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Special Issue Reprint

Critical Perspectives on Mathematics Teacher Education

Edited by
Constantinos Xenofontos and Kathleen T. Nolan

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About the Editors

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Constantinos Xenofontos is a Professor of Mathematics Education at Oslo Metropolitan University. He previously worked as a primary school teacher in Cyprus, and as a mathematics teacher educator in Cyprus and Scotland. His research explores how mathematics teachers' knowledge, beliefs, and practices come to be, as products of the social, cultural, and political environment. Xenofontos is particularly interested in supporting teachers to develop awareness of how mathematics education (research and practice) is framed by sociocultural and sociopolitical factors, and of the challenges many children from minoritized backgrounds face as mathematics learners.

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Kathleen T. Nolan is a Professor of Mathematics Education in the Faculty of Education at the University of Regina, where her teaching and research focus primarily on curriculum studies, mathematics teacher education, and critical and culturally responsive pedagogies. Her theoretical interests range from centering her research on the use of Bourdieu's social field theory to more recent interests in theories based in critical and disruptive pedagogies for mathematics teacher education. Kathleen recently held a Professor 2 position at Oslo Metropolitan University, in Norway (2021–2023), where her collaboration with Xenofontos began.

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Editorial

Special Issue Introduction: Critical Perspectives on Mathematics Teacher Education

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1. Introduction

Nowadays, the field of mathematics education research is more diverse than ever. A quick look at recently published articles in different venues suggests that research topics encompass not only purely cognitive but also cultural, social, and political issues. It can be argued that these last three issues form a unified entity, one that Planas and Valero [1] call the "socio-cultural-political axis" of mathematics education. Attempting to distinguish between them is neither possible nor productive [2]. For this reason, in this Special Issue, we use the term *critical* as a unifying label to address this axis. In doing so, we are fully aware of the diverse understandings and uses of the term critical in the literature [3]. Nevertheless, we refer to the notion from a Freirean point of view, aiming toward inspiring learners and educators of all levels to understand, analyze, and critique social, cultural, and political power structures and patterns of inequality through the cultivation of sociopolitical awareness [4]. To us, *critical* serves as an umbrella under which different philosophical/epistemological approaches intersect, embracing, among other things, discussions on equity, social justice, inclusion, culturally responsive pedagogy, Indigenous education, ethnomathematics, and language diversity [5]. Furthermore, by teacher education, we refer to both initial teacher education (prospective teachers) and continuous professional development (practicing teachers) at all school levels (early years, primary, secondary) [6].

This Special Issue covers a range of critical perspectives in mathematics teacher education. Our initial call for articles was intentionally broad to allow authors to communicate a variety of critical issues, without being restricted to specific frames. Nonetheless, the thirteen contributions to this issue can be organized into three sections. The first section focuses on research being conducted through coursework in initial teacher education with prospective (pre-service) teachers. The second section examines issues of continuous professional development of practicing (in-service) teachers. Finally, the third section comprises theoretical contributions based on authors' reflections on their own research and practices as mathematics teacher educators.

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2. Focus on Coursework with Prospective (Pre-Service) Teachers in Initial Teacher Education

As the authors of articles in this section of the Special Issue will attest, explicit critical work with prospective teachers (PTs) in their initial teacher education courses provides PTs with early-career insights and opportunities not otherwise possible once they begin life as a teacher. In fact, "insights" is the focus of the first article in this Special Issue. In their paper titled "*The role of insights in becoming a culturally responsive mathematics teacher*" (https://doi.org/10.3390/educsci13101028), Kathleen Nolan and Constantinos Xenofontos extend their previous work focused on studying the growth and development of PTs' culturally responsive pedagogy (CRP) by discussing the implementation of their COFRI framework. An acronym standing for challenges, opportunities, fears, resistance, and insights, COFRI offers a way to gauge PTs' understandings of CRP as various points during a course on CRP and mathematics. While their focus is on PTs' insights and understandings, Nolan and Xenofontos also discuss the work of MTEs with regard to modeling CRP and designing courses with the COFRI components and their inter-relationships in mind.

