DEFORMATIONS OF MODULES OF MAXIMAL GRADE AND THE HILBERT SCHEME AT DETERMINANTAL SCHEMES.

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ABSTRACT. Let R be a polynomial ring and M a finitely generated graded R-module of maximal grade (which means that the ideal $I_t(\mathcal{A})$ generated by the maximal minors of a homogeneous presentation matrix, $\mathcal{A}$, of M has maximal codimension in R). Suppose $X := \text{Proj}(R/I_t(\mathcal{A}))$ is smooth in a sufficiently large open subset and $\dim X \geq 1$. Then we prove that the local graded deformation functor of M is isomorphic to the local Hilbert (scheme) functor at $X \subset \text{Proj}(R)$ under a weak assumption which holds if $\dim X \geq 2$. Under this assumption we get that the Hilbert scheme is smooth at (X), and we give an explicit formula for the dimension of its local ring. As a corollary we prove a conjecture of R.M.Miró-Roig and the author that the closure of the locus of standard determinantal schemes with fixed degrees of the entries in a presentation matrix is a generically smooth component V of the Hilbert scheme. Also their conjecture on the dimension of V is proved for $\dim X \geq 1$. The cohomology $H^i_*(\mathcal{N}_X)$ of the normal sheaf of X in $\text{Proj}(R)$ is shown to vanish for $1 \leq i \leq \dim X - 2$. Finally the mentioned results, slightly adapted, remain true replacing R by any Cohen-Macaulay quotient of a polynomial ring.

1. INTRODUCTION

Determinantal objects are central in many areas of mathematics. In algebraic geometry determinantal schemes defined by the vanishing of the $p \times p$-minors of a homogeneous polynomial matrix, may be used to describe classical schemes such as rational normal scrolls and other fibered schemes, Veronese and Segre varieties and Secant schemes to rational normal curves and Segre varieties ([20], [3]). Throughout the years many nice properties are detected for determinantal schemes, e.g. they are arithmetically Cohen-Macaulay with rather well understood free resolutions and singular loci, see [11], [12], [38], [51], and see [6], [5], [13], [16], [31], [37], [41] for history and other important contributions.

In this paper we study the Hilbert scheme along the locus of determinantal schemes. More precisely we study deformations of modules of maximal grade over a polynomial ring R and establish a very strong connection to corresponding deformations of determinantal schemes in $\mathbb{P}^n$. Recall that the grade g of a finitely generated graded R-module M is the grade of its annihilator $I := \text{ann}(M)$, i.e. $g = \text{depth}_I R = \dim R - \dim R/I$. We say a scheme $X \subset \mathbb{P}^n$ of codimension c is *standard determinantal* if its homogeneous saturated ideal is equal to the ideal $I_t(\mathcal{A})$ generated by the $t \times t$ minors of some homogeneous $t \times (t + c - 1)$ matrix $\mathcal{A} = (f_{ij})$, $f_{ij} \in R$. If M is the cokernel of the map determined by $\mathcal{A}$, then $g = c$ because the radicals of I and $I_t(\mathcal{A})$ are equal. Moreover M has maximal grade if and only if $X = \text{Proj}(A), A := R/I_t(\mathcal{A})$ is standard determinantal. In this case $\text{ann}(M) = I_t(\mathcal{A})$ for $c \geq 2$ by [7].

1991 *Mathematics Subject Classification.* Primary 14M12, 14C05, 13D07, 13D10; Secondary 14H10, 14J10, 13D03.

Let $\mathrm{Hilb}^p(\mathbb{P}^n)$ be the Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^n$ of dimension $n-c \geq 0$ and with Hilbert polynomial p. Given integers $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$ and $b_1 \leq ... \leq b_t$, $t \geq 2$, $c \geq 2$, we denote by $W_s(\underline{b};\underline{a}) \subset \mathrm{Hilb}^p(\mathbb{P}^n)$ the stratum of standard determinantal schemes where f_{ij} are homogeneous polynomials of degrees $a_j - b_i$. Inside $W_s(\underline{b};\underline{a})$ we have the open subset $W(\underline{b};\underline{a})$ of determinantal schemes that are generically a complete intersection. The elements are called *good* determinantal schemes. Note that $W_s(\underline{b};\underline{a})$ is irreducible, and $W(\underline{b};\underline{a}) \neq \emptyset$ if we suppose $a_{i-1} - b_i > 0$ for $i \geq 1$, see (2.2).

In this paper we determine the dimension of a non-empty $W(\underline{b};\underline{a})$ provided $a_{i-2} - b_i \geq 0$ for $i \geq 2$ and $n-c \geq 1$ (Theorem 5.5, Corollary 5.6). Indeed

$$\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + K_4 + ... + K_c\,, \tag{1.1}$$

where λ_c and K_i are a large sum of binomials only involving a_j and b_i (see Conjecture 2.2 and (2.10) for the definition of λ_c and K_i). In terms of the Hilbert function, $H_M(-)$, of M, we may alternatively write (1.1) in the form

$$\dim W(\underline{b};\underline{a}) = \sum_{j=0}^{t+c-2} H_M(a_j) - \sum_{i=1}^{t} H_M(b_i) + 1$$

(Remark 3.9). Moreover we prove that the closure $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of the Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^n)$ provided ${}_0\mathrm{Ext}^2_A(M,M)$, the degree zero part of the graded A-module $\mathrm{Ext}^2_A(M,M)$, vanishes for a general $X = \mathrm{Proj}(A)$ of $W(\underline{b};\underline{a})$ (Theorem 5.8, Corollary 5.9). Indeed

$$\dim_{(X)} \mathrm{Hilb}^p(\mathbb{P}^n) - \dim W(\underline{b};\underline{a}) \leq \dim\ {}_0\mathrm{Ext}^2_A(M,M)\,,$$

and $\mathrm{Hilb}^p(\mathbb{P}^n)$ is smooth at (X) if equality holds. We prove that $n-c \geq 2$ implies $\mathrm{Ext}^2_A(M,M) = 0$ (Corollary 4.10), whence $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^n)$ in the case $n-c \geq 2$ and $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$. This proves Conjecture 4.2 of [33]. Moreover our results hold for every $(X) \in W(\underline{b};\underline{a})$ provided a depth condition on the singular locus is fulfilled. A general X of $W(\underline{b};\underline{a})$ satisfies the condition and we get the mentioned results.

The most remarkable finding in this paper is perhaps the method. Indeed an embedded deformation problem for the determinantal scheme $X = \mathrm{Proj}(A)$, $A = R/\mathrm{ann}(M)$ is transfered to a deformation problem for the R-module M where it is handled much more easily because every deformation of M comes from deforming the matrix $\mathcal{A}$. The latter is easy to see from the Buchsbaum-Rim complex. In fact it was in [30] we considered the property "every deformation of X comes from deforming $\mathcal{A}$" for a determinantal *scheme* X, to better understand why $\overline{W(\underline{b};\underline{a})}$ may fail to be an irreducible component (Lemma 2.7, see [30], Ex. 4.1 for schemes not satisfying this property). This leads us in this paper to study deformations of M as a graded R-*module* because the corresponding property holds for M. To compare the graded deformation functor of M with the local Hilbert (scheme) functor of X, we prove that the graded deformation functors of M and $R \to A$ are isomorphic under the assumption ${}_0\mathrm{Ext}^i_A(M,M) = 0$ for $i = 1, 2$ (Theorem 5.2). Note that the graded deformation functor of $R \to A$ is further isomorphic to the local Hilbert functor of X if $n-c \geq 1$. Since we also prove that $n-c \geq i \geq 1$ implies $\mathrm{Ext}^i_A(M,M) = 0$ under mild assumptions (Theorems 4.1 and 4.5), we get our rather algebraic method for studying

a geometric object, the Hilbert scheme. Even the corresponding non-graded deformation functors of M and $R \to A$ are isomorphic for $n - c \geq 2$ (Remark 5.14), which more than indicates that this method holds for local determinantal rings of dimension greater than 2. Hence we expect applications to deformations of determinantal singularities, as well as to multigraded Hilbert schemes. We remark that while the vanishing of $\mathrm{Ext}^i_A(M, M)$ in Theorem 4.1 is mainly known (at least for $i = 1$, see Remark 4.3), the surprise is Theorem 4.5 which reduces the depth assumption of Theorem 4.1 by 1 in important cases. Note that the local deformation functors of M as an A- as well as an R-module were thoroughly studied by R. Ile in [23], [25] and in [24] he studies the case of a determinantal hypersurface X ($\mathcal{A}$ a square matrix) without proving, to our knowledge, the mentioned results (see Remark 4.4). Ile and his paper [24], and the joint papers [31], [32], [33] have, however, served as an inspiration for this work.

We also get further interesting results, e.g. that arbitrary modules of maximal grade are unobstructed (earlier proved by Ile in [23]), and we show that the dimension of their natural deformation spaces is equal to the right hand side of (1.1) (Theorems 3.1 and 3.8, cf. Remark 3.9). Moreover we prove that the cohomology $H^i(\mathcal{N}_X(v))$ of the normal sheaf of $X \subset \mathbb{P}^n$ for a X general in $W(\underline{b}; \underline{a})$ vanishes for $1 \leq i \leq \dim X - 2$ and every v (Theorem 5.11). Even the algebra cohomology groups $\mathrm{H}^i(R, A, A)$ of André-Quillen vanish for $2 \leq i \leq \min\{\dim X - 1, c\}$. This extends a result from T. Svanes' thesis [48] proven there for so-called *generic* determinantal schemes in which the entries of $\mathcal{A}$ are the indeterminates of R, see [6], Thm. 15.10 for details. Finally we remark that the assumption that R is a polynomial ring can be weakened. Indeed all theorems and their proofs generalize at least to the case where $\mathrm{Proj}(R)$ is any arithmetically Cohen-Macaulay k-scheme (and smooth in Theorem 5.11 (ii)), only replacing all $\binom{v+n}{n}$ in (1.1) with $\dim R_v$.

The method of this paper has the power of solving most of the deformation problems the author, together with coauthors (mostly Miró-Roig at Barcelona) has considered in several papers ([31], [32], [33], [30]), mainly:

(1) Determine the dimension of $W(\underline{b}; \underline{a})$ in terms of a_j and b_i (see Conjecture 2.2).
(2) Is $\overline{W(\underline{b}; \underline{a})}$ a generically smooth irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^n)$?

The main method so far has been to delete columns of the matrix $\mathcal{A}$, to get a "flag" of closed subschemes $X = X_c \subset X_{c-1} \subset ... \subset X_2 \subset \mathbb{P}^n$ and to prove the results by considering the smoothness of the Hilbert flag scheme of pairs and its natural projections into the Hilbert schemes. In fact in [32] we solved problem (1) in the cases $2 \leq c \leq 5$ and $n - c \geq 1$ (assuming $char(k) = 0$ if $c = 5$), and recently we almost solved (1) in the remaining cases under the assumption $a_{t+3} > a_{t-2}$ [33]. Concerning problem (2) we gave in [32] an affirmative answer in the range $2 \leq c \leq 4$ and $n - c \geq 2$, (see [14] and [31] for the cases $2 \leq c \leq 3$). We got further improvements in [33] and conjectured a positive answer to problem (2) provided $n - c \geq 2$, but we were not able to solve all technical challenges which increased with the codimension. In this paper we fully prove the conjecture, as well as Conjecture 2.2 for $n - c \geq 1$, with the new approach which is much easier than the older one. For the case $n - c = 0$ we remark that since every element of $W(\underline{b}; \underline{a})$ has the same Hilbert function, problem (2) becomes more natural provided we replace $\mathrm{Hilb}^p(\mathbb{P}^n)$ with $\mathrm{GradAlg}(H)$, see Notations below and [30] for details.

We thank R. Ile, R.M. Miró-Roig, J.A. Christophersen, M. Boij, O.A. Laudal, Johannes Kleppe and U. Nagel for interesting discussions on different aspects of this topic.

Remark. After this paper was put on Math. arXiv, we got aware of the preprint [15] which shows the mentioned conjectures for $n-c \geq 2$, resp. ≥ 1 (cf. Corollaries 5.9, 5.6).

Notations: In this work $R = k[x_0, \ldots, x_n]$ is a polynomial ring over an algebraically closed field, $\mathfrak{m} = (x_0, \ldots, x_n)$ and $\deg x_i = 1$, unless explicitly making other assumptions.

We mainly keep the notations of [32] and [30]. If $X \subset Y$ are closed subschemes of $\mathbb{P} := \mathbb{P}^n := \mathrm{Proj}(R)$, we denote by $\mathcal{I}_{X/Y}$ (resp. $\mathcal{N}_{X/Y}$) the ideal (resp. normal) sheaf of X in Y, and we usually suppress Y when $Y = \mathbb{P}^n$. By the codimension, $\mathrm{codim}_Y X$, of X in Y we simply mean $\dim Y - \dim X$, and we let $\omega_X = \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n})(-n-1)$ if $c = \mathrm{codim}_{\mathbb{P}} X$. When we write $X = \mathrm{Proj}(A)$ we take $A := R/I_X$ and $K_A = \mathrm{Ext}^c_R(A, R)(-n-1)$ where $I_X = H^0_*(\mathcal{I}_X)$ is the saturated homogeneous ideal of $X \subset \mathbb{P}^n$. We denote the group of morphisms between coherent $\mathcal{O}_X$-modules by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ while $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ denotes the corresponding sheaf. Moreover we set $\hom(\mathcal{F}, \mathcal{G}) = \dim_k \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ and we correspondingly use small letters for the dimension, as a k-vector space, of similar groups.

