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Notes on the Kazhdan–Lusztig Theorem on Equivalence of the Drinfeld Category and the Category of $U_q\mathfrak{g}$-Modules

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Abstract We discuss the proof of Kazhdan and Lusztig of the equivalence of the Drinfeld category $\mathcal{D}(\mathfrak{g},\hbar)$ of $\mathfrak{g}$-modules and the category of finite dimensional $U_q\mathfrak{g}$-modules, $q=e^{\pi i\hbar}$, for $\hbar\in\mathbb{C}\setminus\mathbb{Q}^*$. Aiming at operator algebraists the result is formulated as the existence for each $\hbar\in i\mathbb{R}$ of a normalized unitary 2-cochain $\mathcal{F}$ on the dual $\hat{G}$ of a compact simple Lie group G such that the convolution algebra of G with the coproduct twisted by $\mathcal{F}$ is $*$-isomorphic to the convolution algebra of the q-deformation G_q of G, while the coboundary of $\mathcal{F}^{-1}$ coincides with Drinfeld's KZ-associator defined via monodromy of the Knizhnik–Zamolodchikov equations.

Keywords Quantum groups · Drinfeld category · Quasi-bialgebras · Unitary twist

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1 Introduction

One of the most beautiful and important results in quantum groups is the theorem of Drinfeld [4, 5] stating that the category of $U_h\mathfrak{g}$-modules is equivalent to a category of $\mathfrak{g}$-modules with the usual tensor product but with nontrivial associativity morphisms

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defined by the monodromy of the Knizhnik–Zamolodchikov equations from conformal field theory. In defining the latter category, known as the Drinfeld category, Drinfeld was inspired by a result of Kohno which states that the representation of the braid group defined by the universal R-matrix of $U_h\mathfrak{g}$ is equivalent to the monodromy representation of the KZ-equations. Drinfeld proved equivalence of the categories working in the context of quasi-Hopf algebras, which are generalizations of Hopf algebras and are algebraic counterparts of monoidal categories with quasi-fiber functors. In this language the result says that there exists $\mathcal{F} \in (U\mathfrak{g} \otimes U\mathfrak{g})[[h]]$ such that the coproduct $\hat{\Delta}_h$ on $U_h\mathfrak{g} \cong U\mathfrak{g}[[h]]$ is given by $\hat{\Delta}_h = \mathcal{F}\hat{\Delta}(\cdot)\mathcal{F}^{-1}$ and that

$$(\iota \otimes \hat{\Delta})(\mathcal{F}^{-1})(1 \otimes \mathcal{F}^{-1})(\mathcal{F} \otimes 1)(\hat{\Delta} \otimes \iota)(\mathcal{F})$$

coincides with the element Φ_{KZ} defining the associativity morphisms in the Drinfeld category. Drinfeld worked in the formal deformation setting and gave two different proofs. Another proof of the equivalence of the categories that works for all irrational complex parameters was given a few years later by Kazhdan and Lusztig [12, 13]. Their approach was then used by Etingof and Kazhdan [7] to solve the problem of existence of quantization of an arbitrary Lie bialgebra.

The result of Kazhdan and Lusztig can again be formulated in algebraic terms, that is, there exists an analogue of the twist $\mathcal{F}$ in the analytic setting. In [17] we observed that such an element can be used to construct a deformation of the Dirac operator on quantum groups that gives rise to spectral triples. These notes originated from a desire to understand better properties of $\mathcal{F}$ for the study of these quantum Dirac operators. Another motivation is that the result of Kazhdan and Lusztig is not usually formulated in the form we need. Even though the formulation we are using should be obvious to a careful reader, to refer this away to a series of papers totaling several hundred pages seems inappropriate. What makes the situation more complicated is that Kazhdan and Lusztig prove a more general result allowing rational deformation parameters, in which case the Drinfeld category has to be replaced by a category of modules over the affine Lie algebra $\hat{\mathfrak{g}}$.

The notes are organized as follows.

Section 2 contains categorical preliminaries. The main point is Drinfeld's notion of a quasi-Hopf algebra [4]. Since the monoidal categories we are interested in are infinite, one has to understand the coproduct in the multiplier sense, so we talk about discrete quasi-Hopf algebras. Modulo this nuance Section 2 contains the standard dictionary between categorical and algebraic terms: monoidal categories and quasi-bialgebras, equivalence of categories and isomorphism of quasi-bialgebras up to twisting, weak tensor functors and comonoids, rigidity and existence of coinverse.

In Section 3 we introduce the Drinfeld category $\mathcal{D}(\mathfrak{g}, \hbar)$, $\hbar \in \mathbb{C} \setminus \mathbb{Q}^*$. As mentioned above, it is the category of finite dimensional $\mathfrak{g}$-modules with the usual tensor product but with nontrivial associativity morphisms Φ_{KZ} defined via monodromy of the KZ-equations. Alternatively one can think of the associator Φ_{KZ} as a 3-cocycle on the dual discrete group $\hat{G}$. We follow Drinfeld's original argument [4, 5] to prove that $\mathcal{D}(\mathfrak{g}, \hbar)$ is indeed a braided monoidal category. Remark that by specialization and analytic continuation this can be deduced directly from the formal deformation case, which is a bit more convenient to deal with. The simplifications are however not significant, so to avoid confusion we work entirely in the analytic setting. Remark also that there is a somewhat more conceptual proof showing that $\mathcal{D}(\mathfrak{g}, \hbar)$ is the monoidal category which corresponds to a genus zero modular functor, see e.g. [1].

But as everywhere in these notes we sacrifice generality in favor of a hands-on approach.

In Section 4 we formulate the main result, that is, equivalence of $\mathcal{D}(\mathfrak{g}, \hbar)$ and the category $\mathcal{C}(\mathfrak{g}, \hbar)$ of finite dimensional admissible $U_q\mathfrak{g}$-modules, $q = e^{\pi i \hbar}$. Furthermore, the functor $\mathcal{D}(\mathfrak{g}, \hbar) \to \mathcal{C}(\mathfrak{g}, \hbar)$ defining this equivalence can be chosen such that its composition with the forgetful functor $\mathcal{C}(\mathfrak{g}, \hbar) \to \mathcal{V}ec$ is naturally isomorphic to the forgetful functor $\mathcal{D}(\mathfrak{g}, \hbar) \to \mathcal{V}ec$. This means that the equivalence can be expressed in algebraic terms, that is, the corresponding quasi-bialgebras are isomorphic up to twisting. The proof of this theorem occupies the remaining part of the paper. In fact, we prove it only for generic $\hbar$. A simple compactness argument then shows that the result holds at least for all $\hbar \in i\mathbb{R}$, which is the most interesting case from the operator algebra point of view.

The actual proof starts in Section 5. Since we want a functor isomorphic to the forgetful one, we first of all need a tensor structure on the forgetful functor $\mathcal{D}(\mathfrak{g}, \hbar) \to \mathcal{V}ec$. If we have a module M representing this functor then to have a weak tensor structure on the functor is the same thing as having a comonoid structure on M. Clearly, no finite dimensional $\mathfrak{g}$-module can represent the forgetful functor. In Section 5 we define a representing object M in a completion of $\mathcal{D}(\mathfrak{g}, \hbar)$. It can be thought of as an object in an ind-pro-category, but we prefer to think of it as a topological $\mathfrak{g}$-module.

In Section 6 we define a comonoid structure on M thus endowing the functor $\mathrm{Hom}_{\mathfrak{g}}(M, \cdot)$ with a weak tensor structure. We then check that for generic $\hbar$ we in fact get a tensor structure. This already implies that Drinfeld's KZ-associator is a coboundary for generic $\hbar$. It is interesting to note that up to this point the only properties of Φ_{KZ} which have been used are analytic dependence on the parameter $\hbar$ and that the associator acts trivially on the highest weight subspaces. We end the section with an algorithm of how to explicitly find $\mathcal{F}$ such that Φ_{KZ} is a coboundary of $\mathcal{F}^{-1}$. The word explicit should however be taken with a grain of salt, as one has to make choices depending on values of solutions of differential equations.

In Section 7 we show that $U_q\mathfrak{g}$ acts by natural transformations on the functor $\mathrm{Hom}_{\mathfrak{g}}(M, \cdot)$, allowing the latter to be regarded as a functor $\mathcal{D}(\mathfrak{g}, \hbar) \to \mathcal{C}(\mathfrak{g}, \hbar)$. We finally check that this is an equivalence of categories for generic $\hbar$. Although the idea of the definition of this action of $U_q\mathfrak{g}$ is not difficult to convey, the right normalization of the maps involved requires an ingenious choice, which is ultimately dictated by classical identities for hypergeometric functions. This is by far the most technical part of the proof of Kazhdan and Lusztig, and here we omit a couple of the most tedious computations.

2 Quasi-Bialgebras and Monoidal Categories

A monoidal category $\mathcal{C}$ is a category with a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(U, V) \mapsto U \otimes V$, which is associative up to a natural isomorphism

$$\alpha\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

and has an object which is the unit $\mathbf{1}$ up to natural isomorphisms

$$\lambda\colon \mathbf{1} \otimes U \to U, \quad \rho\colon U \otimes \mathbf{1} \to U,$$

such that $\lambda = \rho\colon \mathbf{1}\otimes\mathbf{1}\to\mathbf{1}$ and such that the pentagonal diagram

$$\begin{tikzcd}
(U\otimes(V\otimes W))\otimes X \arrow[d, "\alpha_{1,23,4}"'] & ((U\otimes V)\otimes W)\otimes X \arrow[l, "\alpha\otimes\iota"'] \arrow[r, "\alpha_{12,3,4}"] & (U\otimes V)\otimes(W\otimes X) \arrow[d, "\alpha_{1,2,34}"] \\
U\otimes((V\otimes W)\otimes X) \arrow[rr, "\iota\otimes\alpha"] & & U\otimes(V\otimes(W\otimes X))
\end{tikzcd}$$

and the triangle diagram

$$\begin{tikzcd}
(U\otimes\mathbf{1})\otimes V \arrow[rr, "\alpha"] \arrow[dr, "\rho\otimes\iota"'] & & U\otimes(\mathbf{1}\otimes V) \arrow[dl, "\iota\otimes\lambda"] \\
& U\otimes V &
\end{tikzcd}$$

commute.

We say that $\mathcal{C}$ has strict unit if both λ and ρ are the identity morphisms. If also α is the identity, then $\mathcal{C}$ is called a strict monoidal category.

A braiding in a monoidal category $\mathcal{C}$ is a natural isomorphism $\sigma\colon U\otimes V\to V\otimes U$ such that $\lambda\sigma(U\otimes\mathbf{1}) = \rho(U\otimes\mathbf{1})$ and such that the hexagonal diagram

$$\begin{tikzcd}
(V\otimes U)\otimes W \arrow[d, "\alpha"'] & (U\otimes V)\otimes W \arrow[l, "\sigma\otimes\iota"'] \arrow[r, "\alpha"] & U\otimes(V\otimes W) \arrow[d, "\sigma_{1,23}"] \\
V\otimes(U\otimes W) \arrow[r, "\iota\otimes\sigma"] & V\otimes(W\otimes U) & (V\otimes W)\otimes U \arrow[l, "\alpha"']
\end{tikzcd}$$

and the same diagram with σ replaced by σ^{-1} both commute.

We say that a category is $\mathbb{C}$-linear if it is abelian, the sets $\mathrm{Hom}(U, V)$ are vector spaces over $\mathbb{C}$ and composition of morphisms is bilinear. Of course, when the monoidal category is $\mathbb{C}$-linear the tensor functor $\otimes$ is required to be bilinear on morphisms.

A $\mathbb{C}$-linear category is called semisimple if any object is a finite direct sum of simple objects.

A (weak) quasi-tensor functor between monoidal categories $\mathcal{C}$ and $\mathcal{C}'$ is a functor $F\colon \mathcal{C}\to\mathcal{C}'$ together with a (morphism) isomorphism $F_0\colon \mathbf{1}'\to F(\mathbf{1})$ in $\mathcal{C}'$ and natural (morphisms) isomorphisms

$$F_2\colon F(U)\otimes F(V)\to F(U\otimes V).$$

When the categories are braided then F is called braided if the diagram

$$\begin{tikzcd}
F(U)\otimes F(V) \arrow[r, "F_2"] \arrow[d, "\sigma'"'] & F(U\otimes V) \arrow[d, "F(\sigma)"] \\
F(V)\otimes F(U) \arrow[r, "F_2"] & F(V\otimes U)
\end{tikzcd}$$

commutes.

A (weak) quasi-tensor functor is called a (weak) tensor functor if the diagram

$$\begin{tikzcd}
(F(U)\otimes F(V))\otimes F(W) \arrow[r, "F_2\otimes\iota"] \arrow[d, "\alpha'"'] & F(U\otimes V)\otimes F(W) \arrow[r, "F_2"] & F((U\otimes V)\otimes W) \arrow[d, "F(\alpha)"] \\
F(U)\otimes(F(V)\otimes F(W)) \arrow[r, "\iota\otimes F_2"] & F(U)\otimes F(V\otimes W) \arrow[r, "F_2"] & F(U\otimes(V\otimes W))
\end{tikzcd} \tag{2.1}$$

and the diagrams

$$\begin{tikzcd}
F(\mathbf{1})\otimes F(U) \arrow[r, "F_2"] & F(\mathbf{1}\otimes U) \arrow[d, "F(\lambda)"] \\
\mathbf{1}'\otimes F(U) \arrow[u, "F_0\otimes\iota"] \arrow[r, "\lambda'"] & F(U)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(U)\otimes F(\mathbf{1}) \arrow[r, "F_2"] & F(U\otimes \mathbf{1}) \arrow[d, "F(\rho)"] \\
F(U)\otimes\mathbf{1}' \arrow[u, "\iota\otimes F_0"] \arrow[r, "\rho'"] & F(U)
\end{tikzcd}$$

commute.

We say that a natural isomorphism $\eta\colon F\to G$ between two (weak) (quasi-)tensor functors $\mathcal{C}\to\mathcal{C}'$ is monoidal if the diagrams

$$\begin{tikzcd}
F(U)\otimes F(V) \arrow[r, "F_2"] \arrow[d, "\eta\otimes\eta"'] & F(U\otimes V) \arrow[d, "\eta"] \\
G(U)\otimes G(V) \arrow[r, "G_2"] & G(U\otimes V)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{1}' \arrow[dl, "F_0"'] \arrow[dr, "G_0"] & \\
F(\mathbf{1}) \arrow[rr, "\eta"] & & G(\mathbf{1})
\end{tikzcd}$$

commute.

An equivalence between two monoidal categories is called monoidal if the functors and the natural isomorphisms defining the equivalence are monoidal. If the functors are also ($\mathbb{C}$-linear) (braided) then we speak of a ($\mathbb{C}$-linear) (braided) monoidal equivalence.

According to a theorem of Mac Lane any monoidal category can be strictified, i.e. it is monoidally equivalent to a strict monoidal category, and if the category is ($\mathbb{C}$-linear) (braided) then the equivalence can be chosen to be ($\mathbb{C}$-linear) (braided). This is useful for obtaining new identities for morphisms from known ones: it implies that an identity holds if it can be proved assuming that the associativity morphisms are trivial. As is customary we regard the $\mathbb{C}$-linear monoidal category $\mathcal{V}ec$ of finite dimensional vector spaces as strict.

Consider now a direct sum $A=\oplus_{\lambda\in\Lambda}\mathrm{End}(V_\lambda)$ of full matrix algebras. Define $M(A)$ as the algebraic product $\prod_{\lambda\in\Lambda}\mathrm{End}(V_\lambda)$. If B is another such algebra, we say that a homomorphism $\varphi\colon A\to M(B)$ is nondegenerate if $\varphi(A)B=B$.

Let $A\text{-Mod}_f$ denote the $\mathbb{C}$-linear category of nondegenerate finite dimensional A-modules, so $A\text{-Mod}_f$ is semisimple with simple objects $\{V_\lambda\}_\lambda$. We would like $A\text{-Mod}_f$ to be monoidal with tensor product and strict unit $\mathbb{C}$ defined in the usual way via nondegenerate homomorphisms

$$\Delta\colon A\to M(A\otimes A)=\prod_{\lambda,\mu}\mathrm{End}(V_\lambda\otimes V_\mu),\quad \varepsilon\colon A\to\mathbb{C},$$

and with associativity morphisms $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$ given by acting with an element $\Phi \in M(A \otimes A \otimes A)$. This is indeed the case if and only if Φ is invertible and

$$(\varepsilon \otimes \iota)\Delta = \iota = (\iota \otimes \varepsilon)\Delta, \quad (\iota \otimes \varepsilon \otimes \iota)\Phi = 1 \otimes 1,$$
$$(\iota \otimes \Delta)\Delta = \Phi(\Delta \otimes \iota)\Delta(\cdot)\Phi^{-1},$$
$$(\iota \otimes \iota \otimes \Delta)(\Phi)(\Delta \otimes \iota \otimes \iota)(\Phi) = (1 \otimes \Phi)(\iota \otimes \Delta \otimes \iota)(\Phi)(\Phi \otimes 1). \tag{2.2}$$

We then call A a *discrete quasi-bialgebra* with coproduct Δ, counit ε and associator Φ. Remark that Eq. 2.2 corresponds to the pentagonal diagram. Notice also that by definition A-Mod$_f$ is strict if and only if $\Phi = 1 \otimes 1 \otimes 1$.

If we also have an element $\mathcal{R} \in M(A \otimes A)$ and let $\Sigma\colon U \otimes V \to V \otimes U$ denote the flip, then $\Sigma\mathcal{R}\colon U \otimes V \to V \otimes U$ is a braiding if and only if $\Delta^{op} = \mathcal{R}\Delta(\cdot)\mathcal{R}^{-1}$ and

$$(\Delta \otimes \iota)(\mathcal{R}) = \Phi_{312}\mathcal{R}_{13}\Phi^{-1}_{132}\mathcal{R}_{23}\Phi, \quad (\iota \otimes \Delta)(\mathcal{R}) = \Phi^{-1}_{231}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}\Phi^{-1}. \tag{2.3}$$

In this case we speak of a *quasitriangular discrete quasi-bialgebra* with R-matrix $\mathcal{R}$. Equation 2.3 correspond to the hexagonal diagrams.

Note that the forgetful functor $F\colon A$-Mod$_f \to \mathcal{V}ec$ is a quasi-tensor functor with F_0 and F_2 the identity morphisms. It is a tensor functor if and only if $\Phi = 1 \otimes 1 \otimes 1$.

By a twist in a (quasitriangular) discrete quasi-bialgebra A we mean an invertible element $\mathcal{F}$ in $M(A \otimes A)$ such that $(\varepsilon \otimes \iota)(\mathcal{F}) = (\iota \otimes \varepsilon)(\mathcal{F}) = 1$. The twisting $A_{\mathcal{F}}$ of A by $\mathcal{F}$ is then the (quasitriangular) discrete quasi-bialgebra with comultiplication $\Delta_{\mathcal{F}} = \mathcal{F}\Delta(\cdot)\mathcal{F}^{-1}$, counit $\varepsilon_{\mathcal{F}} = \varepsilon$, associator

$$\Phi_{\mathcal{F}} = (1 \otimes \mathcal{F})(\iota \otimes \Delta)(\mathcal{F})\Phi(\Delta \otimes \iota)(\mathcal{F}^{-1})(\mathcal{F}^{-1} \otimes 1)$$

(and R-matrix $\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$).

Proposition 2.1 *Let A and A' be (quasitriangular) discrete quasi-bialgebras, $F\colon A$-Mod$_f \to \mathcal{V}ec$ and $F\colon A'$-Mod$_f \to \mathcal{V}ec$ the forgetful quasi-tensor functors. Then*

(i) *the (quasitriangular) discrete quasi-bialgebras A' and A are isomorphic if and only if there exists a $\mathbb{C}$-linear (braided) monoidal equivalence $E\colon A$-Mod$_f \to A'$-Mod$_f$ such that $F'E$ and F are monoidally naturally isomorphic;*
(ii) *the (quasitriangular) discrete quasi-bialgebra A' is isomorphic to a twisting $A_{\mathcal{F}}$ of A if and only if there exists a $\mathbb{C}$-linear (braided) monoidal equivalence $E\colon A$-Mod$_f \to A'$-Mod$_f$ such that $F'E$ and F are naturally isomorphic.*

If A and A' are finite dimensional and quasi-Hopf (see below) then one does not need a natural isomorphism of $F'E$ and F in (ii), that is, A' is isomorphic to a twisting of A if and only if the categories A-Mod$_f$ and A'-Mod$_f$ are $\mathbb{C}$-linear (braided) monoidally equivalent [9]. This is no longer true in the infinite dimensional case [2].

Proof of Proposition 2.1 Assume first that we have an isomorphism $\varphi\colon A' \to A_{\mathcal{F}}$. Then by restriction of scalars φ gives a functor $E\colon A$-Mod$_f \to A'$-Mod$_f$. We make it a tensor functor by letting $E_0 = \iota$ and $E_2 = \mathcal{F}^{-1}$. It is easy to see that E is a $\mathbb{C}$-linear (braided) monoidal equivalence. Furthermore, ignoring the quasi-tensor structure we have $F'E = F$, and if $\mathcal{F} = 1 \otimes 1$ then $F'E = F$ as quasi-tensor functors.

Conversely, assume we have a $\mathbb{C}$-linear (braided) monoidal equivalence $E\colon A\text{-Mod}_f \to A'\text{-Mod}_f$ and a natural isomorphism $\eta\colon F \to F'E$. The algebra $M(A)$ can be identified with the algebra $\mathrm{Nat}(F)$ of natural transformations of the forgetful functor F to itself, and similarly $M(A') = \mathrm{Nat}(F')$. The map $\varphi\colon\ \mathrm{Nat}(F') \to \mathrm{Nat}(F)$ defined by $\varphi(a') = \eta^{-1}a'\eta$ is then an isomorphism of algebras.

Identifying $M(A\otimes A)$ with $\mathrm{Nat}(F\otimes F)$, we define $\mathcal{F} \in M(A\otimes A)$ by the diagram

$$\begin{tikzcd} U\otimes V \arrow[r, "\mathcal{F}"] \arrow[d, "\eta"'] & U\otimes V \arrow[d, "\eta\otimes\eta"] \\ E(U\otimes V) & E(U)\otimes E(V) \arrow[l, "E_2"'] \end{tikzcd}$$

In other words, we have $\mathcal{F}_{U,V} = \big(\eta_U^{-1}\otimes\eta_V^{-1}\big)E_2^{-1}\eta_{U\otimes V}$. The element $\mathcal{F}$ is clearly invertible. It is easy to see that it has the property $(\varepsilon\otimes\iota)(\mathcal{F}) = (\iota\otimes\varepsilon)(\mathcal{F}) = 1$ if and only if the maps $E_0, \eta\colon \mathbb{C} \to E(\mathbb{C})$ coincide. This is the case if η is a monoidal natural isomorphism, and can be achieved in general by rescaling η. Furthermore, if η is monoidal then $\mathcal{F}$ is the identity map.

The element $\Delta(a)$ considered as an element of $\mathrm{Nat}(F\otimes F)$ is given by $\Delta(a)_{U,V} = a_{U\otimes V}$. For $a' \in M(A')$ we then have

$$\begin{aligned}(\mathcal{F}^{-1}(\varphi\otimes\varphi)\Delta'(a')\mathcal{F})_{U,V} &= \mathcal{F}_{U,V}^{-1}\big(\eta_U^{-1}\otimes\eta_V^{-1}\big)a'_{E(U)\otimes E(V)}(\eta_U\otimes\eta_V)\mathcal{F}_{U,V}\\ &= \eta_{U\otimes V}^{-1}E_2 a'_{E(U)\otimes E(V)}E_2^{-1}\eta_{U\otimes V} = \eta_{U\otimes V}^{-1}a'_{E(U\otimes V)}\eta_{U\otimes V}\\ &= \varphi(a')_{U\otimes V} = (\Delta\varphi(a'))_{U,V},\end{aligned}$$

so $(\varphi\otimes\varphi)\Delta'\varphi^{-1} = \Delta_{\mathcal{F}}$.

The diagram (2.1) for the tensor functor E reads as

$$\Phi' = \big(\iota\otimes E_2^{-1}\big)E_2^{-1}E(\Phi)E_2(E_2\otimes\iota).$$

Using that $E_2\otimes\iota\colon (E(U)\otimes E(V))\otimes E(W) \to E(U\otimes V)\otimes E(W)$ is $(\eta\otimes\iota)(\mathcal{F}^{-1}\otimes\iota)\,(\eta^{-1}\otimes\eta^{-1}\otimes\iota)$, and that $E_2\colon\ E(U\otimes V)\otimes E(W) \to E((U\otimes V)\otimes W)$ is

$$\eta_{(U\otimes V)\otimes W}\mathcal{F}_{U\otimes V,W}^{-1}\big(\eta_{U\otimes V}^{-1}\otimes\eta_W^{-1}\big) = \eta(\Delta\otimes\iota)(\mathcal{F}^{-1})\big(\eta^{-1}\otimes\eta^{-1}\big),$$

we see that $E_2(E_2\otimes\iota)$ in the expression above equals $\eta(\Delta\otimes\iota)(\mathcal{F}^{-1})(\mathcal{F}^{-1}\otimes\iota)(\eta^{-1}\otimes\eta^{-1}\otimes\eta^{-1})$. Using a similar expression for $\big(\iota\otimes E_2^{-1}\big)E_2^{-1}$ we get

$$\Phi' = (\eta\otimes\eta\otimes\eta)(\iota\otimes\mathcal{F})(\iota\otimes\Delta)(\mathcal{F})\eta^{-1}E(\Phi)\eta(\Delta\otimes\iota)(\mathcal{F}^{-1})(\mathcal{F}^{-1}\otimes\iota)\big(\eta^{-1}\otimes\eta^{-1}\otimes\eta^{-1}\big).$$

Since $\eta^{-1}E(\Phi)\eta = \Phi$, this is exactly the equality $\Phi' = (\varphi^{-1}\otimes\varphi^{-1}\otimes\varphi^{-1})(\Phi_{\mathcal{F}})$.

Finally, if our quasi-bialgebras are quasitriangular and the functor E is braided, we have a commutative diagram

$$\begin{tikzcd} U\otimes V \arrow[r, "\eta"] \arrow[d, "\Sigma\mathcal{R}"'] & E(U\otimes V) \arrow[r, "E_2^{-1}"] \arrow[d, "E(\Sigma\mathcal{R})"'] & E(U)\otimes E(V) \arrow[r, "\eta^{-1}\otimes\eta^{-1}"] \arrow[d, "\Sigma\mathcal{R}'"] & U\otimes V \arrow[d, "\Sigma(\varphi\otimes\varphi)(\mathcal{R}')"] \\ V\otimes U \arrow[r, "\eta"] & E(V\otimes U) \arrow[r, "E_2^{-1}"] & E(V)\otimes E(U) \arrow[r, "\eta^{-1}\otimes\eta^{-1}"] & V\otimes U \end{tikzcd}$$

Therefore $\Sigma(\varphi\otimes\varphi)(\mathcal{R}') = \mathcal{F}\Sigma\mathcal{R}\mathcal{F}^{-1}$, that is, $(\varphi\otimes\varphi)(\mathcal{R}') = \mathcal{R}_{\mathcal{F}}$. $\square$

We will be interested in the case when A' is a bialgebra, so $\Phi' = 1 \otimes 1 \otimes 1$. In this case F' is a tensor functor, so if $E\colon A\text{-Mod}_f \to A'\text{-Mod}_f$ is a monoidal equivalence then $F'E\colon A\text{-Mod}_f \to \mathcal{V}ec$ is a tensor functor. Therefore to show that A' is isomorphic to a twisting of A, by part (ii) of the above proposition, we at least need a tensor functor $A\text{-Mod}_f \to \mathcal{V}ec$ which is naturally isomorphic to the forgetful functor.

We remark the following consequence of the proof of the above proposition: if $E\colon A\text{-Mod}_f \to \mathcal{V}ec$ is a $\mathbb{C}$-linear functor and $\eta\colon F \to E$ is a natural isomorphism then there is a one-to-one correspondence between weak tensor structures on E and elements $\mathcal{G} \in M(A \otimes A)$ such that $(\varepsilon \otimes \iota)(\mathcal{G}) = 1 = (\iota \otimes \varepsilon)(\mathcal{G})$ and

$$\Phi(\Delta \otimes \iota)(\mathcal{G})(\mathcal{G} \otimes 1) = (\iota \otimes \Delta)(\mathcal{G})(1 \otimes \mathcal{G}).$$

Furthermore, E is a tensor functor if and only if $\mathcal{G}$ is invertible, and then $\Phi_{\mathcal{F}} = 1 \otimes 1 \otimes 1$ with $\mathcal{F} = \mathcal{G}^{-1}$.

To define a tensor structure on a functor isomorphic to the forgetful one, it is convenient to use the following notion. An object M in a monoidal category $\mathcal{C}$ with strict unit is called a comonoid if it comes with two morphisms

$$\varepsilon\colon M \to \mathbf{1}, \quad \delta\colon M \to M \otimes M$$

such that $(\varepsilon \otimes \iota)\delta = \iota = (\iota \otimes \varepsilon)\delta$ and $(\iota \otimes \delta)\delta = \alpha(\delta \otimes \iota)\delta$.

Lemma 2.2 *Let A be a discrete quasi-bialgebra, M an object in $A\text{-Mod}_f$. Then there is a one-to-one correspondence between*

(i) *weak tensor structures on the functor* $\mathrm{Hom}(M, \cdot)\colon A\text{-Mod}_f \to \mathcal{V}ec$*;*
(ii) *comonoid structures on M.*

Proof If M is a comonoid then we define $E_2\colon \mathrm{Hom}(M, U) \otimes \mathrm{Hom}(M, V) \to \mathrm{Hom}(M, U \otimes V)$ by $f \otimes g \mapsto (f \otimes g)\delta$ and $E_0\colon \mathbf{1} = \mathbb{C} \to \mathrm{Hom}(M, \mathbb{C})$ by $E_0(1) = \varepsilon$.

