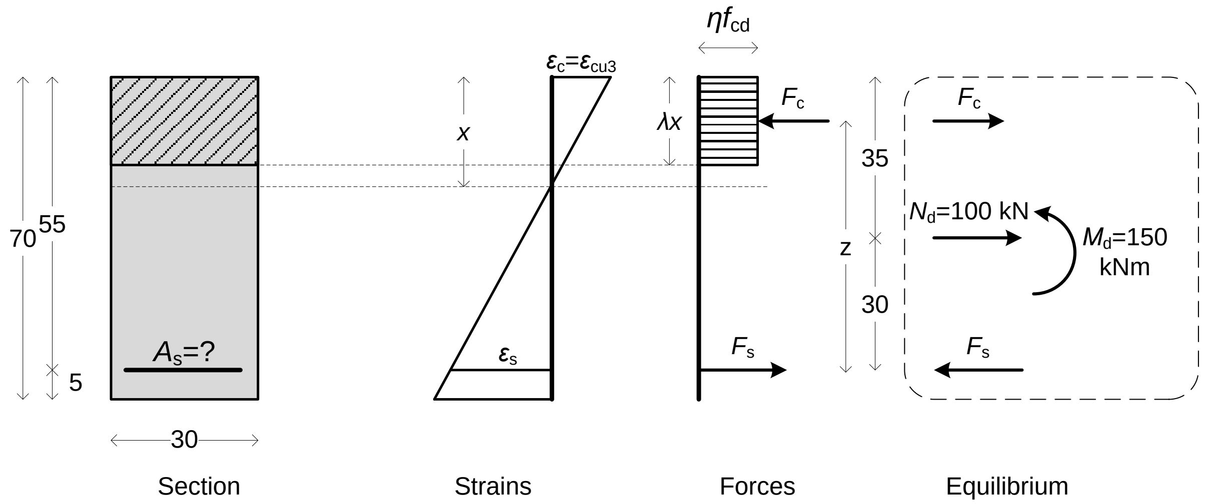


Figure 16. The direct problem of the 3rd numerical example (dimensions in cm).

8.3.1. Direct problem

A. Analytical calculations

1. λ=0.75
2. η=0.90
3. f_{cd} =46667 kPa
4. ε_{cu3}=2.66 ‰
5. f_{yd}=434782.61 kPa
6. ε_{ys}=2.17‰
7. d=0.65 m
8. y_s= 0.30 m
9. M_{sd}=120 kNm
10. $M_{sd,max}$=2495.38 kNm, $M_{sd}<M_{sd,max}$ so proceed with the next calculations
11. $M_{sd,lim}$=1742.81 kNm, $M_{sd}<M_{sd,lim}$ so the design using single steel reinforcement is economic (steel working at full strength)
12. x=0.020 m
13. z=0.643 m
14. F_c=186.74 kN
15. F_s=286.74 kN
16. ε_s=84.71 ‰
17. σ_s=434782.61 kPa
18. **A_s=6.60 cm^2**

B. Using design tables

Using the same methodology as in the previous examples, we have:

1. M_{sd}=120 kNm
2. μ_{sd}=0.0203
3. For μ_{sd1}=0.02, ω_1=0.0202 (Table 13)
4. For μ_{sd2}=0.03, ω_2=0.0305 (Table 13)
5. ω=0.0205 (linear interpolation)
6. σ_s=434.78 MPa

7. A_s=6.59 cm^2

C. Using ω analytic formulas without the use of tables

Using the same methodology as in the previous examples, we have:

1. M_{sd}=120 kNm
2. μ_{sd}=0.0203
3. ω=0.0205
4. ε_s=84.71‰ > ε_{ys}
5. σ_s=434.78 MPa
6. **A_s=6.60 cm^2**

8.3.2. Inverse problem

We assume that we have the same problem as previously, therefore:

- N_d= 100 kN, y_N= h/2=35 cm
- A_s=6.60 cm^2

A. Analytical calculations

1. λ = 0.75
2. η = 0.90

19. f_{cd} =46666.67 kPa

20. ε_{cu3}=2.66‰

3. f_{yd} = 434782.61 kPa
4. ε_{ys} = 2.17‰
5. d = 0.65 m
6. y_s = 0.30 m
7. Criterion of Eq. (84) = 38.92≥1, thus we have Case A, steel working at full strength
8. F_s = 286.96 kN
9. F_c = 186.96 kN
10. x = 0.020 m
11. ε_s = 84.61‰ ≥ε_{ys}
12. M_{sd} = 120.13 kNm
13. **M_d = 150.13 kNm**

B. Using design tables

We assume that steel works at full strength. We calculate ω using Eq. (100)

- ω=0.0205

We read σ_s from the table and we confirm that steel works at full strength (σ_s=434.78 MPa), so we proceed with Case A. We take the value of μ_{sd} from the table (linear interpolation is needed):

- For ω_1=0.0202, μ_{sd1}=0.02 (Table 13)
- For ω_2=0.0305, μ_{sd2}=0.03 (Table 13)

- μ_{sd}=0.0203<$\mu_{sd,lim}$=0.2946

Then we calculate M_{sd} from Eq. (101) and M_d from Eq. (99) as follows:

- M_{sd}=120.28 kNm
- M_d=150.28 kNm

C. Using ω analytic formulas without the use of tables

We assume that steel works at full strength. We calculate μ_{sd} using Eq. (108)

- μ_{sd}=0.0203

It is $\mu_{sd}\leq\mu_{sd,lim}$=0.2946, so indeed steel works at full strength and the assumption was right. We then calculate M_{sd} from Eq. (101) and M_d from Eq. (99):

- M_{sd}=120.13 kNm
- **M_d=150.13 kNm**

8.4 Numerical example 4

The section of the fourth numerical example has the following properties:

- Concrete class C30/37, Height h=50 cm, Width b=25 cm, d_1=5 cm
- For the direct problem, we have: M_d=378 kNm, N_d=-50 kN (compressive), y_N=h/2=25 cm

