BOUNDED VARIATION CONTROL OF ITÔ DIFFUSIONS WITH EXOGENOUSLY RESTRICTED INTERVENTION TIMES

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Abstract. In this paper, bounded variation control of one-dimensional diffusion processes is considered. We assume that the agent is allowed to control the diffusion only at the jump times of an observable, independent Poisson process. The agents objective is to maximize the expected present value of the cumulative payoff generated be the controlled diffusion over its lifetime. We propose a relatively weak set of assumptions on the underlying diffusion and the instantaneous payoff structure under which we solve the problem in closed form. Moreover, we illustrate the main results with an explicit example.

1. Introduction

Consider an agent controlling a one-dimensional but otherwise general diffusion process X which evolves on $\mathbf{R}_+$. Assume that the agent is observing also an independent Poisson process N. The process N imposes an exogenous restriction on the controllability of the diffusion X as follows: at the jump times of N, and at that times *only*, the agent can invoke an instantaneous impulse control on X. Whenever the control is used, the agent gets a payoff which is directly proportional to the magnitude of control. If the state variable is taken to the origin, it is killed and no further payoff will accrue. On the other hand, as long as the underlying process is not killed, it yields cumulative instantaneous payoff with a known, possibly state dependent, rate. Given this setting, the agents objective is to maximize the expected present value of the total payoff generated by the controlled underlying process over its lifetime.

During the last few decades or so, optimal control problems of this form and their applications to economics and finance have attracted a lot of attention. In the classical setting the process N is absent and the agents decisions are based solely on the information on the underlying X. If, in addition, controlling is costless, then the optimal control is typically the local time of X at the optimal exercise boundary, see, e.g., [10], [11], [1], and [23]. While relatively nice to analyze, this setting is highly stylized. One way to make this model more realistic for economic applications is to add a (constant) transaction cost. In this case the optimal control is usually a sequential

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impulse control, see, e.g., [12], [18], [17], [16], and [3]. This is a natural result since the cost of a local time control would be infinite. As we already described, the objective of this paper is to add a different type of "friction" to the model, namely that the controlling is allowed only at the jump times of an independent observable Poisson process N. In contrast to the local time, the control process can no longer follow the underlying X continuously – in some sense, the process N imposes now an exogenous constraint on the controllability of X. From applications point of view, this can correspond to an imperfect flow of information, namely that the relevant information does not accumulate as a continuous flow but rather as information packets with random arrivals. As an example, consider ongoing optimal rotation of a forest stand, see, e.g., [24]. The cumulative instantaneous payoff measures now the amenity value of the forest stand, see [2]. Intuitively, the amenity value consist of the value of the forest stand which is additional to the direct value of the wood as raw material – for example, recreational value. Now, the agents harvesting decisions are based on the information set generated by the process N, which is a subset of full information. This represents the phenomenon that the agent cannot access/monitor all information relevant to the harvesting decision. In fact, the process N gives rise to a simple and, as we will see, tractable way of modeling imperfect information in irreversible decision making. In related studies, a similar type of friction has been used, for example, as a simple model for liquidity effect in the classical investment/consumption optimization problem of Merton, see [21]. In [21], the underlying stock is available for trade and, consequently, the portfolio can be rebalanced only at the jump times of N. The framework of [21] is elaborated further in [15]. For other related papers applying optimal control, see [22] and [20]. In [22], the author studies a classical optimal tracking problem for Brownian noise with quadratic running cost under the assumption that the state variable can be controlled only at the jump times of N. In [20], the authors study utility maximization when the stock price can be observed and traded only at the jump times of N corresponding to the quotes on the market. Related studies on optimal stopping of diffusions can be found in [6], [13], and [9]. In [6], the authors consider a perpetual American call with underlying geometric Brownian motion when the process can be stopped only at the jump times of N. The results of [6] were generalized for a more general diffusion and payoff structure in [13]. Finally, [9] considers optimal stopping of a geometric Brownian motion at its maximum on the jump times of N.

The reminder of the paper is organized as follows. In Section 2 we formalize the stochastic control problem. In Section 3 we derive the closed-form solution for the control problem. In

Section 4 we illustrate the main results with an explicit example and Section 5 concludes the study.

2. The Control Problem

2.1. **The Underlying Dynamics.** Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, be a complete filtered probability space satisfying the usual conditions, see [4], p. 2. We assume that the uncontrolled state process X is defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, evolves on $\mathbf{R}_+$, and follows the regular linear diffusion given as the strongly unique solution of the Itô equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,\ X_0 = x,$$

where the functions μ and $\sigma > 0$ are sufficiently well behaving, cf., [4], p. 45. Here, W is a Wiener process. We denote as $\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$ the second-order linear differential operator associated to X. For a given $r > 0$, we denote as, respectively, $\psi_r > 0$ and $\varphi_r > 0$ the increasing and the decreasing solution of the ordinary second-order linear differential equation $(\mathcal{A} - r)f = 0$ defined on the domain of the characteristic operator of X. Given the boundary classification of X (we pose these assumptions later) and the resulting appropriate boundary conditions, the functions ψ_r and φ_r are determined uniquely by this equation up to a multiplicative constant and they can be identified as the minimal r-excessive functions – for the boundary conditions and further properties of ψ_r and φ_r, see [4], pp. 18–20. Moreover, we define the scale density S' and speed density m' via the formulæ $S'(x) = \exp\left(-\int^x \frac{2\mu(y)}{\sigma^2(y)}dy\right)$ and $m'(x) = \frac{2}{\sigma^2(x)S'(x)}$ for all $x \in \mathbf{R}_+$, see [4], p. 17. Finally, we assume that the filtration $\mathbb{F}$ is rich enough to carry a Poisson process $N = (N_t, \mathcal{F}_t)_{t\geq 0}$ with intensity λ. We call the process N the signal process and assume that X and N are independent.

For $r > 0$, we denote by $\mathcal{L}_1^r$ the class of real valued measurable functions f on $\mathbf{R}_+$ satisfying the integrability condition $\mathbf{E}_x\left[\int_0^{\tau_0} e^{-rt}\left|f(X_t)\right|dt\right] < \infty$, where $\tau_0 = \inf\{t \geq 0 : X_t \notin \mathbf{R}_+\}$. For an arbitrary $f \in \mathcal{L}_1^r$, we define the *resolvent* $R_rf : \mathbf{R}_+ \to \mathbf{R}$ as

$$(R_rf)(x) = \mathbf{E}_x\left[\int_0^{\tau_0} e^{-rs}f(X_s)ds\right],$$

for all $x \in \mathbf{R}_+$. The resolvent R_r and the solutions ψ_r and φ_r are connected in a useful way. Indeed, we know that for a given $f \in \mathcal{L}_1^r$, the resolvent R_rf can be expressed as

$$(R_rf)(x) = B_r^{-1}\varphi_r(x)\int_0^x \psi_r(y)f(y)m'(y)dy + B_r^{-1}\psi_r(x)\int_x^\infty \varphi_r(y)f(y)m'(y)dy, \tag{1}$$

for all $x \in \mathbf{R}_+$, where $B_r = \frac{\psi_r'(x)}{S'(x)}\varphi_r(x) - \frac{\varphi_r'(x)}{S'(x)}\psi_r(x)$ denotes the Wronskian determinant, see [4], pp. 19. We remark that the value of B_r does not depend on the state variable x but depends on the rate r. In addition, we know that the family $(R_r)_{r>0}$ is a semigroup which satisfies the *resolvent equation*

$$R_q - R_r + (q-r)R_qR_r = 0, \tag{2}$$

where $q > r > 0$, cf. [4], p. 4.

2.2. **The Control Problem.** Having the uncontrolled underlying dynamics set up, we formulate now the main stochastic control problem. We study a maximization problem of the expected present value of the total cumulative payoff when the agent is allowed to control the underlying X only at the jump times of the signal process N. Formally speaking, the class of admissible controls $\mathcal{Z}$ consists of the non-decreasing processes ζ which admit the representation

$$\zeta_t = \int_0^{t-} \eta_s dN_s,$$

where N is the signal process and the integrand η is $\mathbb{F}$-predictable. The controlled dynamics X^ζ are given by the Itô integral

$$X_t^\zeta = x + \int_0^t \mu(X_s^\zeta)ds + \int_0^t \sigma(X_s^\zeta)dW_s - \zeta_t,\ 0 \le t \le \tau_0^\zeta, \tag{3}$$

where $\tau_0^\zeta = \inf\{t \ge 0 : X_t^\zeta \notin \mathbf{R}_+\}$.