Conversely, if the functor $E = \mathrm{Hom}(M, \cdot)$ is endowed with a weak tensor structure, we define $\delta\colon M \to M \otimes M$ as the image of $\iota \otimes \iota$ under the map

$$E_2\colon \mathrm{Hom}(M, M) \otimes \mathrm{Hom}(M, M) \to \mathrm{Hom}(M, M \otimes M),$$

and $\varepsilon\colon M \to \mathbb{C}$ as the image of $1 \in \mathbb{C}$ under the map $E_0\colon \mathbb{C} \to \mathrm{Hom}(M, \mathbb{C})$. Using naturality of E_2 one checks that the image of $f \otimes g$ under the map $E_2\colon \mathrm{Hom}(M, U) \otimes \mathrm{Hom}(M, V) \to \mathrm{Hom}(M, U \otimes V)$ is $(f \otimes g)\delta$. It is then straightforward to check that the axioms of a weak tensor functor translate into the defining properties of a comonoid. □

We are of course interested in the case when the functor $\mathrm{Hom}(M, \cdot)$ is naturally isomorphic to the forgetful one. Clearly, no such object M exists in $A\text{-Mod}_f$ unless A is finite dimensional. So one needs to extend the category $A\text{-Mod}_f$ to make the lemma useful. We do not try to do this in general, as depending on the situation different extensions might be useful.

Remark that in the finite dimensional case the unique object up to isomorphism, representing the forgetful functor, is the module A; namely, $\mathrm{Hom}(A, U) \to U$, $f \mapsto f(1)$, is a natural isomorphism. In this case the lemma and the discussion before it show that there exists a one-to-one correspondence between comonoid structures on A and elements $\mathcal{G} \in A \otimes A$ such that $(\varepsilon \otimes \iota)(\mathcal{G}) = 1 = (\iota \otimes \varepsilon)(\mathcal{G})$ and $\Phi(\Delta \otimes \iota)(\mathcal{G})(\mathcal{G} \otimes 1) = (\iota \otimes \Delta)(\mathcal{G})(1 \otimes \mathcal{G})$. Explicitly, given such an element $\mathcal{G}$ one defines $\delta\colon A \to A \otimes A$ by $\delta(a) = \Delta(a)\mathcal{G}$.

Let A be a (quasitriangular) discrete quasi-bialgebra. By a $*$-operation on A we mean an antilinear involutive antihomomorphism $x \mapsto x^*$ on A such that $\Delta(x^*) = \Delta(x)^*$, $\varepsilon(x^*) = \overline{\varepsilon(x)}$, Φ is unitary (and $\mathcal{R}^* = \mathcal{R}_{21}$). We also require any element of the form $1 + x^*x$ to be invertible in $M(A)$, so that A can be completed to a C*-algebra.

Proposition 2.3 *Let A and A' be (quasitriangular) discrete $*$-quasi-bialgebras. Suppose A' is isomorphic to $A_{\mathcal{E}}$ for a twist $\mathcal{E}$. Then there exists a unitary twist $\mathcal{F}$ such that A' and $A_{\mathcal{F}}$ are $*$-isomorphic.*

Proof Let $\varphi\colon A' \to A_{\mathcal{E}}$ be an isomorphism. Since every homomorphism of full matrix algebras (with the standard $*$-operation) is equivalent to a $*$-homomorphism, there exists an invertible element $u \in M(A)$ such that the homomorphism $\varphi_u := u\varphi(\cdot)u^{-1}$ is $*$-preserving. We normalize u such that $\varepsilon(u) = 1$. Then $\mathcal{E}_u = (u \otimes u)\mathcal{E}\Delta(u^{-1})$ is a twist and it is easy to check that $\varphi_u\colon A' \to A_{\mathcal{E}_u}$ is an isomorphism.

Therefore we may assume that φ is $*$-preserving. Consider the polar decomposition $\mathcal{E} = \mathcal{F}|\mathcal{E}|$. Then $\mathcal{F}$ is a unitary twist and we claim that φ is an isomorphism of discrete $*$-quasi-bialgebras A' and $A_{\mathcal{F}}$. As $\varphi\colon A' \to A_{\mathcal{F}}$ is $*$-preserving, we just have to check that $A_{\mathcal{E}} = A_{\mathcal{F}}$.

Applying the $*$-operation to the identity $(\varphi \otimes \varphi)\Delta' = \mathcal{E}\Delta\varphi(\cdot)\mathcal{E}^{-1}$, we get

$$(\varphi \otimes \varphi)\Delta' = (\mathcal{E}^{-1})^*\Delta\varphi(\cdot)\mathcal{E}^*.$$

It follows that $\mathcal{E}^*\mathcal{E}$ commutes with the image of Δ, hence so does $|\mathcal{E}|$. In particular, $\Delta_{\mathcal{E}} = \Delta_{\mathcal{F}}$.

Now apply the map $T(x) = (x^*)^{-1}$ to the identity $(\varphi \otimes \varphi \otimes \varphi)(\Phi') = \Phi_{\mathcal{E}}$. As T preserves Φ' and Φ by unitarity, we get $(\varphi \otimes \varphi \otimes \varphi)(\Phi') = \Phi_{T(\mathcal{E})}$. Therefore

$$(\Phi_{|\mathcal{E}|})_{\mathcal{F}} = \Phi_{\mathcal{E}} = \Phi_{T(\mathcal{E})} = (\Phi_{|\mathcal{E}|^{-1}})_{\mathcal{F}},$$

whence $\Phi_{|\mathcal{E}|} = \Phi_{|\mathcal{E}|^{-1}}$ as $\Delta_{|\mathcal{E}|} = \Delta = \Delta_{|\mathcal{E}|^{-1}}$. Thus

$$\begin{aligned}&(1 \otimes |\mathcal{E}|)(\iota \otimes \hat{\Delta})(|\mathcal{E}|)\Phi(\hat{\Delta} \otimes \iota)(|\mathcal{E}|^{-1})(|\mathcal{E}|^{-1} \otimes 1)\\ &\quad = (1 \otimes |\mathcal{E}|^{-1})(\iota \otimes \hat{\Delta})(|\mathcal{E}|^{-1})\Phi(\hat{\Delta} \otimes \iota)(|\mathcal{E}|)(|\mathcal{E}| \otimes 1).\end{aligned}$$

Since $(\iota \otimes \hat{\Delta})(|\mathcal{E}|)$ and $1 \otimes |\mathcal{E}|$, as well as $|\mathcal{E}| \otimes 1$ and $(\hat{\Delta} \otimes \iota)(|\mathcal{E}|)$, commute, we can write

$$((1 \otimes |\mathcal{E}|)(\iota \otimes \hat{\Delta})(|\mathcal{E}|))^2\Phi = \Phi((|\mathcal{E}| \otimes 1)(\hat{\Delta} \otimes \iota)(|\mathcal{E}|))^2.$$

Consequently

$$(1 \otimes |\mathcal{E}|)(\iota \otimes \hat{\Delta})(|\mathcal{E}|)\Phi = \Phi(|\mathcal{E}| \otimes 1)(\hat{\Delta} \otimes \iota)(|\mathcal{E}|).$$

Thus $\Phi_{|\mathcal{E}|} = \Phi$, and using again $\Delta_{|\mathcal{E}|} = \Delta$ we therefore get $\Phi_{\mathcal{E}} = (\Phi_{|\mathcal{E}|})_{\mathcal{F}} = \Phi_{\mathcal{F}}$.

Finally, assume our quasi-bialgebras are quasitriangular. Applying the $*$-operation and then the flip to the equality $(\varphi \otimes \varphi)(\mathcal{R}') = \mathcal{E}_{21}\mathcal{R}\mathcal{E}^{-1}$ we get

$$\mathcal{E}_{21}\mathcal{R}\mathcal{E}^{-1} = (\mathcal{E}_{21}^*)^{-1}\mathcal{R}\mathcal{E}^*,$$

so that $(\mathcal{E}^*\mathcal{E})_{21}\mathcal{R} = \mathcal{R}\mathcal{E}^*\mathcal{E}$, whence $|\mathcal{E}|_{21}\mathcal{R} = \mathcal{R}|\mathcal{E}|$, or in other words, $\mathcal{R}_{|\mathcal{E}|} = \mathcal{R}$. It follows that $\mathcal{R}_{\mathcal{E}} = (\mathcal{R}_{|\mathcal{E}|})_{\mathcal{F}} = \mathcal{R}_{\mathcal{F}}$. □

We next discuss how the notion of a quasi-bialgebra arises naturally from the Tannakian formalism. This will essentially not be used later.

Let $\mathcal{C}$ be a $\mathbb{C}$-linear monoidal category. A (quasi-)fiber functor is a (quasi-)tensor exact faithful $\mathbb{C}$-linear functor $\mathcal{C} \to \mathcal{V}ec$.

First one has the following reconstruction result [16].

Proposition 2.4 *Let $\mathcal{C}$ be a small $\mathbb{C}$-linear semisimple (braided) monoidal category with simple strict unit. Suppose we have a quasi-fiber functor $F\colon \mathcal{C} \to \mathcal{V}ec$. Then there exists a (quasitriangular) discrete quasi-bialgebra A and a $\mathbb{C}$-linear (braided) monoidal equivalence $E\colon \mathcal{C} \to A\text{-Mod}_f$ such that its composition with the forgetful functor $A\text{-Mod}_f \to \mathcal{V}ec$ is naturally isomorphic to F.*

Remark that by Proposition 2.1 such a quasi-bialgebra A is unique up to isomorphism and twisting. We also remark that, as will be clear from the proof, if F is a fiber functor then A can be chosen to be a discrete bialgebra.

Proof of Proposition 2.4 Let $\{V_\lambda\}_{\lambda\in\Lambda}$ be representatives of isomorphism classes of the simple objects in $\mathcal{C}$. Put $A = \oplus_\lambda \operatorname{End}(F(V_\lambda))$. Then $M(A) = \prod_\lambda \operatorname{End}(F(V_\lambda))$ can be identified with the algebra $\operatorname{Nat}(F)$ of natural transformations of F. Regarding F as a functor $E\colon \mathcal{C} \to A\text{-Mod}_f$, we get an equivalence of $\mathcal{C}$ and $A\text{-Mod}_f$ as $\mathbb{C}$-linear categories, since E is exact and maps the objects V_λ onto all simple objects of $A\text{-Mod}_f$ up to isomorphism.

Identifying $M(A\otimes A)$ with $\operatorname{Nat}(F\otimes F)$ and considering F_2 as a natural transformation from $F\otimes F$ to $F(\cdot\otimes\cdot)$,we define $\Delta\colon M(A) \to M(A\otimes A)$ by $\Delta(a) = F_2^{-1}aF_2$. Define also $\varepsilon\colon M(A) \to \mathbb{C}$ by $\varepsilon(a) = a_{\mathbf{1}} \in \operatorname{End}(F(\mathbf{1})) = \mathbb{C}$. Finally, define $\Phi \in M(A\otimes A\otimes A) = \operatorname{Nat}(F\otimes F\otimes F)$ by

$$\Phi = \left(\iota\otimes F_2^{-1}\right)F_2^{-1}F(\alpha)F_2(F_2\otimes\iota).$$

Then by construction A becomes a discrete quasi-bialgebra and E a monoidal functor.

If $\mathcal{C}$ has braiding σ then define $\mathcal{R} \in M(A\otimes A) = \operatorname{Nat}(F\otimes F)$ by $\mathcal{R} = \Sigma F_2^{-1}F(\sigma)F_2$. Then A is quasitriangular and E is braided. □

A right (resp. left) dual to an object U in a monoidal category $\mathcal{C}$ with strict unit consists of an object $U^\vee$ (resp. ${}^\vee U$) and two morphisms

$$e\colon U^\vee\otimes U \to \mathbf{1},\quad i\colon \mathbf{1} \to U\otimes U^\vee,\quad (\text{resp. } e'\colon U\otimes {}^\vee U \to \mathbf{1},\quad i'\colon \mathbf{1} \to {}^\vee U\otimes U)$$

such that the compositions

$$U \xrightarrow{i\otimes\iota} (U\otimes U^\vee)\otimes U \xrightarrow{\alpha} U\otimes(U^\vee\otimes U) \xrightarrow{\iota\otimes e} U,$$

$$U^\vee \xrightarrow{\iota\otimes i} U^\vee\otimes(U\otimes U^\vee) \xrightarrow{\alpha^{-1}} (U^\vee\otimes U)\otimes U^\vee \xrightarrow{e\otimes\iota} U^\vee$$

(respectively,

$$U \xrightarrow{\iota\otimes i'} U\otimes({}^\vee U\otimes U)\xrightarrow{\alpha^{-1}}(U\otimes{}^\vee U)\otimes U\xrightarrow{e'\otimes\iota} U,$$

$${}^\vee U\xrightarrow{i'\otimes\iota}({}^\vee U\otimes U)\otimes{}^\vee U\xrightarrow{\alpha}{}^\vee U\otimes(U\otimes{}^\vee U)\xrightarrow{\iota\otimes e'}{}^\vee U)$$

are the identity morphisms. The category $\mathcal{C}$ is called rigid if every object has left and right duals.

If $t\in\mathrm{Hom}(U,V)$ then the transpose $t^\vee\colon V^\vee\to U^\vee$ is defined as the composition

$$V^\vee\xrightarrow{\iota\otimes i}V^\vee\otimes(U\otimes U^\vee)\xrightarrow{\alpha^{-1}}(V^\vee\otimes U)\otimes U^\vee\xrightarrow{(\iota\otimes t)\otimes\iota}(V^\vee\otimes V)\otimes U^\vee\xrightarrow{e\otimes\iota}U^\vee.$$

We then have the following identities:

$$(t\otimes\iota)i=(\iota\otimes t^\vee)i\colon \mathbf{1}\to V\otimes U^\vee,\quad e(\iota\otimes t)=e(t^\vee\otimes\iota)\colon V^\vee\otimes U\to\mathbf{1}.$$

This is not difficult to check directly, but is immediate if the category is strict, which we may assume by Mac Lane's theorem. Now if $s\in\mathrm{Hom}(V,W)$, assuming that $\mathcal{C}$ is strict to simplify computations, the morphism $t^\vee s^\vee$ is by definition given by the composition

$$W^\vee\xrightarrow{\iota\otimes i}W^\vee\otimes V\otimes V^\vee\xrightarrow{\iota\otimes\iota\otimes t^\vee}W^\vee\otimes V\otimes U^\vee\xrightarrow{\iota\otimes s\otimes\iota}W^\vee\otimes W\otimes U^\vee\xrightarrow{e\otimes\iota}U^\vee.$$

But as $(t\otimes\iota)i=(\iota\otimes t^\vee)i$, this is exactly the definition of $(st)^\vee$. Therefore $V\mapsto V^\vee$ is a contravariant functor of $\mathcal{C}$ into itself.

Similar arguments show that if $\tilde U^\vee$ is another right dual of U with corresponding morphisms $\tilde i$ and $\tilde e$, then $\gamma=(\tilde e\otimes\iota)\alpha^{-1}(\iota\otimes i)\colon\tilde U^\vee\to U^\vee$ has inverse $(e\otimes\iota)\alpha^{-1}(\iota\otimes\tilde i)$. Also $\tilde e=e(\gamma\otimes\iota)$ and $\tilde i=(\iota\otimes\gamma^{-1})i$. Therefore right duals are unique up to isomorphism. Similar statements hold for left duals. Finally note that

$$(\iota\otimes e)\alpha(i'\otimes\iota)\colon U\to{}^\vee(U^\vee)$$

is an isomorphism with inverse $(\iota\otimes e')\alpha(i\otimes\iota)$, and similarly that $({}^\vee U)^\vee$ is isomorphic to U.

The category $\mathcal{V}ec$ is rigid with $U^\vee={}^\vee U=U^*$ and the morphisms $e=e'$ and $i=i'$ (identifying U^{**} with U), which we shall denote by e_v and i_v, are given by

$$e_v\colon U^*\otimes U\to\mathbb{C},\ f\otimes x\mapsto f(x),\ \text{ and }\ i_v\colon\mathbb{C}\to U\otimes U^*,\ 1\mapsto\sum_i x_i\otimes x^i,$$

where $\{x_i\}_i$ is a basis in U and $\{x^i\}_i$ is the dual basis in U^*. Then $t^\vee$ is the usual dual operator t^*.

Suppose we are given a nondegenerate anti-homomorphism S of a discrete quasi-bialgebra A. Then for any A-module U we can define an A-module structure on the dual space U^* by $af=f(S(a)\,\cdot)$. To make U^* a right dual object we look for morphisms

$$e\colon U^*\otimes U\to\mathbb{C},\quad i\colon\mathbb{C}\to U\otimes U^*$$

in the form $e=e_v(1\otimes\alpha)$ and $i=(\beta\otimes 1)i_v$ for some elements $\alpha,\beta\in M(A)$ (note that if U is simple then any linear maps $U^*\otimes U\to\mathbb{C}$ and $\mathbb{C}\to U\otimes U^*$ must be of this form). Then the maps e and i are morphisms if and only if

$$S(a_{(1)})\alpha a_{(2)}=\varepsilon(a)\alpha,\quad a_{(1)}\beta S(a_{(2)})=\varepsilon(a)\beta \tag{2.4}$$

as endomorphisms of U, and then U^* is a right dual of U in $A\text{-Mod}_f$ if and only if

$$S\big(\Phi_1^{-1}\big)\alpha\Phi_2^{-1}\beta S\big(\Phi_3^{-1}\big)=1,\quad \Phi_1\beta S(\Phi_2)\alpha\Phi_3=1, \tag{2.5}$$

again as endomorphisms of U. If there exists an invertible anti-homomorphism S and elements $\alpha,\beta\in M(A)$ such that Eqs. 2.4 and 2.5 are satisfied, then we say that A is a *discrete quasi-Hopf algebra* with coinverse S. Then $U^\vee=U^*$ with action $af=f(S(a)\,\cdot)$ is a right dual of U, and ${}^\vee U=U^*$ with action $af=f(S^{-1}(a)\,\cdot)$ is a left dual of U with $e'=e_v(S^{-1}(\alpha)\otimes 1)$ and $i'=(1\otimes S^{-1}(\beta))i_v$.

If $\tilde S$ is another coinverse with corresponding elements $\tilde\alpha,\tilde\beta$, then there exists a unique invertible $u\in A$ such that $\tilde S=uS(\cdot)u^{-1}$ and $\tilde\alpha=u\alpha$, $\tilde\beta=\beta u^{-1}$. Conversely, any $\tilde S$ and $\tilde\alpha,\tilde\beta$ defined this way for an invertible u satisfy the same axioms as S and α,β. When $\Phi=1\otimes 1\otimes 1$, then α and β are inverses to each other, and setting $u=\beta$ thus gives $\tilde\alpha=\tilde\beta=1$, so A is a discrete multiplier Hopf algebra with coinverse $\tilde S$ in the sense of [21].

We have explained that if a discrete quasi-bialgebra A has coinverse then $A\text{-Mod}_f$ is rigid. One has the following converse [10, 20, 23].

Proposition 2.5 *Let A be a discrete quasi-bialgebra with $A\text{-Mod}_f$ rigid and such that for every simple module U the dimensions of U and $U^\vee$ as vector spaces coincide. Then A has coinverse.*

Proof Recall that by definition $A=\oplus_{\lambda\in\Lambda}\operatorname{End}(V_\lambda)$. For each λ the module $V_\lambda^\vee$ is simple, so there exists a unique $\bar\lambda\in\Lambda$ such that $V_\lambda^\vee\cong V_{\bar\lambda}$. Fix a linear isomorphism $\eta_\lambda\colon V_\lambda^*\to V_\lambda^\vee$, which exists as the spaces V_λ and $V_\lambda^\vee$ by assumption have the same vector space dimension. Then there exists a unique anti-isomorphism $S_\lambda\colon \operatorname{End}(V_{\bar\lambda})\to\operatorname{End}(V_\lambda)$ such that if we define an action of $\operatorname{End}(V_{\bar\lambda})$ on V_λ^* by $af=f(S_\lambda(a)\cdot)$, then η_λ is an $\operatorname{End}(V_{\bar\lambda})$-module map. Since $V_\lambda\cong({}^\vee V_\lambda)^\vee$, the set $\{\bar\lambda\}_{\lambda\in\Lambda}$ coincides with Λ. Thus our anti-isomorphisms S_λ define an anti-isomorphism S of A onto itself such that for each λ the dual module $V_\lambda^\vee$ is isomorphic to V_λ^* with action $af=f(S(a)\cdot)$. As explained above, the morphisms $e\colon V_\lambda^*\otimes V_\lambda\to\mathbb{C}$ and $i\colon\mathbb{C}\to V_\lambda\otimes V_\lambda^*$ uniquely determine α and β, making S a coinverse.

In more categorical terms the above proof goes as follows. Identify $M(A)$ with the algebra $\operatorname{Nat}(F)$ of natural transformations of the forgetful functor F. Extend isomorphisms $V_\lambda^*\cong V_\lambda^\vee$ to a natural isomorphism η from the functor $U\mapsto F(U)^*$ to the functor $U\mapsto F(U^\vee)$. Then S,α and β are defined by

$$S(a)_U=\eta^*(a_{U^\vee})^*(\eta^*)^{-1},\quad \alpha_U=(\iota\otimes e)(\iota\otimes\eta\otimes\iota)(i_v\otimes\iota),$$
$$\beta_U=(\iota\otimes e_v)(\iota\otimes\eta^{-1}\otimes\iota)(i\otimes\iota). \qquad\square$$

In the case when A is finite dimensional the assumption on the dimensions of U and $U^\vee$ is automatically satisfied [19]. The following example from [18] (see also [24]) shows that this is not the case in general.

Example 2.6 Let G be a discrete group, $B\subset G$ a subgroup such that each double coset BgB contains finitely many right and left cosets of B. Consider the category $\mathcal{C}$ of G-graded B-bimodules $M=\oplus_{g\in G}M_g$ such that M_g is a finite dimensional complex vector space for each g, and $M_g\neq 0$ only for g in finitely many double cosets of B. Define a tensor structure on $\mathcal{C}$ by $M\otimes N=M\otimes_{\mathbb{C}[B]}N$. Note that $\mathbb{C}[B]$ is a unit object. The category is rigid with right and left dual $M^\vee$ given by $M^\vee_g=(M_{g^{-1}})^*$ and B-bimodule structure given by $(b_1fb_2)(x)=f(b_2xb_1)$ for $f\in(M_{g^{-1}})^*$ and $x\in M_{(b_1gb_2)^{-1}}$. The morphism $i\colon\mathbb{C}[B]\to M\otimes M^\vee$ is defined by $i(e)=\sum_{g\in G/B}\sum_{j\in I_g}x_{g,j}\otimes x^*_{g,j}$, where $\{x_{g,j}\}_{j\in I_g}$ is a basis in M_g with dual basis $\{x^*_{g,j}\}_{j\in I_g}$, and $e\colon M^\vee\otimes M\to\mathbb{C}[B]$ is defined by $e(f\otimes x)=f(x(gh)^{-1})gh$ for $f\in M^\vee_g$ and $x\in M_h$ when $gh\in B$, and $e(f\otimes x)=0$ when $gh\notin B$.

The category $\mathcal{C}$ is in general not semisimple. To define a semisimple subcategory consider a functor E from $\mathcal{C}$ to the category of $B\backslash G$-graded finite dimensional right B-modules defined by $E(M)=\mathbb{C}\otimes_{\mathbb{C}[B]}M$. It is not difficult to see that E is an equivalence of categories. Furthermore, using E the simple objects of $\mathcal{C}$ can be described as follows: the modules that are supported on a single double coset BgB (so that $M_h=0$ for $h\notin BgB$), and such that the right action of $B\cap g^{-1}Bg$ on $E(M)_{Bg}$ is irreducible. Consider now only those modules in $\mathcal{C}$ which decompose into simple ones such that the corresponding action of $B\cap g^{-1}Bg$ factors through a finite group. Equivalently, we define a semisimple subcategory $\mathcal{C}_0$ of $\mathcal{C}$ consisting of modules M such that the right action of B on $E(M)$ factors through a finite group. Yet another equivalent condition is that $xb=(gbg^{-1})x$ for all $g\in G$, $x\in M_g$ and b in a finite index subgroup of B (where we use the convention that $(gbg^{-1})x=0$ if $gbg^{-1}\notin B$). Using the latter characterization we see that $\mathcal{C}_0$ is closed under tensor product, and if M is in $\mathcal{C}_0$ then $M^\vee$ is also in $\mathcal{C}_0$.

Consider the functor $F\colon\mathcal{C}_0\to\mathcal{V}ec$ defined by $F(M)=\mathbb{C}\otimes_{\mathbb{C}[B]}M$. To make it a quasi-fiber functor fix a set of representatives R for $B\backslash G$. Then $F(M)\cong\oplus_{g\in R}M_g$. For $g\in G$ denote by $[g]\in R$ the representative of Bg. Then define F_2 as the composition of the canonical isomorphisms

$$\begin{aligned}F(M)\otimes F(N)&\cong\bigoplus_{g,h\in R}M_{[gh^{-1}]}\otimes N_h\cong\bigoplus_{g,h\in R}M_{gh^{-1}}\otimes N_h\\&\cong\bigoplus_{g\in R}(M\otimes N)_g\cong F(M\otimes N),\end{aligned}$$

where in the second step we used the isomorphisms $M_{[gh^{-1}]}\to M_{gh^{-1}}$ given by $x\mapsto(gh^{-1}[gh^{-1}]^{-1})x$. Thus by Proposition 2.4 the functor $F\colon\mathcal{C}_0\to\mathcal{V}ec$ defines a discrete quasi-bialgebra A such that $A\text{-Mod}_f$ is rigid. Notice now that the dimensions of $F(M)$ and $F(M^\vee)$ can be different. Indeed, let $D=BgB$ be a double coset, $M=\mathbb{C}[D]$. Then $M^\vee=\mathbb{C}[D^{-1}]$. We have $\dim F(M)=|B\backslash D|$ and $\dim F(M^\vee)=|B\backslash D^{-1}|=|D/B|$. A simple example where these dimensions can be different is the $ax+b$ groups $G=\begin{pmatrix}\mathbb{Q}^* & \mathbb{Q}\\ 0 & 1\end{pmatrix}$, $B=\begin{pmatrix}1 & \mathbb{Z}\\ 0 & 1\end{pmatrix}$. So in this case the discrete quasi-bialgebra A with rigid monoidal category $A\text{-Mod}_f$ fails to be quasi-Hopf.

3 The Drinfeld Category

Let G be a simply connected simple compact Lie group, $\mathfrak{g}$ its complexified Lie algebra. Consider the tensor category $\mathcal{C}(\mathfrak{g})$ of finite dimensional $\mathfrak{g}$-modules. For each

$\hbar \in \mathbb{C} \setminus \mathbb{Q}$ we shall introduce new associativity morphisms in $\mathcal{C}(\mathfrak{g})$ via monodromy of the Knizhnik–Zamolodchikov equations.

Consider the ad-invariant symmetric form on $\mathfrak{g}$ normalized such that if we choose a maximal torus in G and denote by $\mathfrak{h} \subset \mathfrak{g}$ be the corresponding Cartan subalgebra, then for the dual form on $\mathfrak{h}^*$ we have $(\alpha, \alpha) = 2$ for short roots. In other words, if $(a_{ij})_{1\le i,j\le r}$ is the Cartan matrix of $\mathfrak{g}$, and $d_1, \dots, d_r$ the coprime positive integers such that $(d_i a_{ij})_{i,j}$ is symmetric, then $(\alpha_i, \alpha_j) = d_i a_{ij}$ for a chosen system $\{\alpha_1, \dots, \alpha_r\}$ of simple roots. Let $t = \sum_i x_i \otimes x^i \in \mathfrak{g} \otimes \mathfrak{g}$ be the element defined by this form, so $\{x_i\}_i$ is a basis in $\mathfrak{g}$ and $\{x^i\}_i$ is the dual basis. Since t is defined by an invariant form, it is $\mathfrak{g}$-invariant, that is, $[t, \hat\Delta(x)] = 0$ for all $x \in U\mathfrak{g}$, where $\hat\Delta\colon U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ is the comultiplication. Remark also that by definition of $\hat\Delta$ we have

$$(\hat\Delta \otimes \iota)(t) = t_{13} + t_{23}, \quad (\iota \otimes \hat\Delta)(t) = t_{12} + t_{13}. \tag{3.1}$$

Let $V_1, \dots, V_n$ be finite dimensional $\mathfrak{g}$-modules. Denote by Y_n the set of points $(z_1, \dots, z_n) \in \mathbb{C}^n$ such that $z_i \ne z_j$ for $i \ne j$. The KZ$_n$ equations is the system of differential equations

$$\frac{\partial v}{\partial z_i} = \hbar \sum_{j\ne i} \frac{t_{ij}}{z_i - z_j} v, \quad i = 1, \dots, n,$$

where $v\colon Y_n \to V_1 \otimes \cdots \otimes V_n$. This system is consistent in the sense that the differential operators $\nabla_i = \frac{\partial}{\partial z_i} - \hbar \sum_{j\ne i} \frac{t_{ij}}{z_i - z_j}$ commute with each other, or equivalently, they define a flat holomorphic connection on the trivial vector bundle over Y_n with fiber $V_1 \otimes \cdots \otimes V_n$. This can be checked using that t is symmetric and that $[t_{ij} + t_{jk}, t_{ik}] = 0$, which follows from Eq. 3.1 and $\mathfrak{g}$-invariance of t.

The consistency of the KZ$_n$ equations implies that locally for each $z^0 \in Y_n$ and $v_0 \in V_1 \otimes \cdots \otimes V_n$ there exists a unique holomorphic solution v with $v(z^0) = v_0$. If $\gamma\colon [0, 1] \to Y_n$ is a path starting at $\gamma(0) = z^0$, then this solution can be analytically continued along γ. The map $v_0 \mapsto v(\gamma(1))$ defines a linear isomorphism M_γ of $V_1 \otimes \cdots \otimes V_n$ onto itself. The monodromy operator M_γ depends only on the homotopy class of γ. In particular, for each base point $z^0 \in Y_n$ we get a representation of the fundamental group $\pi_1(Y_n; z^0)$ on $V_1 \otimes \cdots \otimes V_n$ by monodromy operators. Recall that $\pi_1(Y_n; z^0)$ is isomorphic to the pure braid group PB_n, which is the kernel of the homomorphism $B_n \to S_n$. If $V_1 = \cdots = V_n$ then the monodromy representation extends to the whole braid group B_n; we shall briefly return to this a bit later.