We denote the Hilbert scheme by $\mathrm{Hilb}^p(\mathbb{P}^n)$, p the Hilbert polynomial [18], and $(X) \in \mathrm{Hilb}^p(\mathbb{P}^n)$ the point that corresponds to $X \subset \mathbb{P}^n$. Let $\mathrm{GradAlg}(H)$ be the representing object of the functor parametrizing flat families of graded quotients A of R of $\mathrm{depth}_{\mathfrak{m}} A \geq 1$ and with Hilbert function H; $H(i) = \dim A_i$ ([28, Thm. 1.1], [29]). We let (A), or (X) where $X = \mathrm{Proj}(A)$, denote the point of $\mathrm{GradAlg}(H)$ that corresponds to A. Then X (resp. A) is *unobstructed* if $\mathrm{Hilb}^p(\mathbb{P}^n)$ (resp. $\mathrm{GradAlg}(H)$) is smooth at (X). By [14],

$$\mathrm{GradAlg}(H) \simeq \mathrm{Hilb}^p(\mathbb{P}^n) \quad \text{at} \quad (X) \tag{1.2}$$

provided $\mathrm{depth}\, A := \mathrm{depth}_{\mathfrak{m}} A \geq 2$. This implies that if $\dim A \geq 2$ and A is Cohen-Macaulay (CM), then it is equivalent to consider deformations of $X \hookrightarrow \mathbb{P}^n$, or of $R \twoheadrightarrow A$, and moreover that their tangent spaces ${}_0\mathrm{Hom}(I_X, A) \simeq H^0(\mathcal{N}_X)$ are isomorphic where the lower index means the degree zero part of the graded module $\mathrm{Hom}(I_X, A)$. We also deduce that if X is generically a complete intersection, then ${}_0\mathrm{Ext}^1_A(I_X/I_X^2, A)$ is an obstruction space of $\mathrm{Hilb}^p(\mathbb{P}^n)$ at (X) ([29], §1.1). Finally we say that X is *general* in some irreducible subset $W \subset \mathrm{Hilb}^p(\mathbb{P}^n)$ if (X) belongs to a sufficiently small open subset U of W such that any (X) in U has all the openness properties that we want to require.

2. Background

This section recalls basic results on standard and good determinantal schemes needed in the sequel, see [6], [13], [5] and [36] for more details and [12], [8], [11] for background. Let

$$\varphi : F = \bigoplus_{i=1}^{t} R(b_i) \longrightarrow G := \bigoplus_{j=0}^{t+c-2} R(a_j) \tag{2.1}$$

be a graded morphism of free R-modules and let $\mathcal{A} = (f_{ij})_{i=1,\ldots t}^{j=0,\ldots,t+c-2}$, $\deg f_{ij} = a_j - b_i$, be a $t \times (t+c-1)$ homogeneous matrix that represents the dual $\varphi^* := \mathrm{Hom}_R(\varphi, R)$. Let $I_t(\mathcal{A})$ be the ideal of R generated by the maximal minors of $\mathcal{A}$. In this paper we suppose

$$c \geq 2\ , \quad t \geq 2\ , \quad b_1 \leq \ldots \leq b_t \quad \text{and} \quad a_0 \leq a_1 \leq \ldots \leq a_{t+c-2}.$$

Recall that a codimension c subscheme $X \subset \mathbb{P}^n$ is standard determinantal if $I_X = I_t(\mathcal{A})$ for some homogeneous $t \times (t+c-1)$ matrix $\mathcal{A}$ as above. Moreover $X \subset \mathbb{P}^n$ is a *good determinantal* scheme if additionally, $I_{t-1}(\mathcal{A})$ defines a scheme of codimension greater or equal to $c+1$ in $\mathbb{P}^n$. Note that if X is standard determinantal and a generic complete intersection in $\mathbb{P}^n$, then X is *good determinantal*, and conversely [36], Thm. 3.4. We say that $\mathcal{A}$ is minimal if $f_{ij} = 0$ for all i, j with $b_i = a_j$.

Let $W(\underline{b};\underline{a})$ (resp. $W_s(\underline{b};\underline{a})$) be the stratum in $\mathrm{Hilb}^p(\mathbb{P}^n)$ consisting of good (resp. standard) determinantal schemes. By [32], see the end of p. 2877, we get that the closures of these strata in $\mathrm{Hilb}^p(\mathbb{P}^n)$ are equal and irreducible. Moreover since we will not require $\mathcal{A}$ to be minimal for $X = \mathrm{Proj}(R/I_t(\mathcal{A}))$ to belong to $W(\underline{b};\underline{a})$ or $W_s(\underline{b};\underline{a})$ in their definitions (a slight correction to [31] and [32]!), we must reconsider Cor. 2.6 of [32]. Indeed we may use its proof to see (cf. [33] and the text accompanying [33, (2.2)] for details)

$$(2.2)\quad W(\underline{b};\underline{a}) \neq \emptyset \;\Leftrightarrow\; W_s(\underline{b};\underline{a}) \neq \emptyset \;\Leftrightarrow\; a_{i-1} \geq b_i \text{ for all } i \text{ and } a_{i-1} > b_i \text{ for some } i.$$

Let $A = R/I_X$ (i.e. X) be standard determinantal and let $M := \mathrm{coker}(\varphi^*)$. Then one knows that the *Eagon-Northcott complex* yields the following free resolution

$$(2.3)\qquad 0 \longrightarrow \wedge^{t+c-1}G^* \otimes S_{c-1}(F) \otimes \wedge^t F \longrightarrow \wedge^{t+c-2}G^* \otimes S_{c-2}(F) \otimes \wedge^t F \longrightarrow \ldots$$
$$\longrightarrow \wedge^t G^* \otimes S_0(F) \otimes \wedge^t F \longrightarrow R \longrightarrow A \longrightarrow 0$$

of A and that the *Buchsbaum-Rim complex* yields a free resolution of M;

$$(2.4)\qquad 0 \longrightarrow \wedge^{t+c-1}G^* \otimes S_{c-2}(F) \otimes \wedge^t F \longrightarrow \wedge^{t+c-2}G^* \otimes S_{c-3}(F) \otimes \wedge^t F \longrightarrow \ldots$$
$$\longrightarrow \wedge^{t+1}G^* \otimes S_0(F) \otimes \wedge^t F \longrightarrow G^* \longrightarrow F^* \longrightarrow M \longrightarrow 0,$$

(the resolutions are minimal if $\mathcal{A}$ is minimal), see for instance [6], Thm. 2.20 and [13], Cor. A2.12 and Cor. A2.13. Note that (2.3) shows that A is Cohen-Macaulay.

Let $\mathcal{B}$ be the matrix obtained by deleting the last column of $\mathcal{A}$ and let B be the k-algebra given by the maximal minors of $\mathcal{B}$. Let $Y = \mathrm{Proj}(B)$. The transpose of $\mathcal{B}$ induces a map $\phi : F = \oplus_{i=1}^{t} R(b_i) \to G' := \oplus_{j=0}^{t+c-3} R(a_j)$. Let $M_{\mathcal{B}}$ be the cokernel of $\phi^* = \mathrm{Hom}_R(\phi, R)$ and let $M_{\mathcal{A}} = M$ and $c > 2$. In this situation we recall that there is an exact sequence

$$(2.5)\qquad 0 \longrightarrow B \longrightarrow M_{\mathcal{B}}(a_{t+c-2}) \longrightarrow M_{\mathcal{A}}(a_{t+c-2}) \longrightarrow 0$$

in which $B \longrightarrow M_{\mathcal{B}}(a_{t+c-2})$ is a regular section given by the last column of $\mathcal{A}$. Moreover,

$$(2.6)\qquad 0 \longrightarrow M_{\mathcal{B}}(a_{t+c-2})^* := \mathrm{Hom}_B(M_{\mathcal{B}}(a_{t+c-2}), B) \longrightarrow B \longrightarrow A \longrightarrow 0$$

is exact by [36] or [31, (3.1)], i.e. we may put $I_{X/Y} := M_{\mathcal{B}}(a_{t+c-2})^*$. Due to (2.4), M is a maximal Cohen-Macaulay A-module ($\mathrm{depth}\, M = \dim A$), and $I_{X/Y}$ is a maximal Cohen-Macaulay B-module by (2.6). By [13] we have $K_A(n+1) \cong S_{c-1}M_{\mathcal{A}}(\ell_c)$ and hence $K_B(n+1) \cong S_{c-2}M_{\mathcal{B}}(\ell_{c-1})$ where

$$(2.7)\qquad \ell_i := \sum_{j=0}^{t+i-2} a_j - \sum_{k=1}^{t} b_k \quad \text{for} \quad 2 \leq i \leq c.$$

Recall that $\tilde{M}$ is locally free of rank one precisely on $X - V(I_{t-1}(\mathcal{A}))$ ([5], Lem. 1.4.8) and that $X \hookrightarrow \mathbb{P}^n$ is a local complete intersection (l.c.i.) by e.g. [50], Lem. 1.8 provided we restrict to $X - V(I_{t-1}(\mathcal{A}))$. By (2.6) it follows that $X \hookrightarrow Y$ and $Y \hookrightarrow \mathbb{P}^n$ are l.c.i.'s outside $V(I_{t-1}(\mathcal{B}))$. Note that $V(I_{t-1}(\mathcal{B})) \subset V(I_t(\mathcal{A})) = X$.

Remark 2.1. Put $X_c := X$ and $X_{c-1} := Y$, let $c > 2$ and let α be a positive integer. If X is general in $W(\underline{b};\underline{a})$ and $a_{i-\min(\alpha,t)} - b_i \geq 0$ for $\min(\alpha,t) \leq i \leq t$, then

$$\operatorname{codim}_{X_j} Sing(X_j) \geq \min\{2\alpha - 1, j+2\} \text{ for } j = c-1 \text{ and } c\ . \tag{2.8}$$

This follows from Rem. 2.7 of [32] (i.e., from [10]). In particular if $\alpha \geq 3$, we get that the closed embeddings $Y \hookrightarrow \mathbb{P}^n$ and $X \hookrightarrow Y$ are local complete intersections outside some set Z of codimension at least $\min(4,c)$. Indeed we may take $Z = V(I_{t-1}(\mathcal{B}))$.

Moreover we recall the following useful general fact that if L and N are finitely generated A-modules such that $\operatorname{depth}_{I(Z)} L \geq r+1$ and $\tilde{N}$ is locally free on $U := X - Z$, then the natural map

$$\operatorname{Ext}^i_A(N,L) \longrightarrow H^i_*(U, \mathcal{H}om_{\mathcal{O}_X}(\tilde{N},\tilde{L})) \tag{2.9}$$

is an isomorphism, (resp. an injection) for $i < r$ (resp. $i = r$), and $H^i_*(U, \mathcal{H}om_{\mathcal{O}_X}(\tilde{N},\tilde{L})) \simeq H^{i+1}_{I(Z)}(\operatorname{Hom}_A(N,L))$ for $i > 0$, cf. [19], exp. VI. Note that we interpret $I(Z)$ as $\mathfrak{m}$ if $Z = \emptyset$.

In [32] we conjectured the dimension of $W(\underline{b};\underline{a})$ in terms of the invariant

$$\lambda_c := \sum_{i,j} \binom{a_i - b_j + n}{n} + \sum_{i,j} \binom{b_j - a_i + n}{n} - \sum_{i,j} \binom{a_i - a_j + n}{n} - \sum_{i,j} \binom{b_i - b_j + n}{n} + 1. \tag{2.10}$$

Here the indices belonging to a_j (resp. b_i) range over $0 \leq j \leq t+c-2$ (resp. $1 \leq i \leq t$) and we let $\binom{a}{n} = 0$ if a is a negative integer. Since [30, Ex. 3.3] shows that the scheme of $c+1$ general points in $\mathbb{P}^c$ given by the vanishing all 2×2 minors of a general $2 \times (c+1)$ matrix of linear entries is a counterexample to Conjecture 6.1 (and to the special case given in Conjecture 6.2) of [32] for any $c \geq 3$, we slightly changed Conjecture 6.1 in [33] to

Conjecture 2.2. *Given integers $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$ and $b_1 \leq ... \leq b_t$, let $h_{i-3} := 2a_{t+i-2} - \ell_i + n$, for $i = 3,4,...,c$ and assume $a_{i-\min([c/2]+1,t)} \geq b_i$ provided $n > c$ and $a_{i-\min([c/2]+1,t)} > b_i$ provided $n = c$ for $\min([c/2]+1,t) \leq i \leq t$. Except for the family $W(0,0;1,1,...,1)$ of* zero dimensional *schemes above we have, for $W(\underline{b};\underline{a}) \neq \emptyset$, that*

$$\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + K_4 + ... + K_c\ ,$$

where $K_3 = \binom{h_0}{n}$ and $K_4 = \sum_{j=0}^{t+1} \binom{h_1+a_j}{n} - \sum_{i=1}^{t} \binom{h_1+b_i}{n}$ and in general

$$K_{i+3} = \sum_{\substack{r+s=i \\ r,s\geq 0}} \sum_{\substack{0\leq i_1<...<i_r\leq t+i \\ 1\leq j_1\leq...\leq j_s\leq t}} (-1)^{i-r} \binom{h_i + a_{i_1} + \cdots + a_{i_r} + b_{j_1} + \cdots + b_{j_s}}{n} \text{ for } 0 \leq i \leq c-3.$$

In [32], Thm. 3.5 we proved that the right hand side in the formula for $\dim W(\underline{b};\underline{a})$ in the Conjecture is always an upper bound for $\dim W(\underline{b};\underline{a})$, and moreover, that the Conjecture hold in the range

$$2 \leq c \leq 5 \text{ and } n - c \geq 1 \text{ (supposing } chark = 0 \text{ if } c = 5\text{)}. \tag{2.11}$$

Indeed this is mainly [32], Thm. 4.5, Cor. 4.7, Cor. 4.10, Cor. 4.14 and [14] ($c = 2$) and [31] ($c = 3$). Moreover we have by [33] (valid also for $n = c$ without assuming $chark = 0$):

Theorem 2.3 ([33], Thm. 3.2)**.** *Assume $a_0 > b_t$. Then Conjecture 2.2 holds provided $c > 5$ (resp. $2 \leq c \leq 5$) and $a_{t+3} > a_{t-2}$ (resp. $a_{t+c-2} > a_{t-2}$).*

In [33] we stated a Conjecture related to the problem (2) of the Introduction:

Conjecture 2.4. *Given integers* $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$ *and* $b_1 \leq ... \leq b_t$, *we suppose* $n-c \geq 2$, $c \geq 5$ *and* $a_0 > b_t$. *Then* $\overline{W(\underline{b};\underline{a})}$ *is a generically smooth irreducible component of the Hilbert scheme* $\mathrm{Hilb}^p(\mathbb{P}^n)$.

By [33], Cor. 3.8 and Thm. 3.4, Conjecture 2.4 holds provided $a_{t+3} > a_{t-1} + a_t - b_1$ or more generally, if a certain collection of Ext^1-groups vanishes. Note that the conclusion of Conjecture 2.4 holds if $n - c \geq 2$ and $2 \leq c \leq 4$ by [14], [31] and [32].