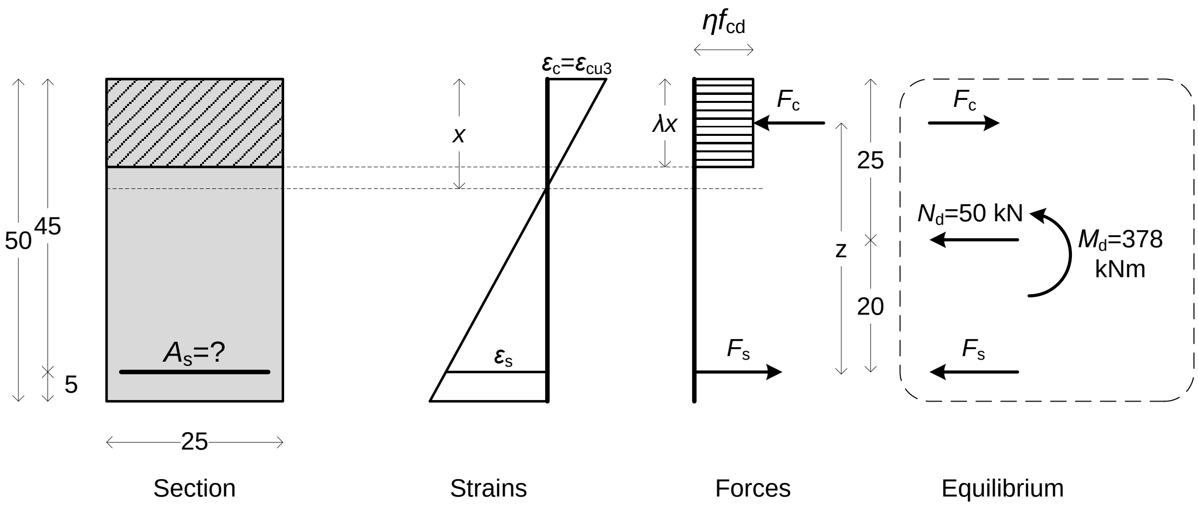


Figure 17. The direct problem of the 4th numerical example (dimensions in cm).

8.4.1. Direct problem

A. Analytical calculations

1. λ=0.8
2. η=1
3. f_{cd} =20000 kPa
4. ε_{cu3}=3.5 ‰
5. f_{yd}=434782.61 kPa
6. ε_{ys}=2.17‰
7. d=0.45 m
8. y_s= 0.20 m
9. M_{sd}=388 kNm
10. $M_{sd,max}$=486.00 kNm, M_{sd}<$M_{sd,max}$ so proceed with the next calculations
11. $M_{sd,lim}$=376.37 kNm, M_{sd}>$M_{sd,lim}$ so the design using single steel reinforcement is not economic (**steel *not* working at full strength**)
12. x=0.291 m
13. z=0.334 m
14. F_c=1162.57 kN
15. F_s=1112.57 kN
16. ε_s=1.92‰ < ε_{ys}
17. σ_s=383803.99 kPa < f_{yd}
18. **A_s=28.99 cm^2**

B. Using design tables

Using the same methodology as in the previous examples, we have:

1. M_{sd}=388 kNm
2. μ_{sd}=0.3832
3. For μ_{sd1}=0.38, ω_1=0.5101, σ_{s1}=397.82 (Table 10)
4. For μ_{sd2}=0.39, ω_2=0.5310, σ_{s2}=354.70 (Table 10)
5. ω=0.5168 (linear interpolation)
6. σ_s=383979.01 kPa (linear interpolation)
7. **A_s=28.98 cm^2**

C. Using *ω* analytic formulas without the use of tables

Using the same methodology as in the previous examples, we have:

1. M_{sd}=388.00 kNm
2. μ_{sd}=0.3832
3. ω=0.5167
4. ε_s=1.92‰<ε_{ys}
5. σ_s= 383803.99 kPa
6. **A_s=28.99 cm^2**

8.4.2. Inverse problem

We assume that we have the same problem as previously, therefore:

- N_d=-50 kN, y_N=h/2=25 cm
- A_s=28.99 cm^2

A. Analytical calculations

1. λ = 0.8
2. η = 1
19. f_{cd} =20000 kPa
20. ε_{cu3}=3.5‰
3. f_{yd} = 434782.61 kPa
4. ε_{ys} = 2.17‰
5. d = 0.45 m
6. y_s = 0.20 m
7. Criterion of Eq. (84) = 0.60<1, thus we have Case B, **steel working *below* full strength**
8. A = 4000, B = 1979.30, C = -913.19
9. Δ = 18528588
10. x = 0.291 m
11. ε_s = 1.92‰ < ε_{ys}
12. σ_s = 383784.87 kPa < f_{yd}
13. M_{sd} = 388.00 kNm
14. **M_d = 378.00 kNm**

B. Using design tables

We first assume that steel works at full strength. Setting σ_s=f_{yd} in Eq. (33) we calculate ω from Eq. (100) as follows

- $\omega_{calc,in}$=0.5824

According to the design table, ω_{lim}=0.4935, so it is $\omega_{calc,in}$>ω_{lim}. Also, if we read the design table for the initially calculated ω=0.5824 we will see that steel works below full strength (σ_s<f_{yd}), which is in conflict with our assumption. This means that the design is not economic and the assumption of steel working at full strength was wrong. We must start the iterative process in order to calculate the real value of ω from the values of the table:

- We start with ω_{lim} which essentially corresponds to σ_s=f_{yd}=434.78 MPa. From this value σ_s=f_{yd} we calculate the new value of ω ($\omega_{calc,in}$) with Eq. (102). For ω_{lim} and σ_s=f_{yd} the calculated value of ω_{calc} should be $\omega_{calc,in}$>ω_{lim}. See the 2nd line of Table 9.
- We continue with the first pair of ω_{table} and $\sigma_{s,table}$ values from the table which correspond to an uneconomic design (first $\sigma_{s,table}$ for which it is $\sigma_{s,table}$<f_{yd}). In our case, this first value is $\sigma_{s1,table}$=397.82 MPa. We calculate ω_{calc} again. In this case, it is again $\omega_{a,calc}$>$\omega_{1,table}$. See the 3rd line of Table 9.
- We repeat the previous calculation with the next pairs until we find a value of ω_{calc} for which ω_{calc}<ω_{table}. In our case this happens in the next pair, as shown in the 4th line of Table 9.