Define the expected present value of the total cumulative payoff as

$$J(x,\zeta) := \mathbf{E}_x\left[\int_0^{\tau_0^\zeta} e^{-rt}\left(\pi(X_t^\zeta)dt + \gamma d\zeta_t\right)\right], \tag{4}$$

where r and γ are exogenously given positive constants. Here, $\pi : \mathbf{R}_+ \to \mathbf{R}$ is the function measuring the instantaneous payoff from continuing the process. The optimal control problem is to find the optimal value function

$$V(x) = \sup_{\zeta \in \mathcal{Z}} J(x,\zeta), \tag{5}$$

and the optimal control ζ^* satisfying $V(x) = J(x,\zeta^*)$ for all $x \in \mathbf{R}_+$.

To set up the framework under which we study Problem (5), define the function $\theta : \mathbf{R}_+ \to \mathbf{R}$ as

$$\theta(x) = \pi(x) + \gamma(\mu(x) - rx). \tag{6}$$

In the economic literature, the function θ is known in as the net convenience yield from holding inventories, cf. [5]. Furthermore, define the auxiliary function $\pi_\gamma : \mathbf{R}_+ \to \mathbf{R}$ as

$$\pi_\gamma(x) = \pi(x) + \lambda\gamma x. \tag{7}$$

Assumption 2.1. *Assume that*

(i) *the upper boundary ∞ is natural and that the lower boundary 0 is either natural, exit or regular for the uncontrolled diffusion X. If the origin is regular, we assume that it is killing,*
(ii) *the functions θ and $\mathrm{id} : x \mapsto x$ are in $\mathcal{L}_1^r$,*
(iii) *the instantaneous payoff π is continuous, non-negative and non-decreasing.*

We make some remarks on Assumption 2.1. First, we assume that the uncontrolled state variable X cannot become infinitely large in finite time and, therefore, that the process can be killed only at 0 – see [4], pp. 18–20, for a characterization of the boundary behavior of diffusions. From the economical point of view, the $\mathcal{L}_1$-condition is natural stating that the expected present value of the total convenience yield must be finite.

We observe that by Assumption 2.1, the function π_γ is non-negative, continuous and in $\mathcal{L}_1^r$. Furthermore, it is linked to the function θ in a convenient way.

Lemma 2.2. *Let Assumption 2.1 hold. Then $(R_{r+\lambda}\pi_\gamma)(x) - \gamma x = (R_{r+\lambda}\theta)(x)$ for all $x \in \mathbf{R}_+$, where the functions π_γ and θ are defined in (7) and (6), respectively.*

Proof. Define the sequence $n \mapsto S_n$ of first exit times as $S_n := \inf\{t \geq 0 : X_t \notin (n^{-1}, n)\}$ for $n \geq 1$. Applying Dynkin's formula to the identity function $\mathrm{id} : x \mapsto x$ yields

$$\mathbf{E}_x\left[e^{-(r+\lambda)S_n} X_{S_n}\right] = x + \mathbf{E}_x\left[\int_0^{S_n} e^{-(r+\lambda)s}(\mu(X_s) - (r+\lambda)X_s)ds\right],$$

for all $x \in \mathbf{R}_+$. Letting $n \to \infty$, we find by bounded convergence that $x - \lambda(R_{r+\lambda}\,\mathrm{id})(x) = R_{r+\lambda}(\mu - r \cdot \mathrm{id})(x)$ for all $x \in \mathbf{R}_+$. Given this identity, we readily verify that

$$\begin{aligned}(R_{r+\lambda}\pi_\gamma)(x) - \gamma x &= (R_{r+\lambda}\pi)(x) - \gamma(x - \lambda R_{r+\lambda}\,\mathrm{id})(x)\\ &= (R_{r+\lambda}\pi)(x) + \gamma R_{r+\lambda}(\mu - r \cdot \mathrm{id})(x)\\ &= (R_{r+\lambda}\theta)(x),\end{aligned}$$

for all $x \in \mathbf{R}_+$. □

It is also worth pointing out that under Assumption 2.1 the function $\psi_r \in \mathcal{L}_1^{r+\lambda}$ for all $r, \lambda > 0$. Indeed, Lemma 2.1 in [13] yields

$$\mathbf{E}_x\left[\int_0^{\tau_0} e^{-(r+\lambda)t}|\psi_r(X_t)|dt\right] = (R_{r+\lambda}\psi_r)(x) = \lambda^{-1}\psi_r(x) < \infty, \tag{8}$$

for all $x \in \mathbf{R}_+$.

We begin the analysis of Problem (5) by first solving a special case. The following proposition is an analogue of Lemma 2 in [1].

Proposition 2.3. *Let $\theta(x) \leq 0$ for all $x \in \mathbf{R}_+$ and Assumption 2.1 hold. Then the optimal control is to take the state variable X to the origin at the first jump time T_1, i.e, to set*

$$\zeta_t^* = \begin{cases} 0, & t < T_1, \\ X_{T_1-}, & t \geq T_1. \end{cases}$$

In this case, the value V reads as

$$V(x) = \mathbf{E}_x\left[\int_0^{T_1} e^{-rs}\pi(X_s)ds + \gamma e^{-rT_1}X_{T_1-}\right] = (R_{r+\lambda}\pi_\gamma)(x),$$

for all $x \in \mathbf{R}_+$.

Proof. Let $x \in \mathbf{R}_+$. Define the family of (almost surely finite) stopping times $\{\tau(\rho)\}_{\rho>0}$ as $\tau(\rho) := \tau_0^\zeta \wedge \rho \wedge \tau_\rho^\zeta$, where $\tau_\rho^\zeta = \inf\{t \geq 0 : X_t^\zeta \geq \rho\}$. Since $(\mathcal{A} - r)(R_{r+\lambda}\pi_\gamma)(x) = \lambda(R_{r+\lambda}\pi_\gamma)(x) - \pi_\gamma(x)$, we find by applying the change of variables formula for general semimartingales, cf., e.g., [8], p.

138, to the process $(t,x) \mapsto e^{-rt}(R_{r+\lambda}\pi_\gamma)(X_t^\zeta)$ that

$$
\begin{aligned}
e^{-r\tau(\rho)}(R_{r+\lambda}\pi_\gamma)(X^\zeta_{\tau(\rho)}) &= (R_{r+\lambda}\pi_\gamma)(x) \\
&+ \mathbf{E}_x\left[\int_0^{\tau(\rho)} e^{-rs}(\lambda(R_{r+\lambda}\pi_\gamma)(X_s^\zeta) - \pi_\gamma(X_s^\zeta))ds\right] \\
&+ \mathbf{E}_x\left[\sum_{T_i \le \tau(\rho)} e^{-rT_i}((R_{r+\lambda}\pi_\gamma)(X^\zeta_{T_i}) - (R_{r+\lambda}\pi_\gamma)(X^\zeta_{T_i-}))\right].
\end{aligned} \tag{9}
$$

To rewrite the right hand side of (9), we observe first that

$$
\begin{aligned}
&\mathbf{E}_x\left[e^{-rT_i}\left((R_{r+\lambda}\pi_\gamma)(X^\zeta_{T_i}) - (R_{r+\lambda}\pi_\gamma)(X^\zeta_{T_i-})\right)\right] = \\
&\mathbf{E}_x\left[e^{-rT_{i-1}}\mathbf{E}_{X_{T_{i-1}}}\left[\int_{T_i-}^{T_i} e^{-rs}\pi(X_s^\zeta)ds + \gamma e^{-r(T_i - T_{i-1})}\Delta\zeta_{T_i}\right]\right],
\end{aligned}
$$

for all $i \ge 1$. Using this and Lemma 2.2, we find after reshuffling the terms of (9) that

$$
\begin{aligned}
&\mathbf{E}_x\left[\int_0^{\tau(\rho)} e^{-rs}(\lambda(R_{r+\lambda}\pi_\gamma)(X_s^\zeta) - \pi_\gamma(X_s^\zeta))ds\right] + \\
&\mathbf{E}_x\left[\sum_{T_i \le \tau(\rho)} e^{-rT_i}((R_{r+\lambda}\pi_\gamma)(X^\zeta_{T_i}) - (R_{r+\lambda}\pi_\gamma)(X^\zeta_{T_i-}))\right] = \\
&\mathbf{E}_x\left[\int_0^{\tau(\rho)} e^{-rs}\lambda(R_{r+\lambda}\theta)(X_s^\zeta)ds\right] - \mathbf{E}_x\left[\int_0^{\tau(\rho)} e^{-rs}\pi(X_s^\zeta)ds + \sum_{T_i \le \tau(\rho)} e^{-rT_i}\gamma\Delta\zeta_{T_i}\right].
\end{aligned}
$$