As in [30] we briefly say "T a local ring" (resp. "T artinian") for a local k-algebra $(T, \mathfrak{m}_T)$ essentially of finite type over $k = T/\mathfrak{m}_T$ (resp. such that $\mathfrak{m}_T^r = 0$ for some integer r). The local deformation functors of this paper will be defined over the category $\underline{\ell}$ of artinian k-algebras. Moreover we say "$T \twoheadrightarrow S$ is small in $\underline{\ell}$" provided there is a surjection $(T, \mathfrak{m}_T) \to (S, \mathfrak{m}_S)$ of artinian k-algebras whose kernel $\mathfrak{a}$ satisfies $\mathfrak{a} \cdot \mathfrak{m}_T = 0$.

If T is a local ring, we denote by $\mathcal{A}_T = (f_{ij,T})$ a matrix of homogeneous polynomials belonging to the graded polynomial algebra $R_T := R \otimes_k T$, satisfying $f_{ij,T} \otimes_T k = f_{ij}$ and $\deg f_{ij,T} = a_j - b_i$. Note that all elements from T are considered to be of degree zero. For short we say $\mathcal{A}_T$ *lifts* $\mathcal{A}$ *to* T. The matrix $\mathcal{A}_T$ induces a morphism

$$\varphi_T : F_T := \oplus_{i=1}^{t} R_T(b_i) \to G_T := \oplus_{j=0}^{t+c-2} R_T(a_j) \ . \tag{2.12}$$

Lemma 2.5. *If* $X = \mathrm{Proj}(A)$, $A = R/I_t(\mathcal{A})$, *is a standard determinantal scheme, then* $A_T := R_T/I_t(\mathcal{A}_T)$ *and* $M_T := \mathrm{coker}\, \varphi_T^*$ *are (flat) graded deformations of* A *and* M *respectively for every choice of* $\mathcal{A}_T$ *as above. In particular* $X_T = \mathrm{Proj}(A_T) \subset \mathbb{P}_T^n := \mathrm{Proj}(R_T)$ *is a deformation of* $X \subset \mathbb{P}^n$ *to* T *with constant Hilbert function.*

Proof ([30], Lem. 4.2, cf. [46], Rem. to Prop. 1). In the Eagon-Northcott and Buchsbaum-Rim complexes over R_T, all free modules and all morphisms in these complexes are determined by $\mathcal{A}_T$. Since these complexes become free resolutions of A and M when we tensor with k over T, it follows that A_T and M_T are T-flat and satisfy $A_T \otimes_T k = A$ and $M_T \otimes_T k = M$. □

Definition 2.6. We say "every deformation of X comes from deforming $\mathcal{A}$" if for every local ring T and every graded deformation $R_T \to A_T$ of $R \to A$ to T, then A_T is of the form $A_T = R_T/I_t(\mathcal{A}_T)$ for some $\mathcal{A}_T$ as above. Note that by (1.2) we can in this definition replace "graded deformations of $R \to A$" by "deformations of $X \hookrightarrow \mathbb{P}^n$" provided $\dim X \geq 1$.

Lemma 2.7. *Let* $X = \mathrm{Proj}(A)$ *be a standard determinantal scheme,* $(X) \in W(\underline{b};\underline{a})$. *If every deformation of* X *comes from deforming* $\mathcal{A}$, *then* A *is unobstructed. Moreover if* $n - c \geq 1$ *then* X *is unobstructed and* $\overline{W(\underline{b};\underline{a})}$ *is an irreducible component of* $\mathrm{Hilb}^p(\mathbb{P}^n)$.

Proof. Let $T \twoheadrightarrow S$ be small in $\underline{\ell}$ and let A_S be a deformation of A to S. By assumption, $A_S = R_S/I_t(\mathcal{A}_S)$ for some matrix $\mathcal{A}_S$. We can lift each $f_{ij,S}$ to a polynomial $f_{ij,T}$ with coefficients in T such that $f_{ij,T} \otimes_T S = f_{ij,S}$. By Lemma 2.5, $A_T := R_T/I_t(\mathcal{A}_T)$ is T-flat, whence A is unobstructed by $A_T \otimes_T S = A_S$. If $\dim X \geq 1$ we get the unobstructedness of X by (1.2). For the remaining part of the proof, see [30], Lem. 4.4. □

Remark 2.8. By these lemmas we get T-flat determinantal schemes by just parameterizing the polynomials of $\mathcal{A}$ over a local ring T, see Rem. 4.5 of [30] and Laksov's papers [38], [37] for somewhat similar results for more general determinantal schemes.

3. DEFORMATIONS OF R-MODULES OF MAXIMAL GRADE

Let M be a finitely generated (torsion) R-module with presentation matrix $\mathcal{A}$, i.e. $M = \operatorname{coker}(\varphi^*)$ with φ as in (2.1). Since the grade of M over R is the grade, or codimension, of the annihilator $I := \operatorname{ann}(M)$ of M, and since the radicals of I and $I_t(\mathcal{A})$ are the same, we get that M has maximal grade if and only if $A := R/I_t(\mathcal{A})$ is standard determinantal. In this case $I = I_t(\mathcal{A})$, see [7] for details. If $M \cong R/I(-b_1)$ is cyclic ($t = 1$), we remark that a module of maximal grade is a complete intersection. The main results of this section is variations of the following

Theorem 3.1. *Let M be a finitely generated graded module over R of maximal grade. Then M is unobstructed. Moreover if $A := R/\operatorname{ann}(M)$ is generically a complete intersection, then*

$$\dim\ {}_0\operatorname{Ext}^1_R(M,M) = \lambda_c + K_3 + K_4 + ... + K_c \quad \text{and}$$

$$\operatorname{depth} \operatorname{Ext}^1_R(M,M) \geq \dim A - 1\ .$$

Remark 3.2. By deformation theory ${}_0\operatorname{Ext}^1_R(M,M)$ (resp. ${}_0\operatorname{Ext}^2_R(M,M)$) is the tangent (resp. the natural obstruction) space of the local deformation functor, $\operatorname{Def}_{M/R}$, of M as a graded R-module (e.g. [47]). Since $c \geq 2$, ${}_0\operatorname{Ext}^2_R(M,M)$ is in many cases non-vanishing.

Remark 3.3. Note that the assumption on A in Theorem 3.1 is equivalent to assuming A good determinantal. By (2.2) good determinantal schemes exist if standard determinantal schemes exist. Hence if we take the polynomials f_{ij} of degrees $a_j - b_i$ in a presentation matrix (f_{ij}) of M general enough, then the assumption on A in Theorem 3.1 is satisfied.

Remark 3.4. While distributing a preliminar version of a paper partially containing Theorem 3.1 to specialists in deformations of modules, we learned that the unobstructedness part of Theorem 3.1 (and hence of Theorem 3.8) was proved in R. Ile's PhD thesis, cf. [23], ch. 6.

Proof. Let $T \twoheadrightarrow S$ be a small in $\underline{\ell}$ and let M_S be any graded deformation of M to the artinian ring S. Let $\mathcal{A} = (f_{ij})$ be a homogeneous matrix that represents φ^*. Since $G^* \xrightarrow{\varphi^*} F^* \to M \to 0$ is exact (cf. (2.1)), we have $M_S = \operatorname{coker}(\varphi^*_S)$ where φ^*_S corresponds to some matrix $\mathcal{A}_S = (f_{ij,S})$, as in (2.12). Since $T \to S$ is surjective, we can lift each $f_{ij,S}$ to a polynomial $f_{ij,T}$ with coefficients in T such that $f_{ij,T} \otimes_T S = f_{ij,S}$. By Lemma 2.5, $M_T := \operatorname{coker}(\varphi^*_T)$ is flat over T and since $M_T \otimes_T S = M_S$ it follows that M is unobstructed.

To see the dimension formula we *claim* that there is an exact sequence

$$(3.1)\quad 0 \to {}_0\operatorname{Hom}_R(M,M) \to {}_0\operatorname{Hom}_R(F^*,M) \to {}_0\operatorname{Hom}_R(G^*,M) \to {}_0\operatorname{Ext}^1_R(M,M) \to 0.$$

Indeed look at the map $d_1 : \wedge^{t+1}G^* \otimes S_0(F) \otimes \wedge^t F \to G^*$ appearing in the Buchsbaum-Rim complex (2.4) and recall that the image of the corresponding map $\wedge^t G^* \otimes S_0(F) \otimes \wedge^t F \to R$ of the Eagon-Northcott complex (2.3) is the ideal $I = \operatorname{ann}(M)$ generated by the maximal minors. It follows that $\operatorname{im} d_1 \subset I \cdot G^*$ and hence that the induced map ${}_0\operatorname{Hom}_R(d_1, M) = 0$. So if we apply ${}_0\operatorname{Hom}_R(-,M)$ to (2.4), we get (3.1) by the definition of ${}_0\operatorname{Ext}^i_R(M,M)$.

Let $E = \operatorname{coker}\varphi$. Then we have an exact sequence

$$0 \longrightarrow E^* \longrightarrow G^* \xrightarrow{\varphi^*} F^* \longrightarrow M \longrightarrow 0\ ,$$

to which we apply the exact functors ${}_0\mathrm{Hom}_R(F^*,-)$ and ${}_0\mathrm{Hom}_R(G^*,-)$. We get

$$ {}_0\hom_R(G^*,M) - {}_0\hom_R(F^*,M) = \lambda_c - 1 + {}_0\hom_R(G^*,E^*) - {}_0\hom_R(F^*,E^*)$$

by using the definition (2.10) of λ_c. Note that ${}_0\hom(M,M) = 1$ by [32], Lem. 3.2 since A is good determinantal by assumption. Hence we get the dimension formula of Theorem 3.1 from (3.1) provided we can prove

$$ {}_0\hom_R(G^*,E^*) - {}_0\hom_R(F^*,E^*) = K_3 + ... + K_c.$$

By [32], Prop. 3.12 we have $1 + K_3 + ... + K_c = {}_0\hom_R(E,E)$ and by the proof of the same proposition we find ${}_0\hom_R(E,E) = 1 + {}_0\hom_R(E,G) - {}_0\hom_R(E,F)$ and whence we get the dimension formula.

Now we consider the depth of $\mathrm{Ext}^1_R(M,M)$. Firstly observe that it is straightforward to see $\operatorname{depth}\mathrm{Ext}^1_R(M,M) \geq \dim A - 2$. Indeed using $\mathrm{Hom}_R(M,M) \simeq A$ ([32], Lem. 3.2) and skipping the lower index 0 in (3.1), we get that all three Hom-modules in (3.1) are maximal CM A-modules. Then if $D := \mathrm{coker}(\mathrm{Hom}_R(M,M) \to \mathrm{Hom}_R(F^*,M))$ we easily conclude since $\operatorname{depth}\mathrm{Ext}^1_R(M,M) \geq \operatorname{depth} D - 1 \geq \dim A - 2$ by [13], Cor. 18.6.

Looking more carefully at the argument, we can show $\operatorname{depth}\mathrm{Ext}^1_R(M,M) \geq \dim A - 1$. Indeed it suffices to prove $\operatorname{depth} D = \dim A$. To see it we use the resolution of A in (2.3) and the resolution of $M \otimes F$ deduced from (2.4). Let $\{f_1, f_2, ..., f_t\}$ be the standard basis of F and $\{y_1, y_2, ...\}$ the standard basis of G^*. The leftmost free modules in these resolutions are $\wedge^{t+c-1}G^* \otimes S_{c-1}(F) \otimes \wedge^t F$ and $\wedge^{t+c-1}G^* \otimes S_{c-2}(F) \otimes \wedge^t F \otimes F$ respectively. We may consider the former as an R-submodule of the latter through the map τ_{c-1} where $\tau_k = id \otimes \tau'_k$ and $\tau'_k : S_k(F) \to S_{k-1}(F) \otimes F$ is induced by sending a symmetric tensor $(f_{i_1} \otimes_s ... \otimes_s f_{i_k}) \in S_k(F)$ onto the "reduced sum" of $\sum_{j=1}^k (f_{i_1} \otimes_s ... \otimes_s \hat{f_{i_j}} \otimes_s ... \otimes_s f_{i_k}) \otimes f_{i_j} \in S_{k-1}(F) \otimes F$. Here "reduced" means sending e.g. $(f_1 \otimes_s f_1 \otimes_s f_1 \otimes_s f_2 \otimes_s f_2 \otimes_s f_3) \in S_6(F)$ onto

$$a(f_1 \otimes_s f_1 \otimes_s f_2 \otimes_s f_2 \otimes_s f_3) \otimes f_1 + b(f_1 \otimes_s f_1 \otimes_s f_1 \otimes_s f_2 \otimes_s f_3) \otimes f_2 + (f_1 \otimes_s f_1 \otimes_s f_1 \otimes_s f_2 \otimes_s f_2) \otimes f_3$$

in $S_5(F) \otimes F$ with $(a,b) = (1,1)$ (and not $(a,b) = (3,2)$!). Then $\tau_1 = id$ and letting $\tau_{-1} : R \to F^* \otimes F$ be the obvious map and $\tau_0 : \wedge^t G^* \otimes S_0(F) \otimes \wedge^t F \to G^* \otimes F$ the map induced by sending $(y_{i_1} \wedge ... \wedge y_{i_t}) \in \wedge^t G^*$ onto $\sum_{j=1}^t (-1)^{j-1}(\varphi^*(y_{i_1}) \wedge ... \wedge \hat{\varphi^*}(y_{i_j}) \wedge ... \wedge \varphi^*(y_{i_t})) \otimes y_{i_j} \in \wedge^{t-1}F^* \otimes G^*$ followed by the natural map $\wedge^{t-1}F^* \otimes G^* \to F \otimes \wedge^t F^* \otimes G^*$, one may check that the collection of maps $\{\tau_i\}_{i \geq -1}$ is actually a map between the free resolutions of A and $M \otimes F$. The explicit description in [26] of the differentials in the resolutions (2.3) and (2.4) may be helpful in checking that the diagrams between the resolutions commute, or see [34], proof of Thm. 3.2 for $\operatorname{depth}\mathrm{Ext}^1_R(M,K_A) = \dim A$ where corresponding (more difficult) details of the commutativity of such diagrams are quite thoroughly explained. Now using the well known mapping cone construction we find a free resolution of D and since $\tau_{c-1} \otimes_R R/\mathfrak{m}$ is injective, the leftmost term $\wedge^{t+c-1}G^* \otimes S_{c-1}(F) \otimes \wedge^t F$ becomes redundant. The minimal R-free resolution of D has therefore the same length as the minimal R-free resolution of A, i.e. D is maximally CM and we are done. □

Remark 3.5. We see from the proof, or Lemma 2.5, that if we arbitrarily lift the polynomials in a presentation matrix of M to polynomials with coefficients in T, we get that $M_T := \mathrm{coker}(\varphi_T^*)$ is *flat* over T. This is not true in general, but for modules of maximal grade it is because the Buchsbaum-Rim complex provides us with a resolution of M_T.