The new associativity morphism $(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$ will be a certain operator which appears naturally in computing the monodromy representations for KZ$_3$, it can be thought of as the monodromy operator from the asymptotic zone $|z_2 - z_1| \ll |z_3 - z_1|$ to the zone $|z_3 - z_2| \ll |z_3 - z_1|$. To proceed rigorously we need to recall a few facts about differential equations with regular singularities. Observe first that if

$$v(z_1, z_2, z_3) = (z_3 - z_1)^{\hbar(t_{12}+t_{23}+t_{13})} w\left(\frac{z_2 - z_1}{z_3 - z_1}\right),$$

then v is a solution of KZ_3 if and only if w is a solution of the equation

$$w'(z) = \hbar \left(\frac{t_{12}}{z} + \frac{t_{23}}{z-1} \right) w(z), \tag{3.2}$$

which we call the modified KZ_3 equation.

Proposition 3.1 *Let V be a finite dimensional vector space, $z \mapsto A(z) \in \mathrm{End}(V)$ a holomorphic function on the unit disc $\mathbb{D}$. Assume $A(0)$ has no eigenvalues that differ by a nonzero integer. Then the equation*

$$xG'(x) = A(x)G(x)$$

for $G\colon (0,1) \to \mathrm{GL}(V)$ has a unique solution such that the function $H(x) = G(x)x^{-A(0)}$ extends to a holomorphic function on $\mathbb{D}$ with value 1 at 0.

Furthermore, if $G(\cdot\,;\hbar)$ is an analogous solution of $xG'(x;\hbar) = \hbar A(x)G(x;\hbar)$, which is well-defined for all $\hbar$ outside the discrete set $\Lambda = \{n(\lambda-\mu)^{-1} \mid n \in \mathbb{N},\ \lambda \neq \mu,\ \lambda$ and μ are eigenvalues of $A(0)\}$, then $H(x;\hbar) = G(x;\hbar)x^{-\hbar A(0)}$ is analytic on $\mathbb{D} \times (\mathbb{C} \setminus \Lambda)$.

Proof We shall give a proof of this standard result (see e.g. [22]), mainly to remind how the assumption on $A(0)$ is used.

Write $A(z) = \sum_{n=0}^{\infty} A_n z^n$. We look for $G(x)$ in the form $H(x)x^{A_0}$, where $H(x) = \sum_{n=0}^{\infty} H_n x^n$ with $H_0 = 1$. Then H must satisfy the equation

$$xH'(x) = A(x)H(x) - H(x)A_0, \tag{3.3}$$

or equivalently, $[A_0, H_n] - nH_n = -\sum_{i=0}^{n-1} A_{n-i}H_i$ for all $n \geq 1$. The operator $\mathrm{ad}_{A_0} - n$ on $\mathrm{End}(V)$ has zero kernel exactly when A_0 has no eigenvalues that differ by n. So by our assumptions there exist unique H_n satisfying the above conditions. We then have to check that the series $\sum_n H_n x^n$ is convergent in the unit disc. Choose $c > 0$ such that $\|(\mathrm{ad}_{A_0} - n)^{-1}\| \leq c$ for all $n \geq 1$. Define numbers h_n recursively by $h_0 = 1$, $h_n = c\sum_{i=0}^{n-1} \|A_{n-i}\| h_i$ for $n \geq 1$. We clearly have $\|H_n\| \leq h_n$. On the other hand, by construction the formal power series $h(x) = \sum_{n=0}^{\infty} h_n x^n$ satisfies the equation $h(x) - 1 = \varphi(x)h(x)$, where $\varphi(x) = c\sum_{n\geq 1} \|A_n\| x^n$. Since φ is analytic on $\mathbb{D}$ and $\varphi(0) = 0$, we see that $h(x) = (1-\varphi(x))^{-1}$ is convergent in a neighbourhood of zero. Hence $\sum_n H_n x^n$ is also convergent in the same neighbourhood. Since a solution of Eq. 3.3 can be continued analytically along any path in $\mathbb{D} \setminus \{0\}$, we conclude that the convergence must hold on the whole disc. Furthermore, as $G(x)$ is invertible for small x, it must be invertible everywhere.

Finally, if $A(z;\hbar)$ is analytic in two variables, then the above argument implies that for any bounded open set U such that the assumption on $A(0;\hbar)$ is satisfied for all $\hbar \in \overline{U}$, there exists a neighbourhood W of zero such that the corresponding solution $H(x;\hbar)$ of Eq. 3.3 with A replaced by $A(\cdot\,;\hbar)$, is analytic on $W \times U$. Fixing $x_0 \in W \setminus \{0\}$, we can consider $H(\cdot\,;\hbar)$ as a solution of a differential equation depending analytically on a parameter and with the analytic initial value $H(x_0;\hbar)$ at $x = x_0$. Hence $H(\cdot\,;\cdot)$ is analytic on $\mathbb{D} \times U$. □

Remark 3.2 Uniqueness of G is equivalent to the following statement: if A is an operator with no eigenvalues that differ by a nonzero integer, and the function $x \mapsto x^A T x^{-A}$ defined for positive x extends to an analytic function in a neighbourhood of zero with value 1 at $x = 0$, then $T = 1$. This is easy to see directly. More generally, if $x^A T x^{-A}$ extends to an analytic function in a neighbourhood of zero then A and T commute.[1]

We will also need a multivariable version of Proposition 3.1.

Proposition 3.3 *Let $A_1, \dots, A_m \colon \mathbb{D}^m \to \mathrm{End}(V)$ be analytic functions. Assume the differential operators $\nabla_i = z_i \frac{\partial}{\partial z_i} - A_i(z)$, $1 \le i \le m$, pairwise commute. Assume also that none of the operators $A_i(0)$ has eigenvalues which differ by a nonzero integer. Then the system of equations*

$$x_i \frac{\partial G}{\partial x_i}(x) = A_i(x) G(x), \quad 1 \le i \le m,$$

has a unique $\mathrm{GL}(V)$-valued solution on $(0, 1)^m$ such that the function $G(x) x_1^{-A_1(0)} \dots x_m^{-A_m(0)}$ extends to an analytic function on $\mathbb{D}^m$ with value 1 at $x = 0$.

Remark that the flatness condition $[\nabla_i, \nabla_j] = 0$ reads as $z_i \frac{\partial A_j}{\partial z_i} - z_j \frac{\partial A_i}{\partial z_j} = [A_i, A_j]$. In particular, it implies that $[A_i(0), A_j(0)] = 0$.

Proof The proposition can be proved by induction on m. To simplify the notation we shall only sketch a proof for $m = 2$, which is actually the only case we shall need later.

The unknown function $H(x_1, x_2) = G(x_1, x_2) x_1^{-A_1(0)} x_2^{-A_2(0)}$ must satisfy the system of equations

$$x_1 \frac{\partial H}{\partial x_1} = A_1 H - H A_1(0), \tag{3.4}$$

$$x_2 \frac{\partial H}{\partial x_2} = A_2 H - H A_2(0). \tag{3.5}$$

By the proof of Proposition 3.1 Eq. 3.4 for $x_2 = 0$ has a unique holomorphic solution H_0 with $H_0(0) = 1$. Using that $[\nabla_1, \nabla_2] = 0$ it is easy to check that $A_2(\cdot, 0) H_0$ is a holomorphic solution of Eq. 3.4 (for $x_2 = 0$) with initial value $A_2(0)$ at $x_1 = 0$, hence $A_2(x_1, 0) H_0(x_1) = H_0(x_1) A_2(0)$ for all x_1 by uniqueness. Then an argument similar to that in the proof of Proposition 3.1 shows that in a neighbourhood of zero there exists a unique holomorphic solution of Eq. 3.5 of the form $H(x_1, x_2) =$

[1] In the formal deformation setting a similar result holds without any assumption on the spectrum of A. Namely, if $x \mapsto x^{hA} T x^{-hA} \in \mathrm{Mat}_n(\mathbb{C})[[h]]$ extends analytically, meaning that every coefficient in the power series extends analytically, then A and T commute. Indeed, we have $x^{hA} T x^{-hA} = T + h[A, T] \log x + \dots$, which forces $[A, T] = 0$. Moreover, we see that already existence of the limit of $x^{hA} T x^{-hA}$ as $x \to 0^+$ implies that A and T commute. As a result replacing analytic functions by formal power series would simplify some of the subsequent arguments.

$\sum_{n=0}^{\infty} H_n(x_1)x_2^n$, so that $H(x_1, 0) = H_0(x_1)$ for small x_1. It remains to show that H also satisfies Eq. 3.4. For this one checks, using $[\nabla_1, \nabla_2] = 0$, that

$$x_1 \frac{\partial H}{\partial x_1} - A_1 H + H A_1(0)$$

is again a solution of Eq. 3.5. Since it is zero at $x_2 = 0$, we conclude that it is zero everywhere. □

Turning to the modified KZ_3 Eq. 3.2, consider more generally the equation

$$w'(z) = \left(\frac{A}{z} + \frac{B}{z-1}\right) w(z), \tag{3.6}$$

where A and B are operators on a finite dimensional vector space V such that neither A nor B has eigenvalues that differ by a nonzero integer. By Proposition 3.1 there is a unique $\mathrm{GL}(V)$-valued solution $G_0(x)$ on $(0, 1)$ such that $G_0(x)x^{-A}$ extends to a holomorphic function on $\mathbb{D}$ with value 1 at 0. Fix $x^0 \in (0, 1)$. If $w_0 \in V$ then $G_0(x)w_0$ is a solution of Eq. 3.6 with initial value $G_0(x^0)w_0$. If we continue it analytically along a loop γ_0 starting at x^0 and turning around 0 counterclockwise then at the end point we get $G_0(x^0)e^{2\pi i A}w_0$. Thus the monodromy operator defined by γ_0 is $G_0(x^0)e^{2\pi i A}G_0(x^0)^{-1}$. Using the change of variables $z \mapsto 1 - z$ we similarly conclude that there is a unique $\mathrm{GL}(V)$-valued solution $G_1(x)$ of Eq. 3.6 such that $G_1(1-x)x^{-B}$ extends to a holomorphic function on $\mathbb{D}$ with value 1 at 0. Then the monodromy operator defined by a loop γ_1 starting at x^0 and turning around 1 counterclockwise is $G_1(x^0)e^{2\pi i B}G_1(x^0)^{-1}$. The fundamental group of $\mathbb{C} \setminus \{0, 1\}$ with the base point x^0 is freely generated by the classes $[\gamma_0]$ and $[\gamma_1]$ of γ_0 and γ_1. Therefore the monodromy representation defined by Eq. 3.6 with the base point x^0 is

$$[\gamma_0] \mapsto G_0(x^0)e^{2\pi i A}G_0(x^0)^{-1}, \quad [\gamma_1] \mapsto G_1(x^0)e^{2\pi i B}G_1(x^0)^{-1}. \tag{3.7}$$

The operator $\Phi(A, B) = G_1(x)^{-1}G_0(x)$ does not depend on x, since a solution of Eq. 3.6 is determined by its initial value. We then see that the above representation is equivalent to the representation

$$[\gamma_0] \mapsto e^{2\pi i A}, \quad [\gamma_1] \mapsto \Phi(A, B)^{-1}e^{2\pi i B}\Phi(A, B),$$

which does not depend on the choice of the base point. In fact it can be interpreted as the monodromy representation with the base point 0 as follows.

Let Γ be the space of solutions of Eq. 3.6 on $(0, 1)$. For each $x^0 \in (0, 1)$ denote by $\pi_{x^0}\colon V \to \Gamma$ the isomorphism such that $\pi_{x^0}(w_0)$ is the solution of Eq. 3.6 with initial value w_0 at x^0. If γ is a curve in $(0, 1)$ then the monodromy operator M_γ is $\pi_{\gamma(1)}^{-1}\pi_{\gamma(0)}$. Define $\pi_{x^0}\colon V \to \Gamma$ for $x^0 = 0, 1$ by letting $\pi_0(w_0) = G_0(\cdot)w_0$ and $\pi_1(w_0) = G_1(\cdot)w_0$. Then $G_0(x^0) = \pi_{x^0}^{-1}\pi_0$, $G_1(x^0) = \pi_{x^0}^{-1}\pi_1$ and $\Phi(A, B) = \pi_1^{-1}\pi_0$ can be thought of as the monodromies from 0 to x^0, from 1 to x^0, and from 0 to 1, respectively. This interpretation agrees with formulas (3.7) since the monodromy operator defined by an infinitesimal loop around zero should of course be $e^{2\pi i A}$.

It is sometimes convenient to define π_0 as follows. Let w_0 be an eigenvector of A with eigenvalue λ. Then $G_0(x)w_0 = x^\lambda G_0(x)x^{-A}w_0$. Therefore $u = \pi_0(w_0)$ is a solution of Eq. 3.6 such that $x^{-\lambda}u(x)$ extends to a holomorphic function on $\mathbb{D}$ with value w_0 at 0. This completely determines π_0 if A is diagonalizable. Similarly, if w_0 is an eigenvector of B with eigenvalue λ then $u = \pi_1(w_0)$ is a solution of Eq. 3.6 such that $x^{-\lambda}u(1-x)$ extends to a holomorphic function on $\mathbb{D}$ with value w_0 at 0.

We remark the following simple properties of $\Phi(A, B)$: if an operator C commutes with A and B then it also commutes with $\Phi(A, B)$, and in addition $\Phi(A, B)$ coincides with $\Phi(A + C, B)$ and $\Phi(A, B + C)$ if the latter operators are well-defined. Indeed, to prove the first claim observe that $e^{sC}G_0(\cdot)e^{-sC}$ has the defining properties of G_0 for every $s \in \mathbb{R}$, hence it coincides with G_0, so G_0 commutes with C and similarly G_1 commutes with C. For the second claim observe that if we replace A by $A + C$ then $G_0(x)$ and $G_1(x)$ get replaced by $G_0(x)x^C$ and $G_1(x)x^C$, whence $\Phi(A + C, B) = \Phi(A, B)$. In particular, if A and B commute then $\Phi(A, B) = \Phi(0, 0) = 1$.

Furthermore, by the second part of Proposition 3.1 for any fixed A and B the function $\mathbb{C} \ni \hbar \mapsto \Phi(\hbar A, \hbar B)$ is well-defined and analytic outside a discrete set. This discrete set does not contain zero, more precisely, $\Phi(\hbar A, \hbar B)$ is defined at least for $|\hbar| < (2\max\{r(A), r(B)\})^{-1}$, where r denotes the spectral radius. It can be shown [5] that the first terms of the Taylor series look like

$$\Phi(\hbar A, \hbar B) = 1 - \hbar^2\zeta(2)[A, B] - \hbar^3\zeta(3)([A, [A, B]] + [B, [A, B]]) + \dots,$$

where ζ is the Riemann zeta function; see [11, 14] for more on this expansion.

Finally, if V is a Hermitian vector space and $A^* = -A$, $B^* = -B$, then $\Phi(A, B)$ is unitary. Indeed, for any $x_0 \in (0, 1)$ the function $G_0(\cdot)G_0(x_0)^{-1}$ with value 1 at $x = x_0$ takes values in the unitary group, being an integral curve of a time-dependent right-invariant vector field on this group. Letting $x_0 \to 0$ in the equality $G_0(x) = (G_0(x)G_0(x_0)^{-1})(G_0(x_0)x_0^{-A})x_0^A$, we conclude that $G_0(x)$ is unitary for any $x \in (0, 1)$. We similarly see that $G_1(x)$ is unitary, and hence $\Phi(A, B)$ is unitary as well.

Returning to the modified KZ_3 equation notice first that the image of the element t in $\mathrm{End}(V_1 \otimes V_2)$ has rational eigenvalues for any finite dimensional $\mathfrak{g}$-modules V_1 and V_2. To see this, we need to recall that

$$t = \frac{1}{2}(\hat{\Delta}(C) - 1 \otimes C - C \otimes 1), \tag{3.8}$$

where $C = \sum_i x_i x^i$ is the Casimir, and that the spectrum of C consists of rational numbers since the image of C under an irreducible representation with highest weight λ is $(\lambda, \lambda + 2\rho)$, where ρ is half the sum of the positive roots. It follows that for any fixed $\hbar \in \mathbb{C} \setminus \mathbb{Q}^*$ and all finite dimensional $\mathfrak{g}$-modules V_1, V_2 and V_3 we have a well-defined natural isomorphism $\Phi(\hbar t_{12}, \hbar t_{23})$ of $V = V_1 \otimes V_2 \otimes V_3$ onto itself. Consider the $\mathrm{GL}(V)$-valued solutions G_0 and G_1 of Eq. 3.2 as described above. Then

$$W_i(x_1, x_2, x_3) = (x_3 - x_1)^{\hbar(t_{12}+t_{23}+t_{13})} G_i\left(\frac{x_2 - x_1}{x_3 - x_1}\right), \quad i = 0, 1,$$

are $\mathrm{GL}(V)$-valued solutions of KZ_3 on $\{x_1 < x_2 < x_3\}$. We have $\Phi(\hbar t_{12}, \hbar t_{23}) = W_1(z^0)^{-1}W_0(z^0)$ for any $z^0 = (x_1^0, x_2^0, x_3^0)$. Furthermore, our considerations imply that $\Phi(\hbar t_{12}, \hbar t_{23})$ can be thought of as the monodromy operator of KZ_3 from the asymptotic zone $x_2 - x_1 \ll x_3 - x_1$ to the zone $x_3 - x_2 \ll x_3 - x_1$, and by conjugating by $W_0(z^0)^{-1}$ the monodromy operators of KZ_3 with the base point z^0 can be written as expressions of $e^{\pi i\hbar t}$ and $\Phi(\hbar t_{12},\hbar t_{23})$,[2] which can be thought of as monodromy operators with the base point at infinity in the asymptotic zone $x_2-x_1\ll x_3-x_1$.

Theorem 3.4 *Let $\hbar\in\mathbb{C}\setminus\mathbb{Q}^*$. Denote by $\mathcal{D}(\mathfrak{g},\hbar)$ the category of finite dimensional $\mathfrak{g}$-modules. Then the standard tensor product, $\alpha=\Phi(\hbar t_{12},\hbar t_{23})$ and $\sigma=\Sigma e^{\pi i\hbar t}$ define on $\mathcal{D}(\mathfrak{g},\hbar)$ a structure of a braided monoidal category.*

By definition $\mathcal{D}(\mathfrak{g},\hbar)$ is the category of non-degenerate finite dimensional $\widehat{\mathbb{C}[G]}$-modules, where $\widehat{\mathbb{C}[G]}$ is the discrete bialgebra of matrix coefficients of finite dimensional representations of G with convolution product, coproduct $\hat\Delta(g)=g\otimes g$ and counit $\hat\varepsilon(g)=1$ for $g\in G\subset M(\widehat{\mathbb{C}[G]})$. We can then reformulate Theorem 3.4 by saying that $(\widehat{\mathbb{C}[G]},\hat\Delta,\hat\varepsilon,\Phi(\hbar t_{12},\hbar t_{23}),e^{\pi i\hbar t})$ is a quasitriangular discrete quasi-bialgebra. Remark that the algebra $M(\widehat{\mathbb{C}[G]})$ can be identified with the algebra $\mathcal{U}(G)$ of closed densely defined operators affiliated with the group von Neumann algebra $W^*(G)$ of G.

The element $\Phi(\hbar t_{12},\hbar t_{23})\in\mathcal{U}(G\times G\times G)$ is called the Drinfeld associator and is often denoted by Φ_{KZ}. Since from now on we are not going to consider any other associativity morphisms apart from the trivial one and $\Phi(\hbar t_{12},\hbar t_{23})$, we write Φ instead of $\Phi(\hbar t_{12},\hbar t_{23})$ if the value of $\hbar$ is clear from the context.

Proof of Theorem 3.4 The only nontrivial relations that we have to check are Eqs. 2.2 and 2.3 with $\mathcal{R}=e^{\pi i\hbar t}$.

[2]To be precise our discussion of the monodromy of the modified KZ_3 equation is not quite enough for this conclusion because the additional factor $(x_3-x_1)^{\hbar(t_{12}+t_{23}+t_{13})}$ has nontrivial monodromy. In other words, the monodromy of the KZ_3 equations does not reduce completely to that of the modified KZ_3 equation. This is not surprising since the map $Y_3\to\mathbb{C}\setminus\{0,1\}$, $z=(x_1,x_2,x_3)\mapsto x=\frac{x_2-x_1}{x_3-x_1}$, induces a surjective homomorphism of the fundamental groups which is however not injective. Namely, consider the standard generators g_1 and g_2 of B_3. It is known that PB_3 is generated by g_1^2, g_2^2 and $g_2g_1^2g_2^{-1}$. For $z^0=(x_1^0,x_2^0,x_3^0)$ with $x_1^0<x_2^0<x_3^0$, represent g_i by a path $\tilde\gamma_i$ interchanging x_i^0 with x_{i+1}^0 such that x_i^0 passes below x_{i+1}^0. Then the images of g_1^2 and g_2^2 in $\pi_1(\mathbb{C}\setminus\{0,1\};x^0)$ can be represented by the curves γ_0 and γ_1 introduced earlier, so the monodromy operators of KZ_3 corresponding to g_1^2 and g_2^2 with the base point z^0 are $W_0(z^0)e^{2\pi i\hbar t_{12}}W_0(z^0)^{-1}$ and $W_1(z^0)e^{2\pi i\hbar t_{23}}W_1(z^0)^{-1}$. But we still have to compute the operator corresponding to $g_2g_1^2g_2^{-1}$. Consider a more general problem. By embedding $V_1\otimes V_2\otimes V_3$ into $(V_1\oplus V_2\oplus V_3)^{\otimes 3}$ we may assume $V_1=V_2=V_3=W$. Extend the representation of PB_3 to a representation of B_3 on $V=W^{\otimes 3}$ defined by $g_1\mapsto\Sigma_{12}M_{\tilde\gamma_1}$ and $g_2\mapsto\Sigma_{23}M_{\tilde\gamma_2}$. If x^0 is the image of $z^0=\tilde\gamma_1(0)$ in $\mathbb{C}\setminus\{0,1\}$ then the image of $\tilde\gamma_1(1)$ is $\frac{x^0}{x^0-1}$. It follows that $M_{\tilde\gamma_1}=W_0(\tilde\gamma(1))W_0(\tilde\gamma(0))^{-1}=(1-x^0)^{\hbar(t_{12}+t_{23}+t_{13})}G_0\left(\frac{x^0}{x^0-1}\right)G_0(x^0)^{-1}$. Here $G_0\left(\frac{x^0}{x^0-1}\right)$ is obtained by analytic continuation of G_0 along the image of $\tilde\gamma_1$, that is, by going through the upper half-plane. It is not difficult to see that $\Sigma_{12}(1-x)^{\hbar(t_{12}+t_{23}+t_{13})}G_0\left(\frac{x}{x-1}\right)\Sigma_{12}=G_0(x)e^{\pi i\hbar t_{12}}$, by checking that the left hand side is a solution of the modified KZ_3 equation. It follows that $\Sigma_{12}M_{\tilde\gamma_1}=G_0(x^0)e^{\pi i\hbar t_{12}}\Sigma_{12}G_0(x^0)^{-1}=W_0(z^0)e^{\pi i\hbar t_{12}}\Sigma_{12}W_0(z^0)^{-1}$. Similarly one checks that $\Sigma_{23}M_{\tilde\gamma_2}=W_1(z^0)e^{\pi i\hbar t_{23}}\Sigma_{23}W_1(z^0)^{-1}$. Thus by conjugating by $W_0(z^0)^{-1}$ we see that the representation of B_3 on V is equivalent to the one given by $g_1\mapsto e^{\pi i\hbar t_{12}}\Sigma_{12}$, $g_2\mapsto\Phi(\hbar t_{12},\hbar t_{23})^{-1}e^{\pi i\hbar t_{23}}\Sigma_{23}\Phi(\hbar t_{12},\hbar t_{23})$.

To prove Eq. 2.2 consider the system KZ_4 in the real simply connected domain $\{x_1 < x_2 < x_3 < x_4\}$. Put

$$T = t_{12} + t_{13} + t_{14} + t_{23} + t_{24} + t_{34}.$$

Note that T commutes with t_{ij} for all i and j. We consider five solutions of KZ_4 in our domain of the form $(x_4 - x_1)^{\hbar T} F(u, v)$, where u and v are certain fractions of $x_j - x_i$ corresponding to five asymptotic zones. Each asymptotic zone is associated to a vertex of the pentagon diagram according to the following rule: if V_i and V_j are between parentheses and V_k is outside, then $|x_j - x_i| \ll |x_k - x_i|$. E.g. the zone corresponding to $((V_1 \otimes V_2) \otimes V_3) \otimes V_4$ is $x_2 - x_1 \ll x_3 - x_1 \ll x_4 - x_1$, and we claim that there exist a unique GL-valued solution W_1 of KZ_4 of the form

$$W_1(x_1, x_2, x_3, x_4) = (x_4 - x_1)^{\hbar T} F_1\left(\frac{x_2 - x_1}{x_3 - x_1}, \frac{x_3 - x_1}{x_4 - x_1}\right),$$

and a function $H_1(\cdot, \cdot)$ analytic on $\mathbb{D}^2$ such that $H_1(0, 0) = 1$ and

$$F_1(u, v) = H_1(u, v) u^{\hbar t_{12}} v^{\hbar(t_{12}+t_{13}+t_{23})} \text{ for } u, v \in (0, 1).$$

Indeed, one checks that F_1 must satisfy the system of equations

$$\begin{aligned} u\frac{\partial F_1}{\partial u} &= \hbar\left(t_{12} + \frac{u}{u-1}t_{23} + \frac{uv}{uv-1}t_{24}\right) F_1, \\ v\frac{\partial F_1}{\partial v} &= \hbar\left(t_{12} + t_{13} + t_{23} + \frac{uv}{uv-1}t_{24} + \frac{v}{v-1}t_{34}\right) F_1. \end{aligned} \tag{3.9}$$

By Proposition 3.3 this system has a unique solution of the required form. Similarly there exist solutions W_2, W_3, W_4, W_5 such that

$$\begin{aligned} W_2(x_1, x_2, x_3, x_4) &= (x_4 - x_1)^{\hbar T} F_2\left(\frac{x_3 - x_2}{x_3 - x_1}, \frac{x_3 - x_1}{x_4 - x_1}\right), \\ W_3(x_1, x_2, x_3, x_4) &= (x_4 - x_1)^{\hbar T} F_3\left(\frac{x_3 - x_2}{x_4 - x_2}, \frac{x_4 - x_2}{x_4 - x_1}\right), \\ W_4(x_1, x_2, x_3, x_4) &= (x_4 - x_1)^{\hbar T} F_4\left(\frac{x_4 - x_3}{x_4 - x_2}, \frac{x_4 - x_2}{x_4 - x_1}\right), \\ W_5(x_1, x_2, x_3, x_4) &= (x_4 - x_1)^{\hbar T} F_5\left(\frac{x_2 - x_1}{x_4 - x_1}, \frac{x_4 - x_3}{x_4 - x_1}\right) \end{aligned}$$

and holomorphic functions $H_i(\cdot, \cdot)$, $i = 2, 3, 4, 5$, in a neighbourhood of zero with $H_i(0, 0) = 1$ and such that for positive u, v we have

$$\begin{aligned} F_2(u, v) &= H_2(u, v) u^{\hbar t_{23}} v^{\hbar(t_{12}+t_{13}+t_{23})}, \\ F_3(u, v) &= H_3(u, v) u^{\hbar t_{23}} v^{\hbar(t_{23}+t_{24}+t_{34})}, \\ F_4(u, v) &= H_4(u, v) u^{\hbar t_{34}} v^{\hbar(t_{23}+t_{24}+t_{34})}, \\ F_5(u, v) &= H_5(u, v) u^{\hbar t_{12}} v^{\hbar t_{34}}. \end{aligned}$$

Explicitly, one checks that F_2, F_3, F_4 and F_5 satisfy

$$\begin{cases} u\dfrac{\partial F_2}{\partial u} = \hbar\left(t_{23} + \dfrac{u}{u-1}t_{12} + \dfrac{uv}{1-v+uv}t_{24}\right)F_2, \\ v\dfrac{\partial F_2}{\partial v} = \hbar\left(t_{12}+t_{13}+t_{23} + \dfrac{uv-v}{1-v+uv}t_{24} + \dfrac{v}{v-1}t_{34}\right)F_2, \end{cases} \tag{3.10}$$

$$\begin{cases} u\dfrac{\partial F_3}{\partial u} = \hbar\left(t_{23} + \dfrac{uv}{uv-v+1}t_{13} + \dfrac{u}{u-1}t_{34}\right)F_3, \\ v\dfrac{\partial F_3}{\partial v} = \hbar\left(t_{23}+t_{24}+t_{34} + \dfrac{v}{v-1}t_{12} + \dfrac{v-uv}{v-uv-1}t_{13}\right)F_3, \end{cases} \tag{3.11}$$

$$\begin{cases} u\dfrac{\partial F_4}{\partial u} = \hbar\left(t_{34} + \dfrac{uv}{uv-1}t_{13} + \dfrac{u}{u-1}t_{23}\right)F_4, \\ v\dfrac{\partial F_4}{\partial v} = \hbar\left(t_{23}+t_{24}+t_{34} + \dfrac{v}{v-1}t_{12} + \dfrac{uv}{uv-1}t_{13}\right)F_4, \end{cases}$$

$$\begin{cases} u\dfrac{\partial F_5}{\partial u} = \hbar\left(t_{12} + \dfrac{u}{u+v-1}t_{23} + \dfrac{u}{u-1}t_{24}\right)F_5, \\ v\dfrac{\partial F_5}{\partial v} = \hbar\left(t_{34} + \dfrac{v}{v-1}t_{13} + \dfrac{v}{u+v-1}t_{23}\right)F_5. \end{cases}$$

It turns out that the solutions W_i are related as follows:

$$W_1 = W_2(\Phi\otimes 1), \tag{3.12}$$

$$W_2 = W_3(\iota\otimes\hat{\Delta}\otimes\iota)(\Phi), \tag{3.13}$$

$$W_3 = W_4(1\otimes\Phi),$$

$$W_4 = W_5(\iota\otimes\iota\otimes\hat{\Delta})(\Phi^{-1}),$$

$$W_5 = W_1(\hat{\Delta}\otimes\iota\otimes\iota)(\Phi^{-1}),$$

which immediately implies Eq. 2.2. We shall only check Eqs. 3.12 and 3.13.