Table 9. Iterative process for the solution of the inverse problem of the 4th example.

ω (from table)	**σ_s (from table)**	**ω (calculated from σ_s with Eq. (102))**
ω_{lim}=0.4935	$\sigma_s=f_{yd}$=434.78	$\omega_{calc,in}$ (from $\sigma_s=f_{yd}$) = 0.5824 > ω_{lim}
$\omega_{1,table}$=0.5101	$\sigma_{s1,table}$=397.82	$\omega_{a,calc}$ (from $\sigma_{s1,table}$) = 0.5348 > $\omega_{1,table}$
$\omega_{2,table}$ =0.5310	$\sigma_{s2,table}$ =354.70	$\omega_{b,calc}$ (from $\sigma_{s2,table}$) = 0.4792 < $\omega_{2,table}$

Then we stop and we use Eq. (103) and Eq. (104) to calculate σ_s and ω as follows:

- σ_s=383894.86 kPa
- ω=0.5168

For the calculation of μ_{sd} we then use linear interpolation:

- For ω_1=0.5101, μ_{sd1}=0.38 (Table 10)
- For ω_2=0.5310, μ_{sd2}=0.39 (Table 10)
- With linear interpolation we obtain: μ_{sd}=0.3832 > $\mu_{sd,lim}$=0.3717

Then we calculate M_{sd} from Eq. (101) and M_d from Eq. (99) as follows:

- M_{sd}=388.02 kNm
- M_d=378.02 kNm

C. Using ω analytic formulas without the use of tables

We first assume that steel works at full strength. We calculate μ_{sd} using Eq. (108)

- μ_{sd}=0.4128

It is $\mu_{sd}>\mu_{sd,lim}$=0.3717, so the assumption was wrong - steel works *below* full strength. We move to Case B. We need to solve a quadratic equation in order to calculate ω. We calculate A_3, B_3, C_3 using Eqs (114), (115), (116). Then we calculate ω using Eq. (119).

- A_3=2250.00, B_3=1979.30, C_3=-1623.44
- Δ_3=18528588
- ω=0.5167
- μ_{sd}=0.3832

We then calculate M_{sd} from Eq. (101) and M_d from Eq. (99) as follows:

- M_{sd}=388.00 kNm
- **M_d=378.00 kNm**

9 CONCLUSIONS

- Eurocode 2-Part 1-1 gives us new tools in order to design concrete cross sections. Three approaches may be used for the stress-strain relation of concrete and another two approaches for the stress-strain relation of the steel reinforcement. In this study we used the rectangular stress distribution for concrete together with the bilinear stress-strain distribution for steel with a horizontal top branch (no hardening, k=1).

- EC2-1-1 allows the designer not to limit the ultimate strain for steel when a horizontal top branch is assumed for its stress-strain diagram. In this case, the concrete zone is assumed to be at the ultimate strain at the ULS and the steel strain can take any value, without any limitation. This approach is followed in the present study - in all the methodologies and the examples, concrete is the critical material in all cases.
- This chapter presents three detailed methodologies for the design of rectangular cross sections with tensile (single) reinforcement, covering all concrete classes, from C12/15 up to C90/105. The purpose in every case is to calculate the necessary tensile steel reinforcement A_s. The first methodology provides an analytical algorithmic procedure that can be easily applied in any programming language. The second methodology is based on design tables that are provided in Appendix A. The third methodology provides again analytic formulas that can replace completely the use of tables and can in fact be used to reproduce these tables.
- Apart from the direct problem, the inverse problem is also studied, where the steel reinforcement is given and the purpose is to find the maximum bending moment that the section can withstand, given also the value and position of the axial force on the section. Again, the inverse problem is solved using the same three methodologies of the direct problem.
- All methodologies provide the same results. The results of the two methodologies based on analytic formulas coincide, while the use of tables incorporates small errors that can affect the decimal digits of the final result. The solution of the inverse problem always leads to the bending moment of the direct problem. Small errors are due to the fact that the steel area is “rounded” in two decimal digits when the inverse problem is defined.
- All Eurocode parameters, such as a_{cc}, γ_c, γ_s, even E_s and many others can be adjusted according to the preferences of the designer, without any limitation. That is with the exception of the Tables of the Appendix where the last columns (steel stress σs) and the limit values have been calculated for E_s=200 GPa and γ_s=1.15. Nevertheless, using the proposed methodology new tables can be generated where the values of these parameters can be different.
- In this study detailed guidelines are provided for reinforced concrete section design accompanied with special design curves for each case. The curves presented are based on equations which are given in closed form.
- The various regions of reinforced concrete section design are explicitly defined. Two limits are defined for the normalized design bending moment: $\mu_{sd,lim}$ and $\mu_{sd,max}>\mu_{sd,lim}$. We have three cases in general:
    1. If for the direct problem, $\mu_{sd}\leq\mu_{sd,lim}$, then the design is economic and this should be the case in practice.
    2. If $\mu_{sd,lim}<\mu_{sd}<\mu_{sd,max}$, then the design is possible, but not economic and it should be avoided, as steel works below its full strength.
    3. If $\mu_{sd}\geq\mu_{sd,max}$ then the design is impossible. The dimensions of the section must be increased and/or compressive reinforcement must be added.

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APPENDIX A: Tables for the design of cross sections with single reinforcement

Assumptions (in accordance with EC2-1-1): E_s=200 GPa and γ_s=1.15, affecting the calculation of $\mu_{sd,lim}$, ω_{lim} and σ_s values, only.

Table 10. Design table for Concrete C12/15 up to C50/60.