Remark 3.6. Let $E = \operatorname{coker}\varphi$, $\varphi = F \to G$ cf. (2.1) and suppose $R/I_t(\mathcal{A})$ is good determinantal. It is stated in [32], Rem. 3.14 that dim ${}_0\mathrm{Ext}^1_R(E,E) = \lambda_c + K_3 + K_4 + ... + K_c$. Indeed one may use the proof above to see ${}_0\mathrm{Ext}^i_R(E,E) \cong {}_0\mathrm{Ext}^i_R(M,M)$ for $i = 1$ while this is not true in general for $i \neq 1$.

Remark 3.7. The theorem admits a *vast generalization* since the assumption that R is a polynomial ring is not necessary. Indeed if R is any commutative graded (resp. local) k-algebra, then a module of maximal grade is unobstructed and the exact sequence (3.1) (resp. where the lower index 0 is removed) holds. In fact all we need for these parts in the proof is the existence and exactness of the Buchsbaum-Rim complex, which hold under almost no assumption on R (cf. [13], Appendix 2).

We will give the details in the graded case of what we claimed in Remark 3.7. This means that we will generalize to *arbitrary* modules of *maximal grade* the following well-known fact for cyclic modules, that a complete intersection R/I is unobstructed and that ${}_0\mathrm{hom}_R(I, R/I) = \sum_{j=1}^q \dim(R/I)_{(a_j)}$ where $\oplus_{j=1}^q R(-a_j) \to I$ is a minimal surjection.

For the remaining part of this section we let $R = \oplus_{v\geq 0} R_v$ be *any* graded k-algebra ($k = R_0$ a not necessarily algebraically closed field), generated by finitely many elements from $\mathfrak{m} := \oplus_{v\geq 1} R_v$. Let

$$G^* := \sum_{j=1}^{q} R(-a_j) \xrightarrow{\varphi^*} F^* := \sum_{j=1}^{p} R(-b_i) \xrightarrow{\pi} M \to 0 \tag{3.2}$$

be a minimal presentation of M and suppose M is of *maximal grade*. Let $N = \ker\pi$. It is known that the tangent space of the graded deformation functor $\mathrm{Def}_M(F^*)$ which deforms the surjection $\pi : F^* \to M$ to artinian k-algebras from $\underline{\ell}$, using trivial deformations of F^*, is isomorphic to ${}_0\mathrm{Hom}_R(N,M)$ and that ${}_0\mathrm{Ext}^1_R(N,M)$ contains all the obstructions of the graded deformations (we may deduce it from [39], Thm. 4.1.14 and Lem. 3.1.7, but [18], Prop. 5.1 and Cor. 5.2 and 5.3 is the classical reference since we here deal with the local deformation functor, adapted to graded deformations, of Grothendieck's Quot scheme). If we apply ${}_0\mathrm{Hom}_R(-,M)$ to $0 \to N \to F^* \to M \to 0$, we get the exact sequence

$$0 \to {}_0\mathrm{Hom}_R(M,M) \to {}_0\mathrm{Hom}_R(F^*,M) \to {}_0\mathrm{Hom}_R(N,M) \to {}_0\mathrm{Ext}^1_R(M,M) \to 0\,, \tag{3.3}$$

and ${}_0\mathrm{Ext}^1_R(N,M) \simeq {}_0\mathrm{Ext}^2_R(M,M)$. We notice that the arguments in the proof of Theorem 3.1 which led to ${}_0\mathrm{Hom}_R(d_1,M) = 0$, where now $d_1 : \wedge^{p+1}G^* \otimes S_0(F) \otimes \wedge^p F \to G^*$, carry over to the general situation we are considering since they relied on how the maps in the Buchsbaum-Rim complex were defined. Hence we get the exact sequence (3.1), and comparing with (3.3), we get that the tangent space of $\mathrm{Def}_M(F^*)$ is

$${}_0\mathrm{Hom}_R(N,M) \simeq {}_0\mathrm{Hom}_R(G^*,M) \simeq \bigoplus_{j=1}^{q} M_{(a_j)}.$$

Since the map d_1 is defined in terms of φ^*, the unobstructedness argument for M in the proof of Theorem 3.1 and the flatness argument of Lemma 2.5 both carry over the general case. Note that also the object $\pi : F^* \to M$ is unobstructed, i.e. $\mathrm{Def}_M(F^*)$ is formally smooth ([44], [39]) because π is easily deformed once M is deformed. Also the proof of the length of an R-free resolution of $\mathrm{Ext}^1_R(M,M)$ holds and we have

Theorem 3.8. *Let M be a finitely generated graded module (as in (3.2)) of maximal grade over a finitely generated graded k-algebra R where $R_0 = k$ is an arbitrary field. Let $N := \ker(F^* \to M)$. Then $\operatorname{pd}_R \operatorname{Ext}^1_R(M,M) \leq c+1$. Moreover M is unobstructed. Indeed $\operatorname{Def}_M(F^*)$ is formally smooth and the dimension of the tangent space of $\operatorname{Def}_M(F^*)$ is*

$$\dim\, {}_0\operatorname{Hom}_R(N,M) = \sum_{j=1}^{q} \dim M_{(a_j)}.$$

Remark 3.9. Under the assumptions of Theorem 3.8, we see from the proof that

$$\dim\, {}_0\operatorname{Ext}^1_R(M,M) - \dim\, {}_0\operatorname{Hom}_R(M,M) = \sum_{j=1}^{q} \dim M_{(a_j)} - \sum_{i=1}^{p} \dim M_{(b_i)}.$$

Now suppose R is any graded Cohen-Macaulay quotient of a polynomial ring $k[x_0,\dots,x_n]$ with the standard grading where k is any field. This will be the natural setting, having algebraic geometry in mind, to which we can generalize the theorems of this paper (sometimes assuming $k = \overline{k}$ to be algebraically closed). Slightly generalizing [32], Lem. 3.2, we get that $\operatorname{Hom}_R(M,M) \simeq A$ if $\operatorname{depth}_{I_{t-1}(\mathcal{A})A} A \geq 1$ (cf. Remark 4.11). Hence ${}_0\operatorname{hom}_R(M,M) = 1$ and the formula above gives an alternative to the formula of Theorem 3.1 for computing ${}_0\operatorname{ext}^1_R(M,M)$. In this general setting one may see that also the formula of Theorem 3.1 holds provided we redefine λ_c and K_i appearing in (2.10) and Conjecture 2.2 in the obvious way, namely by replacing all $\binom{v+n}{n}$ with $\dim R_v$. Indeed this follows from the proof of Theorem 3.1 since the part we use from [32] (Prop. 3.12) also generalize to this setting.

4. The rigidity of modules of maximal grade

In this section we consider a module M of maximal grade as a graded module over $A = R/I$ where $I = \operatorname{ann}(M)$. Recall that $I = I_t(\mathcal{A})$ where $\mathcal{A} = (f_{ij})_{i=1,\dots t}^{j=0,\dots,t+c-2}$ is a $t \times (t+c-1)$ homogeneous presentation matrix of M, cf. (2.1), in which case we put $M = M_{\mathcal{A}}$. A main result in this section is the rigidity of M as an A-module (i.e. $\operatorname{Ext}^1_A(M,M) = 0$) provided $X := \operatorname{Proj}(A)$ is smooth of dimension greater or equal to 1. Furthermore if $\dim X \geq 2$ we also show $\operatorname{Ext}^2_A(M,M) = 0$. More generally we have the following results.

Theorem 4.1. *Let M be a finitely generated graded R-module of maximal grade and let $A := R/\operatorname{ann}(M)$. Let $j \geq 1$ be an integer and suppose $\operatorname{depth}_{I_{t-1}(\mathcal{A})A} A \geq j+1$. Then $\operatorname{Hom}_A(M,M) \simeq A$ and,*

$$\operatorname{Ext}^i_A(M,M) = 0 \quad \text{for} \quad 1 \leq i \leq j-1.$$

Remark 4.2. Let $X = \operatorname{Proj}(A)$, $J := I_{t-1}(\mathcal{A})$ and recall that $\operatorname{depth}_{JA} A = \dim A - \dim A/JA$ and $V(JA) \subset Sing(X)$. We may therefore take j as $j = \operatorname{codim}_X Sing(X) - 1 = \dim X - \dim Sing(X) - 1$ in Theorem 4.1, interpreting $\dim Sing(X)$ as -1 if $Sing(X) = \emptyset$.

Proof. Since $\tilde{M}$ is locally free of rank one over $U := \operatorname{Proj}(A) - V(I_{t-1}(\mathcal{A})A)$ (see the text before Remark 2.1), we can use (2.9) with $L = N = M$ and $i = 0$. We get $\operatorname{Hom}_A(M,M) \simeq$

A. It follows, again by (2.9), that

$$\text{Ext}^i_A(M,M) \simeq H^i_*(U, \mathcal{H}om(\tilde{M},\tilde{M})) \simeq H^{i+1}_{JA}(\text{Hom}_A(M,M)) = 0 \tag{4.1}$$

for $0 < i < j$, whence the result. □

Remark 4.3. We consider the vanishing of $\text{Ext}^i_A(M,M)$ in Theorem 4.1 as mainly known (Schlessinger, see [27], Prop. 2.2.3 for $i = 1$) since it, as in [45] and [49], is rather clear how to generalize [27], Prop. 2.2.3 to a non-smooth X and to $i > 1$ (e.g. [49], Rem. 2.5). Our proof is, however, very short and uses more directly Grothendieck's long exact sequence of Ext-groups appearing in [19], exp. VI, from which (2.9) is deduced.

Remark 4.4. It is clear from the proof that the theorem also holds for $c = 1$. In this case we can only use the result for $j \leq 2$ because the largest possible value of $\text{depth}_{I_{t-1}(\mathcal{A})A} A$ is 3. Thus our proof implies the known rigidity of M ([27], Prop. 2.2.3 and [24], Thm. 2). We continue to restrict ourselves to $c \geq 2$ and refer to [24] for a nice study when $c = 1$.

In running some Macaulay 2 computations ([17]) in the situation of Theorem 4.1 we were surprised to see that also $\text{Ext}^i_A(M,M) = 0$ for $i = j$ in the examples. This observation led us to try to prove Theorem 4.1 under the assumption $\text{depth}_{I_{t-1}(\mathcal{A})A} A \geq j$. The natural case where this happens and where we succeed is as follows. Let $B = R/I_t(\mathcal{B})$ and suppose $\text{depth}_{JB} B \geq j+1$ with $J = I_{t-1}(\mathcal{B})$ where $\mathcal{B}$ is obtained by deleting a column of $\mathcal{A}$. Then since $I_t(\mathcal{A}) \subset I_{t-1}(\mathcal{B}) \subset I_{t-1}(\mathcal{A})$, it follows that $\text{depth}_{JA} A \geq j$. Thus we may take $j = \text{codim}_Y Sing(Y) - 1 = \dim X - \dim Sing(Y)$ in the following theorem.

Theorem 4.5. *Let $B \to A$ be quotients of R defined by the vanishing of the maximal minors of $\mathcal{B}$ and $\mathcal{A}$ respectively where $\mathcal{B}$ is obtained by deleting some column of $\mathcal{A}$. Let M be the finitely generated graded module over R of maximal grade defined by $\mathcal{A}$, i.e. $M := M_{\mathcal{A}}$, cf.* (2.5). *Let $j \geq 2$ be an integer and suppose* $\text{depth}_{I_{t-1}(\mathcal{B})B} B \geq j+1$. *Then* $\text{Hom}_A(M,M) \simeq A$ *and*

$$\text{Ext}^i_A(M,M) = 0 \text{ for } 1 \leq i \leq j-1 \ .$$

Proof. Since M has maximal grade we get that A and hence B are standard determinantal rings (by [4] since we may suppose the matrix $\mathcal{A}$ is minimal). It follows that $N := M_{\mathcal{B}}$ has maximal grade and we can apply Theorem 4.1 to N and M. We get $\text{Ext}^i_B(N,N) = 0$ for $1 \leq i \leq j-1$ and $\text{Hom}_A(M,M) \simeq A$ (and the vanishing of many $\text{Ext}^i_A(M,M)$ which we surprisingly do not use in this proof).

Consider the exact sequence

$$0 \longrightarrow B(-a_{t+c-2}) \longrightarrow N \longrightarrow M \longrightarrow 0 \tag{4.2}$$

induced by (2.5) and put $B_a := B(-a_{t+c-2})$. We *claim* that $\text{Ext}^1_B(M,M)$ is isomorphic to $\text{Hom}_B(B_a, M) \simeq M(a_{t+c-2})$ and that $\text{Ext}^i_B(M,M) = 0$ for $2 \leq i \leq j-1$. To see it we apply $\text{Hom}_B(-,M)$ and $\text{Hom}_B(N,-)$ to (4.2). Their long exact sequences fit into the

following diagram
(4.3)

$$\begin{array}{ccccccccc} & & & & \downarrow & & & & \\ & & & & \mathrm{Ext}^i_B(N,N) & & & & \\ & & & & \downarrow & & & & \\ \to\ \mathrm{Ext}^{i-1}_B(B_a,M) & \to & \mathrm{Ext}^i_B(M,M) & \to & \mathrm{Ext}^i_B(N,M) & \to & \mathrm{Ext}^i_B(B_a,M) & \to & \mathrm{Ext}^{i+1}_B(M,M)\to \\ & & & & \downarrow & & & & \\ & & & & \mathrm{Ext}^{i+1}_B(N,B_a) & & & & \\ & & & & \downarrow & & & & \end{array}$$

where $\mathrm{Ext}^i_B(B_a,M)=0$ for $i>0$. To see that also $\mathrm{Ext}^{i+1}_B(N,B_a)=0$ for $1\le i+1\le j$, we first notice that $\mathrm{Ext}^{i+1}_B(N,B_a)\simeq \mathrm{Ext}^{i+1}_B(N\otimes K_B, K_B(-a_{t+c-2}))$ ([22], Satz 1.2). Moreover since $K_B(n+1)\simeq S_{c-2}N(\ell_{c-1})$, cf. (2.7), we see that $N\otimes K_B$ and $S_{c-1}N$ are closely related. Indeed up to twist they are isomorphic if we restrict to $U_B:=\mathrm{Proj}(B)-V(I_{t-1}(\mathcal{B})B)$. Hence if we let Λ be the kernel of the natural surjective map $S_{c-2}N\otimes_B N\to S_{c-1}N$, it follows that $\mathrm{Supp}\,\Lambda\subset V(I_{t-1}(\mathcal{B})B)$, e.g. we get $\mathrm{Ext}^{i+1}_B(\Lambda,K_B)=0$ for $i+1\le j$ by the assumption $\mathrm{depth}_{I_{t-1}(\mathcal{B})B}B\ge j+1$. Now we recall that $S_{c-1}N$ is a maximal CM B-module ([13]). It follows that $\mathrm{Ext}^{i+1}_B(S_{c-1}N,K_B)=0$ for $i+1>0$. Since the sequence

$$\to \mathrm{Ext}^{i+1}_B(S_{c-1}N,K_B)\to \mathrm{Ext}^{i+1}_B(S_{c-2}N\otimes_B N,K_B)\to \mathrm{Ext}^{i+1}_B(\Lambda,K_B)\to$$

is exact, we deduce that $\mathrm{Ext}^{i+1}_B(N,B_a)=0$ for $0\le i\le j-1$ and hence $\mathrm{Ext}^i_B(N,M)=0$ $1\le i\le j-1$. Then using the big diagram above we get the claim provided we can prove that the surjective map $\mathrm{Hom}_B(B_a,M)\to \mathrm{Ext}^1_B(M,M)$ is an isomorphism. To prove it we continue the horizontal sequence in the big diagram on the left hand side and we get the exact sequence

$$0\to \mathrm{Hom}_B(M,M)\to \mathrm{Hom}_B(N,M)\to \mathrm{Hom}_B(B_a,M)\to \mathrm{Ext}^1_B(M,M)\to 0\ .$$

The two leftmost Hom-modules are easily seen to be isomorphic to A, e.g. $\mathrm{Hom}_B(N,M)\simeq H^0_*(U_B,\mathcal{H}om(\tilde{N},\tilde{M}))\simeq A$ because $\tilde{N}\otimes O_X\simeq\tilde{M}$ is invertible over $U_B\cap X$ and the *claim* is proved.