To prove Eq. 3.12 denote by Θ the operator such that $W_1 = W_2\Theta$. Then

$$F_1(u,v) = F_2(1-u,v)\Theta.$$

For any fixed $u\in(0,1)$ the functions $v\mapsto F_1(u,v)v^{-\hbar(t_{12}+t_{13}+t_{23})}$ and $v\mapsto F_2(1-u,v)v^{-\hbar(t_{12}+t_{13}+t_{23})}$ extend analytically to a neighbourhood of zero. It follows that $v^{\hbar(t_{12}+t_{13}+t_{23})}\Theta v^{-\hbar(t_{12}+t_{13}+t_{23})}$ extends analytically as well. By Remark 3.2 this is possible only when Θ commutes with $t_{12}+t_{13}+t_{23}$. It follows that

$$F_1(u,v)v^{-\hbar(t_{12}+t_{13}+t_{23})} = F_2(1-u,v)v^{-\hbar(t_{12}+t_{13}+t_{23})}\Theta.$$

Letting $v=0$ in this equality and introducing $g_1(u) = F_1(u,v)v^{-\hbar(t_{12}+t_{13}+t_{23})}|_{v=0} = H_1(u,0)u^{\hbar t_{12}}$ and $g_2(u) = F_2(u,v)v^{-\hbar(t_{12}+t_{13}+t_{23})}|_{v=0} = H_2(u,0)u^{\hbar t_{23}}$, we then get

$$g_1(u) = g_2(1-u)\Theta.$$

Furthermore, letting $v = 0$ in Eq. 3.9 and in the first equation of Eq. 3.10, we see that g_1 and g_2 satisfy

$$u\frac{dg_1}{du} = \hbar\left(t_{12} + \frac{u}{u-1}t_{23}\right)g_1,\quad u\frac{dg_2}{du} = \hbar\left(t_{23} + \frac{u}{u-1}t_{12}\right)g_2.$$

The functions $g_1(u)u^{-\hbar t_{12}} = H_1(u,0)$ and $g_2(u)u^{-\hbar t_{23}} = H_2(u,0)$ extend to analytic functions on the unit disc with value 1 at 0. Thus by definition

$$\Theta = \Phi(\hbar t_{12}, \hbar t_{23}) = \Phi \otimes 1.$$

To prove Eq. 3.13 denote again by Θ the element such that $W_2 = W_3\Theta$. Then

$$F_2(u,v) = F_3\left(\frac{uv}{1-v+uv}, 1-v+uv\right)\Theta.$$

As in the argument for Eq. 3.12, but now fixing v instead of u, we first conclude that Θ commutes with t_{23}. Thus

$$\begin{aligned}&F_2(u,v)u^{-\hbar t_{23}}v^{-\hbar t_{23}}\\ &\quad = F_3\left(\frac{uv}{1-v+uv}, 1-v+uv\right)\left(\frac{uv}{1-v+uv}\right)^{-\hbar t_{23}}(1-v+uv)^{-\hbar t_{23}}\Theta.\end{aligned}$$

So letting $u = 0$ and introducing $g_i(v) = F_i(u,v)u^{-\hbar t_{23}}v^{-\hbar t_{23}}|_{u=0}$ for $i = 2, 3$, we get

$$g_2(v) = g_3(1-v)\Theta.$$

Furthermore, from the second equations in Eqs. 3.10 and 3.11 we obtain

$$\begin{aligned}v\frac{dg_2}{dv} &= \hbar\left(t_{12} + t_{13} + \frac{v}{v-1}(t_{24}+t_{34})\right)g_2,\\ v\frac{dg_3}{dv} &= \hbar\left(t_{24} + t_{34} + \frac{v}{v-1}(t_{12}+t_{13})\right)g_3.\end{aligned}$$

The functions $g_2(v)v^{-\hbar(t_{12}+t_{13})} = H_2(0,v)$ and $g_3(v)v^{-\hbar(t_{24}+t_{34})} = H_3(0,v)$ extend to analytic functions in the unit disc with value 1 at 0. Therefore

$$\Theta = \Phi(\hbar t_{12} + \hbar t_{13}, \hbar t_{24} + \hbar t_{34}).$$

As $t_{12} + t_{13} = (\iota \otimes \hat\Delta \otimes \iota)(t_{12})$ and $t_{24} + t_{34} = (\iota \otimes \hat\Delta \otimes \iota)(t_{23})$, we get

$$\Theta = (\iota \otimes \hat\Delta \otimes \iota)(\Phi(\hbar t_{12}, \hbar t_{23})) = (\iota \otimes \hat\Delta \otimes \iota)(\Phi).$$

To prove Eq. 2.3 observe that the second relation in Eq. 2.3 follows from the first one by flipping the first and the third factors and using that $t = t_{21}$ and $\Phi_{321} = \Phi^{-1}$. The latter equality is easily obtained from the change of variables $z \mapsto 1 - z$ in Eq. 3.2.

Turning to the proof of the first identity in Eq. 2.3, consider the system KZ_3 in the simply connected space

$$\Gamma = \{(z_1, z_2, z_3) \in Y_3 \mid \Im z_1 \le \Im z_2 \le \Im z_3\}.$$

Consider the real domain $\{(x_1, x_2, x_3) \mid x_1 < x_2 < x_3\}$ and two GL-valued solutions of KZ_3 in this domain of the form

$$W_0(x_1, x_2, x_3) = (x_3 - x_1)^{\hbar(t_{12}+t_{23}+t_{13})} H_0\left(\frac{x_2 - x_1}{x_3 - x_1}\right)\left(\frac{x_2 - x_1}{x_3 - x_1}\right)^{\hbar t_{12}},$$

$$W_1(x_1, x_2, x_3) = (x_3 - x_1)^{\hbar(t_{12}+t_{23}+t_{13})} H_1\left(\frac{x_3 - x_2}{x_3 - x_1}\right)\left(\frac{x_3 - x_2}{x_3 - x_1}\right)^{\hbar t_{23}}.$$

Similarly we have solutions of KZ_3 in the real domain $\{(x_1, x_2, x_3) \mid x_1 < x_3 < x_2\}$ such that

$$W_2(x_1, x_2, x_3) = (x_2 - x_1)^{\hbar(t_{12}+t_{23}+t_{13})} H_2\left(\frac{x_3 - x_1}{x_2 - x_1}\right)\left(\frac{x_3 - x_1}{x_2 - x_1}\right)^{\hbar t_{13}},$$

$$W_3(x_1, x_2, x_3) = (x_2 - x_1)^{\hbar(t_{12}+t_{23}+t_{13})} H_3\left(\frac{x_2 - x_3}{x_2 - x_1}\right)\left(\frac{x_2 - x_3}{x_2 - x_1}\right)^{\hbar t_{23}},$$

and solutions in the real domain $\{(x_1, x_2, x_3) \mid x_3 < x_1 < x_2\}$ such that

$$W_4(x_1, x_2, x_3) = (x_2 - x_3)^{\hbar(t_{12}+t_{23}+t_{13})} H_4\left(\frac{x_1 - x_3}{x_2 - x_3}\right)\left(\frac{x_1 - x_3}{x_2 - x_3}\right)^{\hbar t_{13}},$$

$$W_5(x_1, x_2, x_3) = (x_2 - x_3)^{\hbar(t_{12}+t_{23}+t_{13})} H_5\left(\frac{x_2 - x_1}{x_2 - x_3}\right)\left(\frac{x_2 - x_1}{x_2 - x_3}\right)^{\hbar t_{12}}.$$

We require the functions H_i to be analytic on the unit disc with value 1 at 0. The functions W_i extend uniquely to solutions of KZ_3 on Γ. By definition of Φ we immediately have

$$W_0 = W_1\Phi, \quad W_2 = W_3\Phi_{132}, \quad W_4 = W_5\Phi_{312}. \tag{3.14}$$

We next compare W_2 and W_4. Consider the set

$$\Omega_2 = \{(z_1, z_2, z_3) \in \Gamma : |z_3 - z_1| < |z_2 - z_1|\}.$$

It has two connected components, Ω_2^+ and Ω_2^-, corresponding to the two possible orientations of the pair of vectors $(z_2 - z_1, z_3 - z_1)$ (if the vectors are colinear, first perturb (z_1, z_2, z_3) in Γ). The initial real domain of definition of W_2 is contained in Ω_2^+, so

$$W_2(z_1, z_2, z_3) = (z_2 - z_1)^{\hbar(t_{12}+t_{23}+t_{13})} H_2\left(\frac{z_3 - z_1}{z_2 - z_1}\right)\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^{\hbar t_{13}}$$
$$\text{for } (z_1, z_2, z_3) \in \Omega_2^+. \tag{3.15}$$

Similarly the set $\Omega_4 = \{(z_1, z_2, z_3) \in \Gamma : |z_3 - z_1| < |z_2 - z_3|\}$ has two connected components Ω_4^+ and Ω_4^-, with Ω_4^+ containing the initial domain of definition of W_4, and

$$W_4(z_1, z_2, z_3) = (z_2 - z_3)^{\hbar(t_{12}+t_{23}+t_{13})} H_4\left(\frac{z_1 - z_3}{z_2 - z_3}\right)\left(\frac{z_1 - z_3}{z_2 - z_3}\right)^{\hbar t_{13}}$$
$$\text{for } (z_1, z_2, z_3) \in \Omega_4^+. \tag{3.16}$$

In the latter expression $(z_1 - z_3)^{\hbar t_{13}}$ means the function on Γ obtained by analytic continuation of $(x_1 - x_3)^{\hbar t_{13}}$ from the real domain $\{x_3 < x_1 < x_2\}$. On the other hand, $(z_3 - z_1)^{\hbar t_{13}}$ in Eq. 3.15 is obtained by analytic continuation from $\{x_1 < x_3 < x_2\}$. Going from the first real domain to the second within Γ changes the argument of $x_1 - x_3$ by $-\pi$, so that $(x_1 - x_3)^{\hbar t_{13}}$ in the second domain is $(x_3 - x_1)^{\hbar t_{13}} e^{-i\pi\hbar t_{13}}$. In other words, we can rewrite Eq. 3.16 as

$$W_4(z_1, z_2, z_3) = (z_2 - z_3)^{\hbar(t_{12}+t_{23}+t_{13})} H_4\left(\frac{z_1 - z_3}{z_2 - z_3}\right)\left(\frac{z_3 - z_1}{z_2 - z_3}\right)^{\hbar t_{13}} e^{-\pi i\hbar t_{13}} \tag{3.17}$$

for $(z_1, z_2, z_3) \in \Omega_4^+$, and now all the power functions on the right hand sides of Eqs. 3.15 and 3.17 are obtained by analytic continuation from the real domain $\{x_1 < x_3 < x_2\}$.

We are now in a position to compute the operator Θ such that $W_2 = W_4\Theta$. For a real point (x_1, x_2, x_3) such that $x_1 < x_3 < x_2$ and $x_3 - x_1 < x_2 - x_3$, which belongs to $\Omega_2^+ \cap \Omega_4^+$, put

$$x = \frac{x_3 - x_1}{x_2 - x_1}.$$

Then by virtue of Eqs. 3.15 and 3.17 the equality $W_2 = W_4\Theta$ implies

$$H_2(x)x^{\hbar t_{13}} = (1 - x)^{\hbar(t_{12}+t_{23}+t_{13})} H_4\left(\frac{x}{x-1}\right)\left(\frac{x}{1-x}\right)^{\hbar t_{13}} e^{-\pi i\hbar t_{13}}\Theta.$$

Since H_2 and H_4 are analytic in a neighbourhood of zero and $H_2(0) = H_4(0) = 1$, we see that the function $x^{\hbar t_{13}} e^{-\pi i\hbar t_{13}}\Theta x^{-\hbar t_{13}}$ extends to an analytic function in a neighbourhood of zero with value 1 at 0. By Remark 3.2 this is possible only when $e^{-\pi i\hbar t_{13}}\Theta = 1$.

Similar considerations apply to the pairs (W_1, W_3) and (W_0, W_5), and we get

$$W_0 = W_5 e^{\pi i\hbar(t_{13}+t_{23})}, \quad W_1 = W_3 e^{\pi i\hbar t_{23}}, \quad W_2 = W_4 e^{\pi i\hbar t_{13}}. \tag{3.18}$$

Equations 3.14 and 3.18 imply

$$e^{-\pi i\hbar(t_{13}+t_{23})}\Phi_{312} e^{\pi i\hbar t_{13}} \Phi_{132}^{-1} e^{\pi i\hbar t_{23}} \Phi = 1.$$

As $(\hat{\Delta} \otimes \iota)(t) = t_{13} + t_{23}$, this is exactly the first identity in Eq. 2.3. □

4 Theorem of Kazhdan and Lusztig

For $q \in \mathbb{C} \setminus \{0\}$ not a root of unity consider the quantized universal enveloping algebra $U_q\mathfrak{g}$. To fix notation recall that it is generated by elements E_i, F_i, K_i, K_i^{-1}, $1 \le i \le r$, satisfying the relations

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i, \quad K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j, \quad K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_i^kE_jE_i^{1-a_{ij}-k} = 0, \quad \sum_{k=0}^{1-a_{ij}}(-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} F_i^kF_jF_i^{1-a_{ij}-k} = 0,$$

where $\begin{bmatrix} m \\ k \end{bmatrix}_{q_i} = \frac{[m]_{q_i}!}{[k]_{q_i}![m-k]_{q_i}!}$, $[m]_{q_i}! = [m]_{q_i}[m-1]_{q_i}\dots[1]_{q_i}$, $[n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and $q_i = q^{d_i}$. This is a Hopf algebra with coproduct $\hat{\Delta}_q$ and counit $\hat{\varepsilon}_q$ defined by

$$\hat{\Delta}_q(K_i) = K_i \otimes K_i,\ \ \hat{\Delta}_q(E_i) = E_i \otimes 1 + K_i \otimes E_i,\ \ \hat{\Delta}_q(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$
$$\hat{\varepsilon}_q(E_i) = \hat{\varepsilon}_q(F_i) = 0,\ \ \hat{\varepsilon}_q(K_i) = 1.$$

If V is a finite dimensional $U_q\mathfrak{g}$-module and $\lambda \in P \subset \mathfrak{h}^*$ is an integral weight, denote by $V(\lambda)$ the space of vectors $v \in V$ of weight λ, so that $K_i v = q_i^{\lambda(h_i)} v$ for all i, where $h_i \in \mathfrak{h}$ is such that $\alpha_j(h_i) = a_{ij}$. Recall that V is called admissible if $V = \oplus_{\lambda \in P} V(\lambda)$. Consider the tensor category of finite dimensional admissible $U_q\mathfrak{g}$-modules. It is a semisimple category with simple objects indexed by dominant integral weights $\lambda \in P_+$. For each $\lambda \in P_+$ we fix an irreducible $U_q\mathfrak{g}$-module V_λ^q with highest weight λ. Denote by $\widehat{\mathbb{C}[G_q]}$ the discrete bialgebra defined by our category, so $\widehat{\mathbb{C}[G_q]} \cong \oplus_{\lambda \in P_+} \mathrm{End}(V_\lambda^q)$. Denote by $\mathcal{U}(G_q)$ the multiplier algebra $M(\widehat{\mathbb{C}[G_q]})$.

The discrete bialgebra $\widehat{\mathbb{C}[G_q]}$ is quasitriangular. The universal R-matrix $\mathcal{R}_\hbar$ depends on the choice of $\hbar \in \mathbb{C}$ such that $q = e^{\pi i \hbar}$. From now on we write q^x instead of $e^{\pi i \hbar x}$, provided the choice of $\hbar$ is clear from the context. The R-matrix $\mathcal{R}_\hbar$ can can be defined by an explicit formula, see e.g. [3, Theorem 8.3.9], but for us it will be enough to remember that it is characterized by $\hat{\Delta}_q^{op} = \mathcal{R}_\hbar \hat{\Delta}_q(\cdot)\mathcal{R}_\hbar^{-1}$ and the following property. Let $\lambda, \mu \in P_+$. Denote by $\bar{\lambda} \in P_+$ the weight $-w_0\lambda$, where w_0 is the longest element in the Weyl group. Then $-\lambda$ is the lowest weight of $V_{\bar{\lambda}}^q$, so there exists a nonzero vector $\zeta_{\bar{\lambda}}^q \in V_{\bar{\lambda}}^q(-\lambda)$ such that $F_i \zeta_{\bar{\lambda}}^q = 0$. Denote also by ξ_μ^q a highest weight vector of V_μ^q, so $E_i \xi_\mu^q = 0$. Then

$$\mathcal{R}_\hbar\big(\zeta_{\bar{\lambda}}^q \otimes \xi_\mu^q\big) = q^{-(\lambda,\mu)} \zeta_{\bar{\lambda}}^q \otimes \xi_\mu^q. \tag{4.1}$$

This indeed characterizes $\mathcal{R}_\hbar$ since $\xi_\mu^q \otimes \zeta_{\bar{\lambda}}^q$ is a cyclic vector in $V_\mu^q \otimes V_{\bar{\lambda}}^q$. Notice that there exists $d \in \mathbb{N}$ such that $d(\lambda, \mu) \in \mathbb{Z}$ for all $\lambda, \mu \in P$. Therefore for each q we get only finitely many different R-matrices $\mathcal{R}_\hbar$.

Denote by $\mathcal{C}(\mathfrak{g}, \hbar)$ the strict braided monoidal category of admissible finite dimensional $U_q\mathfrak{g}$-modules with braiding defined by $\mathcal{R}_\hbar$.

Finally, if $q > 0$ then $\widehat{\mathbb{C}[G_q]}$ is a discrete $*$-bialgebra, with the $*$-operation defined on $U_q\mathfrak{g}$ by for example $K_i^* = K_i$, $E_i^* = F_i K_i$, $F_i^* = K_i^{-1} E_i$. Furthermore, $q = e^{\pi i \hbar}$ for a unique $\hbar \in i\mathbb{R}$. Then $\mathcal{R}_\hbar^* = (\mathcal{R}_\hbar)_{21}$, so $(\widehat{\mathbb{C}[G_q]}, \hat{\Delta}_q, \hat{\varepsilon}_q, \mathcal{R}_\hbar)$ is a quasitriangular discrete $*$-bialgebra.

Since the irreducible $U_q\mathfrak{g}$-modules and $\mathfrak{g}$-modules are both parameterized by dominant integral weights, we have a canonical isomorphism between the centers of $\widehat{\mathbb{C}[G]}$ and $\widehat{\mathbb{C}[G_q]}$. We can now formulate the main result.

Theorem 4.1 *Let $q > 0$ and $\hbar \in i\mathbb{R}$ be such that $q = e^{\pi i \hbar}$. Then there exists a unitary twist $\mathcal{F} \in \mathcal{U}(G \times G)$ such that the quasitriangular discrete $*$-quasi-bialgebras $(\widehat{\mathbb{C}[G]}, \hat{\Delta}, \hat{\varepsilon}, \Phi(\hbar t_{12}, \hbar t_{23}), e^{\pi i \hbar t})_{\mathcal{F}}$ and $(\widehat{\mathbb{C}[G_q]}, \hat{\Delta}_q, \hat{\varepsilon}_q, 1, \mathcal{R}_\hbar)$ are $*$-isomorphic, via an isomorphism extending the canonical identification of the centers.*

We call an element $\mathcal{F}$ in the above theorem a unitary Drinfeld twist.

We shall say that a statement holds for generic $\hbar$ if it holds for $\hbar$ outside a countable set.

Lemma 4.2 *Assume a unitary Drinfeld twist exists for generic $\hbar \in i\mathbb{R}$. Then a unitary Drinfeld twist exists for all $\hbar \in i\mathbb{R}$.*

Proof It suffices to show that if $\hbar_n \to \hbar \in i\mathbb{R}^*$ and a unitary Drinfeld twist exists for every $\hbar_n$ then it exists for $\hbar$.

For each n fix a $*$-isomorphism $\varphi_n\colon \mathcal{U}(G_{q_n}) \to \mathcal{U}(G)$, where $q_n = e^{\pi i \hbar_n}$, and a unitary Drinfeld twist $\mathcal{F}_n$. By compactness of finite dimensional unitary groups, passing to a subsequence we may assume that $\{\mathcal{F}_n\}_n$ converges (in the strong operator topology) to a unitary $\mathcal{F} \in W^*(G)\bar{\otimes}W^*(G)$.

Denote the generators of $U_{q_n}\mathfrak{g}$ by $E_i(q_n)$, $F_i(q_n)$, $K_i(q_n)$. Denote also by $\pi^{q_n}_\lambda\colon U_{q_n}\mathfrak{g} \to \mathrm{End}(V^{q_n}_\lambda)$, resp. $\pi_\lambda\colon U\mathfrak{g} \to \mathrm{End}(V_\lambda)$, an irreducible $*$-representation of $U_{q_n}\mathfrak{g}$, resp. $U\mathfrak{g}$, with highest weight λ. We claim that the sequences $\{(\pi_\lambda \circ \varphi_n)(E_i(q_n))\}_n$ are bounded for any λ. Indeed, since φ_n extends the canonical identification of the centers by assumption, the representation $\pi_\lambda \circ \varphi_n$ is unitarily equivalent to $\pi^{q_n}_\lambda$. Normalize the scalar product on $V^{q_n}_\lambda$ by requiring that the highest weight vector $\xi^{q_n}_\lambda$ has norm one. Then the scalar products

$$\big(\pi^{q_n}_\lambda(F_{i_1}(q_n)\dots F_{i_k}(q_n))\xi^{q_n}_\lambda, \pi^{q_n}_\lambda(F_{j_1}(q_n)\dots F_{j_l}(q_n))\xi^{q_n}_\lambda\big)$$

converge to similar scalar products for $q = e^{\pi i \hbar}$, which can easily be checked by induction on $k+l$ using $F_i^* = K_i^{-1}E_i$ and the quantum Serre relations. Choose a set of multiindices $(i_1,\dots,i_k)$ such that the vectors $(\pi^q_\lambda(F_{i_1}(q)\dots F_{i_k}(q))\xi^q_\lambda$ form a basis in V^q_λ. It then follows that the same expressions for q_n define a basis in $V^{q_n}_\lambda$ whenever n is sufficiently large. By applying the orthonormalization procedure we obtain an orthonormal basis in $V^{q_n}_\lambda$. The matrix coefficients of $\pi^{q_n}_\lambda(E_i(q_n))$ in this basis are determined by the scalar products

$$\big(\pi^{q_n}_\lambda(E_i(q_n))\pi^{q_n}_\lambda(F_{i_1}(q_n)\dots F_{i_k}(q_n))\xi^{q_n}_\lambda, \pi^{q_n}_\lambda(F_{j_1}(q_n)\dots F_{j_l}(q_n))\xi^{q_n}_\lambda\big).$$

It follows that they converge to the corresponding matrix coefficients of $\pi^q_\lambda(E_i(q))$. In particular, the sequence $\{\pi^{q_n}_\lambda(E_i(q_n))\}_n$ is bounded, and hence so is $\{(\pi_\lambda \circ \varphi_n)(E_i(q_n))\}_n$. Similar arguments apply to the other generators of $U_{q_n}\mathfrak{g}$.

By passing to a subsequence, we may therefore assume that the operators $(\pi_\lambda \circ \varphi_n)(T(q_n))$, where $T(q_n)$ is any of the generators $E_i(q_n)$, $F_i(q_n)$, $K_i(q_n)$ of $U_{q_n}\mathfrak{g}$, converge for every dominant integral weight λ. For each λ the operators we get in the limit define a $*$-representation $\tilde\pi_\lambda\colon U_q\mathfrak{g} \to \mathrm{End}(V_\lambda)$. It is a representation with highest weight λ, so for dimension reasons it must be equivalent to the irreducible representation with highest weight λ. The representations $\tilde\pi_\lambda$ define a $*$-isomorphism $\varphi\colon \mathcal{U}(G_q) \to \mathcal{U}(G)$. As $\{(\pi_\lambda \circ \varphi_n)(T(q_n))\}_n$ converges to $(\pi_\lambda \circ \varphi)(T(q))$ for each generator $T(q_n)$ of $U_{q_n}\mathfrak{g}$, the limit $\mathcal{F}$ of $\{\mathcal{F}_n\}_n$ is a unitary Drinfeld twist with respect to φ (e.g. the identity $\Phi(\hbar t_{12}, \hbar t_{23})_{\mathcal{F}} = 1$ holds because $\Phi(\hbar_n t_{12}, \hbar_n t_{23}) \to \Phi(\hbar t_{12}, \hbar t_{23})$). □

Therefore it suffices to prove Theorem 4.1 for generic $\hbar \in i\mathbb{R}$. Furthermore, by Proposition 2.3 it is enough to show that $(\widehat{\mathbb{C}[G]}, \hat\Delta, \hat\varepsilon, \Phi(\hbar t_{12}, \hbar t_{23}), e^{\pi i \hbar t})_{\mathcal{F}}$ and $(\widehat{\mathbb{C}[G_q]}, \hat\Delta_q, \hat\varepsilon_q, 1, \mathcal{R}_\hbar)$ are isomorphic for a (not necessarily unitary) twist $\mathcal{F} \in \mathcal{U}(G \times G)$. By Proposition 2.1(ii) the existence of such an isomorphism can be reformulated in categorical terms as follows, where we now consider complex parameters instead of only purely imaginary ones.

Theorem 4.3 *For generic $\hbar \in \mathbb{C}$ and $q = e^{\pi i \hbar}$ there exists a $\mathbb{C}$-linear braided monoidal equivalence $F\colon \mathcal{D}(\mathfrak{g}, \hbar) \to \mathcal{C}(\mathfrak{g}, \hbar)$ such that F maps an irreducible $\mathfrak{g}$-module with highest weight λ onto an irreducible $U_q\mathfrak{g}$-module with highest weight λ, and the composition of F with the forgetful functor $\mathcal{C}(\mathfrak{g}, \hbar) \to \mathcal{V}ec$ is naturally isomorphic to the forgetful functor $\mathcal{D}(\mathfrak{g}, \hbar) \to \mathcal{V}ec$.*

We will start proving this theorem in the next section. In the remaining part of this section we want to make a few remarks that will not be important later.

The result holds for all $\hbar \notin \mathbb{Q}^*$ by [8, 12, 13]. Recall that since $U_q\mathfrak{g}$ is a Hopf algebra, the category $\mathcal{C}(\mathfrak{g}, \hbar)$ is rigid, with a right dual to V defined by $V^\vee = V^*$, $af = f(\hat{S}_q(a)\,\cdot)$, where $\hat{S}_q$ is the coinverse. It follows that $\mathcal{D}(\mathfrak{g}, \hbar)$ is a rigid tensor category as well. Let us show that rigidity for all $\hbar \notin \mathbb{Q}^*$ follows already from Theorem 4.3;[3] in particular, $(\widehat{\mathbb{C}[G]}, \hat{\Delta}, \hat{\varepsilon}, \Phi(\hbar t_{12}, \hbar t_{23}), e^{\pi i \hbar t})$ is a discrete quasi-Hopf algebra for all $\hbar \notin \mathbb{Q}^*$ by Proposition 2.5. As we have said, this result will not be used later, but it is in fact the first step in extending Theorem 4.3 to all $\hbar \notin \mathbb{Q}^*$.

For an element $\beta = \sum_i n_i \alpha_i$ of the root lattice put $K_\beta = \prod_i K_i^{n_i} \in U_q\mathfrak{g}$ and $h_\beta = \sum_i n_i d_i h_i \in \mathfrak{h}$, so that $\lambda(h_\beta) = (\lambda, \beta)$. For a finite dimensional $\mathfrak{g}$-module V denote by $d(V)$ the dimension of V and by $d_q(V)$ the quantity $\mathrm{Tr}(q^{h_{2\rho}})$, where ρ is half the sum of the positive roots. We use the same notation $d_q(V)$ for the quantum dimension $\mathrm{Tr}(K_{2\rho})$ of a module V in $\mathcal{C}(\mathfrak{g}, \hbar)$.

Recall that we denote by $i_v\colon \mathbb{C} \to V \otimes V^*$ and $e_v\colon V^* \otimes V \to \mathbb{C}$ the standard maps making V^* a right dual of V in $\mathcal{V}ec$.

Corollary 4.4 *Let $\hbar \notin \mathbb{Q}^*$, $q = e^{\pi i \hbar}$, and V be an irreducible $\mathfrak{g}$-module. Then a right dual of V in $\mathcal{D}(\mathfrak{g}, \hbar)$ can be defined by $V^\vee = V^*$ with the usual $\mathfrak{g}$-module structure given by $Xf = -f(X\cdot)$ for $X \in \mathfrak{g}$, and $i_V = i_v$, $e_V = \frac{d_q(V)}{d(V)} e_v$.*

Proof We shall only check that the composition

$$V \xrightarrow{i_V \otimes \iota} (V \otimes V^*) \otimes V \xrightarrow{\Phi} V \otimes (V^* \otimes V) \xrightarrow{\iota \otimes e_V} V$$

is the identity map. By continuity it suffices to prove this for generic $\hbar$.

Assume V is an irreducible module with highest weight λ. The map e_V coincides with the composition

$$V^* \otimes V \xrightarrow{\Sigma q^t} V \otimes V^* \xrightarrow{q^{(\lambda+2\rho,\lambda)} d_q(V) \ell_V} \mathbb{C},$$

where ℓ_V is the unique left inverse of i_V in $\mathcal{D}(\mathfrak{g}, \hbar)$, that is, $\ell_V(v \otimes f) = d(V)^{-1} f(v)$. To see this one just has to check how both maps act on the one-dimensional submodule $i_V(\mathbb{C})$ and then observe that t acts on this submodule as multiplication by $-(\lambda + 2\rho, \lambda)$, which follows from Eq. 3.8.

[3] As well as from the original result of Drinfeld in the formal deformation case.