In the second article in this section, the work of MTEs in designing and facilitating mathematics teacher education courses is front and center in the featured research. In "*Integrating societal issues with mathematical modelling in pre-service teacher education*" (https://doi.org/10.3390/educsci13070721), Lisa Steffensen and Georgia Kasari draw on theoretical perspectives from the field of sociocritical modeling to feature a case study of one MTE who introduces specific problem-posing modeling activities focused on societal issues in courses with pre-service teachers (PTs). Their findings point to the success of integrating these critical-issue mathematical modeling activities in teacher education as one way to move beyond superficial mathematizing problems. Steffensen and Kasari recommend that MTEs work in informal learning communities to support each other as they work to support PTs toward including sociocritical issues in mathematics classrooms.

Also focused on the topic of addressing societal issues in mathematics teacher education courses is the article by Magnus Ödmo, Anna Chronaki, and Lisa Bjorklund Boistrup, titled "*Bringing critical mathematics education and actor-network theory to a statistics course in mathematics teacher education: Actants for articulating complexity in student teachers' foregrounds*" (https://doi.org/10.3390/educsci13121201). In this case, the societal issue being highlighted is climate change, with the context being a required statistics course for student (pre-service) teachers. In the study, student teachers are interviewed about how they see their future selves as teachers in the field of critical mathematics education. By drawing on two theories (critical mathematics education, and actor network theory), the authors illustrate the complex network (actants and connections) encountered by the student teachers when a more traditional statistics course is transformed into a course based on critical perspectives around the context of climate change. In the end, these authors recommend that teacher education programs should deal more explicitly with the complex nature of bringing critical mathematics education practices into mathematics classrooms.

The next article in this section also shares rich stories from interviews with preservice/student teachers as they engaged with specific teacher education courses. In "*Storylines in voices of frustration: Implications for mathematics teacher education in changing times*" (https://doi.org/10.3390/educsci13080816), Annica Andersson, Trine Foyn, Anita Movik Simensen, and David Wagner interview preservice teachers about their teacher education experiences regarding inclusive teaching, asking, for example, what they have learned within the teacher education program about mathematics pedagogy for minoritized and indigenous students. In their aim to better understand the implications these experiences have for teacher education courses and preparation for teaching in diverse classrooms and schools, the authors use storylines as their theoretical construct. The three key storylines constructed out of the data (concerning language, method rigidity, and invisibility) draw attention to important implications for teacher education programs, including creating spaces for deeper and more critical conversations which prepare prospective teachers to challenge and transform current inclusive practices in mathematics classrooms.

The final article in this section on initial teacher education is also focused on the learning of mathematics by students with diverse needs, though this research text shifts our attention from the design side of teacher education courses to approaches to assessment. Specifically, in "*Addressing language diversity in early years mathematics: Proposed classroom practices through a Live Brief Assessment*" (https://doi.org/10.3390/educsci13101025), authors Sinem Hizli Alkan and Derya Sahin Ipek work with Live Brief assessments to address language diversity with Year-1 early years mathematics PTs. After analyzing their data through the lens of Moschkovich's perspectives on the relation between language and pedagogy in mathematics, the authors conclude that within the three pedagogical practice themes (vocabulary teaching, scaffolding, and multi-sensory approaches), the situated sociocultural perspective was under-emphasized. The authors offer critical reflections on the student teachers' proposed practices for addressing language diversity and suggest an important role for teacher education programs to challenge and disrupt student teachers' current thinking on language diversity toward gaining more critical perspectives in the teaching and learning of mathematics.