It remains to compare the groups $\mathrm{Ext}^i_A(M,M)$ and $\mathrm{Ext}^i_B(M,M)$ for which we have a well-known spectral sequence $E_2^{p,q}:=\mathrm{Ext}^p_A(\mathrm{Tor}^B_q(A,M),M)$, converging to $\mathrm{Ext}^{p+q}_B(M,M)$, at our disposal. Since $E_2^{p,0}\simeq \mathrm{Ext}^p_A(M,M)$ we must show

$$E_2^{p,0}=0\ \text{ for } 1\le p<j\ . \tag{4.4}$$

Noticing that $I_{X/Y}\simeq N(a_{t+c-2})^*$ by (2.6) in which $\tilde{N}(a_{t+c-2})|_{U_B}$ is an invertible $\mathcal{O}_Y$-Module over $U_B\subset Y$, we get that the sheafification of $\mathrm{Tor}^B_q(A,M)\simeq \mathrm{Tor}^B_{q-1}(I_{X/Y},M)$ vanishes over $U_B\cap X$ for $q\ge 2$. Hence $\mathrm{Supp}\,\mathrm{Tor}^B_q(A,M)\subset V(I_{t-1}(\mathcal{B})A)$, and we get $E_2^{p,q}=0$ for $q\ge 2$ and $p<j$ by the assumption $\mathrm{depth}_{I_{t-1}(\mathcal{B})B}B\ge j+1$ which leads to $\mathrm{depth}_{I_{t-1}(\mathcal{B})A}A\ge j$. Note also that $E_2^{p,1}=0$ for $0<p<j-1$ because by (2.9),

$$E_2^{p,1}=\mathrm{Ext}^p_A(I_{X/Y}\otimes_B M,M)\simeq H^p_*(U,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_{X/Y}\otimes\tilde{M},\tilde{M}))\simeq H^{p+1}_{I_{t-1}(\mathcal{B})A}M(a_{t+c-2})=0$$

where $U:=U_B\cap X$. Indeed $\mathcal{I}_{X/Y}\otimes\tilde{M}\simeq\tilde{N}(a_{t+c-2})^*\otimes_{\mathcal{O}_Y}\mathcal{O}_X\otimes\tilde{M}\simeq\mathcal{O}_X(-a_{t+c-2})$ over U. In the same way

$$E_2^{0,1}=\mathrm{Hom}_A(I_{X/Y}\otimes_B M,M)\simeq H^0_*(U,\tilde{M}(a_{t+c-2}))\simeq M(a_{t+c-2})\ . \tag{4.5}$$

The spectral sequence leads therefore to an exact sequence

$$0 \to E_2^{1,0} \to \mathrm{Ext}^1_B(M,M) \to E_2^{0,1} \to E_2^{2,0} \to \mathrm{Ext}^2_B(M,M) \to 0 \tag{4.6}$$

and to isomorphisms $E_2^{p,0} \simeq \mathrm{Ext}^p_B(M,M)$ for $2 < p < j$. We already know $\mathrm{Ext}^p_B(M,M) = 0$ for $2 \leq p < j$ by the proven claim. Hence we get (4.4) provided we can show that the "pushforward morphism" $\mathrm{Ext}^1_B(M,M) \to E_2^{0,1} \simeq M(a_{t+c-2})$ of (4.6) is an isomorphism. Since it is quite clear from (4.5) that this morphism is compatible with the isomorphism $\mathrm{Ext}^1_B(M,M) \simeq \mathrm{Hom}_B(B_a, M) \simeq M(a_{t+c-2})$ which we proved in the *claim* (using (4.2)), we get the theorem. □

Remark 4.6. The proof of Theorem 4.5 even shows $\mathrm{Ext}^i_B(M,M) = 0$ for $2 \leq i \leq j-1$.

As we see, the proof of Theorem 4.5 is technically much more complicated that the proof of Theorem 4.1, even though we are only able to weaken the depth assumption on A in some cases (namely when the two algebras in $R/I_{t-1}(\mathcal{B}) \twoheadrightarrow R/I_{t-1}(\mathcal{A})$ have the same dimension). This improvement is, however, important in low dimensional cases in which the radical of $I_{t-1}(\mathcal{B})$ often satisfies

$$\mathfrak{m} = \sqrt{I_{t-1}(\mathcal{B})} \tag{4.7}$$

and hence $\sqrt{I_{t-1}(\mathcal{B})} = \sqrt{I_{t-1}(\mathcal{A})}$. For short we say that we get an l.c.i. scheme by deleting some column if (4.7) holds. We immediately get from the theorems

Corollary 4.7. *Let $X = \mathrm{Proj}(A)$, $A = R/I_t(\mathcal{A})$ be a standard determinantal scheme, let $M = M_{\mathcal{A}}$ and suppose either $\mathrm{depth}_{I_{t-1}(\mathcal{A})A} A \geq 3$, or just $\dim X \geq 1$ provided we get an l.c.i. (e.g. a smooth) scheme by deleting some column of $\mathcal{A}$. Then $\mathrm{Hom}_A(M,M) \simeq A$ and*

$$\mathrm{Ext}^1_A(M,M) = 0 \ .$$

Corollary 4.8. *Let $X = \mathrm{Proj}(A)$, $A = R/I_t(\mathcal{A})$ be a standard determinantal scheme, let $M = M_{\mathcal{A}}$ and suppose $\dim X \geq 1$. Moreover suppose the polynomials f_{ij} of degrees $a_j - b_i$ in a presentation matrix (f_{ij}) of M are chosen general enough and suppose $a_{i-2} \geq b_i$ for $2 \leq i \leq t$. Then $\mathrm{Hom}_A(M,M) \simeq A$ and $\mathrm{Ext}^1_A(M,M) = 0$.*

Proof. We may suppose that the codimension of X in $\mathbb{P}^n$ is $c \geq 3$ since M is a twist of the canonical module of A if $c = 2$ in which case the conclusion is well known. Suppose $\dim X = 1$. Then Remark 2.1 with $\alpha = 2$ shows that both $X = X_c$ and $Y := X_{c-1}$ are smooth because X is general. If, however, $\dim X \geq 2$, then Remark 2.1 still applies to $X = X_c$ and we get $\mathrm{depth}_{I_{t-1}(\mathcal{A})A} A \geq 3$. Hence in any case we conclude by Corollary 4.7. □

In deformation theory it is important to know when $\mathrm{Ext}^2_A(M,M)$ vanishes.

Corollary 4.9. *Let $X = \mathrm{Proj}(A)$, $A = R/I_t(\mathcal{A})$ be a standard determinantal scheme, let $M = M_{\mathcal{A}}$ and suppose either $\mathrm{depth}_{I_{t-1}(\mathcal{A})A} A \geq 4$, or just $\dim X \geq 2$ provided we get an l.c.i. (e.g. a smooth) scheme by deleting some column of $\mathcal{A}$. Then $\mathrm{Hom}_A(M,M) \simeq A$ and*

$$\mathrm{Ext}^i_A(M,M) = 0 \quad \text{for } i = 1 \text{ and } 2 \ .$$

Proof. This follows immediately from Theorem 4.5 and Theorem 4.1 □

Corollary 4.10. *Let $X = \mathrm{Proj}(A)$, $A = R/I_t(\mathcal{A})$ be a standard determinantal scheme, let $M = M_{\mathcal{A}}$ and suppose $\dim X \geq 2$. Moreover suppose the polynomials f_{ij} of degrees $a_j - b_i$ in a presentation matrix (f_{ij}) of M are chosen general enough and suppose $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$. Then $\mathrm{Hom}_A(M,M) \simeq A$ and*

$$\mathrm{Ext}^i_A(M,M) = 0 \quad \text{for } i = 1 \text{ and } 2\ .$$

Proof. We may again suppose that $c \geq 3$. Now if $\dim X = 2$, then Remark 2.1 with $\alpha = 3$ shows that both $X = X_c$ and $Y := X_{c-1}$ are smooth. If, however, $\dim X \geq 3$, then Remark 2.1 still applies to $X = X_c$ and we get $\mathrm{depth}_{I_{t-1}(\mathcal{A})A} A \geq 4$. Thus in any case we conclude by Corollary 4.9. □

Remark 4.11. Also the results of this section admit *substantial generalizations* since the assumption that R is a polynomial ring is not necessary. For instance let R be any graded quotient of a polynomial ring $k[x_0,\ldots,x_n]$ with the standard grading where k is any field. In Theorem 4.1 it suffices to have $\mathrm{depth}_{I_{t-1}(\mathcal{A})A} M = \mathrm{depth}_{I_{t-1}(\mathcal{A})A} A$ and the depth assumption of that theorem to see that the proof works ($\tilde{M}|_U$ locally free of rank one holds in general by [5], Lem. 1.4.8). Moreover in Theorem 4.5, Corollary 4.7 and Corollary 4.9 we use a few places that R is Cohen-Macaulay in which case we get $\mathrm{depth}_{I_{t-1}(\mathcal{A})A} M = \mathrm{depth}_{I_{t-1}(\mathcal{A})A} A$ by [13], Cor. A2.13. So all the mentioned results hold if $\mathrm{Proj}(R)$ is any ACM-scheme (i.e. R is CM). The remaining corollaries hold as well if $\mathrm{Proj}(R)$ is a smooth ACM scheme and $k = \overline{k}$ by Remark 2.1. Indeed Remark 2.1 is really a result for determinantal subschemes of any smooth variety W, not only when $W = \mathbb{P}^n$.

5. Deformations of modules and determinantal schemes

The main goal of this section is to show a close relationship between the local deformation functor, $\mathrm{Def}_{M/R}$, of the graded R-module $M = M_{\mathcal{A}}$ and the corresponding local functor, $\mathrm{Def}_{A/R}$, of deforming the determinantal ring $A = R/\mathrm{ann}(M)$ as a graded quotient of R. We will see that these functors are isomorphic (resp. the first is a natural subfunctor of the other) provided $\dim X \geq 2$ (resp. $\dim X = 1$) and $X = \mathrm{Proj}(A)$ is general. If $\dim X = 1$, the mentioned subfunctor is indeed the functor that corresponds to deforming the determinantal k-algebra A as a *determinantal* quotient of R (Definition 5.1). Combining with results of previous sections and the fact that $\mathrm{Def}_{A/R}$ is the same as the local Hilbert (scheme) functor of X if $\dim X \geq 1$ by (1.2), we get the main results of this paper; the dimension formula for $W(\underline{b};\underline{a})$ and the generically smoothness of $\mathrm{Hilb}^p(\mathbb{P}^n)$ along $W(\underline{b};\underline{a})$. The comparison is mostly to understand well a spectral sequence comparing the tangent and obstruction spaces of the mentioned deformation functors and to use the theorems of the previous sections. This spectral sequence is also important in R. Ile's PhD thesis [23], and in his papers [24] and [25] (see Remark 5.4). In the following we suppose A is generically a complete intersection ($\mathrm{depth}_{I_{t-1}(\mathcal{A})A} A \geq 1$), i.e. that $X = \mathrm{Proj}(A)$ is a good determinantal scheme.

Consider the well-known spectral sequence

$$E_2^{p,q} := \mathrm{Ext}^p_A(\mathrm{Tor}^R_q(A,M),M) \quad \Rightarrow \quad \mathrm{Ext}^{p+q}_R(M,M)\ ,$$

and note that $E_2^{p,0} \simeq \mathrm{Ext}_A^p(M,M)$ and $\mathrm{Tor}_q^R(A,M) \simeq \mathrm{Tor}_{q-1}^R(I_X,M)$ for $q \geq 1$. The spectral sequence leads to the following exact sequence

$$0 \to \mathrm{Ext}_A^1(M,M) \to \mathrm{Ext}_R^1(M,M) \to E_2^{0,1} \to \mathrm{Ext}_A^2(M,M) \to \mathrm{Ext}_R^2(M,M) \to E_2^{1,1} \to \ . \tag{5.1}$$

Indeed $E_2^{0,2} = \mathrm{Hom}(\mathrm{Tor}_2^R(A,M),M) = 0$ because $\mathrm{Tor}_2^R(A,M)$ is supported in $V(I_{t-1}(\mathcal{A})A)$. Moreover

$$E_2^{0,1} \simeq \mathrm{Hom}_A(I_X \otimes_R M, M) \simeq \mathrm{Hom}_R(I_X, \mathrm{Hom}_R(M,M)) \ ,$$

and see [25], Def. 3 for an explicit description of $\mathrm{Ext}_R^1(M,M) \to E_2^{0,1}$. In our situation we recall that $\mathrm{depth}_{I_{t-1}(\mathcal{A})A} A \geq 1$ lead to $\mathrm{Hom}_A(M,M) \simeq A$ by [32], Lem. 3.2. It follows that the edge homomorphism $\mathrm{Ext}_R^1(M,M) \to E_2^{0,1}$ of the spectral sequence above induces a natural map

$$ {}_0\mathrm{Ext}_R^1(M,M) \longrightarrow \ (E_2^{0,1})_0 \simeq \ {}_0\mathrm{Hom}_R(I_X,A) \tag{5.2}$$

between the tangent spaces of the two deformation functors $\mathrm{Def}_{M/R}$ and $\mathrm{Def}_{A/R}$ respectively. Even though we only partially use the spectral sequence in the proof below, Theorem 5.2 is fully motivated by the spectral sequence.