It follows that we equivalently have to show that the composition

$$V \xrightarrow{i_V \otimes \iota} (V \otimes V^*) \otimes V \xrightarrow{\Phi} V \otimes (V^* \otimes V) \xrightarrow{\iota \otimes \Sigma q^t} V \otimes (V \otimes V^*) \xrightarrow{\iota \otimes q^{(\lambda+2\rho,\lambda)} d_q(V) \ell_V} V$$

is the identity map. This computation can be done in the equivalent strict tensor category $\mathcal{C}(\mathfrak{g}, \hbar)$. In other words, we have to check that for an irreducible module V with highest weight λ in $\mathcal{C}(\mathfrak{g}, \hbar)$ the composition

$$V \xrightarrow{i'_V \otimes \iota} V \otimes V^* \otimes V \xrightarrow{\iota \otimes \Sigma \mathcal{R}_\hbar} V \otimes V \otimes V^* \xrightarrow{\iota \otimes q^{(\lambda+2\rho,\lambda)} d_q(V) \ell'_V} V \tag{4.2}$$

is the identity map, where $i'_V : \mathbb{C} \to V \otimes V^*$ is an isomorphism onto the submodule with trivial $U_q\mathfrak{g}$-action and ℓ'_V is the unique left inverse of i'_V. To show this, first of all notice that as i'_V is unique up to a scalar, the composition does not depend on the choice of i'_V. Hence we may assume that i'_V is given by the same formula as i_v. Then the left inverse map ℓ'_V in $\mathcal{C}(\mathfrak{g}, \hbar)$ is given by

$$V \otimes V^* \to \mathbb{C}, \ \ v \otimes f \mapsto d_q(V)^{-1} f(K_{2\rho} v),$$

as can be checked using that the coinverse $\hat{S}_q$ has the property $\hat{S}_q^2(a) = K_{2\rho} a K_{2\rho}^{-1}$. Computing composition (4.2) we are then left to check that $\hat{S}_q((\mathcal{R}_\hbar)_0) K_{2\rho} (\mathcal{R}_\hbar)_1$ acts on V as multiplication by $q^{-(\lambda+2\rho,\lambda)}$. As V is irreducible, we know that $\hat{S}_q((\mathcal{R}_\hbar)_0) K_{2\rho} (\mathcal{R}_\hbar)_1$ acts as a scalar, so it suffices to check how it acts on a highest weight vector, which is easy to compute using the explicit formula for the R-matrix. □

5 Representing the Forgetful Functor

To prove Theorem 4.3 we first of all have to introduce a tensor structure on the forgetful functor $\mathcal{D}(\mathfrak{g}, \hbar) \to \mathcal{V}ec$. The goal is to represent this functor by an object, then by Lemma 2.2 a weak tensor structure on the functor is equivalent to a comonoid structure on the representing object.

It is clear that within $\mathcal{D}(\mathfrak{g}, \hbar)$ we do not have a representing object. If we however allow infinite dimensional modules then there is an obvious choice, the universal enveloping algebra $U\mathfrak{g}$. Namely, for any $\mathfrak{g}$-module V we have a canonical isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g}, V) \to V, \ \ f \mapsto f(1).$$

It is however more convenient to consider the Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{h}$. Viewing $\mathfrak{g}$-modules as $\tilde{\mathfrak{g}}$-modules (with the second copy of $\mathfrak{h}$ acting trivially), the forgetful functor is clearly naturally isomorphic to $\mathrm{Hom}_{\tilde{\mathfrak{g}}}(U\tilde{\mathfrak{g}}, \cdot)$. Recall that $\tilde{\mathfrak{g}}$ comes with a structure of a Manin triple. Namely, denote by $\mathfrak{b}_+$ and $\mathfrak{b}_-$ the Borel subalgebras of $\mathfrak{g}$, and by $\mathfrak{n}_\pm \subset \mathfrak{b}_\pm$ their nilpotent subalgebras. Consider $\mathfrak{b}_+$ and $\mathfrak{b}_-$ as Lie subalgebras of $\tilde{\mathfrak{g}}$ via the embeddings $\eta_\pm : \mathfrak{b}_\pm \to \mathfrak{g} \oplus \mathfrak{h}$, $\eta_\pm(x) = (x, \pm\bar{x})$, where $x \mapsto \bar{x}$ is the projection $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \to \mathfrak{h}$. Then $(\tilde{\mathfrak{g}}, \mathfrak{b}_+, \mathfrak{b}_-)$ is a Manin triple with the symmetric form on $\tilde{\mathfrak{g}}$ given by $\big((x_1, y_1), (x_2, y_2)\big) = (x_1, x_2) - (y_1, y_2)$. Denote by $\tilde{t}$ the element of $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$ defined by this symmetric form.

Identifying $U\mathfrak{b}_+$ with $U\tilde{\mathfrak{g}} \otimes_{U\mathfrak{b}_-} \mathbb{C}$, we consider $U\mathfrak{b}_+$ as a $\tilde{\mathfrak{g}}$-module, which we denote by M_+. Similarly define M_- as $U\tilde{\mathfrak{g}} \otimes_{U\mathfrak{b}_+} \mathbb{C}$. Then $M = M_+ \otimes M_-$ is isomorphic to $U\tilde{\mathfrak{g}}$ as a $\tilde{\mathfrak{g}}$-module by the Poincare-Birkhoff-Witt theorem, so M represents the forgetful functor. We now want to define a comonoid structure on M.

Denote by 1_+ the canonical cyclic vector of M_+. Then there exists a unique $\tilde{\mathfrak{g}}$-module map $\delta_+ \colon M_+ \to M_+ \otimes M_+$ such that $1_+ \mapsto 1_+ \otimes 1_+$. This is nothing else than the comultiplication $\hat{\Delta} \colon U\mathfrak{b}_+ \to U\mathfrak{b}_+ \otimes U\mathfrak{b}_+$. In particular, δ_+ is coassociative. Ignore for the moment that M_+ is infinite dimensional and observe that δ_+ is also coassociative with respect to $\tilde{\Phi} = \Phi(\hbar\tilde{t}_{12}, \hbar\tilde{t}_{23})$, that is, $(\iota \otimes \delta_+)\delta_+ = \tilde{\Phi}(\delta_+ \otimes \iota)\delta_+$. Indeed, formally it is enough to check this on the vector 1_+, and this follows immediately as $\tilde{\Phi}$ acts trivially on the vector $1_+ \otimes 1_+ \otimes 1_+$ since the vector is annihilated by $\tilde{t}_{12}$ and $\tilde{t}_{23}$. We thus see that M_+ is a comonoid. For similar reasons M_- is a comonoid. Now we want to define a comonoid structure on $M = M_+ \otimes M_-$, and there is basically one way to define a morphism $\delta \colon M \to M \otimes M$ using δ_+ and δ_-, namely, as the composition

$$\begin{aligned} M_+ \otimes M_- &\xrightarrow{\delta_+ \otimes \delta_-} (M_+ \otimes M_+) \otimes (M_- \otimes M_-) \\ &\xrightarrow{(\tilde{\Phi} \otimes \iota)\tilde{\Phi}^{-1}_{12,3,4}} (M_+ \otimes (M_+ \otimes M_-)) \otimes M_- \\ &\xrightarrow{\iota \otimes \Sigma e^{\pi i \hbar \tilde{t}} \otimes \iota} (M_+ \otimes (M_- \otimes M_+)) \otimes M_- \\ &\xrightarrow{\tilde{\Phi}_{12,3,4}(\tilde{\Phi}^{-1} \otimes \iota)} (M_+ \otimes M_-) \otimes (M_+ \otimes M_-). \end{aligned} \tag{5.1}$$

As M_+ and M_- are infinite dimensional, it is not obvious how to make sense of this construction. So our first goal is to find a representing module which is approximated by finite dimensional ones.

For every dominant integral weight μ fix an irreducible $\mathfrak{g}$-module V_μ with highest weight μ. Fix also a highest weight vector $\xi_\mu \in V_\mu$. We assume that $V_0 = \mathbb{C}$ and $\xi_0 = 1$. The construction of the representing object is based on the following standard representation theoretic fact, see e.g. [25]: if V is a finite dimensional $\mathfrak{g}$-module and λ an integral weight then the map

$$\mathrm{Hom}_{\mathfrak{g}}(V_{\bar{\mu}} \otimes V_{\lambda+\mu}, V) \to V(\lambda), \quad f \mapsto f(\zeta_{\bar{\mu}} \otimes \xi_{\lambda+\mu}), \tag{5.2}$$

is an isomorphism for sufficiently large dominant integral weights μ, where $V(\lambda) \subset V$ is the subspace of vectors of weight λ and $\zeta_{\bar{\mu}}$ is a lowest weight vector in $V_{\bar{\mu}}$. Remark that the above map is always injective as the vector $\zeta_{\bar{\mu}} \otimes \xi_{\lambda+\mu}$ is cyclic.

We need to make a consistent choice of lowest weight vectors. For this recall that if we fix Chevalley generators e_i, f_i, h_i, $1 \le i \le r$, of $\mathfrak{g}$ then for any $\mathfrak{g}$-module V there is an action of the braid group $B_{\mathfrak{g}}$ associated to $\mathfrak{g}$ on V, see e.g. [15]. Consider the canonical section $W_{\mathfrak{g}} \to B_{\mathfrak{g}}$ and denote by $\theta \in B_{\mathfrak{g}}$ the transformation corresponding to the longest element w_0 in the Weyl group $W_{\mathfrak{g}}$. Then $\theta \colon V \to V$ is a natural isomorphism having the following properties. If V and W are $\mathfrak{g}$-modules then the action of θ on $V \otimes W$ coincides with $\theta \otimes \theta$. Next, θ maps $V(\lambda)$ onto $V(w_0\lambda)$. In particular, $\theta\xi_\mu$ is a lowest weight vector in V_μ, which we denote by ζ_μ. Finally, for all $1 \le i \le r$ we have $\theta f_i = -e_{\bar{i}}\theta$, where $\bar{i}$ is such that $\alpha_{\bar{i}} = \bar{\alpha}_i = -w_0\alpha_i$.

For an integral weight λ and dominant integral weights μ and η such that $\lambda + \mu$ is dominant consider the composition of morphisms

$$\mathrm{tr}^{\eta}_{\mu,\lambda+\mu} \colon V_{\bar{\mu}+\bar{\eta}} \otimes V_{\lambda+\mu+\eta} \xrightarrow{T_{\bar{\mu},\bar{\eta}} \otimes T_{\eta,\lambda+\mu}} V_{\bar{\mu}} \otimes V_{\bar{\eta}} \otimes V_\eta \otimes V_{\lambda+\mu} \xrightarrow{\iota \otimes S_\eta \otimes \iota} V_{\bar{\mu}} \otimes V_{\lambda+\mu}, \tag{5.3}$$

where the morphisms T and S are uniquely determined by

$$T_{\mu,\eta}\colon V_{\mu+\eta}\to V_\mu\otimes V_\eta,\ \ \xi_{\mu+\eta}\mapsto\xi_\mu\otimes\xi_\eta,$$

and

$$S_\eta\colon V_{\bar\eta}\otimes V_\eta\to\mathbb{C},\ \ \zeta_{\bar\eta}\otimes\xi_\eta\mapsto 1.$$

Notice that $T_{\mu,\eta}\zeta_{\mu+\eta}=\zeta_\mu\otimes\zeta_\eta$ by the properties of θ. It follows that

$$\mathrm{tr}^\eta_{\mu,\lambda+\mu}(\zeta_{\bar\mu+\bar\eta}\otimes\xi_{\lambda+\mu+\eta})=\zeta_{\bar\mu}\otimes\xi_{\lambda+\mu},$$

and this completely determines $\mathrm{tr}^\eta_{\mu,\lambda+\mu}$. Using these morphisms define the inverse limit $\mathfrak{g}$-module

$$M_\lambda=\varprojlim_\mu V_{\bar\mu}\otimes V_{\lambda+\mu}.$$

We consider M_λ as a topological $\mathfrak{g}$-module with a base of neighborhoods of zero formed by the kernels of the canonical morphisms $M_\lambda\to V_{\bar\mu}\otimes V_{\lambda+\mu}$. Observe that $\mathrm{tr}^\eta_{\mu,\lambda+\mu}$ is surjective since its image contains the cyclic vector $\zeta_{\bar\mu}\otimes\xi_{\lambda+\mu}$. It follows that the morphisms $M_\lambda\to V_{\bar\mu}\otimes V_{\lambda+\mu}$ are surjective. Hence, if V is a $\mathfrak{g}$-module with discrete topology, then any continuous morphism $M_\lambda\to V$ factors through $V_{\bar\mu}\otimes V_{\lambda+\mu}$ for some μ, so that the space $\mathrm{Hom}_\mathfrak{g}(M_\lambda,V)$ of such morphisms is the inductive limit of $\mathrm{Hom}_\mathfrak{g}(V_{\bar\mu}\otimes V_{\lambda+\mu},V)$.[4] In particular, for any finite dimensional $\mathfrak{g}$-module V the maps (5.2) induce a linear isomorphism

$$\mathrm{Hom}_\mathfrak{g}(M_\lambda,V)\to V(\lambda).$$

Therefore the topological $\mathfrak{g}$-module $M=\oplus_{\lambda\in P}M_\lambda$, where P is the lattice of integral weights, represents the forgetful functor.

There is an obvious deficiency in the construction of the module M_λ: we did not take into account the associativity morphisms in the composition (5.3). So a more natural morphism in $\mathcal{D}(\mathfrak{g},\hbar)$ is the composition

$$\begin{aligned}V_{\bar\mu+\bar\eta}\otimes V_{\lambda+\mu+\eta}&\to(V_{\bar\mu}\otimes V_{\bar\eta})\otimes(V_\eta\otimes V_{\lambda+\mu})\xrightarrow{(\Phi\otimes\iota)\Phi^{-1}_{12,3,4}}(V_{\bar\mu}\otimes(V_{\bar\eta}\otimes V_\eta))\\&\otimes V_{\lambda+\mu}\to V_{\bar\mu}\otimes V_{\lambda+\mu}.\end{aligned}$$

Remark that we could instead use $(\iota\otimes\Phi^{-1})\Phi_{1,2,34}$ as the middle morphism, but by the coherence theorem of Mac Lane we would get the same composition.

The problem now is that we do not get a coherent system of morphisms $V_{\bar\mu+\bar\eta}\otimes V_{\lambda+\mu+\eta}\to V_{\bar\mu}\otimes V_{\lambda+\mu}$. It turns out that this can be rectified by rescaling. First we need a lemma.

Lemma 5.1 *Denote by $g^\hbar_{\mu,\eta}$ the image of $\zeta_{\bar\mu+\bar\eta}\otimes\xi_{\mu+\eta}$ under the composition*

$$V_{\bar\mu+\bar\eta}\otimes V_{\mu+\eta}\xrightarrow{T_{\bar\mu,\bar\eta}\otimes T_{\eta,\mu}}V_{\bar\mu}\otimes V_{\bar\eta}\otimes V_\eta\otimes V_\mu\xrightarrow{(\iota\otimes S_\eta\otimes\iota)B}V_{\bar\mu}\otimes V_\mu\xrightarrow{S_\mu}\mathbb{C},$$

[4] Alternatively one can consider M_λ as an object in the category pro-$\mathcal{C}(\mathfrak{g})$ obtained by free completion of $\mathcal{C}(\mathfrak{g})$ under inverse limits. Then by definition $\mathrm{Hom}(M_\lambda,V)$ is the inductive limit of $\mathrm{Hom}_\mathfrak{g}(V_{\bar\mu}\otimes V_{\lambda+\mu},V)$.

where $B=(\Phi\otimes\iota)\Phi^{-1}_{12,3,4}$. *Then for generic* $\hbar$ *the map* $(\mu,\eta)\mapsto g^\hbar_{\mu,\eta}$ *is a* $\mathbb{C}^*$*-valued symmetric normalized* 2*-cocycle on the semigroup* P_+ *of dominant integral weights, that is,*

$$g^\hbar_{\mu,\eta}=g^\hbar_{\eta,\mu},\quad g^\hbar_{0,\eta}=g^\hbar_{\mu,0}=1,\quad g^\hbar_{\lambda+\mu,\eta}g^\hbar_{\lambda,\mu}=g^\hbar_{\lambda,\mu+\eta}g^\hbar_{\mu,\eta}.$$

In fact using that $\mathcal{D}(\mathfrak{g},\hbar)$ is rigid one can show that $g^\hbar_{\mu,\eta}\neq 0$ for all $\hbar\notin\mathbb{Q}^*$.

Proof It is easy to see that $g^0_{\mu,\eta}=1$. As $g^\hbar_{\mu,\eta}$ is analytic in $\hbar$ outside a discrete set, we conclude that $g^\hbar_{\mu,\eta}\neq 0$ for generic $\hbar$.

That $g^\hbar_{0,\eta}=g^\hbar_{\mu,0}=1$ is immediate as the associator is equal to 1 as long as one of the modules is trivial.

To show that $g^\hbar_{\mu,\eta}$ is a cocycle first observe that the compositions

$$V_{\lambda+\mu+\eta}\xrightarrow{T_{\lambda+\mu,\eta}}V_{\lambda+\mu}\otimes V_\eta\xrightarrow{T_{\lambda,\mu}\otimes\iota}V_\lambda\otimes V_\mu\otimes V_\eta\xrightarrow{\Phi}V_\lambda\otimes V_\mu\otimes V_\eta$$

and

$$V_{\lambda+\mu+\eta}\xrightarrow{T_{\lambda,\mu+\eta}}V_\lambda\otimes V_{\mu+\eta}\xrightarrow{\iota\otimes T_{\mu,\eta}}V_\lambda\otimes V_\mu\otimes V_\eta$$

coincide. To see this we just have to check how these morphisms act on the highest weight vector and then observe that Φ acts trivially on $\xi_\lambda\otimes\xi_\mu\otimes\xi_\eta$, since both t_{12} and t_{23} preserve the one-dimensional space spanned by this vector and in particular commute on this space. Next observe that the composition in the formulation of the lemma coincides with $g^\hbar_{\mu,\eta}S_{\mu+\eta}$ by definition. It turns out that these two properties are enough to establish the cocycle property $g^\hbar_{\lambda+\mu,\eta}g^\hbar_{\lambda,\mu}=g^\hbar_{\lambda,\mu+\eta}g^\hbar_{\mu,\eta}$. To show this we can and shall strictify the category $\mathcal{D}(\mathfrak{g},\hbar)$ and thus omit Φ in all computations. For example the equality of the above two compositions now reads as

$$(T_{\lambda,\mu}\otimes\iota)T_{\lambda+\mu,\eta}=(\iota\otimes T_{\mu,\eta})T_{\lambda,\mu+\eta}. \tag{5.4}$$

Then the morphisms

$$V_{\bar\lambda+\bar\mu+\bar\eta}\otimes V_{\lambda+\mu+\eta}\to\mathbb{C}$$

given by

$$S_\lambda(\iota\otimes S_\mu\otimes\iota)(\iota\otimes\iota\otimes S_\eta\otimes\iota\otimes\iota)(\iota\otimes T_{\bar\mu,\bar\eta}\otimes T_{\eta,\mu}\otimes\iota)\big(T_{\bar\lambda,\bar\mu+\bar\eta}\otimes T_{\mu+\eta,\lambda}\big)$$

and

$$S_\lambda(\iota\otimes S_\mu\otimes\iota)\big(T_{\bar\lambda,\bar\mu}\otimes T_{\mu,\lambda}\big)(\iota\otimes S_\eta\otimes\iota)\big(T_{\bar\lambda+\bar\mu,\bar\eta}\otimes T_{\eta,\lambda+\mu}\big)$$

coincide. On the other hand, the first morphism is equal to

$$S_\lambda\big(\iota\otimes g^\hbar_{\mu,\eta}S_{\mu+\eta}\otimes\iota\big)\big(T_{\bar\lambda,\bar\mu+\bar\eta}\otimes T_{\mu+\eta,\lambda}\big)=g^\hbar_{\mu,\eta}g^\hbar_{\lambda,\mu+\eta}S_{\lambda+\mu+\eta},$$

whereas the second morphism equals

$$g^\hbar_{\lambda,\mu}S_{\lambda+\mu}\big(\iota\otimes S_\eta\otimes\iota\big)\big(T_{\bar\lambda+\bar\mu,\bar\eta}\otimes T_{\eta,\lambda+\mu}\big)=g^\hbar_{\lambda,\mu}g^\hbar_{\lambda+\mu,\eta}S_{\lambda+\mu+\eta}.$$

Since $S_{\lambda+\mu+\eta}\neq 0$ we get $g^\hbar_{\mu,\eta}g^\hbar_{\lambda,\mu+\eta}=g^\hbar_{\lambda,\mu}g^\hbar_{\lambda+\mu,\eta}$.

It remains to check that the cocycle is symmetric. First observe that $T_{\mu,\eta}$ coincides with the composition

$$V_{\mu+\eta} \xrightarrow{T_{\eta,\mu}} V_\eta \otimes V_\mu \xrightarrow{q^{-(\mu,\eta)}\Sigma q^t} V_\mu \otimes V_\eta,$$

where $q = e^{\pi i\hbar}$. To see this we again look at the action on the highest weight vector. Then the claim follows from

$$t(\xi_\eta \otimes \xi_\mu) = (\mu, \eta)\xi_\eta \otimes \xi_\mu, \tag{5.5}$$

which is a consequence of Eq. 3.8 and the fact that C acts on V_λ as multiplication by $(\lambda, \lambda + 2\rho)$. We now strictify $\mathcal{D}(\mathfrak{g}, \hbar)$ and do all computations omitting Φ. Denote by σ the braiding in our new strict category. By definition we have

$$S_\mu\big(\iota \otimes S_\eta \otimes \iota\big)\big(T_{\bar\mu,\bar\eta} \otimes T_{\eta,\mu}\big) = g^\hbar_{\mu,\eta} S_{\mu+\eta}.$$

As

$$T_{\mu,\eta} = q^{(\mu,\eta)}\sigma^{-1} T_{\eta,\mu}, \tag{5.6}$$

we can rewrite this as

$$g^\hbar_{\mu,\eta} S_{\mu+\eta} = S_\mu(\iota \otimes S_\eta \otimes \iota)(\sigma^{-1} \otimes \sigma)\big(T_{\bar\eta,\bar\mu} \otimes T_{\mu,\eta}\big).$$

By the hexagon identities $\sigma_{12,3} = (\sigma \otimes \iota)(\iota \otimes \sigma)$ and $\sigma^{-1}_{1,23} = (\sigma^{-1} \otimes \iota)(\iota \otimes \sigma^{-1})$ we have

$$\sigma^{-1} \otimes \sigma = \big(\sigma^{-1}_{1,23} \otimes \iota\big)(\iota \otimes \sigma_{12,3}).$$

Therefore by naturality of σ we get

$$\begin{aligned} g^\hbar_{\mu,\eta} S_{\mu+\eta} &= S_\mu(\iota \otimes S_\eta \otimes \iota)(\sigma^{-1}_{1,23} \otimes \iota)(\iota \otimes \sigma_{12,3})\big(T_{\bar\eta,\bar\mu} \otimes T_{\mu,\eta}\big) \\ &= S_\mu(S_\eta \otimes \iota \otimes \iota)(\iota \otimes \sigma_{12,3})\big(T_{\bar\eta,\bar\mu} \otimes T_{\mu,\eta}\big) \\ &= S_\eta(\iota \otimes \iota \otimes S_\mu)(\iota \otimes \sigma_{12,3})\big(T_{\bar\eta,\bar\mu} \otimes T_{\mu,\eta}\big) \\ &= S_\eta(\iota \otimes S_\mu \otimes \iota)\big(T_{\bar\eta,\bar\mu} \otimes T_{\mu,\eta}\big) \\ &= g^\hbar_{\eta,\mu} S_{\mu+\eta}. \end{aligned}$$

Hence $g^\hbar_{\mu,\eta} = g^\hbar_{\eta,\mu}$. □

It is well-known that a symmetric cocycle must be a coboundary. We formulate this in the following a bit more precise form.

Lemma 5.2 *Let $(\mu, \eta) \mapsto c_{\mu,\eta}$ be a $\mathbb{C}^*$-valued symmetric normalized 2-cocycle on P_+. Then for any nonzero complex numbers $b_1, \ldots, b_r$ there exists a unique map $P_+ \ni \mu \mapsto b_\mu \in \mathbb{C}^*$ such that*

$$c_{\mu,\eta} = b_{\mu+\eta} b_\mu^{-1} b_\eta^{-1}, \ \ b_0 = 1, \ \ b_{\omega_i} = b_i \ \textit{for} \ i = 1, \ldots, r.$$

Here $\omega_1, \ldots, \omega_r$ are the fundamental weights.

Proof It is clear that the map b is unique if it exists. To show existence, for a weight $\mu \in P_+$, $\mu = k_1\omega_1 + \ldots + k_r\omega_r$, put $|\mu| = k_1 + \ldots + k_r$. Define b_μ by induction on $|\mu|$ as follows. If $\mu - \omega_i$ is dominant for some i then put $b_\mu = c_{\mu-\omega_i,\omega_i} b_{\mu-\omega_i} b_{\omega_i}$. We have to check that b_μ is well-defined. In other words, if $\mu = \nu + \omega_i + \omega_j$ then we must show that

$$c_{\nu+\omega_j,\omega_i} b_{\nu+\omega_j} b_{\omega_i} = c_{\nu+\omega_i,\omega_j} b_{\nu+\omega_i} b_{\omega_j}.$$

Using the cocycle identities

$$c_{\nu+\omega_j,\omega_i} c_{\nu,\omega_j} = c_{\nu,\omega_i+\omega_j} c_{\omega_i,\omega_j} \quad \text{and} \quad c_{\nu+\omega_i,\omega_j} c_{\nu,\omega_i} = c_{\nu,\omega_j+\omega_i} c_{\omega_j,\omega_i}$$

and that $c_{\omega_i,\omega_j} = c_{\omega_j,\omega_i}$, we equivalently have to check that

$$c_{\nu,\omega_i} b_{\nu+\omega_j} b_{\omega_i} = c_{\nu,\omega_j} b_{\nu+\omega_i} b_{\omega_j}.$$

Since $c_{\nu,\omega_i} = b_{\nu+\omega_i} b_\nu^{-1} b_{\omega_i}^{-1}$ and $c_{\nu,\omega_j} = b_{\nu+\omega_j} b_\nu^{-1} b_{\omega_j}^{-1}$ by the inductive assumption, the identity indeed holds.

Therefore we have constructed a map b such that $b_0 = 1$, $b_{\omega_i} = b_i$ and $c_{\mu,\omega_i} = b_{\mu+\omega_i} b_\mu^{-1} b_{\omega_i}^{-1}$ for $i = 1, \ldots, r$ and $\mu \in P_+$. By induction on $|\eta|$ one can easily check that the identity $c_{\mu,\eta} = b_{\mu+\eta} b_\mu^{-1} b_\eta^{-1}$ holds for all $\mu, \eta \in P_+$. $\square$

For generic $\hbar$ fix a map $P_+ \ni \mu \mapsto g_\mu^\hbar \in \mathbb{C}^*$ such that

$$g_{\mu,\eta}^\hbar g_\mu^\hbar g_\eta^\hbar = g_{\mu+\eta}^\hbar.$$

In Section 7 we shall require an additional property of this map, which determines the cochain $g_\mu^\hbar$ up to a character of the quotient P/Q of the weight lattice by the root lattice, but in this section as well as in the next one any $g_\mu^\hbar$ will do.

Define $S_\mu^\hbar = g_\mu^\hbar S_\mu \colon V_{\bar\mu} \otimes V_\mu \to \mathbb{C}$. We modify Eq. 5.3 by introducing the maps

$$\mathrm{tr}_{\mu,\lambda+\mu}^{\eta,\hbar} \colon V_{\bar\mu+\bar\eta} \otimes V_{\lambda+\mu+\eta} \xrightarrow{T_{\bar\mu,\bar\eta} \otimes T_{\eta,\lambda+\mu}} V_{\bar\mu} \otimes V_{\bar\eta} \otimes V_\eta \otimes V_{\lambda+\mu} \xrightarrow{(\iota \otimes S_\eta^\hbar \otimes \iota) B} V_{\bar\mu} \otimes V_{\lambda+\mu}, \tag{5.7}$$

where $B = (\Phi \otimes \iota)\Phi_{12,3,4}^{-1}$.

Lemma 5.3 *The morphisms* (5.7) *are coherent, that is, the composition*

$$V_{\bar\mu+\bar\eta+\bar\nu} \otimes V_{\lambda+\mu+\eta+\nu} \xrightarrow{\mathrm{tr}_{\mu+\eta,\lambda+\mu+\eta}^{\nu,\hbar}} V_{\bar\mu+\bar\eta} \otimes V_{\lambda+\mu+\eta} \xrightarrow{\mathrm{tr}_{\mu,\lambda+\mu}^{\eta,\hbar}} V_{\bar\mu} \otimes V_{\lambda+\mu}$$

coincides with $\mathrm{tr}_{\mu,\lambda+\mu}^{\eta+\nu,\hbar}$.

Proof We strictify $\mathcal{D}(\mathfrak{g}, \hbar)$ once again. Then the composition $\mathrm{tr}_{\mu,\lambda+\mu}^{\eta,\hbar}\, \mathrm{tr}_{\mu+\eta,\lambda+\mu+\eta}^{\nu,\hbar}$ equals

$$\begin{aligned}
&(\iota \otimes S_\eta^\hbar \otimes \iota)\big(T_{\bar\mu,\bar\eta} \otimes T_{\eta,\lambda+\mu}\big)\big(\iota \otimes S_\nu^\hbar \otimes \iota\big)\big(T_{\bar\mu+\bar\eta,\bar\nu} \otimes T_{\nu,\lambda+\mu+\eta}\big) \\
&\quad = \big(\iota \otimes S_\eta^\hbar \otimes \iota\big)\big(\iota \otimes \iota \otimes S_\nu^\hbar \otimes \iota \otimes \iota\big)\big(T_{\bar\mu,\bar\eta} \otimes \iota \otimes \iota \otimes T_{\eta,\lambda+\mu}\big)\big(T_{\bar\mu+\bar\eta,\bar\nu} \otimes T_{\nu,\lambda+\mu+\eta}\big).
\end{aligned}$$

Using Eq. 5.4 this can be written

$$\big(\iota \otimes S^{\hbar}_{\eta} \otimes \iota\big)\big(\iota \otimes \iota \otimes S^{\hbar}_{\nu} \otimes \iota \otimes \iota\big)\big(\iota \otimes T_{\bar\eta,\bar\nu} \otimes T_{\nu,\eta} \otimes \iota\big)\big(T_{\bar\mu,\bar\eta+\bar\nu} \otimes T_{\eta+\nu,\lambda+\mu}\big).$$

By definition of $g^{\hbar}_{\eta,\nu}$ and using that $g^{\hbar}_{\eta,\nu} g^{\hbar}_{\eta} g^{\hbar}_{\nu} = g^{\hbar}_{\eta+\nu}$ we have

$$S^{\hbar}_{\eta}\big(\iota \otimes S^{\hbar}_{\nu} \otimes \iota\big)\big(T_{\bar\eta,\bar\nu} \otimes T_{\nu,\eta}\big) = S^{\hbar}_{\eta+\nu}.$$

Therefore the above expression equals

$$\big(\iota \otimes S^{\hbar}_{\eta+\nu} \otimes \iota\big)\big(T_{\bar\mu,\bar\eta+\bar\nu} \otimes T_{\eta+\nu,\lambda+\mu}\big) = \mathrm{tr}^{\eta+\nu,\hbar}_{\mu,\lambda+\mu}. \qquad \square$$

Using the morphisms $\mathrm{tr}^{\eta,\hbar}_{\mu,\lambda+\mu}$ we can therefore define a $\mathfrak{g}$-module

$$M^{\hbar}_{\lambda} = \varprojlim_{\mu} V_{\bar\mu} \otimes V_{\lambda+\mu}.$$

Again we consider $M^{\hbar}_{\lambda}$ as a topological $\mathfrak{g}$-module with a base of neighbourhoods of zero given by the kernels of the maps $M^{\hbar}_{\lambda} \to V_{\bar\mu} \otimes V_{\lambda+\mu}$, while any module in $\mathcal{D}(\mathfrak{g},\hbar)$ is considered with discrete topology.