3. Focus on Professional Development of Practicing (In-Service) Teachers

The articles in the second section of this Special Issue concern practicing teachers and their engagement in opportunities for continuous professional development. For instance, in their paper titled "*Exploring the interplay between conceptualizing and realizing inquiry—The case of one mathematics teacher's trajectory*" (https://doi.org/10.3390/educsci13080843), Marte Bråtalien, Margrethe Naalsund, and Elisabeta Eriksen focus on how teachers negotiate the interplay between theory and practice regarding inquiry-based teaching. In this work, inquiry is seen as an approach that challenges traditional teaching structures and authority, and places whole-class discussions (as a key practice for equity and pupil empowerment) at the center of mathematics lessons. This study follows the journey of Alex, an experienced high school mathematics teacher who lacked expertise in inquiry-based teaching. By enrolling in a one-semester professional development program Alex's trajectory emphasizes the significance of continuous professional development in teachers' understandings of inquiry. It also highlights how these understandings impact the realization of inquiry-based lessons and how carrying out these lessons can, in turn, shape teachers' conceptions of inquiry.

Angeliki Stylianidou's and Elena Nardi's article "Overcoming obstacles for the inclusion of visually impaired learners through teacher-researcher collaborative design and implementation" (https://doi.org/10.3390/educsci13100973) explores the collaboration between teachers and researchers in designing and implementing mathematics lessons that address the needs of visually impaired learners. In the first phase of the study, the authors used classroom observations, focus group interviews, and individual interviews to document teachers' current practices, while in the second phase, teachers and researchers used data from the previous phase to co-design and implement more inclusive lessons. In closing their article, the authors discuss implications for mathematics teacher education, arguing that ongoing collaboration and professional development are essential for creating inclusive math classrooms sustainably. The article highlights the necessity for mathematics teachers to be better prepared to address and think more critically about the needs of children with disabilities.

In the third article of this section, "*Teacher learning towards equitable mathematics classrooms: Reframing problems of practice*" (https://doi.org/10.3390/educsci13090960), Yvette Solomon, Elisabeta Eriksen, and Annette Hessen Bjerke discuss findings from a year-long professional development program as part of a wider project on inclusive mathematics teaching. The study draws on data from 16 practicing teachers who enrolled in the course. Solomon et al. discuss how teachers' engagement with the program shifted assumptions about mathematics teaching and learning, and participants' professional practice as critical mathematics educators. The authors pinpoint four mechanisms contributing to teachers' enriched conceptions and conclude by articulating the need for future research and professional development to unpack these mechanisms further.

In the last article of this section, "*Teacher development for equitable mathematics classrooms: Reflecting on experience in the context of performativity*" (https://doi.org/10.3390/educsci13100993), Sue Hough and Yvette Solomon follow and discuss the journey of the first author during her development from a teacher who emphasized children's understanding to a teacher with deeper critical understandings of mathematics education. The article highlights the pivotal role of realistic mathematics education (RME) as the theoretical lens that enabled the teacher–researcher to reconceptualize her own pedagogical practice, especially within the challenges set by a wider climate of performativity and tangible measurement of outcomes.

4. Theoretical, Reflective, and/or Sociopolitical Contributions

Four articles approach the topic of the Special Issue from theoretical perspectives and/or comprise authors' reflections on their own work as critical mathematics teacher educators. For example, Marrielle Myers, Kari Kokka, and Rochelle Gutiérrez use braiding as an analogy to introduce a framework with four strands for facilitating the development of candidate teachers' political consciousness. In their article "*Maintaining tensions: Braiding as an analogy for mathematics teacher educators' political work*" (https://doi.org/10.3390/educsci13111100), the authors draw on their work as mathematics teacher educators to illustrate examples of what each strand can look like and to highlight the necessity for colleagues to braid the four strands together.

In their article "*Pedagogical imagination in mathematics teacher education*" (https://doi.org/10.3390/educsci13101059), Ole Skovsmose, Priscila Lima, and Miriam Godoy Penteado build on their previous research focused on the theoretical idea of pedagogical imagination, a concept which was initially developed by their team in post-Apartheid South Africa. The authors deliberately choose not to limit the definition of pedagogical imagination and instead explore its connection to five other broad concepts: dialogue, social justice, mathematics, hope, and sociological imagination. To illustrate these connections, the authors share episodes from discussions between the second author and four prospective teachers in Brazil. The article highlights the adaptability of pedagogical imagination as a theoretical concept that can be applied across different cultural contexts.