Since $\theta(x) \le 0$, we find that $(R_{r+\lambda}\theta)(x) \le 0$. Moreover, since π is non-negative, we find by the definition of π_γ that $(R_{r+\lambda}\pi_\gamma)(x) \ge 0$. These observations yield the inequality

$$
\begin{aligned}
&\mathbf{E}_x\left[\int_0^{\tau(\rho)} e^{-rs}\pi(X_s^\zeta)ds + \sum_{T_i \le \tau(\rho)} e^{-rT_i}\gamma\Delta\zeta_{T_i}\right] = (R_{r+\lambda}\pi_\gamma)(x) - \\
&\mathbf{E}_x\left[e^{-r\tau(\rho)}(R_{r+\lambda}\pi_\gamma)(X^\zeta_{\tau(\rho)})\right] + \mathbf{E}_x\left[\int_0^{\tau(\rho)} e^{-rs}\lambda(R_{r+\lambda}\theta)(X_s^\zeta)ds\right] \le (R_{r+\lambda}\pi_\gamma)(x).
\end{aligned}
$$

Now, by letting $\rho \to \infty$, monotone convergence yields

$$
(R_{r+\lambda}\pi_\gamma)(x) \ge \mathbf{E}_x\left[\int_0^{\tau_0^\zeta} e^{-rs}\pi(X_s^\zeta)ds + \sum_{T_i \le \tau_0^\zeta} e^{-rT_i}\gamma\Delta\zeta_{T_i}\right].
$$

Finally, since the value $(R_{r+\lambda}\pi_\gamma)(x)$ is attainable with the admissible policy described in the claim, the result follows. □

Proposition 2.3 states an intuitively clear result. Indeed, if the net convenience yield θ is non-positive everywhere, there is no incentive to hold an inventory and, therefore, the underlying X should be killed at the first possible occasion, i.e., taken instantaneously to the origin at the time T_1. In this case, the optimal control can be regarded as a threshold stopping rule where the optimal stopping threshold is origin.

The next corollary gives useful bounds for the value function V.

Corollary 2.4. *Let Assumption 2.1 hold. Then the value function V satisfies the inequalities*

$$(R_{r+\lambda}\pi_\gamma)(x) \leq V(x) \leq (R_{r+\lambda}\pi_\gamma)(x) + \frac{\lambda}{r} \sup_{x \in \mathbf{R}_+} (R_{r+\lambda}\theta)(x),$$

for all $x \in \mathbf{R}_+$

Proof. Let $x \in \mathbf{R}_+$. Since

$$\mathbf{E}_x \left[\int_0^{\tau_0^\zeta} e^{-rs} \lambda (R_{r+\lambda}\theta)(X_s^\zeta) ds \right] \leq \frac{\lambda}{r} \sup_{y \in \mathbf{R}_+} (R_{r+\lambda}\theta)(y),$$

the claimed result follows from the proof of Proposition 2.3. □

To close the section, we pose a set of assumptions on the function θ in order to analyze the non-trivial case where θ takes also positive values.

Assumption 2.5. *Assume that*

(i) *there is a unique state $x^* \geq 0$ such that θ is increasing on $(0, x^*)$ and decreasing on (x^*, ∞),*
(ii) *the function θ satisfies the limiting conditions $0 \leq \lim_{x \to 0+} \theta(x) < \infty$ and $\lim_{x \to \infty} \theta(x) < 0$.*

In Assumption 2.5 we restrict our attention to the case where the function θ is a hump-shaped function with a global maximum at x^*. In economic terms, the net convenience yield θ is assumed to be positive for a small reserve x and to become negative for a large value of x. Now, when x is large enough the agent has an incentive to get rid of the excess reserve by using the control to bring the state variable to the region where the yield θ takes more favorable values.

3. The Solution

3.1. **Preliminary Analysis.** Before going into the analysis of Problem (5) under Assumptions 2.1 and 2.5, we prove some auxiliary results. For a given $f \in \mathcal{L}_1^r$, define the function $L_f : \mathbf{R}_+ \to \mathbf{R}$

as

$$L_f(x) = (r+\lambda)\int_x^\infty \varphi_{r+\lambda}(y)f(y)m'(y)dy + \frac{\varphi'_{r+\lambda}(x)}{S'(x)}f(x). \tag{10}$$

The function L_f admits a useful alternative representation given in the next lemma.

Lemma 3.1. *Let $\lambda > 0$ and $f \in C \cap \mathcal{L}_1^{r+\lambda}$. Then the function L_f can be expressed as*

$$L_f(x) = -\frac{m'(x)}{\lambda}\left[\lambda(R_{r+\lambda}f)''(x)\varphi'_{r+\lambda}(x) - \lambda(R_{r+\lambda}f)'(x)\varphi''_{r+\lambda}(x)\right],$$

for all $x \in \mathbf{R}$.

Proof. Let $x \in \mathbf{R}_+$. Using the definition of $B_{r+\lambda}$ and the representation (1), we readily verify that

$$\begin{aligned}-\lambda S'(x)L_f(x) &= \frac{r+\lambda}{B_{r+\lambda}}\left[\lambda\left(\varphi'_{r+\lambda}(x)\psi_{r+\lambda}(x) - \varphi_{r+\lambda}(x)\psi'_{r+\lambda}(x)\right)\int_x^\infty \varphi_{r+\lambda}(y)f(y)m'(y)dy\right] \\ &\quad - \lambda f(x)\varphi'_{r+\lambda}(x) \\ &= (r+\lambda)\left[\lambda(R_{r+\lambda}f)(x)\varphi'_{r+\lambda}(x) - \lambda(R_{r+\lambda}f)'(x)\varphi_{r+\lambda}(x)\right] \\ &\quad - \lambda f(x)\varphi'_{r+\lambda}(x).\end{aligned}$$

Since $\varphi_{r+\lambda}$ is $(r+\lambda)$-harmonic and $(\mathcal{A}-(r+\lambda))(R_{r+\lambda}f)(x) = -f(x)$ (see the proof of Lemma 2.1 in [13]), it is a matter of algebra to show that

$$\begin{aligned}&(r+\lambda)\left[\lambda(R_{r+\lambda}f)(x)\varphi'_{r+\lambda}(x) - \lambda(R_{r+\lambda}f)'(x)\varphi_{r+\lambda}(x)\right] - \lambda f(x)\varphi'_{r+\lambda}(x) = \\ &\frac{1}{2}\sigma^2(x)\left[\lambda(R_{r+\lambda}g)''(x)\varphi'_{r+\lambda}(x) - \lambda(R_{r+\lambda}g)'(x)\varphi''_{r+\lambda}(x)\right]\end{aligned}$$

proving the claim. ☐

The following helpful corollary follows immediately from Lemma 3.1.

Corollary 3.2. *Let $f \in C \cap \mathcal{L}_1^{r+\lambda}$. Furthermore, assume that there exist $\lambda > 0$ and an open $A \subseteq \mathbf{R}_+$ such that $f(x) = \lambda(R_{r+\lambda}f)(x)$ for all $x \in A$. Then*

$$L_f(x) = -\frac{m'(x)}{\lambda}\left[f''(x)\varphi'_{r+\lambda}(x) - f'(x)\varphi''_{r+\lambda}(x)\right],$$

for all $x \in A$.

Define the auxiliary functions $I : \mathbf{R}_+ \to \mathbf{R}$ and $J : \mathbf{R}_+ \to \mathbf{R}$ as

$$I(x) = \frac{(R_r\pi)'(x) - \gamma}{\psi'_r(x)}, \quad J(x) = \frac{(R_{r+\lambda}\pi_\gamma)'(x) - \gamma}{\varphi'_{r+\lambda}(x)}, \tag{11}$$

where π_γ is defined in (7). Next lemma provides us with the needed monotonicity properties of I and J under our standing assumptions.