Definition 5.1. Let $X = \mathrm{Proj}(A)$, $A = R/I_t(\mathcal{A})$, be a good determinantal scheme and let $\underline{\ell}$ be the category of artinian k-algebras (cf. the text before (2.12)). Then the local deformation functor $\mathrm{Def}_{A \in W(\underline{b};\underline{a})}$, defined on $\underline{\ell}$, is the subfunctor of $\mathrm{Def}_{A/R}$ given by:

$$\mathrm{Def}_{A \in W(\underline{b};\underline{a})}(T) = \left\{ A_T \in \mathrm{Def}_{A/R}(T) | A_T = R_T/I_t(\mathcal{A}_T) \text{ for some matrix } \mathcal{A}_\mathrm{T} \text{ lifting } \mathcal{A} \text{ to } T \right\}.$$

Note that there is a natural map $\mathrm{Def}_{M/R} \to \mathrm{Def}_{A \in W(\underline{b};\underline{a})}$ because for every graded deformation M_T of M to T there exists a matrix $\mathcal{A}_T$ whose induced morphism has M_T as cokernel (see the first part of the proof of Theorem 3.1) and because different matrices inducing the same M_T define the same ideal of maximal minors (Fittings lemma, [13], Cor. 20.4). The map is surjective since we can use the matrix $\mathcal{A}_T$ in Definition 5.1 to define $M_T \in \mathrm{Def}_{M/R}(T)$.

The condition "for some matrix $\mathcal{A}_T$ lifting $\mathcal{A}$ to T" above which means that there exists a homogeneous matrix $\mathcal{A}_T$ lifting $\mathcal{A}$ to T, may be insufficient for forcing $\mathrm{Def}_{A \in W(\underline{b};\underline{a})}$ to have nice properties. For instance we do not know whether $\mathrm{Def}_{A \in W(\underline{b};\underline{a})}$ is pro-representable, or even has a hull, since we have no proof for the surjectivity of

$$\mathrm{Def}_{A \in W(\underline{b};\underline{a})}(T_1 \times_S T_2) \longrightarrow \mathrm{Def}_{A \in W(\underline{b};\underline{a})}(T_1) \times_{\mathrm{Def}_{A \in W(\underline{b};\underline{a})}(S)} \mathrm{Def}_{A \in W(\underline{b};\underline{a})}(T_2) \tag{5.3}$$

for every pair of morphisms $T_i \to S$, $i = 1,2$, in $\underline{\ell}$ with $T_2 \twoheadrightarrow S$ small (see Schlessinger's main theorem in [44]). In [46] Schaps solves a related problem by assuming that $\mathcal{A}$ has the unique lifting property and she gets some results on the existence of a hull for determinantal non-embedded deformations. In our context, assuming ${}_0\mathrm{Ext}_A^1(M,M) = 0$, then we shall see that $\mathrm{Def}_{A \in W(\underline{b};\underline{a})}$ behaves well because for every element of $\mathrm{Def}_{A \in W(\underline{b};\underline{a})}(T)$ there exists a *unique* module $M_{\mathcal{A}_T}$ even though $\mathcal{A}_T$ is not unique.

Indeed let $D := k[\epsilon]/(\epsilon^2)$ be the dual numbers and let

$$\lambda := \dim \ {}_0\mathrm{Ext}_R^1(M,M) = \lambda_c + K_3 + K_4 + ... + K_c \ ,$$

cf. Theorem 3.1. Recalling that $W(\underline{b};\underline{a})$ is a certain quotient of an open irreducible set in the affine scheme $\mathbb{V} = \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{G}^*, \mathcal{F}^*)$ whose rational points correspond to $t \times (t+c-1)$ matrices and that $\dim W(\underline{b};\underline{a}) \leq \lambda$ ([32], p. 2877 and Thm. 3.5), we get

Theorem 5.2. *Let $X = \mathrm{Proj}(A)$, $A = R/I_t(\mathcal{A})$ be a good determinantal scheme. If ${}_0\mathrm{Ext}^1_A(M,M) = 0$ then the functor $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}$ is pro-representable, the pro-representing object has dimension $\dim W(\underline{b};\underline{a})$ and*

$$\mathrm{Def}_{A\in W(\underline{b};\underline{a})} \simeq \mathrm{Def}_{M/R}\ .$$

Hence $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}$ is formally smooth. Moreover the tangent space of $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}$ is the subvector space of ${}_0\mathrm{Hom}_R(I_X, A)$ that corresponds to graded deformation $R_D \to A_D$ of $R \to A$ to D of the form $A_D = R_D/I_t(\mathcal{A}_D)$ for some matrix $\mathcal{A}_D$ which lifts $\mathcal{A}$ to D.

If in addition ${}_0\mathrm{Ext}^2_A(M,M) = 0$, then $\mathrm{Def}_{M/R} \simeq \mathrm{Def}_{A\in W(\underline{b};\underline{a})} \simeq \mathrm{Def}_{A/R}$ and $\mathrm{Def}_{A/R}$ is formally smooth. Moreover every deformation of X comes from deforming $\mathcal{A}$ (cf. Definition 2.6).

Proof. We already know that $\mathrm{Def}_{M/R}(T) \to \mathrm{Def}_{A\in W(\underline{b};\underline{a})}(T)$ is well defined and surjective. To see that it is injective, we will construct an inverse. Suppose therefore that there are two matrices $(\mathcal{A}_T)_1$ and $(\mathcal{A}_T)_2$ lifting $\mathcal{A}$ to T and such that $I_t((\mathcal{A}_T)_1) = I_t((\mathcal{A}_T)_2)$. The two matrices define two graded deformations M_1 and M_2 of the R-modules M to R_T by Lemma 2.5. Since, however, the two matrices define the *same* graded deformation $A_T := R_T/I_t((\mathcal{A}_T)_1)$ of A to T, we get that M_1 and M_2 are two graded deformations of the A-module M to A_T! Due to ${}_0\mathrm{Ext}^1_A(M,M) = 0$, $\mathrm{Hom}_A(M,M) \simeq A$ and deformation theory, we conclude that $M_1 = M_2$ up to multiplication with a unit of T, i.e. we get a well defined map which clearly is an inverse.

Since we have $\mathrm{Def}_{A\in W(\underline{b};\underline{a})} \simeq \mathrm{Def}_{M/R}$ and we know that $\mathrm{Def}_{M/R}$ has a hull ([47]), it follows that $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}$ has a hull (or one may easily show the surjectivity of (5.3) directly by using the uniqueness of $M_{\mathcal{A}_T}$). Note that the injectivity of (5.3) follows from $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}$ being a subfunctor of the pro-representable functor $\mathrm{Def}_{A/R}$ ([29], Prop. 9), whence $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}$ is pro-representable by [44]. Moreover using $\mathrm{Def}_{A\in W(\underline{b};\underline{a})} \simeq \mathrm{Def}_{M/R}$ and Theorem 3.1 we get that $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}$ is formally smooth and that $\dim H = \lambda$ where H is the pro-representing object of $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}$. The description of its tangent space follows from Definition 5.1 and (5.1)-(5.2) since ${}_0\mathrm{Hom}_R(I_X, A)$ is the tangent space of $\mathrm{Def}_{A/R}$.

So far we know $\dim W(\underline{b};\underline{a}) \leq \lambda = \dim H$. To see that $\dim H = \dim W(\underline{b};\underline{a})$, it suffices to see that the family of determinantal rings over H, corresponding to the "universal object" of $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}$, is algebraizable. This is clear in our context, (see the explicit description of H in the proof of [39], Thm. 4.2.4). Indeed take λ independent elements of ${}_0\mathrm{Ext}^1_R(M,M) \simeq \mathrm{Def}_{A\in W(\underline{b};\underline{a})}(D)$, let $\mathcal{A} + \epsilon\mathcal{A}_1, ..., \mathcal{A} + \epsilon\mathcal{A}_\lambda$ be corresponding presentation matrices of the elements (i.e. modules), and let $\mathcal{A}_T := \mathcal{A} + t_1\mathcal{A}_1 + ... + t_\lambda\mathcal{A}_\lambda$ (linear combination in the parameters t_k) where T be the polynomial ring $T = k[t_1, ..., t_\lambda]$. Then the algebraic family $A_T := R_T/I_t(\mathcal{A}_T)$ is T-flat at $(0, ..., 0) \in \mathrm{Spec}(T)$ (Lemma 2.5) and hence flat in a neighborhood and we get what we want.

Finally we suppose ${}_0\mathrm{Ext}^2_A(M,M) = 0$. Using $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}(D) \simeq {}_0\mathrm{Ext}^1_R(M,M)$ and the spectral sequence (5.1) we get isomorphisms $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}(D) \simeq {}_0\mathrm{Hom}_R(I_X, A) \simeq \mathrm{Def}_{A/R}(D)$ of tangent spaces. To show $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}(T) \simeq \mathrm{Def}_{A/R}(T)$ for any $(T, \mathfrak{m}_T)$ in $\underline{\ell}$, we may by induction suppose $\mathfrak{m}_T^{r+1} = 0$ and $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}(T/\mathfrak{m}_T^r) \simeq \mathrm{Def}_{A/R}(T/\mathfrak{m}_T^r)$.

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Def}_{A\in W(\underline{b};\underline{a})}(T) & \hookrightarrow & \mathrm{Def}_{A/R}(T) \\ \downarrow & & \downarrow \\ \mathrm{Def}_{A\in W(\underline{b};\underline{a})}(T/\mathfrak{m}_T^r) & \simeq & \mathrm{Def}_{A/R}(T/\mathfrak{m}_T^r) \end{array} \tag{5.4}$$

and notice that the leftmost vertical map is surjective since $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}$ is formally smooth. Hence for a given $A_T \in \mathrm{Def}_{A/R}(T)$ there exists $A'_T \in \mathrm{Def}_{A/R}(T)$ such that $A'_T \simeq R_T/I_t(\mathcal{A}_T)$ for some matrix $\mathcal{A}_T$ which lifts a matrix $\mathcal{A}_{T/\mathfrak{m}_T^r}$ defining $A_T \otimes_T T/\mathfrak{m}_T^r$ to T. The difference of A_T and A'_T belongs to $\mathrm{Def}_{A/R}(D) \otimes_k \mathfrak{m}_T^r \simeq \mathrm{Def}_{A\in W(\underline{b};\underline{a})}(D) \otimes_k \mathfrak{m}_T^r$, and "adding" it to A'_T we get that $A_T \in \mathrm{Def}_{A\in W(\underline{b};\underline{a})}(T)$, whence $\mathrm{Def}_{A\in W(\underline{b};\underline{a})}(T) \simeq \mathrm{Def}_{A/R}(T)$. It follows that the completion of the local ring $\mathcal{O}_{\mathrm{Hilb},(X)}$ of $\mathrm{Hilb}^p(\mathbb{P}^n)$ at (X) is isomorphic to H. Since we in the preceding paragraph explicitly constructed an algebraic determinantal family over some neighborhood of $(0,...,0)$ in $\mathrm{Spec}(k[t_1,...,t_\lambda])$ (thinking about it, we must have $\mathcal{O}_{\mathrm{Hilb},(X)} \simeq k[t_1,...,t_\lambda]_{(t_1,...,t_\lambda)}$ since $k=\overline{k}$), we get that "every deformation of X comes from deforming $\mathcal{A}$" and we are done. □

Remark 5.3. Let us endow the closed subset $\overline{W(\underline{b};\underline{a})}$ of $\mathrm{Hilb}^p(\mathbb{P}^n)$ with the reduced scheme structure (this is natural since "the part $W(\underline{b};\underline{a})$ of $\mathrm{Hilb}^p(\mathbb{P}^n)$ is unobstructed" by the proof of Lemma 2.7). Let $X=\mathrm{Proj}(A)$, $A=R/I_t(\mathcal{A})$ belong to $W(\underline{b};\underline{a})$. Then the proof related to $\dim W(\underline{b};\underline{a})=\lambda$ above imply that the Zariski tangent space, $(\mathfrak{m}_W/\mathfrak{m}_W^2)^\vee$, of $W(\underline{b};\underline{a})$ satisfies

$$(\mathfrak{m}_W/\mathfrak{m}_W^2)^\vee \;=\; \mathrm{Def}_{A\in W(\underline{b};\underline{a})}(D)\ . \tag{5.5}$$

In the proof we used $\dim W(\underline{b};\underline{a}) \le \lambda$ ([32], Thm. 3.5) to show $\dim W(\underline{b};\underline{a})=\lambda$. We will now explain this inequality by a direct argument. Indeed take any $(X') \in W(\underline{b};\underline{a})$. Then there is a matrix $t\times(t+c-1)$ matrix $\mathcal{A}'$ whose maximal minors define X'. By Lemma 2.5 the matrix $\mathcal{A}+x(\mathcal{A}'-\mathcal{A})$, x a parameter, defines a flat family of good determinantal schemes over some open set $U \subset \mathrm{Spec}(k[x]) \simeq \mathbb{A}^1$ containing $x=0$ and $x=1$. Thus to any $(X') \in W(\underline{b};\underline{a})$ there is a tangent direction, i.e. an element t_X of $(\mathfrak{m}_W/\mathfrak{m}_W^2)^\vee \subset {}_0\mathrm{Hom}_R(I_X,A) = \mathrm{Def}_{A/R}(D)$, given by the matrix $\mathcal{A}'-\mathcal{A}$. By Definition 5.1, $t_X \in \mathrm{Def}_{A\in W(\underline{b};\underline{a})}(D)$, thus $(\mathfrak{m}_W/\mathfrak{m}_W^2)^\vee \subset \mathrm{Def}_{A\in W(\underline{b};\underline{a})}(D)$ by the relationship between $W(\underline{b};\underline{a})$ and its Zariski tangent space. Taking dimensions we have shown $\dim W(\underline{b};\underline{a}) \le \lambda$. Then the proof of Theorem 5.2 implies $\dim W(\underline{b};\underline{a})=\lambda$ and hence we get (5.5).