Kay Owens draws on her extensive research experiences in the context of Papua New Guinea to discuss the vital role of pre- and in-service mathematics teacher education in illuminating the impact of neocolonialism on mathematics teaching and learning, as well as the need for mathematics education to focus more explicitly on culturally relevant approaches. In doing so, her article, titled "*The role of mathematics teacher education in overcoming narrow neocolonial views of mathematics*" (https://doi.org/10.3390/educsci13090868), uses a variety of data sources to exemplify the impacts of colonization, post-colonial aid and globalization on mathematics education. In light of the impact of First Nations' colonization, and multiculturalism in certain areas, there is a growing importance for teacher education programs to include Ethnomathematics as a mandatory subject. Owens strongly advocates for this viewpoint, suggesting its importance in ensuring a more comprehensive and inclusive mathematics education.

In the last article, "*The Fascists are coming! Teacher education for when right-wing activism micro-governs classroom practice*" (https://doi.org/10.3390/educsci13090883), Peter Appelbaum uses the US context as an example of how research-informed policies and practices based on liberal multiculturalism have fallen short of realizing their promise. In highlighting the fascistization of many so-called Western societies, Appelbaum uses the Currere methodology and its four phases of inquiry to stress the need for mathematics teacher educators to reconsider the content and processes of teacher education so as to better prepare teachers to recognize and address right-wing narratives.

Conflicts of Interest: The authors declare no conflict of interest.

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Article

The Role of *Insights* in Becoming a Culturally Responsive Mathematics Teacher

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Abstract: This paper extends earlier research on prospective and practicing teachers' (PPTs') developing understandings of culturally responsive pedagogy (CRP) while enrolled in a teacher education course for CRP and mathematics. Here, we take as our starting point a framework we refer to as COFRI, which describes five integral components of PPTs' perspectives on CRP: Challenges, Opportunities, Fears, Resistance, and Insights. Viewing PPTs' reflective journal entries through the lens of this framework, we noticed interesting relationships between the five components that had not been evident in our initial analysis. Specifically, we observed that, as we coded participants' reflections according to C, O, F, R, and I, each I (insight) appeared to be related to one (or more) of the other components in quite different ways. Additionally, careful study of the insights expressed by PPTs lead to our categorization of insights according to one of three types: mathematical, pedagogical, or ideological. As a result, this paper offers a new way to interpret the five components, specifically their relationships to new insights into CRP and the corresponding types of insights that PPTs produce over the course of one semester. In closing, this paper discusses implications for mathematics teacher educators in understanding and processing PPTs' evolving understandings of CRP.

Keywords: culturally responsive pedagogy (CRP); mathematics teacher education; prospective and practicing teachers; course-based research; insights

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1. Introduction

In mathematics teacher education, culturally responsive pedagogy (CRP) is increasingly recognized as a critical approach toward realizing a vision of equitable and socially just mathematics classrooms [1,2]. Such an approach involves the concept of culture, a particularly complex concept that is loaded with different meanings [3,4]. Our work is aligned with Hofstede, who defines culture as "part of our conditioning that we share with other members of our nation, region, or group but not with members of other nations, regions, or groups" [5] (p. 76). We find this definition useful because it does not equate culture to nations. Quite the opposite, it acknowledges that culture has structural and dynamic dimensions [6]. Regarding the former, culture can be formed and expressed at different levels nested within one another: cultural representation within the individual person, groups (e.g., classrooms), organizations (e.g., schools, teacher education programs), nations, and the global culture. The dynamic dimension pertains to the interrelationships among the various levels of culture and how they impact each other.

Conceptualizations of CRP in mathematics education have included a focus on Indigenous education, in addition to social justice, critical mathematics, ethnomathematics, language diversity, and equity-based research [7]. In fact, even though the published literature on CRP is extensive, Young points out: "The void in scholarly research is not in the knowledge of theories but in the knowledge of how to implement them, particularly in a way that has a wide-reaching and sustainable impact on teacher education" [8] (p. 259). This paper aims to make that wide-reaching and sustainable impact on mathematics teacher education through its focus on studying prospective and practicing teachers' (PPTs') perspectives on CRP as students enrolled in an undergraduate course on CRP in mathematics. The primary goal of that course-based research study was to document, through a reflective journal assignment, how PPTs' ideas, understandings, experiences, and knowledge of CRP in mathematics change/grow/evolve throughout the semester.