Lemma 3.3. *Let Assumptions 2.1 and 2.5 hold. Then*

(i) *there exists a unique $\tilde{x} > x^*$ such that $I'(x) \gtreqless 0$ when $x \gtreqless \tilde{x}$,*

(ii) *there exists a unique $\hat{x}_\lambda < x^*$ such that $J'(x) \lesseqgtr 0$ when $x \lesseqgtr \hat{x}_\lambda$.*

Proof. For the proof of the claim on I, see [3], Lemma 3.2. To prove the second claim, we first note that using Lemma 2.2 we can write

$$J'(x) = \frac{d}{dx}\left[\frac{(R_{r+\lambda}\pi_\gamma)'(x) - \gamma}{\varphi'_{r+\lambda}(x)}\right] = \frac{d}{dx}\left[\frac{(R_{r+\lambda}\theta)'(x)}{\varphi'_{r+\lambda}(x)}\right],$$

for all $x \in \mathbf{R}_+$. Consider the expected cumulative present value $(R_{r+\lambda}\theta)(x)$. Using the representation (1), we find that

$$\frac{(R_{r+\lambda}\theta)'(x)}{\varphi'_{r+\lambda}(x)} = B^{-1}_{r+\lambda}\int_0^x \psi_{r+\lambda}(y)\theta(y)m'(y)dy + B^{-1}_{r+\lambda}\frac{\psi'_{r+\lambda}(x)}{\varphi'_{r+\lambda}(x)}\int_x^\infty \varphi_{r+\lambda}(y)\theta(y)m'(y)dy.$$

Since $\varphi''_{r+\lambda}(x)\psi'_{r+\lambda}(x) - \varphi'_{r+\lambda}(x)\psi''_{r+\lambda}(x) = \frac{2(r+\lambda)B_{r+\lambda}S'(x)}{\sigma^2(x)}$, it is a matter of differentiation to show that

$$\frac{d}{dx}\left[\frac{(R_{r+\lambda}\theta)'(x)}{\varphi'_{r+\lambda}(x)}\right] = -\frac{2S'(x)}{\sigma^2(x)\varphi'^2_{r+\lambda}(x)}L_\theta(x),$$

where $x \in \mathbf{R}_+$ and the function L_θ is defined using (10). Now, in order to prove the claimed result on J, it is sufficient to show that there is a unique $\hat{x}_\lambda$ such that $L_\theta(x) \gtreqless 0$ when $x \lesseqgtr \hat{x}_\lambda$.

First, let $z > x > x^*$. Since the function θ is non-increasing on (x^*, ∞), we find that

$$\begin{aligned}
\frac{1}{r+\lambda}(L_\theta(z) - L_\theta(x)) &= -\int_x^z \varphi_{r+\lambda}(y)\theta(y)m'(y)dy + \frac{\theta(z)}{r+\lambda}\frac{\varphi'_{r+\lambda}(z)}{S'(z)} - \frac{\theta(x)}{r+\lambda}\frac{\varphi'_{r+\lambda}(x)}{S'(x)} \\
&> \frac{\theta(x)}{r+\lambda}\left[\frac{\varphi'_{r+\lambda}(x)}{S'(x)} - \frac{\varphi'_{r+\lambda}(z)}{S'(z)}\right] + \frac{\theta(z)}{r+\lambda}\frac{\varphi'_{r+\lambda}(z)}{S'(z)} - \frac{\theta(x)}{r+\lambda}\frac{\varphi'_{r+\lambda}(x)}{S'(x)} \\
&= \frac{\varphi'_{r+\lambda}(z)}{S'(z)}\left[\frac{\theta(z)-\theta(x)}{r+\lambda}\right] \geq 0,
\end{aligned}$$

proving that L_θ is increasing on (x^*, ∞). Similarly, we find that when $z < x < x^*$,

$$\begin{aligned}
\frac{1}{r+\lambda}(L_\theta(x) - L_\theta(z)) &= -\int_z^x \varphi_{r+\lambda}(y)\theta(y)m'(y)dy + \frac{\theta(x)}{r+\lambda}\frac{\varphi'_{r+\lambda}(x)}{S'(x)} - \frac{\theta(z)}{r+\lambda}\frac{\varphi'_{r+\lambda}(z)}{S'(z)} \\
&< \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\left[\frac{\theta(x)-\theta(z)}{r+\lambda}\right] \leq 0,
\end{aligned}$$

proving that L_θ is decreasing on $(0, x^*)$.

Since the boundary ∞ is natural for the underlying X, we find that $\lim_{x\to\infty} L_\theta(x) = 0$ and that

$$\begin{aligned} L_\theta(x) &= (r+\lambda)\int_x^\infty \varphi_{r+\lambda}(y)\theta(y)m'(y)dy + \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\theta(x) \\ &< \theta(x)\left[\frac{\varphi'_{r+\lambda}(x)}{S'(x)} - \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\right] = 0. \end{aligned}$$

for all $x \geq x^*$. On the other hand, mean value theorem implies that for all $x < x^*$, the equality

$$\begin{aligned} L_\theta(x) &= (r+\lambda)\int_x^{x^*} \varphi_{r+\lambda}(y)\theta(y)m'(y)dy + \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\theta(x) \\ &+ (r+\lambda)\int_{x^*}^\infty \varphi_{r+\lambda}(y)\theta(y)m'(y)dy \\ &= \theta(\xi)\left[\frac{\varphi'_{r+\lambda}(x^*)}{S'(x^*)} - \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\right] + \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\theta(x) \\ &+ (r+\lambda)\int_{x^*}^\infty \varphi_{r+\lambda}(y)\theta(y)m'(y)dy \\ &= [\theta(x)-\theta(\xi)]\frac{\varphi'_{r+\lambda}(x)}{S'(x)} + \frac{\varphi'_{r+\lambda}(x^*)}{S'(x^*)}\theta(\xi) + (r+\lambda)\int_{x^*}^\infty \varphi_{r+\lambda}(y)\theta(y)m'(y)dy, \end{aligned}$$

holds for some $\xi \in (x, x^*)$. Since the lower boundary 0 is non-entrance, the function $\frac{\varphi'_{r+\lambda}(x)}{S'(x)} \to -\infty$, and, consequently, $L_\theta(x) \to \infty$ as $x \to 0$. This proves the claimed result on J. □

In order to simplify the subsequent notation, define the auxiliary function $g : \mathbf{R}_+ \to \mathbf{R}$ as

(12) $$g(x) = \gamma x - (R_r\pi)(x).$$

Moreover, recall the definition (10). Using these, define the function $Q : \mathbf{R}_+ \to \mathbf{R}$ as the ratio

$$Q(x) = \frac{L_g(x)}{L_{\psi_r}(x)}.$$

We remark that under our assumptions the function Q is well-defined. This function is the key quantity when determining the optimal control ζ^*. Next lemma provides us with the needed monotonicity properties of Q under our standing assumptions.

Lemma 3.4. *Let Assumptions 2.1 and 2.5 hold. Then there exist a unique* $\hat{x} = \operatorname{argmax}\{Q(x)\} \in (\hat{x}_\lambda, \tilde{x})$ *such that* $Q'(x) \gtreqless 0$ *whenever* $x \lesseqgtr \hat{x}$.