<table>
<tr><th colspan="8">Concretes from C12/15 up to C50/60 - $\mu_{sd,max}$=0.4800</th></tr>
<tr><th>μ_{sd}</th><th>ω</th><th>$\xi=x/d$</th><th>$\zeta=z/d$</th><th>ε_s (‰)</th><th>σ_s (B400)
$\mu_{sd,lim}$=0.3916
ω_{lim}=0.5344</th><th>σ_s (B500)
$\mu_{sd,lim}$=0.3717
ω_{lim}=0.4935</th><th>σ_s (B600)
$\mu_{sd,lim}$=0.3533
ω_{lim}=0.4584</th></tr>
<tr><td>0.01</td><td>0.0101</td><td>0.0126</td><td>0.9950</td><td>275.09</td><td rowspan="39">347.83</td><td rowspan="37">434.78</td><td rowspan="35">521.74</td></tr>
<tr><td>0.02</td><td>0.0202</td><td>0.0253</td><td>0.9899</td><td>135.09</td></tr>
<tr><td>0.03</td><td>0.0305</td><td>0.0381</td><td>0.9848</td><td>88.41</td></tr>
<tr><td>0.04</td><td>0.0408</td><td>0.0510</td><td>0.9796</td><td>65.07</td></tr>
<tr><td>0.05</td><td>0.0513</td><td>0.0641</td><td>0.9743</td><td>51.06</td></tr>
<tr><td>0.06</td><td>0.0619</td><td>0.0774</td><td>0.9690</td><td>41.72</td></tr>
<tr><td>0.07</td><td>0.0726</td><td>0.0908</td><td>0.9637</td><td>35.05</td></tr>
<tr><td>0.08</td><td>0.0835</td><td>0.1044</td><td>0.9583</td><td>30.04</td></tr>
<tr><td>0.09</td><td>0.0945</td><td>0.1181</td><td>0.9528</td><td>26.14</td></tr>
<tr><td>0.10</td><td>0.1056</td><td>0.1320</td><td>0.9472</td><td>23.02</td></tr>
<tr><td>0.11</td><td>0.1168</td><td>0.1460</td><td>0.9416</td><td>20.47</td></tr>
<tr><td>0.12</td><td>0.1282</td><td>0.1603</td><td>0.9359</td><td>18.34</td></tr>
<tr><td>0.13</td><td>0.1398</td><td>0.1747</td><td>0.9301</td><td>16.53</td></tr>
<tr><td>0.14</td><td>0.1515</td><td>0.1893</td><td>0.9243</td><td>14.99</td></tr>
<tr><td>0.15</td><td>0.1633</td><td>0.2042</td><td>0.9183</td><td>13.64</td></tr>
<tr><td>0.16</td><td>0.1754</td><td>0.2192</td><td>0.9123</td><td>12.47</td></tr>
<tr><td>0.17</td><td>0.1876</td><td>0.2345</td><td>0.9062</td><td>11.43</td></tr>
<tr><td>0.18</td><td>0.2000</td><td>0.2500</td><td>0.9000</td><td>10.50</td></tr>
<tr><td>0.19</td><td>0.2126</td><td>0.2657</td><td>0.8937</td><td>9.67</td></tr>
<tr><td>0.20</td><td>0.2254</td><td>0.2818</td><td>0.8873</td><td>8.92</td></tr>
<tr><td>0.21</td><td>0.2384</td><td>0.2980</td><td>0.8808</td><td>8.24</td></tr>
<tr><td>0.22</td><td>0.2517</td><td>0.3146</td><td>0.8742</td><td>7.63</td></tr>
<tr><td>0.23</td><td>0.2652</td><td>0.3314</td><td>0.8674</td><td>7.06</td></tr>
<tr><td>0.24</td><td>0.2789</td><td>0.3486</td><td>0.8606</td><td>6.54</td></tr>
<tr><td>0.25</td><td>0.2929</td><td>0.3661</td><td>0.8536</td><td>6.06</td></tr>
<tr><td>0.26</td><td>0.3072</td><td>0.3840</td><td>0.8464</td><td>5.62</td></tr>
<tr><td>0.27</td><td>0.3218</td><td>0.4022</td><td>0.8391</td><td>5.20</td></tr>
<tr><td>0.28</td><td>0.3367</td><td>0.4208</td><td>0.8317</td><td>4.82</td></tr>
<tr><td>0.29</td><td>0.3519</td><td>0.4399</td><td>0.8240</td><td>4.46</td></tr>
<tr><td>0.30</td><td>0.3675</td><td>0.4594</td><td>0.8162</td><td>4.12</td></tr>
<tr><td>0.31</td><td>0.3836</td><td>0.4794</td><td>0.8082</td><td>3.80</td></tr>
<tr><td>0.32</td><td>0.4000</td><td>0.5000</td><td>0.8000</td><td>3.50</td></tr>
<tr><td>0.33</td><td>0.4169</td><td>0.5211</td><td>0.7915</td><td>3.22</td></tr>
<tr><td>0.34</td><td>0.4343</td><td>0.5429</td><td>0.7828</td><td>2.95</td></tr>
<tr><td>0.35</td><td>0.4523</td><td>0.5653</td><td>0.7739</td><td>2.69</td></tr>
<tr><td>0.36</td><td>0.4708</td><td>0.5886</td><td>0.7646</td><td>2.45</td><td>489.34</td></tr>
<tr><td>0.37</td><td>0.4901</td><td>0.6126</td><td>0.7550</td><td>2.21</td><td>442.63</td></tr>
<tr><td>0.38</td><td>0.5101</td><td>0.6376</td><td>0.7449</td><td>1.99</td><td>397.82</td><td>397.82</td></tr>
<tr><td>0.39</td><td>0.5310</td><td>0.6637</td><td>0.7345</td><td>1.77</td><td>354.70</td><td>354.70</td></tr>
<tr><td>0.40</td><td>0.5528</td><td>0.6910</td><td>0.7236</td><td>1.57</td><td>313.05</td><td>313.05</td><td>313.05</td></tr>
<tr><td>0.41</td><td>0.5757</td><td>0.7197</td><td>0.7121</td><td>1.36</td><td>272.67</td><td>272.67</td><td>272.67</td></tr>
<tr><td>0.42</td><td>0.6000</td><td>0.7500</td><td>0.7000</td><td>1.17</td><td>233.33</td><td>233.33</td><td>233.33</td></tr>
<tr><td>0.43</td><td>0.6258</td><td>0.7823</td><td>0.6871</td><td>0.97</td><td>194.81</td><td>194.81</td><td>194.81</td></tr>
<tr><td>0.44</td><td>0.6536</td><td>0.8170</td><td>0.6732</td><td>0.78</td><td>156.81</td><td>156.81</td><td>156.81</td></tr>
<tr><td>0.45</td><td>0.6838</td><td>0.8547</td><td>0.6581</td><td>0.59</td><td>118.99</td><td>118.99</td><td>118.99</td></tr>
<tr><td>0.46</td><td>0.7172</td><td>0.8964</td><td>0.6414</td><td>0.40</td><td>80.86</td><td>80.86</td><td>80.86</td></tr>
<tr><td>0.47</td><td>0.7551</td><td>0.9438</td><td>0.6225</td><td>0.21</td><td>41.67</td><td>41.67</td><td>41.67</td></tr>
</table>

Table 11. Design table for Concrete C55/60.