To begin, we describe this paper's important relationship with two previous research articles, both of which serve to set the stage for the key extension of what is presented here. In Nolan and Xenofontos [9], we used Ladson-Billings' [10,11] three elements of CRP (academic achievement, cultural competence, and sociopolitical consciousness) as a starting point and a lens to analyze course data provided by PPTs. Subsequently, we built upon those elements to develop a new framework, which we refer to as COFRI. The COFRI framework highlights five integral components of PPTs' perspectives on CRP: Challenges, Opportunities, Fears, Resistance, and Insights. Following this, in Nolan and Xenofontos [12], we utilized the COFRI framework to analyze five PPTs' first two journal entries, in the form of narrative case studies. As we sought to build upon the second paper's findings, and essentially "test" our COFRI framework on the remaining journal responses for these same five case study participants, we noticed interesting relationships between the five components that had not been evident in our initial analysis. Simply put, we observed that, as we coded PPTs' journal reflections according to C, O, F, R, and I, each I (insight) appeared to be related to one (or more) of the other components in quite different ways. As a result, we aim here to offer a new way to understand the five components and, in turn, a tool for mathematic teachers and teacher educators to understand and process their students' (PPTs') evolving understandings of CRP. Specifically, we aim to address the following research questions:

(1) In what ways are insights connected to challenges, opportunities, fears, and resistance in PPTs' reflections throughout a semester-long CRP course?
(2) What types of CRP-related insights do PPTs have as a result of their engagement with the course?

Before we attend to these questions, we present the theoretical considerations that frame our work. Subsequently, we turn to this study and its methods, followed by a presentation and discussion of our findings. In closing, we address important implications for mathematics teacher education.

2. Theoretical Considerations

The three key areas of scholarly research that provide the theoretical foundation for this paper are: meanings and intentions of CRP; the COFRI framework; and "aha" (insightful) moments in mathematical learning.

2.1. Meanings and Intentions of CRP

Gay proposes that CRP "is the behavioral expressions of knowledge, beliefs, and values that recognize the importance of racial and cultural diversity in learning" [13] (pp. 36–37). More specifically, Ladson-Billings [14] delineates three components of CRP: academic success (i.e., student learning), cultural competence, and sociopolitical consciousness. Ladson-Billings' work has been influential for CRP research in general. Yet, we observe that mathematics-specific CRP studies do not place as much emphasis on the third component, i.e., sociopolitical consciousness, as they do with the other two components. Of course, we do not claim that issues of sociopolitical nature are overlooked by colleagues in mathematics education; on the contrary, it appears that developing sociopolitical consciousness in mathematics has been typically associated with a different direction of research, that of *critical mathematics education* (e.g., [15,16]), or what other colleagues refer to as *teaching mathematics for social justice* (e.g., [17,18]). In our own work, we have found Ladson-Billings's three components to be highly informative and productive for our analysis and subsequent development of the COFRI framework [9,12].

Research focused on developing the culturally responsive practices and beliefs of PPTs ranges from addressing issues of equity and inclusion to challenging dominant narratives of teaching, learning, and curriculum [19]. Some of the published literature describes the actions and dispositions teachers should develop in their work with diverse students and cultures toward becoming culturally responsive, specifically toward developing critical consciousness [20,21]. Rychly and Graves offer four characteristics of teacher practice that, they believe, are essential to being and becoming a culturally responsive teacher: be empathetic and caring; be reflective about one's own beliefs and dispositions; be reflective about one's own worldview and potential cultural blind spots; and seek to grow in one's knowledge of culture and cultural practices [22] (p. 45).