Proposition 5.4 *For $\lambda \in P$ and generic $\hbar \in \mathbb{C}$ the topological module $M^{\hbar}_{\lambda}$ is isomorphic to M_{λ}. In particular, for any such $\hbar$ the functor $\mathcal{D}(\mathfrak{g},\hbar) \to \mathcal{V}ec$, $V \mapsto V(\lambda)$, is naturally isomorphic to* $\mathrm{Hom}_{\mathfrak{g}}\big(M^{\hbar}_{\lambda}, \cdot\big)$.

Proof Fix a regular dominant integral weight μ (that is, μ lies in the interior of the Weyl chamber). Then $n\mu$ dominates any other weight for sufficiently large n. Choose $n_0 \in \mathbb{N}$ such that $n_0\mu + \lambda \ge 0$. Then M_{λ} is isomorphic to the inverse limit of

$$V_{n_0\bar\mu} \otimes V_{\lambda+n_0\mu} \xleftarrow{\mathrm{tr}^{\mu}_{n_0\mu,\lambda+n_0\mu}} V_{(n_0+1)\bar\mu} \otimes V_{\lambda+(n_0+1)\mu} \xleftarrow{\mathrm{tr}^{\mu}_{(n_0+1)\mu,\lambda+(n_0+1)\mu}} V_{(n_0+2)\bar\mu} \otimes V_{\lambda+(n_0+2)\mu} \longleftarrow \dots,$$

and $M^{\hbar}_{\lambda}$ is the inverse limit of

$$V_{n_0\bar\mu} \otimes V_{\lambda+n_0\mu} \xleftarrow{\mathrm{tr}^{\mu,\hbar}_{n_0\mu,\lambda+n_0\mu}} V_{(n_0+1)\bar\mu} \otimes V_{\lambda+(n_0+1)\mu} \xleftarrow{\mathrm{tr}^{\mu,\hbar}_{(n_0+1)\mu,\lambda+(n_0+1)\mu}} V_{(n_0+2)\bar\mu} \otimes V_{\lambda+(n_0+2)\mu} \longleftarrow \dots.$$

It is therefore enough to find isomorphisms f_n of $V_{n\bar\mu} \otimes V_{\lambda+n\mu}$ onto itself such that for all $n \ge n_0$ we have

$$f_n\, \mathrm{tr}^{\mu}_{n\mu,\lambda+n\mu} = \mathrm{tr}^{\mu,\hbar}_{n\mu,\lambda+n\mu}\, f_{n+1}.$$

We construct f_n by induction on n. Take f_{n_0} to be the identity map. Assuming that f_n is constructed, observe that $\mathrm{tr}^{\mu}_{n\mu,\lambda+n\mu}$ is surjective since it maps the vector $\zeta_{(n+1)\bar\mu} \otimes \xi_{\lambda+(n+1)\mu}$ onto the cyclic vector $\zeta_{n\bar\mu} \otimes \xi_{\lambda+n\mu}$. It follows that for generic $\hbar$ the map $\mathrm{tr}^{\mu,\hbar}_{n\mu,\lambda+n\mu}$ is surjective as well. Therefore both maps $f_n\,\mathrm{tr}^{\mu}_{n\mu,\lambda+n\mu}$ and $\mathrm{tr}^{\mu,\hbar}_{n\mu,\lambda+n\mu}$ are surjective. This is enough to conclude that f_{n+1} exists. Indeed, the claim is that if g_1 and g_2 are surjective morphisms $V \to W$ of finite dimensional $\mathfrak{g}$-modules then there exists an isomorphism f of V onto itself such that $g_1 f = g_2$. To see this we can reduce to the situation when $V = U \otimes \mathbb{C}^n$ and $W = U \otimes \mathbb{C}^m$ for some irreducible $\mathfrak{g}$-module U. Then $g_i = \iota \otimes h_i$, where $h_i\colon \mathbb{C}^n \to \mathbb{C}^m$ is a linear surjective map. Clearly we can find an invertible linear map $h\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $h_1 h = h_2$, and then put $f = \iota \otimes h$. □

6 A Comonoid Structure on the Representing Object

In the previous section we showed that for generic $\hbar \in \mathbb{C}$ the topological $\mathfrak{g}$-module

$$M^\hbar = \oplus_{\lambda \in P} M^\hbar_\lambda$$

represents the forgetful functor $\mathcal{D}(\mathfrak{g}, \hbar) \to \mathcal{V}ec$. In this section we shall turn the functor $\mathrm{Hom}_{\mathfrak{g}}(M^\hbar, \cdot)$ into a tensor functor. To do this we introduce a comonoid structure on $M^\hbar$.

Define

$$M^\hbar_{\lambda_1} \hat{\otimes} M^\hbar_{\lambda_2} = \varprojlim_{\mu_1, \mu_2} \big(V_{\bar\mu_1} \otimes V_{\lambda_1+\mu_1}\big) \otimes \big(V_{\bar\mu_2} \otimes V_{\lambda_2+\mu_2}\big)$$

and then

$$M^\hbar \hat{\otimes} M^\hbar = \prod_{\lambda_1, \lambda_2 \in P} M^\hbar_{\lambda_1} \hat{\otimes} M^\hbar_{\lambda_2}.$$

Higher tensor powers of $M^\hbar$ are defined similarly. We want to define

$$\delta^\hbar\colon M^\hbar \to M^\hbar \hat{\otimes} M^\hbar.$$

The restriction of $\delta^\hbar$ to $M^\hbar_\lambda$ composed with the projection $M^\hbar \hat{\otimes} M^\hbar \to M^\hbar_{\lambda_1} \hat{\otimes} M^\hbar_{\lambda_2}$ will be nonzero only if $\lambda = \lambda_1 + \lambda_2$, so $\delta^\hbar$ is determined by maps

$$\delta^\hbar_{\lambda_1,\lambda_2}\colon M^\hbar_{\lambda_1+\lambda_2} \to M^\hbar_{\lambda_1} \hat{\otimes} M^\hbar_{\lambda_2}.$$

Motivated by Eq. 5.1 we define these morphisms using the compositions

$$\begin{aligned} m^\hbar_{\mu,\eta,\lambda_1,\lambda_2}\colon V_{\bar\mu+\bar\eta} \otimes V_{\lambda_1+\lambda_2+\mu+\eta} &\xrightarrow{T_{\bar\mu,\bar\eta} \otimes T_{\lambda_1+\mu,\lambda_2+\eta}} (V_{\bar\mu} \otimes V_{\bar\eta}) \otimes (V_{\lambda_1+\mu} \otimes V_{\lambda_2+\eta}) \\ &\xrightarrow{q^{(\lambda_1+\mu,\eta)} B^{-1}(\iota \otimes \Sigma q^t \otimes \iota) B} (V_{\bar\mu} \otimes V_{\lambda_1+\mu}) \otimes (V_{\bar\eta} \otimes V_{\lambda_2+\eta}), \end{aligned} \tag{6.1}$$

where $q = e^{\pi i \hbar}$ and $B = (\Phi \otimes \iota)\Phi^{-1}_{12,3,4}$.

Lemma 6.1 *The morphisms $m^\hbar$ are consistent with the morphisms $\mathrm{tr}^{\cdot,\hbar}$ defining the inverse limits, so they define morphisms $\delta^\hbar_{\lambda_1,\lambda_2}\colon M^\hbar_{\lambda_1+\lambda_2} \to M^\hbar_{\lambda_1} \hat{\otimes} M^\hbar_{\lambda_2}$.*

Proof We have to check that

$$\big(\mathrm{tr}^{\nu,\hbar}_{\mu,\lambda_1+\mu} \otimes \mathrm{tr}^{\omega,\hbar}_{\eta,\lambda_2+\eta}\big) m^\hbar_{\mu+\nu,\eta+\omega,\lambda_1,\lambda_2} = m^\hbar_{\mu,\eta,\lambda_1,\lambda_2}\, \mathrm{tr}^{\nu+\omega,\hbar}_{\mu+\eta,\lambda_1+\lambda_2+\mu+\eta}. \tag{6.2}$$

Since

$$\mathrm{tr}^{\nu,\hbar}_{\mu,\lambda_1+\mu} \otimes \mathrm{tr}^{\omega,\hbar}_{\eta,\lambda_2+\eta} = \big(\mathrm{tr}^{\nu,\hbar}_{\mu,\lambda_1+\mu} \otimes \iota \otimes \iota\big)\big(\iota \otimes \iota \otimes \mathrm{tr}^{\omega,\hbar}_{\eta,\lambda_2+\eta}\big),$$

it suffices to check this assuming that either ν or ω is zero. We shall only consider the case $\omega = 0$. We therefore have to check that

$$\big(\mathrm{tr}^{\nu,\hbar}_{\mu,\lambda_1+\mu} \otimes \iota \otimes \iota\big) m^{\hbar}_{\mu+\nu,\eta,\lambda_1,\lambda_2} = m^{\hbar}_{\mu,\eta,\lambda_1,\lambda_2}\, \mathrm{tr}^{\nu,\hbar}_{\mu+\eta,\lambda_1+\lambda_2+\mu+\eta}\,. \tag{6.3}$$

We strictify $\mathcal{D}(\mathfrak{g},\hbar)$. Denote by σ the braiding in the strict tensor category. In the computation below we omit subindices of the morphisms T since they are completely determined by the target modules. We will keep track of some of them to get the right power of q. Thus by definition of $\mathrm{tr}^{\cdot,\hbar}$ and $m^{\hbar}$ the left hand side of Eq. 6.3 is equal to

$$\begin{aligned} &q^{(\lambda_1+\mu+\nu,\eta)}\big(\iota\otimes S^{\hbar}_{\nu}\otimes\iota\otimes\iota\otimes\iota\big)(T_{\bar\mu,\bar\nu}\otimes T\otimes\iota\otimes\iota)(\iota\otimes\sigma\otimes\iota)(T_{\bar\mu+\bar\nu,\bar\eta}\otimes T)\\ &\quad= q^{(\lambda_1+\mu+\nu,\eta)}\big(\iota\otimes S^{\hbar}_{\nu}\otimes\iota\otimes\iota\otimes\iota\big)(\iota\otimes\iota\otimes\sigma_{1,23}\otimes\iota)(T_{\bar\mu,\bar\nu}\otimes\iota\otimes T\otimes\iota)(T_{\bar\mu+\bar\nu,\bar\eta}\otimes T). \end{aligned}$$

Using the identity $(T_{\bar\mu,\bar\nu}\otimes\iota)T_{\bar\mu+\bar\nu,\bar\eta} = (\iota\otimes T_{\bar\nu,\bar\eta})T_{\bar\mu,\bar\nu+\bar\eta}$, see Eq. 5.4, the above expression can be written as

$$q^{(\lambda_1+\mu+\nu,\eta)}\big(\iota\otimes S^{\hbar}_{\nu}\otimes\iota\otimes\iota\otimes\iota\big)(\iota\otimes\iota\otimes\sigma_{1,23}\otimes\iota)(\iota\otimes T_{\bar\nu,\bar\eta}\otimes\iota\otimes T)(T_{\bar\mu,\bar\nu+\bar\eta}\otimes T).$$

By Eq. 5.6 we have $T_{\bar\nu,\bar\eta} = q^{-(\bar\nu,\bar\eta)}\sigma T_{\bar\eta,\bar\nu} = q^{-(\nu,\eta)}\sigma T_{\bar\eta,\bar\nu}$, so we get

$$q^{(\lambda_1+\mu,\eta)}\big(\iota\otimes S^{\hbar}_{\nu}\otimes\iota\otimes\iota\otimes\iota\big)(\iota\otimes\iota\otimes\sigma_{1,23}\otimes\iota)(\iota\otimes\sigma T\otimes\iota\otimes T)(T\otimes T).$$

By the hexagon identity we have $(\iota\otimes\sigma_{1,23})(\sigma\otimes\iota\otimes\iota) = \sigma_{1,234}$, so the above expression equals

$$q^{(\lambda_1+\mu,\eta)}\big(\iota\otimes S^{\hbar}_{\nu}\otimes\iota\otimes\iota\otimes\iota\big)(\iota\otimes\sigma_{1,234}\otimes\iota)(\iota\otimes T\otimes\iota\otimes T)(T\otimes T).$$

Using again that $(T\otimes\iota)T = (\iota\otimes T)T$, we get

$$\begin{aligned} &q^{(\lambda_1+\mu,\eta)}\big(\iota\otimes S^{\hbar}_{\nu}\otimes\iota\otimes\iota\otimes\iota\big)(\iota\otimes\sigma_{1,234}\otimes\iota)(T\otimes\iota\otimes\iota\otimes T)(T\otimes T)\\ &\quad= q^{(\lambda_1+\mu,\eta)}(\iota\otimes\sigma\otimes\iota)\big(\iota\otimes\iota\otimes S^{\hbar}_{\nu}\otimes\iota\otimes\iota\big)(T\otimes\iota\otimes\iota\otimes T)(T\otimes T)\\ &\quad= q^{(\lambda_1+\mu,\eta)}(\iota\otimes\sigma\otimes\iota)(T\otimes T)\big(\iota\otimes S^{\hbar}_{\nu}\otimes\iota\big)(T\otimes T), \end{aligned}$$

which is exactly the right hand side of Eq. 6.3. □

Using the morphisms $\delta^{\hbar}_{\lambda_1,\lambda_2}$ we can in an obvious way define morphisms

$$(\delta^{\hbar}\otimes\iota)\delta^{\hbar},\ (\iota\otimes\delta^{\hbar})\delta^{\hbar}\colon M^{\hbar}\to M^{\hbar}\hat\otimes M^{\hbar}\hat\otimes M^{\hbar}.$$

Lemma 6.2 *We have* $\Phi(\delta^{\hbar}\otimes\iota)\delta^{\hbar} = (\iota\otimes\delta^{\hbar})\delta^{\hbar}$.

Proof For $\lambda_1,\lambda_2,\lambda_3\in P$ we have to check that

$$\Phi\big(\delta^{\hbar}_{\lambda_1,\lambda_2}\otimes\iota\big)\delta^{\hbar}_{\lambda_1+\lambda_2,\lambda_3} = \big(\iota\otimes\delta^{\hbar}_{\lambda_2,\lambda_3}\big)\delta^{\hbar}_{\lambda_1,\lambda_2+\lambda_3}.$$

This reduces to showing that

$$\begin{aligned} &\Phi_{12,34,56}\big(m^{\hbar}_{\mu_1,\mu_2,\lambda_1,\lambda_2}\otimes\iota\otimes\iota\big)m^{\hbar}_{\mu_1+\mu_2,\mu_3,\lambda_1+\lambda_2,\lambda_3}\\ &\quad= \big(\iota\otimes\iota\otimes m^{\hbar}_{\mu_2,\mu_3,\lambda_2,\lambda_3}\big)m^{\hbar}_{\mu_1,\mu_2+\mu_3,\lambda_1,\lambda_2+\lambda_3}. \end{aligned}$$

Let us first check that the powers of q in the definition of $m^{\hbar}$ match. On the left hand side we have $q^{(\lambda_1+\mu_1,\mu_2)+(\lambda_1+\lambda_2+\mu_1+\mu_2,\mu_3)}$, whereas on the right hand side we get

$q^{(\lambda_2+\mu_2,\mu_3)+(\lambda_1+\mu_1,\mu_2+\mu_3)}$, which obviously coincide. Strictifying and omitting subindices in T, as we did in the proof of the previous lemma, it remains to show that

$$(\iota\otimes\sigma\otimes\iota\otimes\iota\otimes\iota)(T\otimes T\otimes\iota\otimes\iota)(\iota\otimes\sigma\otimes\iota)(T\otimes T)$$
$$=(\iota\otimes\iota\otimes\iota\otimes\sigma\otimes\iota)(\iota\otimes\iota\otimes T\otimes T)(\iota\otimes\sigma\otimes\iota)(T\otimes T).$$

By naturality of the braiding, the left hand side equals

$$(\iota\otimes\sigma\otimes\iota\otimes\iota\otimes\iota)(\iota\otimes\iota\otimes\sigma_{1,23}\otimes\iota)(T\otimes\iota\otimes T\otimes\iota)(T\otimes T),$$

whereas the right hand side equals

$$(\iota\otimes\iota\otimes\iota\otimes\sigma\otimes\iota)(\iota\otimes\sigma_{12,3}\otimes\iota\otimes\iota)(\iota\otimes T\otimes\iota\otimes T)(T\otimes T).$$

As $(T\otimes\iota)T=(\iota\otimes T)T$, we thus only need to check that

$$(\sigma\otimes\iota\otimes\iota)(\iota\otimes\sigma_{1,23})=(\iota\otimes\iota\otimes\sigma)(\sigma_{12,3}\otimes\iota),$$

which is immediate from the hexagon identities $\sigma_{1,23}=(\iota\otimes\sigma)(\sigma\otimes\iota)$ and $\sigma_{12,3}=(\sigma\otimes\iota)(\iota\otimes\sigma)$. □

We next introduce a morphism $\varepsilon^{\hbar}\colon M^{\hbar}\to\mathbb{C}$ by requiring it to be nonzero only on $M_0^{\hbar}$, where we set it to be the canonical morphism $M_0^{\hbar}\to V_{\bar{0}}\otimes V_0=\mathbb{C}$, so that $\varepsilon^{\hbar}\colon M_0^{\hbar}\to\mathbb{C}$ is determined by the morphisms

$$\mathrm{tr}_{0,0}^{\mu,\hbar}=S_0^{\hbar}\colon V_{\bar{\mu}}\otimes V_{\mu}\to\mathbb{C}.$$

Lemma 6.3 *We have* $(\varepsilon^{\hbar}\otimes\iota)\delta^{\hbar}=\iota=(\iota\otimes\varepsilon^{\hbar})\delta^{\hbar}$.

Proof We have to check that on $M_\lambda^{\hbar}$ we have $(\varepsilon^{\hbar}\otimes\iota)\delta_{0,\lambda}^{\hbar}=\iota=(\iota\otimes\varepsilon^{\hbar})\delta_{\lambda,0}^{\hbar}$. This follows from the fact that $m_{0,\eta,0,\lambda}^{\hbar}$ and $m_{\mu,0,\lambda,0}^{\hbar}$ are the identity maps. □

Therefore $M^{\hbar}$ is a comonoid, so $\mathrm{Hom}_{\mathfrak{g}}(M^{\hbar},\cdot)$ becomes a weak tensor functor $\mathcal{D}(\mathfrak{g},\hbar)\to\mathcal{V}ec$.

Proposition 6.4 *For generic* $\hbar\in\mathbb{C}$ *the weak tensor functor* $\mathrm{Hom}_{\mathfrak{g}}(M^{\hbar},\cdot)\colon\mathcal{D}(\mathfrak{g},\hbar)\to\mathcal{V}ec$ *is a tensor functor.*

Proof Let V and W be finite dimensional $\mathfrak{g}$-modules. We have to show that for generic $\hbar$ the map

$$\mathrm{Hom}_{\mathfrak{g}}(M^{\hbar},V)\otimes\mathrm{Hom}_{\mathfrak{g}}(M^{\hbar},W)\to\mathrm{Hom}_{\mathfrak{g}}(M^{\hbar},V\otimes W),\quad f\otimes g\mapsto(f\otimes g)\delta^{\hbar},$$

is a linear isomorphism. As $\mathrm{Hom}_{\mathfrak{g}}(M^{\hbar},V)=\oplus_{\lambda\in P}\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^{\hbar},V)$ (notice that the direct sum is finite, because $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^{\hbar},V)\ne 0$ only if $V(\lambda)\ne 0$), we equivalently have to check that for any $\lambda\in P$ the map

$$\bigoplus_{\lambda_1+\lambda_2=\lambda}\mathrm{Hom}_{\mathfrak{g}}\left(M_{\lambda_1}^{\hbar},V\right)\otimes\mathrm{Hom}_{\mathfrak{g}}\left(M_{\lambda_2}^{\hbar},W\right)\to\mathrm{Hom}_{\mathfrak{g}}\left(M_\lambda^{\hbar},V\otimes W\right),$$
$$f_{\lambda_1}\otimes g_{\lambda_2}\mapsto(f_{\lambda_1}\otimes g_{\lambda_2})\delta_{\lambda_1,\lambda_2}^{\hbar},$$

is an isomorphism for generic $\hbar$. As $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^\hbar, V)$ is the inductive limit of $\mathrm{Hom}_{\mathfrak{g}}(V_{\bar\mu} \otimes V_{\lambda+\mu}, V)$, it suffices to check that for fixed $\lambda \in P$ and all sufficiently large μ_1 and μ_2 the map

$$\bigoplus_{\lambda_1+\lambda_2=\lambda} \mathrm{Hom}_{\mathfrak{g}}(V_{\bar\mu_1} \otimes V_{\lambda_1+\mu_1}, V) \otimes \mathrm{Hom}_{\mathfrak{g}}(V_{\bar\mu_2} \otimes V_{\lambda_2+\mu_2}, W)$$
$$\to \mathrm{Hom}_{\mathfrak{g}}(V_{\bar\mu_1+\bar\mu_2} \otimes V_{\lambda_1+\lambda_2+\mu_1+\mu_2}, V \otimes W),$$

which maps $f_{\lambda_1} \otimes g_{\lambda_2}$ onto $(f_{\lambda_1} \otimes g_{\lambda_2})m^\hbar_{\mu_1,\mu_2,\lambda_1,\lambda_2}$, is an isomorphism for generic $\hbar$. As the map is analytic in $\hbar$ outside a discrete set, it suffices to check that it is an isomorphism for $\hbar = 0$. For sufficiently large μ_1 we have isomorphisms $\mathrm{Hom}_{\mathfrak{g}}(V_{\bar\mu_1} \otimes V_{\lambda_1+\mu_1}, V) \to V(\lambda_1)$, $f \mapsto f(\zeta_{\bar\mu_1} \otimes \xi_{\lambda_1+\mu_1})$, and similar isomorphisms for W and $V \otimes W$. It is then easy to verify that under these isomorphisms the above map (for $\hbar = 0$) becomes

$$\bigoplus_{\lambda_1+\lambda_2=\lambda} V(\lambda_1) \otimes W(\lambda_2) \to (V \otimes W)(\lambda), \quad v \otimes w \mapsto v \otimes w,$$

which is clearly an isomorphism. □

Recall that the construction of the comonoid $M^\hbar$ depends on the choice of a 1-cochain $g^\hbar_\mu$ with coboundary $g^\hbar_{\mu,\eta}$.

Lemma 6.5 *Up to an isomorphism the comonoid $M^\hbar$ does not depend on the choice of $g^\hbar_\mu$.*

Proof Assume that $\tilde g^\hbar_\mu$ is another cochain. Denote by $\tilde M^\hbar$ the new comonoid. The map $\chi\colon P_+ \to \mathbb{C}^*$ defined by $\chi(\mu) = \tilde g^\hbar_\mu (g^\hbar_\mu)^{-1}$ is a homomorphism. Then it is straightforward to check that the morphisms $V_{\bar\mu} \otimes V_{\lambda+\mu} \to V_{\bar\mu} \otimes V_{\lambda+\mu}$ given by multiplication with $\chi(\mu)$ induce an isomorphism of $\tilde M^\hbar$ and $M^\hbar$ which respects their comonoid structures. □

So far we have constructed for generic $\hbar \in \mathbb{C}$ a tensor functor $\mathcal{D}(\mathfrak{g}, \hbar) \to \mathcal{V}ec$. Up to natural isomorphisms of tensor functors the construction is canonical. Furthermore, disregarding the tensor structure the functor is naturally isomorphic to the forgetful functor. By the discussion after Proposition 2.1 (or by combining Propositions 2.4 and 2.1(ii)) it already follows that for generic $\hbar$ a twisting of $(\widehat{\mathbb{C}[G]}, \hat\Delta, \hat\varepsilon, \Phi(\hbar t_{12}, \hbar t_{23}), e^{\pi i \hbar t})$ is isomorphic to a discrete bialgebra, or equivalently, there exists a twist $\mathcal{F}^\hbar \in \mathcal{U}(G \times G)$ such that $\Phi(\hbar t_{12}, \hbar t_{23})_{\mathcal{F}^\hbar} = 1$. In the next section we will show that this bialgebra is isomorphic to $\widehat{\mathbb{C}[G_q]}$ by turning the tensor functor $\mathcal{D}(\mathfrak{g}, \hbar) \to \mathcal{V}ec$ into an equivalence of the braided monoidal categories $\mathcal{D}(\mathfrak{g}, \hbar)$ and $\mathcal{C}(\mathfrak{g}, \hbar)$.

In the remaining part of the section we will summarize how one gets a twist $\mathcal{F}^\hbar$ such that $\Phi(\hbar t_{12}, \hbar t_{23})_{\mathcal{F}^\hbar} = 1$ in the form of an "algorithm".

1. For $\mu, \eta \in P_+$ compute the image $g^\hbar_{\mu,\eta}$ of $\zeta_{\bar\mu+\bar\eta} \otimes \xi_{\mu+\eta}$ under the composition

$$V_{\bar\mu+\bar\eta} \otimes V_{\mu+\eta} \xrightarrow{T_{\bar\mu,\bar\eta} \otimes T_{\eta,\mu}} V_{\bar\mu} \otimes V_{\bar\eta} \otimes V_\eta \otimes V_\mu \xrightarrow{(\iota \otimes S_\eta \otimes \iota)B} V_{\bar\mu} \otimes V_\mu \xrightarrow{S_\mu} \mathbb{C},$$

where B is $(\Phi\otimes\iota)\Phi^{-1}_{12,3,4}$. Fix nonzero numbers $z_1,\dots,z_r$. For $\mu=\omega_{i_1}+\dots+\omega_{i_k}$ put

$$g^\hbar_\mu = z_{i_1}\prod_{l=1}^{k-1} g^\hbar_{\omega_{i_1}+\dots+\omega_{i_l},\omega_{i_{l+1}}} z_{i_{l+1}}.$$

2. Fix a regular dominant integral weight μ. For each $\lambda\in P$ choose $n_\lambda\in\mathbb{N}$ such that $n_\lambda\mu+\lambda\ge 0$. Then inductively choose isomorphisms f^λ_n, $n\ge n_\lambda$, of $V_{n\bar\mu}\otimes V_{\lambda+n\mu}$ onto itself such that $f^\lambda_{n_\lambda}$ is the identity map and for each $n\ge n_\lambda$ the following diagram commutes:

$$\begin{tikzcd}
V_{n\bar\mu}\otimes V_{\lambda+n\mu} \arrow[d, "f^\lambda_n"'] & V_{(n+1)\bar\mu}\otimes V_{\lambda+(n+1)\mu} \arrow[l, "\mathrm{tr}^\mu_{n\mu,\lambda+n\mu}"'] \arrow[d, "f^\lambda_{n+1}"] \\
V_{n\bar\mu}\otimes V_{\lambda+n\mu} & V_{(n+1)\bar\mu}\otimes V_{\lambda+(n+1)\mu}, \arrow[l, "\mathrm{tr}^{\mu,\hbar}_{n\mu,\lambda+n\mu}"']
\end{tikzcd}$$

where $\mathrm{tr}^{\mu,\hbar}_{n\mu,\lambda+n\mu}$ is the composition

$$V_{(n+1)\bar\mu}\otimes V_{\lambda+(n+1)\mu} \xrightarrow{T_{n\bar\mu,\bar\mu}\otimes T_{\mu,\lambda+n\mu}} V_{n\bar\mu}\otimes V_{\bar\mu}\otimes V_\mu\otimes V_{\lambda+n\mu}$$
$$\xrightarrow{(\iota\otimes g^\hbar_\mu S_\mu\otimes\iota)B} V_{n\bar\mu}\otimes V_{\lambda+n\mu}$$

with $B=(\Phi\otimes\iota)\Phi^{-1}_{12,3,4}$, and $\mathrm{tr}^\mu_{n\mu,\lambda+n\mu}$ is defined similarly with $g^\hbar_\mu$ and Φ trivial.