<table>
<tr><th colspan="8">Concrete C55/67 - $\mu_{sd,max}$=0.4655</th></tr>
<tr><th>μ_{sd}</th><th>ω</th><th>$\xi=x/d$</th><th>$\zeta=z/d$</th><th>ε_s (‰)</th><th>σ_s (B400)
$\mu_{sd,lim}$=0.3685
ω_{lim}=0.4933</th><th>σ_s (B500)
$\mu_{sd,lim}$=0.3477
ω_{lim}=0.4528</th><th>σ_s (B600)
$\mu_{sd,lim}$=0.3287
ω_{lim}=0.4185</th></tr>
<tr><td>0.01</td><td>0.0101</td><td>0.0131</td><td>0.9948</td><td>235.60</td><td rowspan="36">347.83</td><td rowspan="34">434.78</td><td rowspan="32">521.74</td></tr>
<tr><td>0.02</td><td>0.0202</td><td>0.0263</td><td>0.9896</td><td>115.61</td></tr>
<tr><td>0.03</td><td>0.0305</td><td>0.0397</td><td>0.9844</td><td>75.61</td></tr>
<tr><td>0.04</td><td>0.0409</td><td>0.0532</td><td>0.9790</td><td>55.61</td></tr>
<tr><td>0.05</td><td>0.0514</td><td>0.0669</td><td>0.9737</td><td>43.60</td></tr>
<tr><td>0.06</td><td>0.0620</td><td>0.0807</td><td>0.9682</td><td>35.60</td></tr>
<tr><td>0.07</td><td>0.0727</td><td>0.0947</td><td>0.9627</td><td>29.88</td></tr>
<tr><td>0.08</td><td>0.0836</td><td>0.1089</td><td>0.9571</td><td>25.58</td></tr>
<tr><td>0.09</td><td>0.0946</td><td>0.1232</td><td>0.9515</td><td>22.24</td></tr>
<tr><td>0.10</td><td>0.1057</td><td>0.1377</td><td>0.9458</td><td>19.57</td></tr>
<tr><td>0.11</td><td>0.1170</td><td>0.1524</td><td>0.9400</td><td>17.38</td></tr>
<tr><td>0.12</td><td>0.1285</td><td>0.1673</td><td>0.9341</td><td>15.55</td></tr>
<tr><td>0.13</td><td>0.1401</td><td>0.1824</td><td>0.9282</td><td>14.01</td></tr>
<tr><td>0.14</td><td>0.1518</td><td>0.1977</td><td>0.9221</td><td>12.68</td></tr>
<tr><td>0.15</td><td>0.1638</td><td>0.2133</td><td>0.9160</td><td>11.53</td></tr>
<tr><td>0.16</td><td>0.1759</td><td>0.2290</td><td>0.9098</td><td>10.52</td></tr>
<tr><td>0.17</td><td>0.1882</td><td>0.2451</td><td>0.9035</td><td>9.63</td></tr>
<tr><td>0.18</td><td>0.2006</td><td>0.2613</td><td>0.8971</td><td>8.83</td></tr>
<tr><td>0.19</td><td>0.2133</td><td>0.2779</td><td>0.8906</td><td>8.12</td></tr>
<tr><td>0.20</td><td>0.2263</td><td>0.2947</td><td>0.8840</td><td>7.48</td></tr>
<tr><td>0.21</td><td>0.2394</td><td>0.3118</td><td>0.8772</td><td>6.90</td></tr>
<tr><td>0.22</td><td>0.2528</td><td>0.3292</td><td>0.8704</td><td>6.37</td></tr>
<tr><td>0.23</td><td>0.2664</td><td>0.3469</td><td>0.8634</td><td>5.88</td></tr>
<tr><td>0.24</td><td>0.2803</td><td>0.3650</td><td>0.8563</td><td>5.44</td></tr>
<tr><td>0.25</td><td>0.2945</td><td>0.3835</td><td>0.8490</td><td>5.02</td></tr>
<tr><td>0.26</td><td>0.3089</td><td>0.4024</td><td>0.8416</td><td>4.64</td></tr>
<tr><td>0.27</td><td>0.3238</td><td>0.4217</td><td>0.8340</td><td>4.29</td></tr>
<tr><td>0.28</td><td>0.3389</td><td>0.4414</td><td>0.8262</td><td>3.96</td></tr>
<tr><td>0.29</td><td>0.3544</td><td>0.4616</td><td>0.8182</td><td>3.65</td></tr>
<tr><td>0.30</td><td>0.3703</td><td>0.4823</td><td>0.8101</td><td>3.35</td></tr>
<tr><td>0.31</td><td>0.3867</td><td>0.5036</td><td>0.8017</td><td>3.08</td></tr>
<tr><td>0.32</td><td>0.4035</td><td>0.5255</td><td>0.7931</td><td>2.82</td></tr>
<tr><td>0.33</td><td>0.4208</td><td>0.5481</td><td>0.7842</td><td>2.58</td><td>515.41</td></tr>
<tr><td>0.34</td><td>0.4387</td><td>0.5714</td><td>0.7750</td><td>2.34</td><td>468.92</td></tr>
<tr><td>0.35</td><td>0.4572</td><td>0.5954</td><td>0.7655</td><td>2.12</td><td>424.66</td><td>424.66</td></tr>
<tr><td>0.36</td><td>0.4764</td><td>0.6204</td><td>0.7557</td><td>1.91</td><td>382.39</td><td>382.39</td></tr>
<tr><td>0.37</td><td>0.4963</td><td>0.6464</td><td>0.7455</td><td>1.71</td><td>341.89</td><td>341.89</td><td>341.89</td></tr>
<tr><td>0.38</td><td>0.5172</td><td>0.6735</td><td>0.7348</td><td>1.51</td><td>302.96</td><td>302.96</td><td>302.96</td></tr>
<tr><td>0.39</td><td>0.5390</td><td>0.7020</td><td>0.7236</td><td>1.33</td><td>265.39</td><td>265.39</td><td>265.39</td></tr>
<tr><td>0.40</td><td>0.5619</td><td>0.7319</td><td>0.7118</td><td>1.15</td><td>229.00</td><td>229.00</td><td>229.00</td></tr>
<tr><td>0.41</td><td>0.5863</td><td>0.7635</td><td>0.6994</td><td>0.97</td><td>193.57</td><td>193.57</td><td>193.57</td></tr>
<tr><td>0.42</td><td>0.6122</td><td>0.7973</td><td>0.6861</td><td>0.79</td><td>158.88</td><td>158.88</td><td>158.88</td></tr>
<tr><td>0.43</td><td>0.6401</td><td>0.8337</td><td>0.6717</td><td>0.62</td><td>124.65</td><td>124.65</td><td>124.65</td></tr>
<tr><td>0.44</td><td>0.6707</td><td>0.8735</td><td>0.6561</td><td>0.45</td><td>90.55</td><td>90.55</td><td>90.55</td></tr>
<tr><td>0.45</td><td>0.7046</td><td>0.9177</td><td>0.6387</td><td>0.28</td><td>56.09</td><td>56.09</td><td>56.09</td></tr>
<tr><td>0.46</td><td>0.7434</td><td>0.9682</td><td>0.6188</td><td>0.10</td><td>20.50</td><td>20.50</td><td>20.50</td></tr>
</table>