While Rychly and Graves [22], and many others (see, for example, [14,23,24]), write more generally about CRP for teachers, there is also substantial research available which focuses specifically on developing as culturally responsive *mathematics* teachers [25–28]. Kelley, whose research explores the role of funds of knowledge in developing as a culturally responsive mathematics teacher, highlights how teacher education programs are called to address more complex issues to move the field toward challenging "what constitutes mathematical knowledge and who possesses this knowledge" [26] (p. 1). Some scholars have taken up this call by proposing critically reflective frameworks that can be used by PPTs as they develop mathematics lesson plans [29], as they revise existing curriculum materials [30], and as they conduct self-study into growing their own practices as culturally responsive mathematics teachers [31].

Research on CRP in mathematics teacher education points to the need to not only focus on the beliefs and actions of PPTs, but also on those of mathematics teacher educators (MTEs) as a productive way forward in developing PPTs' CRP. In other words, MTEs must grow and develop their own culturally responsive practices [32] so that they are better able to model these practices in their teacher education courses with PPTs. A recent literature review, which set out to identify, synthesize, and analyze key scholarly texts in the field of teacher educator CRP, not only identified a dearth of studies specifically focused on MTE practices, but it pointed to the absence of an explicit critical lens for examining the CRP practices of teacher educators in general [2]. As it turns out, research with/on MTEs has predominantly been conducted through self-study [33] where MTEs reflect on, for instance, how they model (or not) CRP in their own work with PPTs [32,34,35] as well as their efforts to design teacher education courses or field experience highlighting CRP, equity, and/or anti-racist practices [36–39]. These efforts on the part of MTEs are grounded in a belief that students from diverse cultural backgrounds will continue to suffer inequities and injustices within mathematics classrooms until teacher educators themselves take on the responsibility to grow and model their own CRP. Willey and Drake urge both mathematics teachers and teacher educators "to sharpen our sociopolitical lenses in order to notice and disrupt manifestations of privilege and oppression in mathematics education" [40] (p. 68).

Research studies in mathematics teacher education point to the slow development and lack of sustainable impact regarding CRP [8]. For example, when it comes to PPTs' ideas for implementing CRP, O'Keeffe et al. [41] report on PPTs' fear and apprehension that they will practice CRP in ways that are insensitive or even offensive to certain cultures. Castro [20] suggests that such views continue because there is a serious absence of complex thinking around issues of cultural diversity and equitable classrooms. He proposes that further research on teaching practices, curricular components, and intercultural relationships will hopefully prompt the necessary changes in PPTs' attitudes and beliefs toward deeper critical awareness about issues of inequity and injustice.

Returning to the characteristics of culturally responsive teachers (as stated earlier in this section), we highlight here the importance of the reflective piece, and support the belief that "teachers will be unable to fully do the work of culturally responsive pedagogy if they do not first investigate their own attitudes and beliefs" [22] (p. 46). In line with this, the research highlighted in this paper is grounded in a course-based study where the data were drawn from a reflective journal assignment. The primary goal of that assignment was to provide PPTs with opportunities to reflect on their own beliefs and dispositions about mathematics and CRP, while also being reflective on their own worldviews and embedded ideologies that could be serving as barriers. Given this, we claim that the assignment reflections serve two very important roles: first, improving PPTs' own awareness of their growth as culturally responsive mathematics teachers and, second, informing MTEs about their own practices as CRP models/educators as they strive for better/richer/deeper course experiences for their PPT students.

2.2. The COFRI Framework

Table 1 presents our COFRI framework, as first introduced and implemented in Nolan and Xenofontos [9,12]. While the framework appears to neatly compartmentalize each of the five integral components, we note, however:

> . . . a key learning for us in this data analysis was around the idea that a best approach to analysis did not involve extracting evidence of individual COFRI components from the participants' journal responses. Instead, utilizing a case study approach to analysis meant that response narratives remained intact and illustrated to us something quite significant: that COFRI components could exist side by side and even overlap/intersect. In other words, in one journal response it was possible to see evidence of, for example, participants expressing fear and resistance, while also looking ahead to the opportunities that might be available to them as they learn more about CRP. This is significant for mathematics teacher education in providing an entry point for PPTs themselves to reflect on the juxta positioning of very different perspectives in their journey to becoming culturally responsive teachers. [12] (pp. 314–315)

Table 1. The COFRI components—brief descriptions.