Remark 5.4. If the assumption ${}_0\mathrm{Ext}^1_A(M,M)=0$ of Theorem 5.2 is not satisfied, then the local deformation functor $\mathrm{Def}_{M/A}$ of deforming M *as a graded A-module* and its connection to $\mathrm{Def}_{M/R}$ may be quite complicated, see [25] which compares the corresponding non-graded functors using (5.1). However, by the results of the preceding section, $\mathrm{Ext}^1_A(M,M)=0$ and $\mathrm{Hom}_A(M,M)\simeq A$ are weak assumptions for modules of maximal grade.

We now deduce the main theorems of the paper. In the first theorem we let

$$\mathrm{ext}^2(M,M) := \dim\ker(\ {}_0\mathrm{Ext}^2_A(M,M) \to\ {}_0\mathrm{Ext}^2_R(M,M)\ )\,,\quad \text{cf. (5.1)},$$

and *notice* that we write $\mathrm{Hilb}(\mathbb{P}^n)$ for $\mathrm{Hilb}^p(\mathbb{P}^n)$ (resp. $\mathrm{GradAlg}(H)$) if $n-c\ge 1$ (resp. $n-c=0$), cf. the text accompanying (1.2) for explanations and notations.

Theorem 5.5. *Let $X = \mathrm{Proj}(A) \subset \mathbb{P}^n$, $A = R/I_t(\mathcal{A})$ be a good determinantal scheme of $W(\underline{b};\underline{a})$ of dimension $n-c \geq 0$, let $M = M_{\mathcal{A}}$ and suppose ${}_0\mathrm{Ext}^1_A(M,M) = 0$. Then*

$$\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + K_4 + ... + K_c \ .$$

Moreover, for the codimension of $W(\underline{b};\underline{a})$ in $\mathrm{Hilb}(\mathbb{P}^n)$ in a neighborhood of (X) we have

$$\dim_{(X)} \mathrm{Hilb}(\mathbb{P}^n) - \dim W(\underline{b};\underline{a}) \leq \mathrm{ext}^2(M,M) \ ,$$

with equality if and only if $\mathrm{Hilb}(\mathbb{P}^n)$ is smooth at (X). In particular these conclusions hold if $\mathrm{depth}_{I_{t-1}(\mathcal{A})A} A \geq 3$, or if $n-c \geq 1$ and we get an l.c.i. (e.g. a smooth) scheme by deleting some column of $\mathcal{A}$.

Proof. This follows from Theorem 5.2, Theorem 3.1, (5.1)-(5.2) and Corollary 4.7. □

Corollary 5.6. *Given integers $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$ and $b_1 \leq ... \leq b_t$, we suppose $n-c \geq 1$ and $a_{i-2} - b_i \geq 0$ for $2 \leq i \leq t$. Then*

$$\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + K_4 + ... + K_c \ .$$

provided $\dim W(\underline{b};\underline{a}) \neq \emptyset$. In particular Conjecture 4.1 of [33] *holds in the case $n-c \geq 1$.*

Proof. This follows from Theorem 5.5 and Corollary 4.8 since Conjecture 4.1 of [33] is Conjecture 2.2 of this paper (and remember that we always suppose $c \geq 2$ and $t \geq 2$). □

Remark 5.7. Even for zero-dimensional determinantal schemes ($n - c = 0$) the assumption ${}_0\mathrm{Ext}^1_A(M,M) = 0$ seems very weak, and hence we almost always have the conjectured value of $\dim W(\underline{b};\underline{a})$. Thus Theorem 5.5 completes Theorem 4.19 of [30] in the zero-dimensional case. Indeed in computing many examples by Macaulay 2 we have so far only found ${}_0\mathrm{Ext}^1_A(M,M) \neq 0$ for examples outside the range of Conjecture 2.2.

Note that $\overline{W(\underline{b};\underline{a})}$ is not always an irreducible component of $\mathrm{Hilb}(\mathbb{P}^n)$. An example showing this was given in [31], Ex. 10.5, and many more were found in [30], Ex. 4.1, in which there are examples for every $c \geq 3$ (the matrix is linear except for the last column). All examples satisfy $n - c \leq 1$. Indeed [30] contains exact formulas for the codimension of $\overline{W(\underline{b};\underline{a})}$ in $\mathrm{Hilb}(\mathbb{P}^n)$ under some assumptions. Further investigations in [33] led us to conjecture that $\overline{W(\underline{b};\underline{a})}$ is an irreducible component provided $n - c \geq 2$. Now we can prove it!

Theorem 5.8. *Let $X = \mathrm{Proj}(A) \subset \mathbb{P}^n$, $A = R/I_t(\mathcal{A})$ be a good determinantal scheme of $W(\underline{b};\underline{a})$ of dimension $n-c \geq 1$, let $M = M_{\mathcal{A}}$ and suppose ${}_0\mathrm{Ext}^i_A(M,M) = 0$ for $i = 1$ and 2. Then the Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^n)$ is smooth at (X),*

$$\dim_{(X)} \mathrm{Hilb}^p(\mathbb{P}^n) = \lambda_c + K_3 + K_4 + ... + K_c \ ,$$

and every deformation of X comes from deforming $\mathcal{A}$. In particular this conclusion holds if $\mathrm{depth}_{I_{t-1}(\mathcal{A})A} A \geq 4$, or if $n - c \geq 2$ and we get an l.c.i. (e.g. a smooth) scheme by deleting some column of $\mathcal{A}$.

Proof. This follows immediately from Theorem 5.2, Theorem 3.1 and Corollary 4.9. □

Corollary 5.9. *Given integers* $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$ *and* $b_1 \leq ... \leq b_t$, *we suppose* $n-c \geq 1$, $a_{i-2} - b_i \geq 0$ *for* $2 \leq i \leq t$ *and* ${}_0\mathrm{Ext}^2_A(M,M) = 0$ *for a general* $X = \mathrm{Proj}(A)$ *of* $W(\underline{b};\underline{a})$. *Then the closure* $\overline{W(\underline{b};\underline{a})}$ *is a generically smooth irreducible component of the Hilbert scheme* $\mathrm{Hilb}^p(\mathbb{P}^n)$ *of dimension*

$$\lambda_c + K_3 + K_4 + ... + K_c \ .$$

In particular this conclusion holds if $n-c \geq 2$, $a_{i-\min(3,t)} \geq b_i$ *for* $\min(3,t) \leq i \leq t$ *and* $\dim W(\underline{b};\underline{a}) \neq \emptyset$. *It follows that Conjecture 4.2 of* [33] *holds.*

Proof. This follows from Corollary 4.8, Theorem 5.8, Lemma 2.7 and Corollary 4.10, and note that Conjecture 4.2 of [33] is the same as Conjecture 2.4 of this paper. □

Even in the one-dimensional case ($n-c=1$) the assumption ${}_0\mathrm{Ext}^2_A(M,M) = 0$ seems rather weak, and we can often conclude as in Corollary 5.9. Note that if ${}_0\mathrm{Ext}^2_A(M,M) = 0$ for a general X of $W(\underline{b};\underline{a})$ and $a_{i-2} - b_i \geq 0$ for $2 \leq i \leq t$, we get

$$ {}_0\mathrm{hom}_R(I_X, A) = \ \lambda_c + K_3 + K_4 + ... + K_c \tag{5.6}$$

by Corollary 4.8 and (5.1)-(5.2). So one may alternatively skip assuming ${}_0\mathrm{Ext}^2_A(M,M) = 0$ and instead compute ${}_0\mathrm{hom}_R(I_X, A)$ and check if (5.6) holds, to conclude as in Corollary 5.9 (cf. Theorem 5.2). In [30], Prop. 4.15 we gave several criteria for describing $\overline{W(\underline{b};\underline{a})}$ in the one-dimensional case. None of them apply in the following example.

Example 5.10 (determinantal curves in $\mathbb{P}^4$, i.e. with $c=3$)**.**

Let $\mathcal{A} = (f_{ij})$ be a 2×4 matrix whose entries are general polynomials of the same degree $\deg f_{ij} = 2$. The vanishing of all 2×2 minors of $\mathcal{A}$ defines a smooth curve X of degree 32 and genus 65 in $\mathbb{P}^4$. A Macaulay 2 computation shows ${}_0\mathrm{Ext}^2_A(M,M) = 0$. It follows from Corollary 5.9 that $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^4)$ of dimension $\lambda_3 + K_3 = 101$.

Note that our previous method was to delete a column to get a matrix $\mathcal{B}$ and an algebra $B := R/J$, $J := I_t(\mathcal{B})$ and to verify ${}_0\mathrm{Ext}^1_B(J/J^2, I/J) = 0$ with $I = I_t(\mathcal{A})$. However, by Macaulay 2, ${}_0\mathrm{Ext}^1_B(J/J^2, I/J)$ as well as ${}_0\mathrm{Ext}^1_A(I/I^2, A)$, are 5-dimensional and the approach of using Prop. 4.15 (i) does not apply (since ${}_0\mathrm{Ext}^1_B(I/J, A) \neq 0$), neither do Prop. 4.15 (ii) nor (iii), and certainly not our earliest result in [31] on this topic since Prop. 4.15 generalizes [31, Cor. 10.15] for curves.

It is known that the vanishing of the cohomology group $H^1(\mathcal{N}_X)$ (resp. $\mathrm{Ext}^1_A(I_X/I_X^2, A)$) of a locally (resp. generically) complete intersection $X \hookrightarrow \mathbb{P}$ implies that X is unobstructed, and that the converse is not true, e.g. we may have $H^1(\mathcal{N}_X) \neq 0$ for X unobstructed. Since we by Theorem 5.8 get that X is unobstructed by mainly assuming $n-c \geq 2$, one may wonder if we can prove a little more, namely $H^1(\mathcal{N}_X) = 0$. Indeed we can if $n-c \geq 3$. More precisely recalling $\mathrm{depth}_J A = \dim A - \dim A/J$ we have

Theorem 5.11. *Let* $X = \mathrm{Proj}(A) \subset \mathbb{P}^n$, $A = R/I_t(\mathcal{A})$ *be a standard determinantal scheme.*

(i) *If* $\mathrm{depth}_{I_{t-1}(\mathcal{A})A} A \geq 4$ *or equivalently,* $\dim X \geq 3 + \dim R/I_{t-1}(\mathcal{A})$, *then*

$$\mathrm{Ext}^i_A(I_X/I_X^2, A) = 0 \ \text{ for } 1 \leq i \leq \dim X - 2 - \dim R/I_{t-1}(\mathcal{A}), \ \text{ and}$$
$$H^i(\mathcal{N}_X(v)) = 0 \ \text{ for } \ 1 \leq i \leq \dim X - 2 \ \text{ and every } v.$$

(ii) *In particular if* $\dim X \geq 3$, $a_{i-\min(3,t)} \geq b_i$ *for* $\min(3,t) \leq i \leq t$ *and* X *is general in* $W(\underline{b};\underline{a})$, *then conclusions of* (i) *hold. If furthermore* $a_j \geq b_i$ *for every* j *and* i, *then*

$$\mathrm{Ext}^i_A(I_X/I_X^2, A) = 0 \quad \text{for } 1 \leq i \leq \min\{\dim X - 2, c-1\}\ .$$

Proof. (i) Using (5.1) -(5.2) and Corollary 4.9 we get $\mathrm{Ext}^1_R(M,M) \simeq \mathrm{Hom}_R(I_X, A)$. It follows that

$$\operatorname{depth} \mathrm{Hom}_R(I_X, A) \geq \dim X \tag{5.7}$$

by Theorem 3.1. Thus the local cohomology group $H^i_{\mathfrak{m}}(\mathrm{Hom}_R(I_X, A))$ vanishes for $i < \dim X$. Recalling that the sheafification of $\mathrm{Hom}_R(I_X, A) \simeq \mathrm{Hom}_A(I_X/I_X^2, A)$ is $\mathcal{N}_X$, we get $H^i_*(\mathcal{N}_X) = 0$ for $1 \leq i < \dim X - 1$, whence we have the second vanishing of (i).

Next let $r := \operatorname{depth}_J A - 1$ where J is the ideal $I_{t-1}(\mathcal{A})A$ of A. It is known that (5.7) also implies $\operatorname{depth}_J \mathrm{Hom}_R(I_X, A) \geq r$ (e.g. [35], Lem. 28). Thus the local cohomology group $H^i_J(\mathrm{Hom}_A(I_X/I_X^2, A))$ vanishes for $i < r$ and we get the first vanishing of (i) by (2.9) (letting $N = I_X/I_X^2$ and $L = A$).

(ii) Finally we use Remark 2.1 with $\alpha = 3$ (resp. $\alpha \geq (c+3)/2$) to see that $\operatorname{depth}_J A \geq 4$ (resp. $\operatorname{codim}_X V(J) \geq c+2$). In particular (i) applies to get the first statement. For the final statement, we recall the well known fact that $c+2$ is the largest possible value of the height of J in A, whence $\operatorname{codim}_X V(J) = c+2$ with the usual interpretation that $c+2 = \operatorname{codim}_X V(J) > \dim X$ implies $V(J) = \emptyset$. This implies the theorem. □

For the algebra cohomology groups $\mathrm{H}^i(R, A, A)$ of André-Quillen (cf. [1]) we deduce

Corollary 5.12. *Let* $A = R/I_t(\mathcal{A})$ *be a standard determinantal graded* k*-algebra.*

(i) *If* $\operatorname{depth}_{I_{t-1}(\mathcal{A})A} A \geq 4$ *then*

$$\mathrm{H}^i(R, A, A) = 0 \quad \text{for} \quad 2 \leq i \leq \operatorname{depth}_{I_{t-1}(\mathcal{A})A} A - 2\,.$$

(ii) *If* $\dim A \geq 4$, $a_j \geq b_i$ *for every* j, i *and* $\mathrm{Proj}(A)$ *is general in* $W(\underline{b};\underline{a})$, *then*

$$\mathrm{H}^i(R, A, A) = 0 \quad \text{for} \quad 2 \leq i \leq \min\{\dim A - 2, c\}\,.$$

Proof. The spectral sequence relating algebra cohomology to algebra homology ([1], Prop 16.1 or [39]), implies, under the sole assumption $\operatorname{depth}_{I_{t-1}(\mathcal{A})A} A \geq 1$, that

$$\mathrm{Ext}^i_A(I_X/I_X^2, A) \simeq \mathrm{H}^{i+1}(R, A, A)\,.$$

□

Remark 5.13. (i) The vanishing of $H^i_*(\mathcal{N}_X)$ of Theorem 5.11 is known if $c = 3$ ([35], Lem. 35) or $c = 4$ ([32], Cor. 5.5). It $c = 2$ even more is true by [9] (or see [31], Cor. 6.5).