Table 12. Design table for Concrete C60/75.

Concrete C60/75 - $\mu_{sd,max}$=0.4510							
μ_{sd}	ω	ξ=x/d	ζ=z/d	ε_s (‰)	σ_s (B400) $\mu_{sd,lim}$=0.3482 ω_{lim}=0.4593	σ_s (B500) $\mu_{sd,lim}$=0.3270 ω_{lim}=0.4198	σ_s (B600) $\mu_{sd,lim}$=0.3079 ω_{lim}=0.3865
0.01	0.0101	0.0137	0.9947	208.29	347.83	434.78	521.74
0.02	0.0202	0.0275	0.9894	102.14			
0.03	0.0305	0.0414	0.9840	66.75			
0.04	0.0409	0.0555	0.9785	49.05			
0.05	0.0514	0.0698	0.9730	38.43			
0.06	0.0620	0.0842	0.9674	31.34			
0.07	0.0728	0.0989	0.9617	26.28			
0.08	0.0837	0.1137	0.9560	22.48			
0.09	0.0947	0.1287	0.9501	19.53			
0.10	0.1059	0.1438	0.9443	17.16			
0.11	0.1172	0.1592	0.9383	15.23			
0.12	0.1287	0.1748	0.9323	13.61			
0.13	0.1404	0.1907	0.9261	12.24			
0.14	0.1522	0.2067	0.9199	11.07			
0.15	0.1642	0.2230	0.9136	10.05			
0.16	0.1764	0.2396	0.9072	9.15			
0.17	0.1888	0.2564	0.9007	8.36			
0.18	0.2013	0.2735	0.8940	7.66			
0.19	0.2141	0.2908	0.8873	7.03			
0.20	0.2272	0.3085	0.8804	6.46			
0.21	0.2404	0.3266	0.8735	5.95			
0.22	0.2539	0.3449	0.8663	5.48			
0.23	0.2677	0.3636	0.8591	5.05			
0.24	0.2818	0.3827	0.8517	4.65			
0.25	0.2962	0.4023	0.8441	4.28			
0.26	0.3109	0.4222	0.8364	3.95			
0.27	0.3259	0.4426	0.8285	3.63			
0.28	0.3413	0.4636	0.8204	3.34			
0.29	0.3571	0.4851	0.8120	3.06			
0.30	0.3734	0.5071	0.8035	2.80			
0.31	0.3901	0.5298	0.7947	2.56			*511.76*
0.32	0.4073	0.5532	0.7856	2.33			*465.71*
0.33	0.4251	0.5774	0.7763	2.11		*422.07*	*422.07*
0.34	0.4435	0.6024	0.7666	1.90		*380.58*	*380.58*
0.35	0.4627	0.6284	0.7565	1.71	*341.03*	*341.03*	*341.03*
0.36	0.4826	0.6554	0.7460	1.52	*303.18*	*303.18*	*303.18*
0.37	0.5033	0.6837	0.7351	1.33	*266.85*	*266.85*	*266.85*
0.38	0.5251	0.7133	0.7236	1.16	*231.83*	*231.83*	*231.83*
0.39	0.5481	0.7445	0.7115	0.99	*197.93*	*197.93*	*197.93*
0.40	0.5725	0.7776	0.6987	0.82	*164.94*	*164.94*	*164.94*
0.41	0.5986	0.8130	0.6850	0.66	*132.64*	*132.64*	*132.64*
0.42	0.6267	0.8513	0.6701	0.50	*100.77*	*100.77*	*100.77*
0.43	0.6576	0.8932	0.6539	0.34	*68.98*	*68.98*	*68.98*
0.44	0.6921	0.9401	0.6357	0.18	*36.77*	*36.77*	*36.77*
0.45	0.7321	0.9943	0.6147	0.02	*3.30*	*3.30*	*3.30*

Table 13. Design table for Concrete C70/85.