Characteristic	Brief Description
Challenge	The idea of challenge involves awareness of one's lack or partial development of competence to address an issue. Challenge is based on a person's perception that new knowledge, dispositions, skills, or tools (KDST) are required, which they are inspired to move forward and acquire.
Opportunity	Opportunity refers to the identification of space for something "good" to happen. A person sees the space as already existing; things are in place to move forward (i.e., the person has the KDST to move forward) to make good things happen.
Fear	The feeling that attempting something might lead to failure. A person might be inclined to stop in their tracks (or even move backward), and to rationalize this (non)movement by saying they do not have (and cannot easily obtain) the KDST to achieve it.
Resistance	The expression of dispositions against or disbelief in the importance, feasibility, or possibility of specific ideas. Resistance can manifest itself through "rationalizing discourses" which have the property of projecting how *others* will act or respond to a situation—an "it's not me, it's them" approach to resisting an idea.
Insight	An understanding or realization of what is currently happening and/or how things could be. In addition to seeing what is currently happening, a person will generate new ideas for extending, adapting, and/or improving. Generally, when a person has "insight", this will affect the other four components. That is, an insight suggests a new direction which could create a new challenge, opportunity, or even fear or resistance depending on the "tools" one has. Consequently, an insight might lead to gaining new tools (challenge), moving forward with what one has (opportunity), halting/moving backwards (fear), or disbelief (resistance).

2.3. Insightful Moments and Mathematics Teacher Education Research

We now turn our attention to the academic literature on one of the COFRI components, the insight, as we seek to unpack this further and gain a deeper understanding of its relationships with the other four components. In doing so, we are fully aware of the variation in labels used by colleagues to discuss similar issues. For example, Brody and Hadar refer to "critical moments", defined as "significant events marking change in" one's "professional narrative" [42] (p. 53), whereas Rahmawati and Taylor [43] (2015) write about "moments of critical realisation". In turn, Yacek and Gary (2020) base their work on the concept of "epiphany. . . a special kind of transformational catalyst that calls us to become a better version of ourselves", adding that such epiphanic experiences "involve establishing contact with one or more substantive ethical goods or values that had previously remained out of our field of vision" [44] (p. 219). From a different perspective, Liljedahl introduces the concept of "AHA! experience", arguing that, "[a]lthough it defies logic and resists explanation, it requires neither logic nor explanation to define it. The AHA! experience is self-defining" [45] (p. 220). While we agree that all these labels are equally relevant, we decided to continue using the term insight, primarily for consistency with our previous work [9,12].

Despite the various labels that can be found in the literature, a relatively recent Delphi study involving academics working in this area [46] concludes that, in educational settings, these insightful moments typically result from reflections on real-life experiences, systematic questioning and reflective activities, the use of analogies, and team discussions and problem solving. Yet, our readings of the relevant literature suggest that various scholars exploring such moments within (mathematics) teacher education focus almost exclusively on a specific type of insight in their work. For instance, the work of Liljedahl [45] investigates insights from a subject-matter point of view, and specifically how prospective elementary teachers experience mathematical realizations during their undergraduate studies. Such insights, claims the author, typically come with expression of anxiety or pleasure, and a change in one's beliefs and attitudes. Drawing on the work of Liljedahl [45], Caniglia et al. [47] approach insightful moments with emphasis on pedagogy. In their study, opportunities to reflect on such moments allowed prospective mathematics and science teachers to develop greater awareness of the self as a teacher, the importance of knowing one's students, inconsistencies between one's own beliefs and students' beliefs, and the importance of anticipating students' misconceptions. Finally, reflecting on her own culturally diverse background, upbringing, schooling, and education, and experiences as a teacher and a teacher educator, Nieto [48] writes about a different type of insight, that of ideological, sociopolitical realizations that have had significant impact on her becoming and being an educator. In acknowledging these different types of insights, we do not wish to place emphasis on any specific type. On the contrary, we anticipate that all three of these types can be found in our work with PPTs and CRP. Therefore, we are particularly interested in understanding how insights related to CRP and mathematics may be channeled through mathematical, pedagogical, and/or ideological lenses.