3. Let $\eta,\nu\in P_+$. Then $\mathcal{F}^\hbar$ is defined by requiring that it acts on the space $V_\eta\otimes V_\nu$ by the operator $\mathcal{F}^\hbar_{\eta,\nu}$ such that for weights λ_1 and λ_2 with $V_\eta(\lambda_1)\ne 0$, $V_\nu(\lambda_2)\ne 0$ and $\lambda=\lambda_1+\lambda_2$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak g}(V_{n\bar\mu}\otimes V_{\lambda_1+n\mu},V_\eta)\otimes\mathrm{Hom}_{\mathfrak g}(V_{m\bar\mu}\otimes V_{\lambda_2+m\mu},V_\nu) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathfrak g}(V_{(n+m)\bar\mu}\otimes V_{\lambda+(n+m)\mu},V_\eta\otimes V_\nu) \arrow[d] \\
V_\eta(\lambda_1)\otimes V_\nu(\lambda_2) \arrow[r, "(\mathcal{F}^\hbar_{\eta,\nu})^{-1}"] & (V_\eta\otimes V_\nu)(\lambda)
\end{tikzcd}$$

where the left vertical arrow is the map

$$f\otimes g\mapsto f f^{\lambda_1}_n(\zeta_{n\bar\mu}\otimes\xi_{\lambda_1+n\mu})\otimes g f^{\lambda_2}_m(\zeta_{m\bar\mu}\otimes\xi_{\lambda_2+m\mu}),$$

the right vertical arrow is the map

$$f\mapsto f f^{\lambda_1+\lambda_2}_{n+m}(\zeta_{(n+m)\bar\mu}\otimes\xi_{\lambda+(n+m)\mu})$$

and finally the top horizontal arrow maps $f\otimes g$ onto the composition

$$V_{(n+m)\bar\mu}\otimes V_{\lambda+(n+m)\mu} \xrightarrow{T_{n\bar\mu,m\bar\mu}\otimes T_{\lambda_1+n\mu,\lambda_2+m\mu}} \left(V_{n\bar\mu}\otimes V_{m\bar\mu}\right)\otimes\left(V_{\lambda_1+n\mu}\otimes V_{\lambda_2+m\mu}\right)$$
$$\xrightarrow{q^{(\lambda_1+n\mu,m\mu)}B^{-1}(\iota\otimes\Sigma q^t\otimes\iota)B} \left(V_{n\bar\mu}\otimes V_{\lambda_1+n\mu}\right)\otimes\left(V_{m\bar\mu}\otimes V_{\lambda_2+m\mu}\right)$$
$$\xrightarrow{f\otimes g} V_\eta\otimes V_\nu$$

with $q = e^{\pi i \hbar}$ and $B = (\Phi \otimes \iota)\Phi^{-1}_{12,3,4}$. Here n and m can be any natural numbers large enough so that $n \ge n_{\lambda_1}, m \ge n_{\lambda_2}, n + m \ge n_{\lambda_1+\lambda_2}$ and the vertical arrows are isomorphisms.

7 Representing $U_q\mathfrak{g}$ by Endomorphisms of the Functor

In this section we will show that $U_q\mathfrak{g}$, $q = e^{\pi i \hbar}$, can be represented by endomorphisms of the functor $\mathrm{Hom}_{\mathfrak{g}}(M^\hbar, \cdot)$, so $\mathrm{Hom}_{\mathfrak{g}}(M^\hbar, \cdot)$ can be considered as a functor $\mathcal{D}(\mathfrak{g}, \hbar) \to \mathcal{C}(\mathfrak{g}, \hbar)$. For this it is natural to try to define an action of the opposite algebra $(U_q\mathfrak{g})^{op}$ on $M^\hbar$. We will show a bit less, namely, that there is an action of a larger algebra $U_q\tilde{\mathfrak{g}}$ such that the corresponding action on the functor factors through $U_q\mathfrak{g}$.

Denote by $U_q\tilde{\mathfrak{g}}$ the universal algebra generated by elements E_i, F_i, K_i, K_i^{-1}, $1 \le i \le r$, such that

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i, \quad K_iE_jK_i^{-1} = q_i^{-a_{ij}}E_j,$$
$$K_iF_jK_i^{-1} = q_i^{a_{ij}}F_j, \quad E_iF_j - F_jE_i = -\delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}.$$

This is a Hopf algebra with coproduct $\hat{\Delta}_q$ defined by

$$\hat{\Delta}_q(K_i) = K_i \otimes K_i, \quad \hat{\Delta}_q(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \hat{\Delta}_q(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i.$$

The action of $U_q\tilde{\mathfrak{g}}$ on $M^\hbar = \oplus_{\lambda\in P} M^\hbar_\lambda$ will be such that

$$E_iM^\hbar_\lambda \subset M^\hbar_{\lambda-\alpha_i}, \quad F_iM^\hbar_\lambda \subset M^\hbar_{\lambda+\alpha_i}, \quad K_i|_{M^\hbar_\lambda} = q_i^{\lambda(h_i)}.$$

From now on we shall write $\lambda(i)$ instead of $\lambda(h_i)$ to simplify notation. Therefore $\lambda(1), \dots, \lambda(r)$ are the coefficients of λ in the basis $\omega_1, \dots, \omega_r$.

Recalling that $M^\hbar_\lambda$ is the inverse limit of $V_{\bar{\mu}} \otimes V_{\lambda+\mu}$, to define F_i we need consistent morphisms

$$V_{\bar{\mu}+\bar{\eta}} \otimes V_{\lambda+\mu+\eta} \to V_{\bar{\mu}} \otimes V_{\lambda+\alpha_i+\mu}.$$

These will be defined using morphisms $V_{\lambda+\mu+\eta} \to V_\eta \otimes V_{\lambda+\alpha_i+\mu}$, or in other words, morphisms

$$V_{\mu+\eta-\alpha_i} \to V_\mu \otimes V_\eta.$$

Up to a scalar there exists only one such morphism. Indeed, if $\mu(i), \eta(i) \ge 1$, then the weight space $(V_\mu \otimes V_\eta)(\mu + \eta - \alpha_i)$ is spanned by the vectors $f_i\xi_\mu \otimes \xi_\eta$ and $\xi_\mu \otimes f_i\xi_\eta$. The vector

$$\mu(i)\xi_\mu \otimes f_i\xi_\eta - \eta(i)f_i\xi_\mu \otimes \xi_\eta$$

is the only vector in this space, up to a scalar, which is killed by e_i. The corresponding morphism is defined by

$$\tau_{i;\mu,\eta} \colon V_{\mu+\eta-\alpha_i} \to V_\mu \otimes V_\eta, \quad \xi_{\mu+\eta-\alpha_i} \mapsto \mu(i)\xi_\mu \otimes f_i\xi_\eta - \eta(i)f_i\xi_\mu \otimes \xi_\eta. \tag{7.1}$$

Remark that we also have

$$\tau_{i;\mu,\eta}(\zeta_{\mu+\eta-\alpha_i}) = -\mu(i)\zeta_\mu \otimes e_{\bar{i}}\zeta_\eta + \eta(i)e_{\bar{i}}\zeta_\mu \otimes \zeta_\eta, \tag{7.2}$$

as can be easily checked using the properties of θ discussed in Section 5.

Up to a scalar the morphism $V_{\bar{\mu}+\bar{\eta}} \otimes V_{\lambda+\mu+\eta} \to V_{\bar{\mu}} \otimes V_{\lambda+\alpha_i+\mu}$ will be defined as the composition

$$V_{\bar{\mu}+\bar{\eta}} \otimes V_{\lambda+\mu+\eta} \xrightarrow{T_{\bar{\mu},\bar{\eta}}\otimes\tau_{i;\eta,\lambda+\alpha_i+\mu}} V_{\bar{\mu}} \otimes V_{\bar{\eta}} \otimes V_\eta \otimes V_{\lambda+\alpha_i+\mu} \xrightarrow{(\iota\otimes S^{\hbar}_{\eta}\otimes\iota)B} V_{\bar{\mu}} \otimes V_{\lambda+\alpha_i+\mu},$$

where $B = (\Phi \otimes \iota)\Phi^{-1}_{12,3,4}$. To find the right normalization we want these maps to define the usual action of $\mathfrak{g}$ on the forgetful functor for $\hbar = 0$. It is not difficult to check that for $\hbar = 0$ we have to divide the above map by $\eta(i)$. More importantly, we want the above maps to be consistent with $\mathrm{tr}^{\cdot,\hbar}$ for all $\hbar$. We are then forced to find out how the associator Φ composes with morphisms $V_{\mu+\eta+\nu-\alpha_i} \to V_\mu \otimes V_\eta \otimes V_\nu$ obtained by combining the maps τ and T. The space of all possible morphisms is isomorphic to the two-dimensional subspace of $V_\mu \otimes V_\eta \otimes V_\nu$ of vectors of weight $\mu + \eta + \nu - \alpha_i$ killed by e_i. Therefore we have to compute the operator $\Phi(A, B)$ for two-by-two matrices A and B.

Lemma 7.1 *Let* $A = \begin{pmatrix} a+b & 0 \\ c & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} -b-c & a \\ 0 & 0 \end{pmatrix}$ *be such that the numbers* $a, b, c, a+b, a+c, b+c, a+b+c$ *are non-integral. Consider the eigenvectors*

$$e_1 = \begin{pmatrix} a+b \\ c \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

of A *and the eigenvectors*

$$f_1 = \begin{pmatrix} a \\ b+c \end{pmatrix}, \quad f_2 = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

of B*. Then*

$$\begin{pmatrix} b & -c \\ 0 & b+c \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & a+b \\ b & -a \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

and

$$\begin{aligned}
&\Phi(A, B)\begin{pmatrix} \sin\pi b & -\sin\pi c \\ 0 & \sin\pi(b+c) \end{pmatrix} \\
&\quad\times \begin{pmatrix} \frac{1}{\Gamma(1+a+b)\Gamma(1+c)\Gamma(1-(a+b+c))} & 0 \\ 0 & \frac{1}{\Gamma(1+a)\Gamma(1+b)\Gamma(1-(a+b))} \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sin\pi(a+b) \\ \sin\pi b & -\sin\pi a \end{pmatrix} \\
&\quad\times \begin{pmatrix} \frac{1}{\Gamma(1+a)\Gamma(1+b+c)\Gamma(1-(a+b+c))} & 0 \\ 0 & \frac{1}{\Gamma(1+b)\Gamma(1+c)\Gamma(1-(b+c))} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.
\end{aligned}$$

Proof The equation $v' = \left(\frac{A}{x} + \frac{B}{x-1}\right) v$, where $v\colon (0,1) \to \mathbb{C}^2$, has the form

$$\begin{cases} v_0' = \left(\dfrac{a+b}{x} - \dfrac{b+c}{x-1}\right) v_0 + \dfrac{a}{x-1} v_1 \\ v_1' = \dfrac{c}{x} v_0. \end{cases}$$

It follows that $u = v_1$ satisfies the Gauss differential equation

$$x(1-x)u'' + (\gamma - (\alpha+\beta+1)x)u' - \alpha\beta u = 0, \tag{7.3}$$

where $\alpha = -a$, $\beta = c$, $\gamma = 1 - a - b$. Denote by Γ the space of solutions of this equation on $(0, 1)$. By our discussion on page 14 the operator $\Phi(A, B)$ can be written as $\pi_1^{-1}\pi_0$, where the linear isomorphisms $\pi_0, \pi_1 \colon \mathbb{C}^2 \to \Gamma$ are defined as follows. If ξ is an eigenvector of A with eigenvalue λ then $\pi_0(\xi)$ is the unique solution $u \in \Gamma$ such that the vector valued function

$$(0,1) \ni x \mapsto x^{-\lambda} \begin{pmatrix} \frac{x}{c} u'(x) \\ u(x) \end{pmatrix}$$

extends to a holomorphic function on the unit disc with value ξ at $x = 0$. Similarly, if ξ is an eigenvector of B with eigenvalue λ then $\pi_1(\xi)$ is the unique solution $u \in \Gamma$ such that the vector valued function

$$(0,1) \ni x \mapsto x^{-\lambda} \begin{pmatrix} \frac{1-x}{c} u'(1-x) \\ u(1-x) \end{pmatrix}$$

extends to a holomorphic function on the unit disc with value ξ at $x = 0$.

Recall that the Euler hypergeometric function $F(\alpha, \beta, \gamma; \cdot)$ is the unique solution u of Eq. 7.3 which is analytic on the unit disc and such that $u(0) = 1$, $u'(0) = \alpha\beta/\gamma$. Consider the following four solutions of Eq. 7.3:

$$\begin{aligned} u_1 &= x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta, 2-\gamma; x), \\ u_2 &= F(\alpha, \beta, \gamma; x), \\ u_3 &= F(\alpha, \beta, 1+\alpha+\beta-\gamma; 1-x), \\ u_4 &= x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta, 1-\alpha-\beta+\gamma; 1-x). \end{aligned}$$

Then it is immediate that the isomorphisms π_0 and π_1 are given by

$$\pi_0(e_1) = cu_1,\ \ \pi_0(e_2) = bu_2,\ \ \pi_1(f_1) = (b+c)u_3,\ \ \pi_1(f_2) = -\frac{bc}{1-b-c} u_4.$$

We have the following identity, see e.g. [6]:

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta-\gamma+1)} u_3 = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} u_2 + \Gamma(\gamma-1)u_1.$$

Substituting x for $1-x$ and γ for $1+\alpha+\beta-\gamma$ we also get

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} u_2 = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} u_3 + \Gamma(\alpha+\beta-\gamma)u_4.$$

A direct but tedious computation using these identities together with the identities $\Gamma(1+x) = x\Gamma(x)$ and $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$ yields the result. □

Define morphisms

$$\tau^{\hbar}_{i;\mu,\eta} = \frac{\tau_{i;\mu,\eta}}{\Gamma(1+\hbar d_i\mu(i))\Gamma(1+\hbar d_i\eta(i))\Gamma(1-\hbar d_i(\mu(i)+\eta(i)))} : V_{\mu+\eta-\alpha_i} \to V_\mu \otimes V_\eta, \tag{7.4}$$

where $\tau_{i;\mu,\eta}$ is defined by Eq. 7.1.

The subspace of $V_\mu \otimes V_\eta \otimes V_\nu$ of vectors of weight $\mu+\eta+\nu-\alpha_i$ killed by e_i is spanned by the vectors

$$\eta(i)\xi_\mu \otimes \xi_\eta \otimes f_i\xi_\nu - \nu(i)\xi_\mu \otimes f_i\xi_\eta \otimes \xi_\nu \ \text{ and } \ \mu(i)\xi_\mu \otimes f_i\xi_\eta \otimes \xi_\nu - \eta(i) f_i\xi_\mu \otimes \xi_\eta \otimes \xi_\nu.$$

This space is invariant under the operators t_{12} and t_{23}. In the above basis these operators have the form

$$t_{12} = \begin{pmatrix} (\mu,\eta) & 0 \\ d_i\nu(i) & (\mu,\eta) - d_i\mu(i) - d_i\eta(i) \end{pmatrix}, \quad t_{23} = \begin{pmatrix} (\eta,\nu) - d_i\eta(i) - d_i\nu(i) & d_i\mu(i) \\ 0 & (\eta,\nu) \end{pmatrix}.$$

To see this first recall that $t(\xi_\mu \otimes \xi_\eta) = (\mu,\eta)\xi_\mu \otimes \xi_\eta$ by Eq. 5.5. Using $\mathfrak{g}$-invariance of t we therefore get

$$t(f_i\xi_\mu \otimes \xi_\eta + \xi_\mu \otimes f_i\xi_\eta) = f_i t(\xi_\mu \otimes \xi_\eta) = (\mu,\eta)(f_i\xi_\mu \otimes \xi_\eta + \xi_\mu \otimes f_i\xi_\eta). \tag{7.5}$$

Using $(\mu,\alpha_i) = d_i\mu(i)$ and $(\alpha_i,\rho) = d_i$ and arguing as for Eq. 5.5 we get

$$t|_{\tau_{i;\mu,\eta}(V_{\mu+\eta-\alpha_i})} = (\mu,\eta) - d_i\mu(i) - d_i\eta(i), \tag{7.6}$$

whence

$$\begin{aligned} &t(\mu(i)\xi_\mu \otimes f_i\xi_\eta - \eta(i) f_i\xi_\mu \otimes \xi_\eta) \\ &\quad = ((\mu,\eta) - d_i\mu(i) - d_i\eta(i))\big(\mu(i)\xi_\mu \otimes f_i\xi_\eta - \eta(i) f_i\xi_\mu \otimes \xi_\eta\big). \end{aligned}$$

By virtue of this identity and Eq. 7.5 we conclude that

$$t\big(f_i\xi_\mu \otimes \xi_\eta\big) = ((\mu,\eta) - d_i\eta(i)) f_i\xi_\mu \otimes \xi_\eta + d_i\mu(i)\xi_\mu \otimes f_i\xi_\eta.$$

Applying the flip we also get

$$t\big(\xi_\eta \otimes f_i\xi_\mu\big) = ((\mu,\eta) - d_i\eta(i))\xi_\eta \otimes f_i\xi_\mu + d_i\mu(i) f_i\xi_\eta \otimes \xi_\mu.$$

These two identities and Eq. 5.5 imply the above matrix forms of t_{12} and t_{23}.

Recall now that $\Phi(A,B) = \Phi(A-\alpha, B-\beta)$ for any scalars α and β. So replacing t_{12} by $t_{12} - ((\mu,\eta) - d_i\mu(i) - d_i\eta(i))1$ and t_{23} by $t_{23} - (\eta,\nu)1$ we are in a position to apply Lemma 7.1 with $a = \hbar d_i\mu(i)$, $b = \hbar d_i\eta(i)$ and $c = \hbar d_i\nu(i)$. One checks that the vectors e_1, e_2, f_1, f_2 in the lemma are exactly the images of $\hbar d_i\eta(i)\xi_{\mu+\eta+\nu-\alpha_i}$ under the morphisms $(T_{\mu,\eta} \otimes \iota)\tau_{i;\mu+\eta,\nu}$, $(\tau_{i;\mu,\eta} \otimes \iota)T_{\mu+\eta-\alpha_i,\nu}$, $(\iota \otimes T_{\eta,\nu})\tau_{i;\mu,\eta+\nu}$ and $(\iota \otimes \tau_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}$, respectively. As

$$\sin \pi\hbar d_i x = \frac{q_i^x - q_i^{-x}}{2\sqrt{-1}} = \frac{q_i - q_i^{-1}}{2\sqrt{-1}}[x]_{q_i},$$

Lemma 7.1 can therefore be reformulated as the following identity between morphisms $V_{\mu+\eta+\nu-\alpha_i} \to V_\mu \otimes V_\eta \otimes V_\nu$:

$$\Phi(\hbar t_{12}, \hbar t_{23}) \begin{pmatrix} [\eta(i)]_{q_i} & -[\nu(i)]_{q_i} \\ 0 & [\eta(i)+\nu(i)]_{q_i} \end{pmatrix} \begin{pmatrix} (T_{\mu,\eta}\otimes\iota)\tau^{\hbar}_{i;\mu+\eta,\nu} \\ (\tau^{\hbar}_{i;\mu,\eta}\otimes\iota)T_{\mu+\eta-\alpha_i,\nu} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & [\mu(i)+\eta(i)]_{q_i} \\ [\eta(i)]_{q_i} & -[\mu(i)]_{q_i} \end{pmatrix} \begin{pmatrix} (\iota\otimes T_{\eta,\nu})\tau^{\hbar}_{i;\mu,\eta+\nu} \\ (\iota\otimes\tau^{\hbar}_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i} \end{pmatrix}. \tag{7.7}$$

It is remarkable that the proof of this identity is the first and only place where one uses nontrivial specific properties of Φ beyond being an associator; the only special property which we used before Lemma 7.1 was that Φ acts trivially on the highest weight space of $V_\mu \otimes V_\eta \otimes V_\nu$.

Proposition 7.2 *The morphisms*

$$V_{\bar\mu+\bar\eta}\otimes V_{\lambda+\mu+\eta} \xrightarrow{[\eta(i)]_{q_i}^{-1} T_{\bar\mu,\bar\eta}\otimes\tau^{\hbar}_{i;\eta,\lambda+\alpha_i+\mu}} V_{\bar\mu}\otimes V_{\bar\eta}\otimes V_\eta\otimes V_{\lambda+\alpha_i+\mu}$$
$$\xrightarrow{(\iota\otimes S^{\hbar}_\eta\otimes\iota)B} V_{\bar\mu}\otimes V_{\lambda+\alpha_i+\mu}, \tag{7.8}$$

where $B=(\Phi\otimes\iota)\Phi^{-1}_{12,3,4}$, *are consistent with* $\mathrm{tr}^{\cdot,\hbar}$ *and hence define a morphism* $F_i\colon M^{\hbar}_\lambda \to M^{\hbar}_{\lambda+\alpha_i}$. *Similarly the morphisms*

$$V_{\bar\mu+\bar\eta}\otimes V_{\lambda+\mu+\eta} \xrightarrow{[\eta(i)]_{q_i}^{-1}\tau^{\hbar}_{\bar i;\bar\mu+\bar\alpha_i,\bar\eta}\otimes T_{\eta,\lambda+\mu}} V_{\bar\mu+\bar\alpha_i}\otimes V_{\bar\eta}\otimes V_\eta\otimes V_{\lambda+\mu} \xrightarrow{(\iota\otimes S^{\hbar}_\eta\otimes\iota)B} V_{\bar\mu+\bar\alpha_i}\otimes V_{\lambda+\mu} \tag{7.9}$$

define a morphism $E_i\colon M^{\hbar}_\lambda\to M^{\hbar}_{\lambda-\alpha_i}$.

Furthermore, for generic $\hbar$ *we can choose the* 1*-cochain* $g^{\hbar}_\mu$ *such that for each i the composition*

$$V_{2\bar\omega_i-\bar\alpha_i}\otimes V_{2\omega_i-\alpha_i} \xrightarrow{\tau^{\hbar}_{\bar i;\bar\omega_i,\bar\omega_i}\otimes\tau^{\hbar}_{i;\omega_i,\omega_i}} V_{\bar\omega_i}\otimes V_{\bar\omega_i}\otimes V_{\omega_i}\otimes V_{\omega_i} \xrightarrow{(\iota\otimes S_{\omega_i}\otimes\iota)B} V_{\bar\omega_i}\otimes V_{\omega_i} \xrightarrow{S_{\omega_i}} \mathbb{C} \tag{7.10}$$

coincides with $-\frac{g^{\hbar}_{2\omega_i-\alpha_i}}{\left(g^{\hbar}_{\omega_i}\right)^2}[2]_{q_i}S_{2\omega_i-\alpha_i}$. *If* $g^{\hbar}_\mu$ *is chosen this way then the morphisms* E_i *and* F_i *together with the morphism* $K_i\colon M^{\hbar}\to M^{\hbar}$ *acting on* $M^{\hbar}_\lambda$ *as multiplication by* $q_i^{\lambda(i)}$, *define an action of the algebra* $U_q\tilde{\mathfrak g}$ *on* $M^{\hbar}$. *This action respects the comonoid structure of* $M^{\hbar}$ *in the sense that* $\delta^{\hbar}(\omega x)=\hat\Delta_q(\omega)\delta^{\hbar}(x)$ *and* $\varepsilon^{\hbar}(\omega x)=\hat\varepsilon_q(\omega)\varepsilon^{\hbar}(x)$ *for all* $\omega\in U_q\tilde{\mathfrak g}$ *and* $x\in M^{\hbar}$.

Proof Denote the morphism Eq. 7.8 by $\Psi^{\eta,\hbar}_{i;\mu,\lambda+\alpha_i+\mu}$. For consistency we have to check that

$$\mathrm{tr}^{\eta,\hbar}_{\mu,\lambda+\alpha_i+\mu}\,\Psi^{\nu,\hbar}_{i;\mu+\eta,\lambda+\alpha_i+\mu+\eta}=\Psi^{\eta+\nu,\hbar}_{i;\mu,\lambda+\alpha_i+\mu} \quad\text{and}\quad \Psi^{\eta,\hbar}_{i;\mu,\lambda+\alpha_i+\mu}\,\mathrm{tr}^{\nu,\hbar}_{\mu+\eta,\lambda+\alpha_i+\mu+\eta}=\Psi^{\eta+\nu,\hbar}_{i;\mu,\lambda+\alpha_i+\mu}.$$

We shall only check the first identity. Once again we strictify $\mathcal{D}(\mathfrak{g},\hbar)$. As we have done before, we shall often skip the lower indices of maps when they are determined by the target modules. By definition we have

$$\Psi^{\eta+\nu,\hbar}_{i;\mu,\lambda+\alpha_i+\mu}=\frac{1}{[\eta(i)+\nu(i)]_{q_i}}\big(\iota\otimes S^{\hbar}_{\eta+\nu}\otimes\iota\big)(T\otimes\tau^{\hbar}_i).$$

Using that $S^{\hbar}_{\eta+\nu}=S^{\hbar}_{\eta}(\iota\otimes S^{\hbar}_{\nu}\otimes\iota)(T\otimes T_{\nu,\eta})$ by definition, we can rewrite the right hand side as

$$\frac{1}{[\eta(i)+\nu(i)]_{q_i}}(\iota\otimes S^{\hbar}_{\eta}\otimes\iota)(\iota\otimes\iota\otimes S^{\hbar}_{\nu}\otimes\iota\otimes\iota)(\iota\otimes T\otimes T_{\nu,\eta}\otimes\iota)(T\otimes\tau^{\hbar}_i). \tag{7.11}$$

On the other hand,

$$\begin{aligned}
&\mathrm{tr}^{\eta,\hbar}_{\mu,\lambda+\alpha_i+\mu}\,\Psi^{\nu,\hbar}_{i;\mu+\eta,\lambda+\alpha_i+\mu+\eta}\\
&\quad=\frac{1}{[\nu(i)]_{q_i}}\big(\iota\otimes S^{\hbar}_{\eta}\otimes\iota\big)(T\otimes T)\big(\iota\otimes S^{\hbar}_{\nu}\otimes\iota\big)\big(T\otimes\tau^{\hbar}_i\big)\\
&\quad=\frac{1}{[\nu(i)]_{q_i}}\big(\iota\otimes S^{\hbar}_{\eta}\otimes\iota\big)\big(\iota\otimes\iota\otimes S^{\hbar}_{\nu}\otimes\iota\otimes\iota\big)(T\otimes\iota\otimes\iota\otimes T)\big(T\otimes\tau^{\hbar}_i\big)
\end{aligned}$$

Using that $(T\otimes\iota)T=(\iota\otimes T)T$ by Eq. 5.4 we can rewrite this as

$$\frac{1}{[\nu(i)]_{q_i}}\big(\iota\otimes S^{\hbar}_{\eta}\otimes\iota\big)\big(\iota\otimes\iota\otimes S^{\hbar}_{\nu}\otimes\iota\otimes\iota\big)(\iota\otimes T\otimes\iota\otimes T)\big(T\otimes\tau^{\hbar}_i\big). \tag{7.12}$$

It follows from Eq. 7.7 that up to a scalar factor the difference

$$\frac{1}{[\eta(i)+\nu(i)]_{q_i}}(T_{\nu,\eta}\otimes\iota)\tau^{\hbar}_i-\frac{1}{[\nu(i)]_{q_i}}(\iota\otimes T)\tau^{\hbar}_i$$

(in our strictified category) is equal to $(\tau^{\hbar}_i\otimes\iota)T$. Therefore to show that Eqs. 7.11 and 7.12 are equal we have to check that the morphism

$$\begin{gathered}
\big(\iota\otimes S^{\hbar}_{\eta}\otimes\iota\big)\big(\iota\otimes\iota\otimes S^{\hbar}_{\nu}\otimes\iota\otimes\iota\big)\big(\iota\otimes T\otimes\tau^{\hbar}_i\otimes\iota\big)(T\otimes T)\colon\\
V_{\bar\mu+\bar\eta+\bar\nu}\otimes V_{\lambda+\mu+\eta+\nu}\to V_{\bar\mu}\otimes V_{\lambda+\alpha_i+\mu}
\end{gathered}$$

is zero. In fact already $S^{\hbar}_{\eta}\big(\iota\otimes S^{\hbar}_{\nu}\otimes\iota\big)\big(T\otimes\tau^{\hbar}_i\big)=0$ since zero is the only morphism from $V_{\bar\eta+\bar\nu}\otimes V_{\eta+\nu-\alpha_i}$ to $\mathbb{C}$.

Thus F_i is well-defined. Similarly one proves that E_i is well-defined.

Next we have to check that under a specific choice of $g^{\hbar}_{\mu}$ the morphisms E_i, F_i, K_i satisfy the defining relations of $U_q\tilde{\mathfrak{g}}$. The only nontrivial relation is

$$E_iF_j-F_jE_i=-\delta_{ij}\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}}. \tag{7.13}$$

The rest clearly holds without any assumptions on the cochain by using that $\alpha_j(i)=a_{ij}$.

The composition Eq. 7.10 coincides with $S_{2\omega_i-\alpha_i}$ up to a scalar factor since the space of morphisms $V_{2\bar\omega_i-\bar\alpha_i}\otimes V_{2\omega_i-\alpha_i}\to\mathbb{C}$ is one-dimensional. This factor is nonzero for generic $\hbar$ since it is equal to -2 for $\hbar = 0$. Indeed, by virtue of Eqs. 7.1 and 7.2 we have to show that the image of

$$(-\zeta_{\bar{\omega}_i} \otimes e_i\zeta_{\bar{\omega}_i} + e_i\zeta_{\bar{\omega}_i} \otimes \zeta_{\bar{\omega}_i}) \otimes (\xi_{\omega_i} \otimes f_i\xi_{\omega_i} - f_i\xi_{\omega_i} \otimes \xi_{\omega_i})$$

under the map $S_{\omega_i}(\iota \otimes S_{\omega_i} \otimes \iota)$ equals -2. This follows from $S_{\omega_i}(e_i\zeta_{\bar{\omega}_i} \otimes f_i\xi_{\omega_i}) = -1$ used twice, which in turn follows from the identities

$$e_i\zeta_{\bar{\omega}_i} \otimes f_i\xi_{\omega_i} = e_i(\zeta_{\bar{\omega}_i} \otimes f_i\xi_{\omega_i}) - \zeta_{\bar{\omega}_i} \otimes e_i f_i\xi_{\omega_i} = e_i(\zeta_{\bar{\omega}_i} \otimes f_i\xi_{\omega_i}) - \zeta_{\bar{\omega}_i} \otimes \xi_{\omega_i}.$$

So to show that we can make the specific choice of the cochain $g^{\hbar}_{\mu}$ stated in the formulation we just have to check that the ratios $g^{\hbar}_{2\omega_i-\alpha_i}/\big(g^{\hbar}_{\omega_i}\big)^2$ can take arbitrary values. As we already remarked in the proof of Lemma 6.5, the cochain $g^{\hbar}_{\mu}$ is defined up to multiplication by a homomorphism $\chi\colon P \to \mathbb{C}^*$. If we replace $g^{\hbar}_{\mu}$ by $g^{\hbar}_{\mu}\chi(\mu)$ then $g^{\hbar}_{2\omega_i-\alpha_i}/\big(g^{\hbar}_{\omega_i}\big)^2$ changes by the factor $\chi(\alpha_i)^{-1}$. Therefore the claim follows from the fact that any homomorphism from the root lattice Q into $\mathbb{C}^*$ can be extended to the weight lattice P. This is well-known and easy to see using infinite divisibility of $\mathbb{C}^*$.