Proof. Let $x \in \mathbf{R}_+$. By standard differentiation, we find that

$$L^2_{\psi_r}(x)Q'(x) = L_{\psi_r}(x)\times$$
$$\left[g'(x)\frac{\varphi'_{r+\lambda}(x)}{S'(x)} + g(x)\frac{\varphi''_{r+\lambda}(x)S'(x) - \varphi'_{r+\lambda}(x)S''(x)}{S'^2(x)} - (r+\lambda)\varphi_{r+\lambda}(x)g(x)m'(x)\right] -$$
$$L_g(x)\times$$
$$\left[\psi'_r(x)\frac{\varphi'_{r+\lambda}(x)}{S'(x)} + \psi_r(x)\frac{\varphi''_{r+\lambda}(x)S'(x) - \varphi'_{r+\lambda}(x)S''(x)}{S'^2(x)} - (r+\lambda)\varphi_{r+\lambda}(x)\psi_r(x)m'(x)\right]$$
$$= L_{\psi_r}(x)\left[g'(x)\frac{\varphi'_{r+\lambda}(x)}{S'(x)} + \mathcal{A}\varphi_{r+\lambda}(x)g(x)m'(x) - (r+\lambda)\varphi_{r+\lambda}(x)g(x)m'(x)\right]$$
$$- L_g(x)\left[\psi'_r(x)\frac{\varphi'_{r+\lambda}(x)}{S'(x)} + \mathcal{A}\varphi_{r+\lambda}(x)\psi_r(x)m'(x) - (r+\lambda)\varphi_{r+\lambda}(x)\psi_r(x)m'(x)\right]$$
$$= \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\left[g'(x)L_{\psi_r}(x) - \psi'_r(x)L_g(x)\right],$$

and, consequently, that

$$Q'(x) \lesseqqgtr 0 \text{ if and only if } g'(x)L_{\psi_r}(x) \gtreqqless \psi'_r(x)L_g(x).$$

Assume that $x > \tilde{x}$. Since $\varphi'_{r+\lambda}(x) < 0$ and $g''(x)\psi'_r(x) < g'(x)\psi''_r(x)$, we find using Corollary 3.2 for the function ψ_r (this is justified by (8)), the resolvent equation (2), and Lemma 3.3 that

$$g'(x)L_{\psi_r}(x) - \psi'_r(x)L_g(x) >$$
$$\frac{m'(x)\psi'_r(x)}{\lambda}\left(\varphi'_{r+\lambda}(x)(\lambda(R_{r+\lambda}g)''(x) - g''(x)) - \varphi''_{r+\lambda}(x)(\lambda(R_{r+\lambda}g)'(x) - g'(x))\right) =$$
$$\frac{m'(x)\psi'_r(x)\varphi'^2_{r+\lambda}(x)}{\lambda}J'(x) > 0.$$

We conclude that the function Q is non-decreasing on $(\tilde{x}, \infty)$. On the other hand, since $g''(x)\psi'_r(x) > g'(x)\psi''_r(x)$ on $(0, x^*)$ and $\hat{x}_\lambda < x^*$, we find using the same argument that

$$g'(x)L_{\psi_r}(x) - \psi'_r(x)L_g(x) < \frac{m'(x)\psi'_r(x)\varphi'^2_{r+\lambda}(x)}{\lambda}J'(x) < 0,$$

and, consequently, that Q is non-decreasing on $(0, \tilde{x}_\lambda)$. By continuity, Q must have a turning point $\hat{x}$ in the interval $(\hat{x}_\lambda, \tilde{x})$. Finally, since $g'(\hat{x})L_{\psi_r}(\hat{x}) = \psi'_r(\hat{x})L_g(\hat{x})$, the uniqueness of $\hat{x}$ follows from Lemma 3.3. □

In Lemma 3.4 we proved that the function $Q : x \mapsto \frac{L_g(x)}{L_{\psi_r}(x)}$ has a unique global maximum $\hat{x}$. We remark the this maximum is characterized by the condition

$$g'(\hat{x})L_{\psi_r}(\hat{x}) = \psi'_r(\hat{x})L_g(\hat{x}). \tag{13}$$

3.2. **Necessary Conditions.** Having the necessary auxiliary results at our disposal, we proceed to the study of Problem (5) under Assumptions 2.1 and 2.5. We start by restricting our attention to a specific subclass of admissible control policies and derive a unique candidate for the optimal value – denote the candidate as F. Given the infinite time horizon, the time homogeneity of the process X, and the constant jump rate of the signal process N, we assume that the optimal value exists and is constituted by the threshold control policy defined as follows: If the state variable X^ζ is above the fixed threshold y^* when the Poisson process N jumps, exert the impulse control to return the state variable to the boundary y^* and restart the evolution. On the other hand, if $X^\zeta_{T_i-} < y^*$ for a given i, do note intervene the evolution of X^ζ. Formally, this can be put as follows: if $X^\zeta_{T_i-} \geq y^*$ for some $i \geq 0$, exert the impulse $\Delta\zeta_{T_i} = X^\zeta_{T_i-} - y^*$ and start the process anew from y^*. Now, for the given threshold y^*, the state space $\mathbf{R}_+$ is partitioned into the *waiting region* $(0, y^*)$ and the *action region* $[y^*, \infty)$. At every jump time T_i, the agent chooses between two alternatives: she either exerts the control or waits.

In the continuation region $(0, y^*)$, the Bellman principle implies that the candidate F should satisfy the balance condition

$$F(x) = \mathbf{E}_x\left[\int_0^U e^{-rs}\pi(X_s)ds + e^{-rU}F(X_U)\right], \tag{14}$$

where U is an independent exponentially distributed random variable with mean λ^{-1}. Since the underlying X is strong Markov, we find that on the waiting region $(0, y^*)$

$$\begin{aligned}
&\mathbf{E}_x\left[\int_0^U e^{-rs}\pi(X_s)ds + e^{-rU}F(X_U)\right] = \\
&\quad (R_r\pi)(x) + \lambda(R_{r+\lambda}F)(x) - \mathbf{E}_x\left[\int_U^\infty e^{-rs}\pi(X_s)ds\right] = \\
&\quad (R_r\pi)(x) + \lambda(R_{r+\lambda}F)(x) - \mathbf{E}_x\left[e^{-rU}(R_r\pi)(X_U)\right] \\
&\quad (R_r\pi)(x) + \lambda(R_{r+\lambda}F)(x) - \lambda(R_{r+\lambda}R_r\pi)(x).
\end{aligned}$$

By coupling this with (14), Lemma 2.1 in [13] implies that the function $x \mapsto F(x) - (R_r\pi)(x)$ coincides with an r-harmonic function on $(0, y^*)$, i.e., the function F satisfies the ODE

$$(\mathcal{A} - r)F(x) + \pi(x) = 0, \tag{15}$$

for all $x < y^*$. Since we are looking for a function that is bounded in the origin, we conclude that $F(x) = (R_r\pi)(x) + c\psi_r(x)$ for all $x < y^*$ for some constant c.

Assume that $x \geq y^*$. Now, the agent will use the impulse control given that the Poisson process N jumps. In an infinitesimal time dt, the process N jumps with probability λdt. In this case, the agent invokes the control which yields the payoff $\gamma(x - y^*) + F(y^*)$. On the other hand, the process N does not jump with probability $1-\lambda dt$. In this case, the added expected present value is $\pi(x)dt + \mathbf{E}_x[e^{-rdt}F(X_{dt})]$. Now, the Bellman principle coupled with a heuristic usage of Dynkin's formula suggests that

$$\begin{aligned} F(x) &= \lambda dt(\gamma(x - y^*) + F(y^*)) + (1 - \lambda dt)(\pi(x)dt + \mathbf{E}_x[e^{-rdt}F(X_{dt})]) \\ &= \lambda dt(\gamma(x - y^*) + F(y^*)) + \pi(x)dt + F(x) + (\mathcal{A} - r)F(x)dt - \lambda F(x)dt, \end{aligned}$$

and, consequently, that the candidate F should satisfy the ODE

$$(\mathcal{A} - (r + \lambda))F(x) = -(\pi(x) + \lambda(\gamma(x - y^*) + F(y^*))), \tag{16}$$

for all $x \geq y^*$. Using the representation (1) and partial integration, it is straightforward to show that a particular solution to (16) can be written as

$$(R_{r+\lambda}\pi_\gamma)(x) + \frac{\lambda}{\lambda + r}(F(y^*) - \gamma y^*)\left[1 + \delta\varphi_{r+\lambda}(x)\right], \tag{17}$$

where the function π_γ is defined in (7) and

$$\delta = \begin{cases} 0, & \text{when } 0 \text{ is natural}, \\ \frac{\lambda\psi'(0)}{B_{r+\lambda}(\lambda+r)S'(0)}(\gamma y^* - F(y^*)), & \text{when } 0 \text{ is exit or regular}. \end{cases}$$

Using Corollary 2.4, we conclude that the candidate F admits the representation

$$F(x) = (R_{r+\lambda}\pi_\gamma)(x) + d\varphi_{r+\lambda}(x) + \frac{\lambda}{\lambda + r}(F(y^*) - \gamma y^*),$$

for all $x \geq y^*$, where d is a constant. By substituting $x = y^*$, solving $F(y^*)$ and plugging it back to the previous expression, an elementary simplification yields

$$F(x) = \lambda(R_{r+\lambda}\pi_\gamma)(x) + d\varphi_{r+\lambda}(x) + \frac{\lambda}{r}(\lambda(R_{r+\lambda}\pi_\gamma)(y^*) + d\varphi_{r+\lambda}(y^*) - \gamma y^*),$$

for all $x \geq y^*$.