Concrete C70/85 - $\mu_{sd,max}$=0.4219							
μ_{sd}	ω	$\xi=x/d$	$\zeta=z/d$	ε_s (‰)	σ_s (B400) $\mu_{sd,lim}$=0.3155 ω_{lim}=0.4079	σ_s (B500) $\mu_{sd,lim}$=0.2946 ω_{lim}=0.3712	σ_s (B600) $\mu_{sd,lim}$=0.2761 ω_{lim}=0.3405
0.01	0.0101	0.0149	0.9944	175.62			
0.02	0.0202	0.0300	0.9888	85.98			
0.03	0.0305	0.0452	0.9830	56.09			
0.04	0.0409	0.0606	0.9773	41.14			
0.05	0.0515	0.0763	0.9714	32.17			
0.06	0.0621	0.0921	0.9655	26.19			
0.07	0.0730	0.1081	0.9595	21.92			
0.08	0.0839	0.1243	0.9534	18.71			
0.09	0.0950	0.1408	0.9472	16.21			
0.10	0.1063	0.1574	0.9410	14.21			
0.11	0.1177	0.1744	0.9346	12.58			
0.12	0.1293	0.1915	0.9282	11.21			
0.13	0.1411	0.2090	0.9216	10.05			
0.14	0.1530	0.2267	0.9150	9.06			521.74
0.15	0.1652	0.2447	0.9082	8.20		434.78	
0.16	0.1775	0.2630	0.9014	7.44	347.83		
0.17	0.1901	0.2816	0.8944	6.78			
0.18	0.2029	0.3005	0.8873	6.18			
0.19	0.2159	0.3198	0.8801	5.65			
0.20	0.2292	0.3395	0.8727	5.17			
0.21	0.2427	0.3596	0.8651	4.73			
0.22	0.2566	0.3801	0.8575	4.33			
0.23	0.2707	0.4011	0.8496	3.97			
0.24	0.2852	0.4225	0.8416	3.63			
0.25	0.3000	0.4444	0.8333	3.32			
0.26	0.3152	0.4670	0.8249	3.03			
0.27	0.3308	0.4901	0.8162	2.76			
0.28	0.3468	0.5138	0.8073	2.51			*502.63*
0.29	0.3633	0.5383	0.7981	2.28			*455.63*
0.30	0.3804	0.5635	0.7887	2.06		*411.42*	*411.42*
0.31	0.3980	0.5896	0.7789	1.85		*369.70*	*369.70*
0.32	0.4163	0.6167	0.7687	1.65	*330.18*	*330.18*	*330.18*
0.33	0.4352	0.6448	0.7582	1.46	*292.62*	*292.62*	*292.62*
0.34	0.4550	0.6741	0.7472	1.28	*256.80*	*256.80*	*256.80*
0.35	0.4757	0.7048	0.7357	1.11	*222.50*	*222.50*	*222.50*
0.36	0.4975	0.7370	0.7236	0.95	*189.51*	*189.51*	*189.51*
0.37	0.5205	0.7712	0.7108	0.79	*157.64*	*157.64*	*157.64*
0.38	0.5450	0.8075	0.6972	0.63	*126.67*	*126.67*	*126.67*
0.39	0.5714	0.8465	0.6826	0.48	*96.35*	*96.35*	*96.35*
0.40	0.6000	0.8889	0.6667	0.33	*66.40*	*66.40*	*66.40*
0.41	0.6317	0.9358	0.6491	0.18	*36.44*	*36.44*	*36.44*
0.42	0.6676	0.9891	0.6291	0.03	*5.87*	*5.87*	*5.87*

Table 14. Design table for Concrete C80/95.

Concrete C80/95 - $\mu_{sd,max}$=0.3929							
μ_{sd}	ω	$\xi=x/d$	$\zeta=z/d$	ε_s (‰)	σ_s (B400) $\mu_{sd,lim}$=0.2892 ω_{lim}=0.3695	σ_s (B500) $\mu_{sd,lim}$=0.2695 ω_{lim}=0.3358	σ_s (B600) $\mu_{sd,lim}$=0.2521 ω_{lim}=0.3078
0.01	0.0101	0.0163	0.9941	156.89			
0.02	0.0202	0.0328	0.9881	76.66			
0.03	0.0305	0.0496	0.9820	49.92			
0.04	0.0410	0.0665	0.9759	36.54			
0.05	0.0516	0.0837	0.9697	28.51			
0.06	0.0623	0.1011	0.9634	23.16			
0.07	0.0731	0.1187	0.9570	19.33			
0.08	0.0842	0.1366	0.9505	16.46			
0.09	0.0953	0.1547	0.9439	14.22			
0.10	0.1067	0.1731	0.9372	12.43			
0.11	0.1182	0.1918	0.9305	10.97			
0.12	0.1299	0.2108	0.9236	9.74			
0.13	0.1418	0.2302	0.9166	8.71		434.78	521.74
0.14	0.1539	0.2498	0.9094	7.82			
0.15	0.1663	0.2698	0.9022	7.05	347.83		
0.16	0.1788	0.2902	0.8948	6.37			
0.17	0.1916	0.3109	0.8873	5.77			
0.18	0.2046	0.3321	0.8796	5.24			
0.19	0.2179	0.3537	0.8718	4.76			
0.20	0.2315	0.3757	0.8638	4.33			
0.21	0.2454	0.3983	0.8556	3.93			
0.22	0.2597	0.4214	0.8473	3.58			
0.23	0.2742	0.4450	0.8387	3.25			
0.24	0.2892	0.4693	0.8299	2.94			
0.25	0.3046	0.4942	0.8208	2.66			
0.26	0.3204	0.5199	0.8115	2.40			*480.87*
0.27	0.3367	0.5463	0.8020	2.16		*432.38*	*432.38*
0.28	0.3535	0.5737	0.7921	1.93		*386.99*	*386.99*
0.29	0.3709	0.6019	0.7818	1.72	*344.35*	*344.35*	*344.35*
0.30	0.3890	0.6313	0.7712	1.52	*304.14*	*304.14*	*304.14*
0.31	0.4078	0.6618	0.7601	1.33	*266.07*	*266.07*	*266.07*
0.32	0.4275	0.6937	0.7485	1.15	*229.89*	*229.89*	*229.89*
0.33	0.4481	0.7272	0.7364	0.98	*195.35*	*195.35*	*195.35*
0.34	0.4699	0.7625	0.7236	0.81	*162.22*	*162.22*	*162.22*
0.35	0.4929	0.7999	0.7100	0.65	*130.27*	*130.27*	*130.27*
0.36	0.5176	0.8399	0.6955	0.50	*99.26*	*99.26*	*99.26*
0.37	0.5442	0.8831	0.6799	0.34	*68.91*	*68.91*	*68.91*
0.38	0.5734	0.9305	0.6627	0.19	*38.90*	*38.90*	*38.90*
0.39	0.6061	0.9835	0.6435	0.04	*8.74*	*8.74*	*8.74*

Table 15. Design table for Concrete C90/105.