Assuming now that the cochain $g^{\hbar}_{\mu}$ is chosen as stated we want to check Eq. 7.13. Denoting the composition Eq. 7.9 by $\Phi^{\eta,\hbar}_{i;\mu+\alpha_i,\lambda+\mu}$, to prove Eq. 7.13 for $i = j$ it suffices to show that

$$\begin{aligned}&\Phi^{\omega_i,\hbar}_{i;\mu+\alpha_i,\lambda+\alpha_i+\mu}\Psi^{\omega_i,\hbar}_{i;\mu+\omega_i,\lambda+\alpha_i+\mu+\omega_i} - \Psi^{\omega_i,\hbar}_{i;\mu+\alpha_i,\lambda+\alpha_i+\mu}\Phi^{\omega_i,\hbar}_{i;\mu+\alpha_i+\omega_i,\lambda+\mu+\omega_i}\\ &\quad= -[\lambda(i)]_{q_i}\,\mathrm{tr}^{2\omega_i-\alpha_i,\hbar}_{\mu+\alpha_i,\lambda+\mu+\alpha_i}\,.\end{aligned} \tag{7.14}$$

The first term on the left hand side in our strictified category is

$$\begin{aligned}&\big(\iota \otimes S^{\hbar}_{\omega_i} \otimes \iota\big)\big(\tau^{\hbar}_{\bar{i}} \otimes T\big)\big(\iota \otimes S^{\hbar}_{\omega_i} \otimes \iota\big)\big(T \otimes \tau^{\hbar}_i\big)\\ &\quad= \big(\iota \otimes S^{\hbar}_{\omega_i} \otimes \iota\big)(\iota \otimes \iota \otimes S^{\hbar}_{\omega_i} \otimes \iota \otimes \iota)\big(\tau^{\hbar}_{\bar{i}} \otimes \iota \otimes \iota \otimes T\big)\big(T \otimes \tau^{\hbar}_i\big).\end{aligned}$$

Expressing similarly the second term we get that the left hand side of Eq. 7.14 equals

$$\big(\iota \otimes S^{\hbar}_{\omega_i} \otimes \iota\big)\big(\iota \otimes \iota \otimes S^{\hbar}_{\omega_i} \otimes \iota \otimes \iota\big)\big(\big(\tau^{\hbar}_{\bar{i}} \otimes \iota \otimes \iota \otimes T\big)\big(T \otimes \tau^{\hbar}_i\big) - \big(T \otimes \iota \otimes \iota \otimes \tau^{\hbar}_i\big)\big(\tau^{\hbar}_{\bar{i}} \otimes T\big)\big).$$

Next we use identities Eq. 7.7 to express $\big(\tau^{\hbar}_{\bar{i}} \otimes \iota \otimes \iota \otimes T\big)\big(T \otimes \tau^{\hbar}_i\big)$ and $(T \otimes \iota \otimes \iota \otimes \tau^{\hbar}_i)(\tau^{\hbar}_{\bar{i}} \otimes T)$ in the form $(\iota \otimes * \otimes * \otimes \iota)(* \otimes *)$. A tedious but straightforward computation keeping track of subindices shows that the terms $(\iota \otimes T \otimes T \otimes \iota)\big(\tau^{\hbar}_{\bar{i}} \otimes \tau^{\hbar}_i\big)$ cancel, and what is left is the term $\frac{[\lambda(i)]_{q_i}}{[2]_{q_i}}\big(\iota \otimes \tau^{\hbar}_{\bar{i}} \otimes \tau^{\hbar}_i \otimes \iota\big)(T \otimes T)$ and scalar multiples of $\big(\iota \otimes T \otimes \tau^{\hbar}_i \otimes \iota\big)\big(\tau^{\hbar}_{\bar{i}} \otimes T\big)$ and $\big(\iota \otimes \tau^{\hbar}_{\bar{i}} \otimes T \otimes \iota\big)\big(T \otimes \tau^{\hbar}_i\big)$. The last two terms vanish when composed with $\big(\iota \otimes S^{\hbar}_{\omega_i} \otimes \iota\big)\big(\iota \otimes \iota \otimes S^{\hbar}_{\omega_i} \otimes \iota \otimes \iota\big)$ for the same reason as in the proof of consistency of $\Psi^{\cdot,\hbar}$. Therefore the left hand side of Eq. 7.14 equals

$$\frac{[\lambda(i)]_{q_i}}{[2]_{q_i}}\big(\iota \otimes S^{\hbar}_{\omega_i} \otimes \iota\big)\big(\iota \otimes \iota \otimes S^{\hbar}_{\omega_i} \otimes \iota \otimes \iota\big)\big(\iota \otimes \tau^{\hbar}_{\bar{i}} \otimes \tau^{\hbar}_i \otimes \iota\big)(T \otimes T).$$

We have $S^\hbar_{\omega_i}\big(\iota\otimes S^\hbar_{\omega_i}\otimes\iota\big)\big(\tau^\hbar_{\bar i}\otimes\tau^\hbar_i\big)=-[2]_{q_i}S^\hbar_{2\omega_i-\alpha_i}$ by our choice of the cochain $g^\hbar_\mu$, so the above expression is equal to

$$-[\lambda(i)]_{q_i}\big(\iota\otimes S^\hbar_{2\omega_i-\alpha_i}\otimes\iota\big)(T\otimes T),$$

which by definition is the right hand side of Eq. 7.14.

The relation $E_iF_j-F_jE_i=0$ for $i\neq j$ is proved similarly by showing that

$$\Phi^{\eta,\hbar}_{i;\mu+\alpha_i,\lambda+\alpha_j+\mu}\Psi^{\nu,\hbar}_{j;\mu+\eta,\lambda+\alpha_j+\mu+\eta}-\Psi^{\eta,\hbar}_{j;\mu+\alpha_i,\lambda+\alpha_j+\mu}\Phi^{\nu,\hbar}_{i;\mu+\alpha_i+\eta,\lambda+\mu+\eta}=0.$$

We only remark that in this case the morphism $S^\hbar_\eta\big(\iota\otimes S^\hbar_\nu\otimes\iota\big)\big(\tau^\hbar_{\bar i}\otimes\tau^\hbar_j\big)$ vanishes as there are no nonzero morphisms $V_{\bar\eta+\bar\nu-\bar\alpha_i}\otimes V_{\eta+\nu-\alpha_j}\to\mathbb{C}$.

It remains to show that the action of $U_q\tilde{\mathfrak g}$ respects the comonoid structure of $M^\hbar$. We shall only check that $\delta^\hbar(F_ix)=\hat\Delta_q(F_i)\delta^\hbar(x)$, that is,

$$\delta^\hbar_{\lambda_1,\lambda_2}F_i=q_i^{-\lambda_2(i)}(F_i\otimes\iota)\delta^\hbar_{\lambda_1-\alpha_i,\lambda_2}+(\iota\otimes F_i)\delta^\hbar_{\lambda_1,\lambda_2-\alpha_i}.$$

The morphisms $\delta^\hbar$ are induced by the morphisms $m^\hbar$ defined by Eq. 6.1. Therefore it suffices to check that

$$\begin{aligned}&m^\hbar_{\mu,\eta,\lambda_1,\lambda_2}\Psi^{\nu,\hbar}_{i;\mu+\eta,\lambda_1+\lambda_2+\mu+\eta}\\&\quad=q_i^{-\lambda_2(i)}(\Psi^{\nu,\hbar}_{i;\mu,\lambda_1+\mu}\otimes\iota\otimes\iota)m^\hbar_{\mu+\nu,\eta,\lambda_1-\alpha_i,\lambda_2}+(\iota\otimes\iota\otimes\Psi^{\nu,\hbar}_{i;\eta,\lambda_2+\eta})m^\hbar_{\mu,\eta+\nu,\lambda_1,\lambda_2-\alpha_i}.\end{aligned}\tag{7.15}$$

The left hand side multiplied by $[\nu(i)]_{q_i}$ in our strictified category with braiding σ is

$$\begin{aligned}&q^{(\lambda_1+\mu,\eta)}(\iota\otimes\sigma\otimes\iota)(T_{\bar\mu,\bar\eta}\otimes T_{\lambda_1+\mu,\lambda_2+\eta})\big(\iota\otimes S^\hbar_\nu\otimes\iota\big)\big(T_{\bar\mu+\bar\eta,\bar\nu}\otimes\tau^\hbar_{i;\nu,\lambda_1+\lambda_2+\mu+\eta}\big)\\&\quad=q^{(\lambda_1+\mu,\eta)}(\iota\otimes\sigma\otimes\iota)\big(\iota\otimes\iota\otimes S^\hbar_\nu\otimes\iota\otimes\iota\big)\big(T_{\bar\mu,\bar\eta}\otimes\iota\otimes\iota\otimes T_{\lambda_1+\mu,\lambda_2+\eta}\big)\\&\qquad\times\big(T_{\bar\mu+\bar\eta,\bar\nu}\otimes\tau^\hbar_{i;\nu,\lambda_1+\lambda_2+\mu+\eta}\big).\end{aligned}$$

We claim that

$$\begin{aligned}\big(\iota\otimes T_{\lambda_1+\mu,\lambda_2+\eta}\big)\tau^\hbar_{i;\nu,\lambda_1+\lambda_2+\mu+\eta}&=q_i^{-\lambda_2(i)-\eta(i)}\big(\tau^\hbar_{i;\nu,\lambda_1+\mu}\otimes\iota\big)T_{\lambda_1-\alpha_i+\mu+\nu,\lambda_2+\eta}\\&\quad+q^{(\lambda_1+\mu,\nu)}(\sigma^{-1}\otimes\iota)\big(\iota\otimes\tau^\hbar_{i;\nu,\lambda_2+\eta}\big)T_{\lambda_1+\mu,\lambda_2-\alpha_i+\eta+\nu}.\end{aligned}\tag{7.16}$$

We postpone the proof of this equality. Using it we see that the left hand side of Eq. 7.15 multiplied by $[\nu(i)]_{q_i}$ is the sum of the term

$$\begin{aligned}&q^{(\lambda_1+\mu,\eta)}q_i^{-\lambda_2(i)-\eta(i)}(\iota\otimes\sigma\otimes\iota)\big(\iota\otimes\iota\otimes S^\hbar_\nu\otimes\iota\otimes\iota\big)\\&\quad\times\big(T_{\bar\mu,\bar\eta}\otimes\iota\otimes\tau^\hbar_{i;\nu,\lambda_1+\mu}\otimes\iota\big)\big(T_{\bar\mu+\bar\eta,\bar\nu}\otimes T_{\lambda_1-\alpha_i+\mu+\nu,\lambda_2+\eta}\big)\\&\quad=q^{(\lambda_1+\mu+\nu,\eta)}q_i^{-\lambda_2(i)-\eta(i)}(\iota\otimes\sigma\otimes\iota)\big(\iota\otimes\iota\otimes S^\hbar_\nu\otimes\iota\otimes\iota\big)\\&\quad\times\big(\iota\otimes\sigma^{-1}T_{\bar\nu,\bar\eta}\otimes\tau^\hbar_i\otimes\iota\big)\big(T_{\bar\mu,\bar\eta+\bar\nu}\otimes T\big)\\&\quad=q^{(\lambda_1+\mu+\nu,\eta)}q_i^{-\lambda_2(i)-\eta(i)}(\iota\otimes\sigma\otimes\iota)\big(\iota\otimes\iota\otimes S^\hbar_\nu\otimes\iota\otimes\iota\big)(\iota\otimes\sigma^{-1}\otimes\iota\otimes\iota\otimes\iota)\\&\quad\times\big(T\otimes\iota\otimes\tau^\hbar_i\otimes\iota\big)(T\otimes T),\end{aligned}\tag{7.17}$$

where we have used $(T\otimes\iota)T=(\iota\otimes T)T$ twice and that $T_{\bar\eta,\bar\nu}=q^{(\eta,\nu)}\sigma^{-1}T_{\bar\nu,\bar\eta}$ by Eqs. 5.4 and 5.6, and the term

$$q^{(\lambda_1+\mu,\eta+\nu)}(\iota\otimes\sigma\otimes\iota)\big(\iota\otimes\iota\otimes S^\hbar_\nu\otimes\iota\otimes\iota\big)(\iota\otimes\iota\otimes\iota\otimes\sigma^{-1}\otimes\iota)\big(T\otimes\iota\otimes\iota\otimes\tau^\hbar_i\big)(T\otimes T)$$
$$=q^{(\lambda_1+\mu,\eta+\nu)}(\iota\otimes\sigma\otimes\iota)\big(\iota\otimes\iota\otimes S^\hbar_\nu\otimes\iota\otimes\iota\big)(\iota\otimes\iota\otimes\iota\otimes\sigma^{-1}\otimes\iota)\big(\iota\otimes T\otimes\iota\otimes\tau^\hbar_i\big)(T\otimes T). \tag{7.18}$$

On the other hand, the first term on the right hand side of Eq. 7.15 multiplied by $[\nu(i)]_{q_i}$ equals

$$q_i^{-\lambda_2(i)}q^{(\lambda_1-\alpha_i+\mu+\nu,\eta)}\big(\iota\otimes S^\hbar_\nu\otimes\iota\otimes\iota\otimes\iota\big)\big(T\otimes\tau^\hbar_i\otimes\iota\otimes\iota\big)(\iota\otimes\sigma\otimes\iota)(T\otimes T).$$

By naturality of σ this expression can be written as

$$q_i^{-\lambda_2(i)}q^{(\lambda_1-\alpha_i+\mu+\nu,\eta)}\big(\iota\otimes S^\hbar_\nu\otimes\iota\otimes\iota\otimes\iota\big)\big(\iota\otimes\iota\otimes\sigma_{1,23}\otimes\iota\big)(T\otimes\iota\otimes\tau^\hbar_i\otimes\iota)(T\otimes T).$$

As $(\alpha_i,\eta)=d_i\eta(i)$, to see that this is equal to Eq. 7.17 we just have to check that

$$\sigma\big(\iota\otimes S^\hbar_\nu\otimes\iota\big)\big(\sigma^{-1}\otimes\iota\otimes\iota\big)=\big(S^\hbar_\nu\otimes\iota\otimes\iota\big)(\iota\otimes\sigma_{1,23}).$$

Writing $\sigma\colon U\otimes V\to V\otimes U$ as $(\iota\otimes\sigma)(\sigma\otimes\iota)\colon U\otimes\mathbb{C}\otimes V\to\mathbb{C}\otimes V\otimes U$ and using naturality of σ we have

$$\sigma\big(\iota\otimes S^\hbar_\nu\otimes\iota\big)=\big(S^\hbar_\nu\otimes\iota\otimes\iota\big)(\iota\otimes\iota\otimes\sigma)(\sigma_{1,23}\otimes\iota).$$

As $(\iota\otimes\iota\otimes\sigma)(\sigma_{1,23}\otimes\iota)=(\iota\otimes\sigma_{1,23})(\sigma\otimes\iota\otimes\iota)$ by the hexagon identities, we get the required equality.

Similarly it is proved that Eq. 7.18 coincides with the second term on the right hand side of Eq. 7.15 multiplied by $[\nu(i)]_{q_i}$.

Therefore it remains to check identity Eq. 7.16. Replacing $\lambda_1+\mu$ by μ and $\lambda_2+\eta$ by η, we have to show that

$$(\iota\otimes T_{\mu,\eta})\tau^\hbar_{i;\nu,\mu+\eta}=q_i^{-\eta(i)}\big(\tau^\hbar_{i;\nu,\mu}\otimes\iota\big)T_{\mu+\nu-\alpha_i,\eta}+q^{(\mu,\nu)}(\sigma^{-1}\otimes\iota)\big(\iota\otimes\tau^\hbar_{i;\nu,\eta}\big)T_{\mu,\eta+\nu-\alpha_i}.$$

It follows from identities Eq. 7.7 that

$$[\mu(i)+\nu(i)]_{q_i}(\iota\otimes T_{\mu,\eta})\tau^\hbar_{i;\nu,\mu+\eta}$$
$$=[\nu(i)]_{q_i}(T_{\nu,\mu}\otimes\iota)\tau^\hbar_{i;\mu+\nu,\eta}+[\mu(i)+\eta(i)+\nu(i)]_{q_i}\big(\tau^\hbar_{i;\nu,\mu}\otimes\iota\big)T_{\mu+\nu-\alpha_i,\eta}.$$

Therefore we equivalently have to check that

$$[\nu(i)]_{q_i}(\sigma T_{\nu,\mu}\otimes\iota)\tau^\hbar_{i;\mu+\nu,\eta}+[\mu(i)+\eta(i)+\nu(i)]_{q_i}\big(\sigma\tau^\hbar_{i;\nu,\mu}\otimes\iota\big)T_{\mu+\nu-\alpha_i,\eta}$$
$$=q_i^{-\eta(i)}[\mu(i)+\nu(i)]_{q_i}\big(\sigma\tau^\hbar_{i;\nu,\mu}\otimes\iota\big)T_{\mu+\nu-\alpha_i,\eta}$$
$$+q^{(\mu,\nu)}[\mu(i)+\nu(i)]_{q_i}\big(\iota\otimes\tau^\hbar_{i;\nu,\eta}\big)T_{\mu,\eta+\nu-\alpha_i}.$$

But up to the factor $q^{(\mu,\nu)}$ this is exactly the identity

$$[\nu(i)]_{q_i}(T_{\mu,\nu}\otimes\iota)\tau^\hbar_{i;\mu+\nu,\eta}-[\eta(i)]_{q_i}\big(\tau^\hbar_{i;\mu,\nu}\otimes\iota\big)T_{\mu+\nu-\alpha_i,\eta}$$
$$=[\mu(i)+\nu(i)]_{q_i}\big(\iota\otimes\tau^\hbar_{i;\nu,\eta}\big)T_{\mu,\eta+\nu-\alpha_i}$$

from Eq. 7.7, if we take into account that $\sigma T_{\nu,\mu} = q^{(\mu,\nu)} T_{\mu,\nu}$ by Eq. 5.6 and

$$\sigma \tau^{\hbar}_{i;\nu,\mu} = -q^{(\mu,\nu)} q_i^{-\mu(i)-\nu(i)} \tau^{\hbar}_{i;\mu,\nu},$$

which in turn follows from Eq. 7.6 and $\Sigma \tau_{i;\nu,\mu} = -\tau_{i;\mu,\nu}$. □

Remark 7.3 If we replace the cochain $g^{\hbar}_{\mu}$ by the cochain $g^{\hbar}_{\mu}\chi(\mu)$, where $\chi\colon P \to \mathbb{C}^*$ is a homomorphism, then by Lemma 6.5 the comonoid remains unaltered up to an isomorphism. One can easily check that if use the same formulas to define the morphisms F_i and E_i with the new cochain then the morphism F_i remains unchanged, while E_i changes to $\chi(\alpha_i)E_i$.

Lemma 7.4 *Let V be a finite dimensional $\mathfrak{g}$-module. Assume the cochain $g^{\hbar}_{\mu}$ is chosen as in Proposition 7.2. Then for generic $\hbar$ the action of $U_q\tilde{\mathfrak{g}}$ on $M^{\hbar}$ defines an action of $U_q\mathfrak{g}$ on $\mathrm{Hom}_{\mathfrak{g}}(M^{\hbar}, V)$.*

Proof The action of $U_q\tilde{\mathfrak{g}}$ on $M^{\hbar}$ by $\mathfrak{g}$-endomorphisms defines an action of $(U_q\tilde{\mathfrak{g}})^{op}$ on $\mathrm{Hom}_{\mathfrak{g}}(M^{\hbar}, V)$. To show that this action defines an action of $U_q\mathfrak{g}$ we just have to check that the relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0 \ \text{ and } \ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0 \tag{7.19}$$

are satisfied for $i \neq j$.

We may assume that $V = V_\lambda$ for some λ. The morphisms $\mathrm{tr}^{\mu,\hbar}_{0,\lambda}\colon V_{\bar\mu} \otimes V_{\lambda+\mu} \to V_0 \otimes V_\lambda = V_\lambda$ define a morphism $\xi^{\hbar}_{\lambda}\colon M^{\hbar}_{\lambda} \to V_\lambda$, which we consider as a vector in $\mathrm{Hom}_{\mathfrak{g}}(M^{\hbar}, V_\lambda)$. We have $E_i\xi^{\hbar}_{\lambda} = \xi^{\hbar}_{\lambda} \circ E_i = 0$ as there are no nonzero morphisms $M^{\hbar}_{\lambda+\alpha_i} \to V_\lambda$, so $\xi^{\hbar}_{\lambda}$ is a highest weight vector in $\mathrm{Hom}_{\mathfrak{g}}(M^{\hbar}, V_\lambda)$. In particular, if we denote by $G_{ij} \in (U_q\tilde{\mathfrak{g}})^{op}$ the left hand side of the first equation in Eq. 7.19 then $G_{ij}\xi^{\hbar}_{\lambda} = 0$. Using the relations in $U_q\tilde{\mathfrak{g}}$ it can be checked that G_{ij} commutes with F_l for all l. Therefore to prove that $G_{ij} = 0$ on $\mathrm{Hom}_{\mathfrak{g}}(M^{\hbar}, V_\lambda)$ it suffices to show that $\mathrm{Hom}_{\mathfrak{g}}(M^{\hbar}, V_\lambda)$ is spanned by $F_{i_1} \dots F_{i_m}\xi^{\hbar}_{\lambda} = \xi^{\hbar}_{\lambda} \circ F_{i_m} \circ \dots \circ F_{i_1}$. By Remark 7.3 the latter property is independent of the choice of $g^{\hbar}_{\mu}$, so we may assume that $g^{\hbar}_{\mu}$ is an analytic function in $\hbar$ with $g^{0}_{\mu} = 1$, e.g. by choosing $g^{\hbar}_{\omega_k} = 1$ for all k.

Choose a finite set I of multiindices $(i_1, \dots, i_m)$ such that the vectors $f_{i_1} \dots f_{i_m}\xi_\lambda$ form a basis of V_λ. Since $\dim \mathrm{Hom}_{\mathfrak{g}}(M^{\hbar}, V_\lambda) \le \dim V_\lambda$ it then suffices to check that for generic $\hbar$ the vectors $F_{i_1} \dots F_{i_m}\xi^{\hbar}_{\lambda}$, $(i_1, \dots, i_m) \in I$, are linearly independent. The vectors

$$F_{i_1} \dots F_{i_m}\xi^{\hbar}_{\lambda} \in \mathrm{Hom}_{\mathfrak{g}}\left(M^{\hbar}_{\lambda-\alpha_{i_1}-\dots-\alpha_{i_m}}, V_\lambda\right)$$

are defined by morphisms $V_{\bar\mu} \otimes V_{\lambda-\alpha_{i_1}-\dots-\alpha_{i_m}+\mu} \to V_\lambda$. Therefore it suffices to check that the latter morphisms are linearly independent for generic $\hbar$. Since they depend analytically on $\hbar$, it is enough to check linear independence for $\hbar = 0$. Under the injective maps $\mathrm{Hom}_{\mathfrak{g}}(V_{\bar\mu} \otimes V_{\lambda-\eta+\mu}, V_\lambda) \to V_\lambda(\lambda-\eta)$, $f \mapsto f(\zeta_{\bar\mu} \otimes \xi_{\lambda-\eta+\mu})$, the morphisms are mapped onto the vectors $f_{i_1} \dots f_{i_m}\xi_\lambda$, which are linearly independent by assumption. To see that we indeed get the vectors $f_{i_1}\dots f_{i_m}\xi_\lambda$ we just have to observe that $\mathrm{tr}^{\mu}_{0,\lambda}\colon V_{\bar\mu}\otimes V_{\lambda+\mu}\to V_\lambda$ is mapped onto ξ_λ and that the diagrams

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathfrak g}(V_{\bar\mu}\otimes V_{\nu+\alpha_k+\mu}, V_\lambda) & \xrightarrow{\circ\Psi^{\eta,0}_{k;\mu,\nu+\alpha_k+\mu}} & \mathrm{Hom}_{\mathfrak g}(V_{\bar\mu+\bar\eta}\otimes V_{\nu+\mu+\eta}, V_\lambda)\\
\downarrow & & \downarrow\\
V_\lambda(\nu+\alpha_k) & \xrightarrow{f_k} & V_\lambda(\nu)
\end{array}$$

commute, where the top arrow is defined by the morphism

$$\Psi^{\eta,0}_{k;\mu,\nu+\alpha_k+\mu}\colon V_{\bar\mu+\bar\eta}\otimes V_{\nu+\mu+\eta}\to V_{\bar\mu}\otimes V_{\nu+\alpha_k+\mu}$$

given by Eq. 7.8 (with $\hbar=0$ and $g^0_\mu=1$).

Therefore we have proved the first relation in Eq. 7.19. The second is proved similarly by considering the lowest weight vector $\zeta^\hbar_\lambda\in\mathrm{Hom}_{\mathfrak g}(M^\hbar_{-\bar\lambda}, V_\lambda)$ defined by $\mathrm{tr}^{\mu-\bar\lambda,\hbar}_{\bar\lambda,0}\colon V_{\bar\mu}\otimes V_{-\bar\lambda+\mu}\to V_\lambda\otimes V_0=V_\lambda$. □

Thus for generic $\hbar$ we have a well-defined action of $U_q\mathfrak g$ on $\mathrm{Hom}_{\mathfrak g}(M^\hbar, V)$, so $\mathrm{Hom}_{\mathfrak g}(M^\hbar,\cdot)$ can be considered as a functor $\mathcal D(\mathfrak g,\hbar)\to\mathcal C(\mathfrak g,\hbar)$. By Proposition 6.4 and the last part of Proposition 7.2 it is a tensor functor. Furthermore, by Proposition 5.4 for generic $\hbar$ the module $M^\hbar$ is isomorphic to the module M representing the forgetful functor. Therefore the following theorem finishes the proof of Theorem 4.3 and thus also of Theorem 4.1.

Theorem 7.5 *If the cochain $g^\hbar_\mu$ is chosen as in Proposition* 7.2 *then for generic $\hbar$ and $q=e^{\pi i\hbar}$ the functor* $\mathrm{Hom}_{\mathfrak g}(M^\hbar,\cdot)$ *is a $\mathbb C$-linear braided monoidal equivalence of the categories $\mathcal D(\mathfrak g,\hbar)$ and $\mathcal C(\mathfrak g,\hbar)$. This functor maps an irreducible $\mathfrak g$-module with highest weight λ onto an irreducible $U_q\mathfrak g$-module with highest weight λ.*

Proof We have already proved that for generic $\hbar$ the functor $F^\hbar=\mathrm{Hom}_{\mathfrak g}(M^\hbar,\cdot)$ is a tensor functor. Furthermore, by the proof of Lemma 7.4 for any $\lambda\in P$ the $U_q\mathfrak g$-module $F^\hbar(V_\lambda)$ has a highest weight vector $\xi^\hbar_\lambda$ of weight λ. Since the dimension of this module is not bigger than that of V_λ, we conclude that $F^\hbar(V_\lambda)$ must be an irreducible $U_q\mathfrak g$-module with highest weight λ. Therefore the image of the functor contains all simple objects in $\mathcal C(\mathfrak g,\hbar)$ up to isomorphism. Since the functor $F^\hbar$ respects direct sums, we conclude that it is an equivalence of tensor categories.

It remains to check that the functor respects braiding, that is, the diagram

$$\begin{array}{ccc}
F^\hbar(U)\otimes F^\hbar(V) & \xrightarrow{\Sigma\mathcal R_\hbar} & F^\hbar(V)\otimes F^\hbar(U)\\
{\scriptstyle F^\hbar_2}\downarrow & & \downarrow{\scriptstyle F^\hbar_2}\\
F^\hbar(U\otimes V) & \xrightarrow{F^\hbar(\Sigma q^t)} & F^\hbar(V\otimes U)
\end{array}$$

commutes. It suffices to consider $U=V_{\bar\lambda}$ and $V=V_\mu$. Consider the lowest weight vector $\zeta^\hbar_{\bar\lambda}\in F^\hbar(V_{\bar\lambda})$ and the highest weight vector $\xi^\hbar_\mu\in F^\hbar(V_\mu)$ defined in the proof of Lemma 7.4. It suffices to compute how the morphisms in the above diagram act on $\zeta^{\hbar}_{\bar{\lambda}} \otimes \xi^{\hbar}_{\mu}$. By Eq. 4.1 we have $\mathcal{R}_{\hbar}(\zeta^{\hbar}_{\bar{\lambda}} \otimes \xi^{\hbar}_{\mu}) = q^{-(\lambda,\mu)}\zeta^{\hbar}_{\bar{\lambda}} \otimes \xi^{\hbar}_{\mu}$. Recalling that $F^{\hbar}_2$ is defined using $\delta^{\hbar}\colon M^{\hbar} \to M^{\hbar}\hat{\otimes}M^{\hbar}$, we just have to check that

$$q^{-(\lambda,\mu)}\big(\xi^{\hbar}_{\mu} \otimes \zeta^{\hbar}_{\bar{\lambda}}\big)\delta^{\hbar} = \Sigma q^{t}\big(\zeta^{\hbar}_{\bar{\lambda}} \otimes \xi^{\hbar}_{\mu}\big)\delta^{\hbar}$$

as morphisms $M^{\hbar} \to V_{\mu} \otimes V_{\bar{\lambda}}$. Recall that $\delta^{\hbar}$ is induced by the morphisms $m^{\hbar}$ defined by Eq. 6.1. Since $\xi^{\hbar}_{\mu}$ and $\zeta^{\hbar}_{\bar{\lambda}}$ are defined by the morphisms $\mathrm{tr}^{\eta,\hbar}_{0,\mu}$ and $\mathrm{tr}^{\nu,\hbar}_{\lambda,0}$, respectively, by equality Eq. 6.2 it suffices to show the following equality of endomorphisms of $V_{\mu} \otimes V_{\bar{\lambda}}$:

$$q^{-(\lambda,\mu)}m^{\hbar}_{0,\lambda,\mu,-\lambda} = \Sigma q^{t} m^{\hbar}_{\lambda,0,-\lambda,\mu}.$$

This is immediate by definition Eq. 6.1, since the associator Φ acts trivially on a tensor product of three modules if at least one module is trivial. □

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