The next task is to determine the constants c and d. To this end, we assume *a priori* that the candidate F is twice continuously differentiable over the boundary y^*. Now, we find first that

$$(\mathcal{A}-r)F(x)+\pi(x)=\begin{cases}0, & x<y^*,\\ -\lambda((F(y^*)-\gamma y^*)-(F(x)-\gamma x)), & x\geq y^*.\end{cases} \tag{18}$$

Since the coefficients μ and σ and the payoff π are continuous and $F\in C^2$, we observe that the left hand side of (18) is continuous over the threshold y^*. Thus, we find from the right hand side of (18) that the function $x\mapsto F(x)-\gamma x$ has a turning point in y^* and, consequently, that $F'(y^*)=\gamma$. This allows us to determine the constants c and d. Indeed, a simple computation yields $(R_r\pi)'(y^*)+c\psi_r'(y^*)=\gamma=(R_{r+\lambda}\pi_\gamma)'(y^*)+d\varphi_{r+\lambda}'(y^*)$ and, consequently,

$$c=\frac{\gamma-(R_r\pi)'(y^*)}{\psi_r'(y^*)},\ d=\frac{\gamma-(R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi_{r+\lambda}'(y^*)}.$$

Define the function $F:\mathbf{R}_+\to\mathbf{R}$ as

$$F(x)=\begin{cases}(R_{r+\lambda}\pi_\gamma)(x)+\frac{\gamma-(R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi_{r+\lambda}'(y^*)}\varphi_{r+\lambda}(x)+A(y^*), & x\geq y^*,\\ (R_r\pi)(x)+\frac{\gamma-(R_r\pi)'(y^*)}{\psi_r'(y^*)}\psi_r(x), & x<y^*,\end{cases} \tag{19}$$

where

$$A(y^*)=\frac{\lambda}{r}\left(\frac{\gamma-(R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi_{r+\lambda}'(y^*)}-\frac{\gamma y-(R_{r+\lambda}\pi_\gamma)(y^*)}{\varphi_{r+\lambda}(y^*)}\right)\varphi_{r+\lambda}(y^*).$$

This function is our candidate for the optimal value of Problem (5).

To have a complete description of the candidate solution for Problem (5), we derive a characterizing condition for the threshold y^*. First, since $F\in C^2$, the threshold y^* must satisfy the condition

$$\begin{aligned}(R_{r+\lambda}\pi_\gamma)''(y^*)&+\frac{\gamma-(R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi_{r+\lambda}'(y^*)}\varphi_{r+\lambda}''(y^*)\\ &-(R_r\pi)''(y^*)+\frac{\gamma-(R_r\pi)'(y^*)}{\psi_r'(y^*)}\psi_r''(y^*)=0.\end{aligned} \tag{20}$$

Using the definition (12) and the resolvent equation (2), we find that this condition can expressed as

$$
\begin{aligned}
0 &= \lambda(R_{r+\lambda}g)''(y^*) + \frac{\gamma-(R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi'_{r+\lambda}(y^*)}\varphi''_{r+\lambda}(y^*) - \frac{\gamma-(R_r\pi)'(y^*)}{\psi'_r(y^*)}\psi''_r(y^*) \\
&= \lambda(R_{r+\lambda}g)''(y^*) + \frac{\gamma-(R_r\pi)'(y^*)-\lambda(R_{r+\lambda}g)'(y^*)}{\varphi'_{r+\lambda}(y^*)}\varphi''_{r+\lambda}(y^*) \\
&- \frac{\gamma-(R_r\pi)'(y^*)}{\psi'_r(y^*)}\psi''_r(y^*) \\
&= \frac{\lambda(R_{r+\lambda}g)''(y^*)\varphi'_{r+\lambda}(y^*)-\lambda(R_{r+\lambda}g)'(y^*)\varphi''_{r+\lambda}(y^*)}{\varphi'_{r+\lambda}(y^*)} \\
&- \frac{g'(y^*)}{\psi'_r(y^*)}\left(\frac{\psi''_r(y^*)\varphi'_{r+\lambda}(y^*)-\psi'_r(y^*)\varphi''_{r+\lambda}(y^*)}{\varphi'_{r+\lambda}(y^*)}\right).
\end{aligned}
$$

Finally, by using Lemma 3.1 and Corollary 3.2, we find that the condition (20) can be rewritten as

$$
\begin{aligned}
&g'(y^*)\left((r+\lambda)\int_{y^*}^{\infty}\varphi_{r+\lambda}(y)\psi_r(y)m'(y)dy+\psi_r(y^*)\frac{\varphi'_{r+\lambda}(y^*)}{S'(y^*)}\right) = \\
&\psi'_r(y^*)\left((r+\lambda)\int_{y^*}^{\infty}\varphi_{r+\lambda}(y)g(y)m'(y)dy+g(y^*)\frac{\varphi'_{r+\lambda}(y^*)}{S'(y^*)}\right).
\end{aligned} \tag{21}
$$

We established in Lemma 3.4 that under Assumptions 2.1 and 2.5, there is a unique threshold $\hat{x}$ satisfying the condition (21) – in the following, we identify y^* with $\hat{x}$. This unique threshold gives rise to the twice continuously differentiable function F defined in (19).

To summarize, we collect the findings on the candidate F and the threshold y^* to the next proposition.

Proposition 3.5. *Let Assumptions 2.1 and 2.5 hold. Then the function F defined in* (19)*, where the threshold y^* is characterized uniquely by* (21)*, is the unique solution to the free boundary problem*

$$
\begin{cases}
F \in C^2, \\
(\mathcal{A}-r)F(x)+\pi(x)=0, & x<y^*, \\
(\mathcal{A}-(r+\lambda))F(x) = -(\pi(x)+\lambda(\gamma(x-y^*)+F(y^*))), & x \geq y^*.
\end{cases}
$$

3.3. **Sufficient Conditions.** In Proposition 3.5 we presented our main results on the candidate F and the threshold y^*. To prove that F and y^* give rise to the optimal value and control of Problem (5), we first make some further computations. Let $x<y^*$. Since $y^*<\tilde{x}$, Lemma 3.3 implies that

$$
F'(x)-\gamma = \psi'_r(x)\left[\frac{(R_r\pi)'(x)-\gamma}{\psi'_r(x)} - \frac{(R_r\pi)'(y^*)-\gamma}{\psi'_r(y^*)}\right] > 0.
$$

On the other hand, when $x \geq y^*$, Lemma 3.3 implies that

$$F'(x) - \gamma = \varphi'_{r+\lambda}(x) \left[\frac{(R_{r+\lambda}\pi_\gamma)'(x) - \gamma}{\varphi'_{r+\lambda}(x)} - \frac{(R_{r+\lambda}\pi_\gamma)'(y^*) - \gamma}{\varphi'_{r+\lambda}(y^*)} \right] \leq 0,$$

since $y^* > \hat{x}_\lambda$. Thus, we conclude that under Assumptions 2.1 and 2.5, the function $x \mapsto F(x) - \gamma x$ has a unique global maximum at y^* and, consequently, that F satisfies the variational principle

$$(\mathcal{A} - r)F(x) + \pi(x) + \lambda \left[\sup_{y \leq x} \{(F(y) - \gamma y) - (F(x) - \gamma x)\} \right] = 0, \tag{22}$$

for all $x \in \mathbf{R}_+$. For brevity, denote

$$\begin{aligned} \Phi(x) : &= \sup_{y \leq x} \{(F(y) - \gamma y) - (F(x) - \gamma x)\} \\ &= \{F(y^*) + \gamma(x - y^*) - F(x)\} \mathbf{1}_{[y^*,\infty)}(x), \end{aligned} \tag{23}$$

for all $x \in \mathbf{R}_+$. Using these observations, we prove our main result on Problem (5).