<table>
<tr><th colspan="8">Concrete C90/105 - $\mu_{sd,max}$=0.3640</th></tr>
<tr><th>μ_{sd}</th><th>ω</th><th>$\xi=x/d$</th><th>$\zeta=z/d$</th><th>ε_s (‰)</th><th>σ_s (B400)
$\mu_{sd,lim}$=0.2652
ω_{lim}=0.3356</th><th>σ_s (B500)
$\mu_{sd,lim}$=0.2469
ω_{lim}=0.3050</th><th>σ_s (B600)
$\mu_{sd,lim}$=0.2307
ω_{lim}=0.2795</th></tr>
<tr><td>0.01</td><td>0.0101</td><td>0.0180</td><td>0.9937</td><td>142.08</td><td rowspan="26">347.83</td><td rowspan="24">434.78</td><td rowspan="23">521.74</td></tr>
<tr><td>0.02</td><td>0.0203</td><td>0.0362</td><td>0.9873</td><td>69.28</td></tr>
<tr><td>0.03</td><td>0.0306</td><td>0.0546</td><td>0.9809</td><td>45.01</td></tr>
<tr><td>0.04</td><td>0.0411</td><td>0.0733</td><td>0.9743</td><td>32.87</td></tr>
<tr><td>0.05</td><td>0.0517</td><td>0.0923</td><td>0.9677</td><td>25.58</td></tr>
<tr><td>0.06</td><td>0.0624</td><td>0.1115</td><td>0.9610</td><td>20.72</td></tr>
<tr><td>0.07</td><td>0.0734</td><td>0.1310</td><td>0.9541</td><td>17.25</td></tr>
<tr><td>0.08</td><td>0.0845</td><td>0.1508</td><td>0.9472</td><td>14.64</td></tr>
<tr><td>0.09</td><td>0.0957</td><td>0.1709</td><td>0.9402</td><td>12.61</td></tr>
<tr><td>0.10</td><td>0.1072</td><td>0.1914</td><td>0.9330</td><td>10.98</td></tr>
<tr><td>0.11</td><td>0.1188</td><td>0.2122</td><td>0.9257</td><td>9.65</td></tr>
<tr><td>0.12</td><td>0.1307</td><td>0.2333</td><td>0.9183</td><td>8.54</td></tr>
<tr><td>0.13</td><td>0.1427</td><td>0.2549</td><td>0.9108</td><td>7.60</td></tr>
<tr><td>0.14</td><td>0.1550</td><td>0.2768</td><td>0.9031</td><td>6.79</td></tr>
<tr><td>0.15</td><td>0.1675</td><td>0.2992</td><td>0.8953</td><td>6.09</td></tr>
<tr><td>0.16</td><td>0.1803</td><td>0.3220</td><td>0.8873</td><td>5.47</td></tr>
<tr><td>0.17</td><td>0.1934</td><td>0.3453</td><td>0.8791</td><td>4.93</td></tr>
<tr><td>0.18</td><td>0.2067</td><td>0.3691</td><td>0.8708</td><td>4.44</td></tr>
<tr><td>0.19</td><td>0.2203</td><td>0.3935</td><td>0.8623</td><td>4.01</td></tr>
<tr><td>0.20</td><td>0.2343</td><td>0.4184</td><td>0.8536</td><td>3.61</td></tr>
<tr><td>0.21</td><td>0.2486</td><td>0.4440</td><td>0.8446</td><td>3.26</td></tr>
<tr><td>0.22</td><td>0.2633</td><td>0.4703</td><td>0.8354</td><td>2.93</td></tr>
<tr><td>0.23</td><td>0.2785</td><td>0.4973</td><td>0.8260</td><td>2.63</td></tr>
<tr><td>0.24</td><td>0.2940</td><td>0.5251</td><td>0.8162</td><td>2.35</td><td>470.36</td></tr>
<tr><td>0.25</td><td>0.3101</td><td>0.5538</td><td>0.8062</td><td>2.10</td><td>419.05</td><td>419.05</td></tr>
<tr><td>0.26</td><td>0.3267</td><td>0.5834</td><td>0.7958</td><td>1.86</td><td>371.30</td><td>371.30</td></tr>
<tr><td>0.27</td><td>0.3439</td><td>0.6142</td><td>0.7850</td><td>1.63</td><td>326.68</td><td>326.68</td><td>326.68</td></tr>
<tr><td>0.28</td><td>0.3618</td><td>0.6461</td><td>0.7739</td><td>1.42</td><td>284.82</td><td>284.82</td><td>284.82</td></tr>
<tr><td>0.29</td><td>0.3805</td><td>0.6794</td><td>0.7622</td><td>1.23</td><td>245.36</td><td>245.36</td><td>245.36</td></tr>
<tr><td>0.30</td><td>0.4000</td><td>0.7143</td><td>0.7500</td><td>1.04</td><td>208.00</td><td>208.00</td><td>208.00</td></tr>
<tr><td>0.31</td><td>0.4205</td><td>0.7509</td><td>0.7372</td><td>0.86</td><td>172.46</td><td>172.46</td><td>172.46</td></tr>
<tr><td>0.32</td><td>0.4422</td><td>0.7897</td><td>0.7236</td><td>0.69</td><td>138.48</td><td>138.48</td><td>138.48</td></tr>
<tr><td>0.33</td><td>0.4653</td><td>0.8310</td><td>0.7092</td><td>0.53</td><td>105.78</td><td>105.78</td><td>105.78</td></tr>
<tr><td>0.34</td><td>0.4902</td><td>0.8753</td><td>0.6936</td><td>0.37</td><td>74.09</td><td>74.09</td><td>74.09</td></tr>
<tr><td>0.35</td><td>0.5172</td><td>0.9235</td><td>0.6768</td><td>0.22</td><td>43.08</td><td>43.08</td><td>43.08</td></tr>
<tr><td>0.36</td><td>0.5470</td><td>0.9768</td><td>0.6581</td><td>0.06</td><td>12.34</td><td>12.34</td><td>12.34</td></tr>
</table>