(ii) Note that Corollary 5.12 for so-called *generic* determinantal schemes is proved by Svanes (see [6], Thm. 15.10) while [2], (1.4.3) shows the corollary for some non-generic determinantal schemes as well.

(iii) As for $c = 2$ one may hope that $H^i_*(\mathcal{N}_X) = 0$ also for $i = \dim X - 1$. This is not true, as one may see through examples, using e.g. Macaulay 2. We have checked it for some surfaces in the range $3 \leq c < 6$ and always found it to be non-zero (cf. [6], 15.11).

Remark 5.14. In proving Theorem 5.11 we used Corollary 4.10 to see that not only ${}_0\mathrm{Ext}^i_A(M,M)$ vanishes for $i=1,2$, but in fact that the whole $\mathrm{Ext}^i_A(M,M)$-group vanishes for $i=1$ and 2. Arguing as in Theorem 5.2 and using the vanishing of the whole $\mathrm{Ext}^i_A(M,M)$-group for $i=1$ and 2, we may see that the non-graded deformation functor;

$$\mathrm{Def}^{\mathrm{non-gr}}_{A/R}(T)=\{R_T\to A_T | A_T \text{ is } T-\text{flat and } \ A_T\otimes_T k\simeq A\}$$

in which a deformation A_T of $R\to A$ to an artinian T in $\underline{\ell}$ is possibly non-graded, is formally smooth provided $\mathrm{depth}_{I_{t-1}(\mathcal{A})A}A\geq 4$, or $\dim X\geq 2$ and we get an l.c.i. scheme by deleting some column of $\mathcal{A}$. This result is the best possible with regard to $\dim X\geq 2$ because one knows that $\mathrm{Def}^{\mathrm{non-gr}}_{A/R}$ is non-smooth for a one-dimensional rational normal scroll $\mathrm{Proj}(A)\subset\mathbb{P}^n$ for $n\geq 4$ ([43]). Note also that we may deduce the result above for generic determinantal schemes satisfying $\dim X\geq 3$ by works of Svanes ([6], Thm. 15.10)

Remark 5.15. The results so far of this section admit *substantial generalizations* with respect to R being a polynomial ring. Indeed we may let R be any graded CM quotient of a polynomial ring $k[x_0,\dots,x_n]$, $k=\overline{k}$, with the standard grading provided we in all results replace $\mathbb{P}^n$ by $\mathrm{Proj}(R)$ and interpret the assumption "A good determinantal" by "A standard determinantal satisfying $\mathrm{depth}_{I_{t-1}(\mathcal{A})A}A\geq 1$" ([31], Prop. 3.2). Then the proof of Theorem 5.2 works (we need Remark 5.3) since we have $\mathrm{Hom}_R(M,M)\simeq A$ by Remark 3.9. Using Remark 4.11 we get that Theorem 5.5, Theorem 5.8 and Theorem 5.11(i) are valid in this generality while it for the corollaries and Theorem 5.11 (ii) suffices to suppose that $\mathrm{Proj}(R)$ is a smooth ACM-scheme (in the case $c>2$, see the next theorem for $c=2$). Note that the assumption $k=\overline{k}$ allows us to keep the definition $W(\underline{b};\underline{a})$ as a certain locus in $\mathrm{Hilb}^p(\mathbb{P}^n)$.

Finally we will illustrate the results mentioned in the last remark to see that, in addition to reproving and generalizing Ellingsrud's codimension 2 result ([14]) a little, we can enlighten the differences between the cases $c=2$ and $c>2$. Indeed the main ingredient is that if $c=2$ and $X=\mathrm{Proj}(A)$ is standard determinantal in an ACM scheme $Y=\mathrm{Proj}(R)$, then $M\simeq K_A(s)$ for some integer s where K_A is the canonical module of A (cf. the line before (2.7)). It follows that we do not need the results of section 4 at all to conclude that ${}_0\mathrm{Ext}^i_A(M,M)=0$ for $i>0$ because this is well known. Moreover in section 5 we needed the weak assumption $\mathrm{depth}_{I_{t-1}(\mathcal{A})A}A\geq 1$ to get $\mathrm{Hom}_A(M,M)\simeq A$ which was central in (5.1) -(5.2) and hence in the proof of Theorem 5.2. Now this isomorphism always holds, again by $M\simeq K_A(s)$, and we get $\mathrm{Def}_{M/R}\simeq\mathrm{Def}_{A/R}$ without requiring $\mathrm{depth}_{I_{t-1}(\mathcal{A})A}A\geq 1$. These functors are formally smooth (Theorem 3.1, Remarks 3.2 and 3.9) and we deduce the theorem below where we interpret $\mathrm{Hilb}(Y)$ as $\mathrm{Hilb}^p(Y)$ (resp. $\mathrm{GradAlg}(H)$) if $\dim X\geq 1$ (resp. $\dim X=0$) as in Theorem 5.5. Notice that we now deal with standard determinantal schemes X of codimension 2 *in* $Y=\mathrm{Proj}(R)$ (they are usually not determinantal schemes in $\mathbb{P}^n$). With $\underline{b}$, $\underline{a}$ as in (3.2) and $X\in W_s(\underline{b};\underline{a})$ we get

Theorem 5.16. *Let $Y=\mathrm{Proj}(R)\subset\mathbb{P}^n_k$ be an ACM scheme where k is any field and let $X=\mathrm{Proj}(A)\subset Y$, $A=R/I_t(\mathcal{A})$, be any standard determinantal scheme of codimension 2 in Y. Then* $\mathrm{Hilb}(Y)$ *is smooth at* (X) *and* $\dim_{(X)}\mathrm{Hilb}(Y)=\lambda(R)_2$ *where*

$$\lambda(R)_2:=\sum_{i,j}\dim R_{(a_i-b_j)}+\sum_{i,j}\dim R_{(b_j-a_i)}-\sum_{i,j}\dim R_{(a_i-a_j)}-\sum_{i,j}\dim R_{(b_i-b_j)}+1.$$

Moreover every deformation of X comes from deforming $\mathcal{A}$. In particular if $k = \overline{k}$, then $\mathrm{Hilb}(Y)$ *is smooth along* $W_s(\underline{b};\underline{a})$ *and the closure* $\overline{W_s(\underline{b};\underline{a})}$ *in* $\mathrm{Hilb}(Y)$ *is an irreducible component of dimension* $\lambda(R)_2$.

Indeed there are no singular points (X) of $\mathrm{Hilb}(Y)$, $(X) \in W_s(\underline{b};\underline{a})$ while singular points of $\mathrm{Hilb}(Y)$ for $c > 2$ at $(X) \in W_s(\underline{b};\underline{a})$ are quite common (see [40] and Rem. 3.6 of [33]).

References

[1] M. André. Homologie des algbres commutatives. Die Grundlehren der mathematischen Wissenschaften, Band 206. Springer-Verlag, Berlin-New York, 1974.

[2] K. Behnke and J.A. Christophersen, *Hypersurface sections and obstructions (rational surface singularities*, Compos. Math. **77** n. 3 (1991), 233-268.

[3] G.M. Besana, M.L. Fania *The Dimension of the Hilbert Scheme of Special Threefolds*, Comm. in Alg., **33** (2005), 3811-3829.

[4] W. Bruns, *The Eisenbud-Evans generalized principal ideal theorem and determinantal ideals*, Proc. Amer. Math. Soc. **83** (1981), 19-24.

[5] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, **39**. Cambridge University Press, Cambridge, 1993.

[6] W. Bruns and U. Vetter, *Determinantal rings*, Springer-Verlag, Lectures Notes in Mathematics **1327**, New York/Berlin, 1988.

[7] D.A. Buchsbaum and D. Eisenbud, *What Annihilates a Module?* J. Algebra **47**, (1977), 231-243.

[8] D.A. Buchsbaum and D.S. Rim, *A generalized Koszul complex, II. Depth and multiplicity*, Trans Amer. Math. Soc. **111** (1964), 457–473.

[9] R.O. Buchweitz, *Contributions a la theorie des singularites,* Thesis l'Université Paris VII (1981).

[10] M.C. Chang, *A filtered Bertini-type theorem*, Crelle J. **397** (1989), 214-219.

[11] J.A. Eagon and M. Hochster, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020-1058.

[12] J.A. Eagon and D.G. Northcott, *Ideals defined by matrices and a certain complex associated with them*, Proc. Roy. Soc. London **269** (1962), 188-204.

[13] D. Eisenbud, *Commutative Algebra. With a view toward algebraic geometry*, Springer-Verlag, Graduate Texts in Mathematics **150** (1995).

[14] G. Ellingsrud, *Sur le schéma de Hilbert des variétés de codimension 2 dans* $\mathbb{P}^e$ *a cône de Cohen-Macaulay*, Ann. Scient. Éc. Norm. Sup. **8** (1975), 423-432.

[15] D. Faenzi and M.L. Fania, *On the Hilbert scheme of determinantal subvarities*, ArXiv: math.AG/1012.4692.

[16] E. Gorla, *A generalized Gaeta's Theorem*, Compos. Math. **144** (2008), no. 3, 689-704.

[17] D. Grayson and M. Stillman, *Macaulay 2-a software system for algebraic geometry and commutative algebra*, available at http://www.math.uiuc.edu/Macaulay2/ .

[18] A. Grothendieck, *Les schémas de Hilbert*, Séminaire Bourbaki, exp. **221** (1960).

[19] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschetz locaux et globaux*, North Holland, Amsterdam (1968).

[20] J. Harris, *Algebraic Geometry: a first course*, GTM **133** Springer–Verlag, New York, 1992.

[21] R. Hartshorne, *Algebraic Geometry*, GTM **52**, Springer-Verlag, New York, 1983.

[22] J. Herzog, *Deformationen von Cohen-Macaulay Algebren*, Crelle J. **318** (1980), 83-105.

[23] R. Ile, *Obstructions to deforming modules*, PhD thesis, University of Oslo, 2001.

[24] R. Ile, *Deformation theory of rank one maximal Cohen-Macaulay modules on hypersurface singularities and the Scandinavian complex*, Compos. Math. **140** (2004), 435-446.

[25] R. Ile, *Change of rings in Deformation theory of modules*, Trans Amer. Math. Soc. **356** no. 12 (2004), 4873-4896.

[26] D. Kirby, *A sequence of complexes associated with a matrix*, J. London Math. Soc. (2), **7** (1973), 523-530.

[27] S.L. Kleiman and J. Landolfi, *Geometry and deformation of special Schubert varities*, Compos. Math. **23** (1971), no. 3, 407-434.

[28] J.O. Kleppe, *The smoothness and the dimension of* $\mathrm{PGor}(H)$ *and of other strata of the punctual Hilbert scheme,* J. Algebra **200** no. 2 (1998), 606–628.

[29] J.O. Kleppe, *Maximal families of Gorenstein algebras*, Trans Amer. Math. Soc. **358** no. 7 (2006), 3133–3167.

[30] J.O. Kleppe, *Families of low dimensional determinantal schemes*, Journal of Pure and Applied Algebra, 215 no. 7 (2011) 1711-1725. doi 10.1016/j.jpaa.2010.10.007
[31] J.O. Kleppe, J. Migliore, R.M. Miró-Roig, U. Nagel and C. Peterson, *Gorenstein liaison, complete intersection liaison invariants and unobstructedness*, Memoirs A.M.S **732**, 2001.
[32] J.O. Kleppe and R.M. Miró-Roig, *Dimension of families of determinantal schemes*, Trans. Amer. Math. Soc. **357** (2005), 2871-2907.
[33] J.O. Kleppe and R.M. Miró-Roig, *Families of determinantal schemes*, Proc. Amer. Math. Soc. 139 (2011), 3831-3843.
[34] J.O. Kleppe and R.M. Miró-Roig, *On the normal sheaf of determinantal varieties.* Preprint, see http://www.iu.hio.no/ jank/papers.htm.
[35] J.O. Kleppe and C. Peterson. *Sheaves with Canonical Determinant on Cohen-Macaulay schemes.* J. Algebra **256** (2002), 250-279.
[36] J. Kreuzer, J. Migliore, U. Nagel and C. Peterson, *Determinantal schemes and Buchsbaum-Rim sheaves*, JPAA **150** (2000), 155-174.
[37] D. Laksov, *Deformation and transversality*, Lect. Notes in Math., Proceedings Copenhagen, Springer-Verlag **732** (1978), 300-316.
[38] D. Laksov, *Deformation of determinantal schemes*, Compos. Math. **30** (1975), no. 3, 273-292.
[39] O.A. Laudal, *Formal Moduli of Algebraic Structures*, Lect. Notes in Math., **754**, Springer–Verlag, New York, 1979.
[40] M. Martin-Deschamps and R. Piene, *Arithmetically Cohen-Macaulay curves in $\mathbb{P}^4$ of degree 4 and genus 0*, Manuscripta Math. **93** (1997), 391-408.
[41] R.M. Miró-Roig, *Determinantal ideals*, Progress in Mathematics, **264** Birkhuser Verlag, Basel, 2008.
[42] Ch. Peskine, L. Szpiro, *Liaison des varietes algebrique*, Invent. Math. **26** (1974), 271-302.
[43] H.C. Pinkham, *Deformations of algebraic varieties with G_m action*, Astrisque **20** (1974).
[44] M. Schlessinger, *Functors of Artin rings.* Trans. Amer. Math. Soc. **130** (1968), 208-222.
[45] M. Schlessinger, *On rigid singularities.* Proc. of the Rice University Conference, 1972.
[46] M. Schaps, *Versal determinantal deformations.* Pacific J. Math. **107**, no. 1, (1983), 213-221.
[47] A. Siqveland, *The Method of Computing Formal Moduli.* J. Algebra **241** (2001), 292-327.
[48] T. Svanes, *Coherent Cohomology on flag manifolds and rigidity*, Thesis, MIT, Cambridge Mass., 1972.
[49] T. Svanes, *Some Criteria for Rigidity of Noetherian Rings*, Math. Z. **144** (1975), 135-145.
[50] B. Ulrich, *Ring of invariants and linkage of determinantal ideals*, Math. Ann. **274** (1986), 1-17.
[51] J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, no. 149, Cambridge University Press, 2003.

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