Theorem 3.6. *Let Assumptions 2.1 and 2.5 hold. Then, for all $i \geq 1$, the optimal control policy ζ^* is to take the state variable X^{ζ^*} instantaneously to the state y^* characterized uniquely by* (21) *whenever $X^{\zeta^*}_{T_i-} > y^*$, i.e., the size of the impulse is $\Delta\zeta^*_{T_i} = (X^{\zeta^*}_{T_i-} - y^*)^+$ for all i. Moreover, the value V of the optimal control problem* (5) *reads as*

$$V(x) = F(x) = \begin{cases} (R_{r+\lambda}\pi_\gamma)(x) + \frac{\gamma - (R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi'_{r+\lambda}(y^*)} \varphi_{r+\lambda}(x) + A(y^*), & x \geq y^*, \\ (R_r\pi)(x) + \frac{\gamma - (R_r\pi)'(y^*)}{\psi'_r(y^*)} \psi_r(x), & x < y^*, \end{cases} \tag{24}$$

where

$$A(y^*) = \frac{\lambda}{r} \left(\frac{\gamma - (R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi'_{r+\lambda}(y^*)} - \frac{\gamma y - (R_{r+\lambda}\pi_\gamma)(y^*)}{\varphi_{r+\lambda}(y^*)} \right) \varphi_{r+\lambda}(y^*).$$

Proof. Let $x \in \mathbf{R}_+$. We prove first that $F(x) \geq J(x, \zeta)$ for all $\zeta \in \mathcal{Z}$. Recall the definition of the family $\{\tau(\rho)\}_{\rho>0}$ from the proof of Proposition 2.3. Application of the change of variables formula for general semimartingales, cf. [8], p. 138, to the stopped process $(t, x) \mapsto e^{-r(t\wedge\tau(\rho))} F(X^\zeta_{t\wedge\tau(\rho)})$ yields

$$\begin{aligned} e^{-r(t\wedge\tau(\rho))} F(X^\zeta_{t\wedge\tau(\rho)}) = F(x) &+ \int_0^{t\wedge\tau(\rho)} e^{-rs}(\mathcal{A} - r)F(X^\zeta_s)ds \\ &+ \int_0^{t\wedge\tau(\rho)} e^{-rs}\sigma(X^\zeta_s)F'(X^\zeta_s)dW_s \\ &+ \sum_{s \leq t\wedge\tau(\rho)} e^{-rs}[F(X^\zeta_s) - F(X^\zeta_{s-})], \end{aligned}$$

for all $\rho > 0$. On the other hand, since the control ζ can jump only if the Poisson process N jumps, the expression (22) implies that $F(X_s^\zeta) + \gamma(\Delta\zeta_s) - F(X_{s-}^\zeta) \leq \Phi(X_{s-}^\zeta)$, where the function Φ is defined in (23). Coupling this with (22) yields

$$
\begin{aligned}
e^{-r(t\wedge\tau(\rho))}F(X^\zeta_{t\wedge\tau(\rho)}) + \int_0^{t\wedge\tau(\rho)} e^{-rs}\left(\pi(X_s^\zeta)ds + \gamma d\zeta_s\right) \leq& \\
& F(x) + \int_0^{t\wedge\tau(\rho)} e^{-rs}\sigma(X_s^\zeta)F'(X_s^\zeta)dW_s \\
& - \lambda\int_0^{t\wedge\tau(\rho)} e^{-rs}\Phi(X_{s-}^\zeta)ds + \int_0^{t\wedge\tau(\rho)} e^{-rs}\Phi(X_{s-}^\zeta)dN_s = \\
& F(x) + M_{t\wedge\tau(\rho)} + Z_{t\wedge\tau(\rho)},
\end{aligned}
\tag{25}
$$

where M and Z are local martingales defined as

$$
M_t := \int_0^t e^{-rs}\sigma(X_s^\zeta)F'(X_s^\zeta)dW_s,\ Z_t := \int_0^t e^{-rs}\Phi(X_{s-}^\zeta)d\tilde{N}_s.
$$

Here, $\tilde{N} = (N_t - \lambda t)_{t\geq 0}$ is the compensated Poisson process. Moreover, we observe from the expression (25) that the local martingale part $(M_{t\wedge\tau(\rho)} + Z_{t\wedge\tau(\rho)})$ is bounded uniformly from below by $-F(x)$. Hence $(M_{t\wedge\tau(\rho)} + Z_{t\wedge\tau(\rho)})$ is a supermartingale and, in particular, $\mathbf{E}_x[M_{t\wedge\tau(\rho)} + Z_{t\wedge\tau(\rho)}] \leq 0$ for all $t, \rho > 0$. By taking expectations sidewise in (25), we find that

$$
\mathbf{E}_x\left[e^{-r(t\wedge\tau(\rho))}F(X^\zeta_{t\wedge\tau(\rho)})\right] + \mathbf{E}_x\left[\int_0^{t\wedge\tau(\rho)} e^{-rs}\left(\pi(X_s^\zeta)ds + \gamma d\zeta_s\right)\right] \leq F(x),
$$

for all $t, \rho > 0$. By letting t and ρ tend to infinity, we obtain

$$
F(x) \geq \lim_{t,\rho\to\infty}\mathbf{E}_x\left[e^{-r(t\wedge\tau(\rho))}F(X^\zeta_{t\wedge\tau(\rho)})\right] + J(x,\zeta).
$$

Since F is non-negative, we conclude that $F(x) \geq J(x,\zeta)$.

To show that the value F is attainable with the admissible policy ζ^*, it suffices to show that $J(x,\zeta^*) \geq F(x)$. First, since N jumps only upwards and F is non-negative and non-decreasing, we find using (23) that

$$
\begin{aligned}
Z_{t\wedge\tau(\rho)} &\leq \int_0^{t\wedge\tau(\rho)} e^{-rs}\Phi(X_{s-}^{\zeta^*})dN_s \\
&\leq \gamma\int_0^{t\wedge\tau(\rho)} e^{-rs}(X_{s-}^{\zeta^*} - y^*)\mathbf{1}_{[y^*,\infty)}(X_{s-}^{\zeta^*})dN_s \leq \gamma\int_0^\infty e^{-rs}d\zeta_s^*,
\end{aligned}
$$

for all $t, \rho > 0$. Thus the process Z is bounded uniformly from above by an integrable random variable and, consequently, is a submartingale. On the other hand, since the functions σ and F' are continuous and the stopped process $X^{\zeta^*}_{\cdot\wedge\tau(\rho)}$ is bounded, we find that the integrand of M is also bounded. This implies that the local martingale M is a martingale and, consequently, that $\mathbf{E}_x[M_{t\wedge\tau(\rho)} + Z_{t\wedge\tau(\rho)}] \geq 0$ for all $t, \rho > 0$. We observe that for the control ζ^*, the inequality (25) holds in fact as an equality. Therefore it follows from (25) that

$$\mathbf{E}_x\left[e^{-r(t\wedge\tau(\rho))}F(X^{\zeta^*}_{t\wedge\tau(\rho)})\right] + \mathbf{E}_x\left[\int_0^{t\wedge\tau(\rho)} e^{-rs}\left(\pi(X^{\zeta^*}_s)ds + \gamma d\zeta^*_s\right)\right] \geq F(x),$$

for all $t, \rho > 0$. By letting t and ρ tend to infinity, we find by bounded convergence that

$$F(x) \leq \liminf_{t,\rho\to\infty} \mathbf{E}_x\left[e^{-r(t\wedge\tau(\rho))}F(X^{\zeta^*}_{t\wedge\tau(\rho)})\right] + J(x, \zeta^*).$$

Now, recall that y^* is the global maximum of $x \mapsto F(x) - \gamma x$. Thus

$$0 \leq \mathbf{E}_x\left[e^{-r(t\wedge\tau(\rho))}F(X^{\zeta^*}_{t\wedge\tau(\rho)})\right] \leq \mathbf{E}_x\left[e^{-r(t\wedge\tau(\rho))}(F(y^*) + \gamma(X^{\zeta^*}_{t\wedge\tau(\rho)} - y^*))\right].$$

Since $\mathrm{id} \in \mathcal{L}^r_1$, we conclude that $\liminf_{t,\rho\to\infty} \mathbf{E}_x\left[e^{-r(t\wedge\tau(\rho))}F(X^{\zeta^*}_{t\wedge\tau(\rho)})\right] = 0$ and, consequently, that $V(x) = J(x, \zeta^*)$. □

We proved in Theorem 3.6 that the unique solution to the free boundary problem described in Proposition 3.5 constitutes the optimal solution to Problem (5) under Assumptions 2.1 and 2.5. It is worth pointing out that in Lemma 3.4, we proved that the optimal trigger threshold y^* is dominated by the state $\tilde{x} = \operatorname{argmax}\{I(x)\}$ for all $\lambda > 0$, where I is defined in (11). On the other hand, we know that $\tilde{x}$ coincides with the optimal reflection threshold of the associated bounded variation control problem where the functional (4) is maximized over all $\mathbb{F}$-adapted non-decreasing controls under Assumptions 2.1 and 2.5, see [3], Lemma 3.4. Intuitively, this associated problem should correspond to the limit $\lambda \to \infty$. Indeed, it seems clear that as the rate λ increases the opportunities to control appear on average more frequently in time. Since it is costless to control, the controller should be more inclined to use it. This should result into a higher threshold and on the average smaller but more frequent controls in the neighborhood of this threshold. Unfortunately, a rigorous proof of the property $y^* \to \tilde{x}$ as $\lambda \to \infty$ remains open. In the next section, we consider an example where this property holds. However, Lemma 3.4 shows that the restriction of admissible intervention times to the jump times of the Poisson process N unambiguously lowers the optimal threshold to exercise the control.

Regarding the limit $\lambda \to 0$, we observe from Corollary 3.2 that $L_{\psi_r}(x) \to \infty$ for all $x \in \mathbf{R}_+$ as $\lambda \to 0$. Consequently, the characterization (21) reduces to $g'(y^*) = 0$ on the limit $\lambda \to 0$. This can be rewritten as $(R_r\pi)'(y^*) = \gamma$. By simply plugging this into the expression (24), we obtain the limiting value $V(x) = (R_r\pi)(x)$ for all $x \in \mathbf{R}_+$. This is a natural result. Indeed, the limit $\lambda \to 0$ corresponds to the case when the Poisson process N does not jump and, consequently, there will be no opportunities to control the diffusion X. Hence, the value of the control problem consists only of the expected cumulative present value of the instantaneous payoff, namely the resolvent $(R_r\pi)$.

4. An Illustration

We illustrate some of the main results of the paper with an explicit example. To this end, we assume that the uncontrolled underlying dynamics X follow a geometric Brownian motion given by the Itô equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

where $\mu \in \mathbf{R}$ and $\sigma \in \mathbf{R}_+$ are exogenously given constants. We assume that $\mu - \frac{1}{2}\sigma^2 > 0$. In this case the process $X_t \to \infty$ almost surely as $t \to \infty$, see [19], p. 63. The differential operator associated to X reads as $\mathcal{A} = \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx}$. A straightforward computation yields the scale density $S'(x) = x^{-\frac{2\mu}{\sigma^2}}$ and, consequently, the speed density $m'(x) = \frac{2}{(\sigma x)^2} x^{\frac{2\mu}{\sigma^2}}$ for all $x \in \mathbf{R}_+$. Now, fix the constants $r, \lambda > 0$. It is well known that the minimal excessive functions $\psi_\cdot$ and $\varphi_\cdot$ can now be written as

$$\begin{cases} \psi_r(x) = x^b, \\ \varphi_r(x) = x^a, \end{cases} \quad \begin{cases} \psi_{r+\lambda}(x) = x^\beta, \\ \varphi_{r+\lambda}(x) = x^\alpha, \end{cases}$$

where the constants

$$\begin{cases} b = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \\ a = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0, \end{cases}$$

$$\begin{cases} \beta = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1, \\ \alpha = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} < 0. \end{cases}$$

Furthermore, we find that the Wronskian $B_{r+\lambda} = 2\sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}}$.

To set up the control problem, define

$$\pi(x) = x^\delta, \ 0 < \delta < 1,$$

and fix the constant γ. To check the validity of Assumptions 2.1 and 2.5, we find that the net convenience yield θ reads as $\theta(x) = x^\delta - \gamma(r - \mu)x$. For Assumptions 2.1 and 2.5 to hold, it suffices to assume that $\mu < r$. In particular, we find that

$$x^* = \operatorname{argmax}\{\theta(x)\} = \left(\frac{\delta}{\gamma(r-\mu)}\right)^{\frac{1}{1-\delta}}.$$

Using the representation (1), it is a matter of straightforward integration to show that $(R_r\pi)(x) = \frac{x^\delta}{\iota(\delta)}$, where $\iota(\delta) = r - \delta\mu - \frac{\sigma^2}{2}\delta(\delta-1)$, for all $x \in \mathbf{R}_+$. Using this, we find that

$$\tilde{x} = \left\{\frac{\delta(b-\delta)}{\gamma\iota(\delta)(b-1)}\right\}^{\frac{1}{1-\delta}}.$$

To proceed, recall the definition of the operator L_f from (10) and that $g(x) := \gamma x - (R_r\pi)(x) = \gamma x - \frac{x^\delta}{\iota(\delta)}$. To determine the optimal exercise threshold y^*, we need to find the functions L_g and L_{ψ_r} – see the condition (21). For L_g, we find first that

$$\int_x^\infty \varphi_{r+\lambda}(y)g(y)m'(y)dy = \frac{2}{\sigma^2 x^\beta}\left\{\frac{x^\delta}{\iota(\delta)(\delta-\beta)} - \frac{\gamma x}{1-\beta}\right\},$$

and, consequently, that

$$\begin{aligned} L_g(x) &= \frac{2(r+\lambda)}{\sigma^2 x^\beta}\left\{\frac{x^\delta}{\iota(\delta)(\delta-\beta)} - \frac{\gamma x}{1-\beta}\right\} + \left\{\gamma x - \frac{x^\delta}{\iota(\delta)}\right\}\frac{\alpha x^{\alpha-1}}{x^{-\frac{2\mu}{\sigma^2}}} \\ &= \frac{x^{\delta-\beta}}{\iota(\delta)}\left\{\frac{2(r+\lambda)}{\sigma^2(\delta-\beta)} - \alpha\right\} + \gamma x^{1-\beta}\left\{\alpha - \frac{2(r+\lambda)}{\sigma^2(1-\beta)}\right\}. \end{aligned} \tag{26}$$

For L_{ψ_r}, we verify readily that

$$\int_x^\infty \varphi_{r+\lambda}(y)\psi_r(y)m'(y)dy = \frac{2}{\sigma^2(\beta-b)}x^{b-\beta},$$

and, consequently, that

$$L_{\psi_r}(x) = \left\{\frac{2(r+\lambda)}{\sigma^2(\beta-b)} + \alpha\right\}x^{b-\beta}. \tag{27}$$

By inserting the expressions (26) and (27) into the condition (21), we find after a straightforward simplification that

$$y^* = \left\{\frac{\delta(b-\delta)}{\gamma\iota(\delta)(b-1)}\frac{\beta-1}{\beta-\delta}\right\}^{\frac{1}{1-\delta}}. \tag{28}$$

We observe immediately from the expression (28) that the optimal exercise threshold y^* is dominated by the optimal reflection threshold $\tilde{x}$. Furthermore, in this particular example the threshold y^* converges to $\tilde{x}$ as the intensity λ tends to infinity. On the other hand, we observe from (28) that $y^* \to \left\{\frac{\delta}{\gamma\iota(\delta)}\right\}^{\frac{1}{1-\delta}}$ as $\lambda \to 0$. We verify readily that this state is the unique solution of the condition $(R_r\pi)'(y^*) = \gamma$.

5. Concluding Remarks

In this paper, we studied bounded variation control of one-dimensional diffusions. In particular, we set up a class of control problems where the admissible controls are sequential impulse controls which can be exerted only at the jump times of an independent, observable Poisson process N. We proposed a set of weak assumptions on the underlying diffusion and instantaneous payoff structure under which we derived a closed-form solution to the problem. The main result, which is new to our best knowledge is proved using a combination of results from stochastic calculus and the Markov theory of diffusions. In comparison to the classical singular stochastic control setting, we also showed that the restriction of the admissible intervention times to the jump times of the Poisson process N lowers the optimal trigger threshold in comparison to the classical local time control.

This study has at least two interesting generalizations. First, it would be interesting to make the controlling costly. In this case, it seem reasonable to guess that the resulting exercise threshold and the regeneration point, i.e. the point where the process is started anew after the control, are no longer the same. Secondly, we considered in this paper time homogeneous case. It would be interesting to see if some of the results of this study could generalized to case where, for example, λ and r are given dynamic structures. These questions are left for future research.

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