3. Methods

This study was conceptualized around a course entitled Culturally Responsive Pedagogy (CRP) in the Mathematics Classroom, offered as part of a Teaching Elementary School Mathematics (TESM) certificate program at a Canadian university. The course, designed and taught by one of the authors (Nolan), has an overarching goal of deepening understanding of mathematics concepts while developing a critical cultural consciousness. Nolan designed the course to reflect the many intersecting fields of research that can be seen to shape CRP, such as ethnomathematics [EM], language diversity [LD], equity-based [E-b], social justice [SJ], Indigenous education [IE], and critical mathematics education [CM].

One of the course assignments asked students (PPTs) to maintain a reflective journal where they would respond to questions designed to stimulate their growth and development of CRP. The questions were assigned by the instructor at the end of each class. The reflective journal assignment was used in each of three offerings of the course, from 2017 to 2021, and consisted of 10–13 reflective questions (depending on the offering). The course was designed to be "responsive" to the specific students enrolled in the course and their experiences/interests, in addition to the selected readings assigned in each offering. Hence, the journal questions posed each day over each offering varied considerably. That said, several questions did not change significantly between offerings and are presented below to provide a sense of what PPTs were asked to reflect on:

- What concerns you the most about today's discussions on bringing culture, responsiveness, and mathematics together? Is this the start of something different for you, as a mathematics teacher and learner?
- As culturally responsive mathematics teachers, how do you promote culturally inclusive and culturally appropriate mathematics in your classroom? How do you tell the difference between culturally *appropriate* and cultural *appropriation*?
- With the individual student seminars now complete, take a few moments to reflect on the seven topics/issues used to structure this course (EM, LD, E-b, SJ, IE, CM, and CRP). What have you noticed regarding overlaps and intersections between the seven topics?

As offered by Rychly and Graves, "the work of becoming 'culturally responsive' is quite personal, and may best begin with individuals engaging in reflection as a process" [22] (p. 48), which necessarily includes digging deeply into one's own beliefs, preconceived ideas, and expectations around students from diverse cultures and their engagement with mathematics.

This paper draws on that reflective journal assignment as data for the study. Specifically, this paper draws on the data from the same five case study participants as focused on in [12]: Cindy, Olive, Felix, Raymond, and Iris. For this paper, however, the analysis has been extended to include all journal entries for each of these five cases. As alluded to above, these five participants were selected for our COFRI-driven analysis due to the way in which their data illustrated how expressions of challenges, opportunities, fears, resistance, and/or insights could exist side by side, potentially even overlapping/intersecting with each other within one journal response. In fact, this side-by-side positioning of the COFRI components is what led us, in our analysis, to discovering the significant role played by I—insights.

4. Analysis and Results

Following from the two research questions presented earlier, our analysis is divided into two sections.

4.1. Connections between COFRI Components

Our first research question asks: In what ways are insights connected to challenges, opportunities, fears, and resistance in PPTs' reflections throughout a semester-long CRP course? Our aim here is to begin with our COFRI framework, specifically the definition for the Insight component, and to zoom in on the accuracy and completeness of this definition in light of our continuing analysis of PPTs' journal entries.

We draw attention to the way in which we defined an insight in our initial conceptualization (see Table 1). We not only describe the strong potential for an insight to be closely connected to at least one other COFRI component, but we also include that an insight might "lead to" one (or more) of the other components. In other words, in our initial data analysis and COFRI conceptualization, we noticed that the expression of an insight seemed to appear first, followed by the expression of what this insight might mean in terms of a future challenge, opportunity, fear, or resistance. This definition seemed clearly reflective of our initial data analysis (the first two journal entries). However, as we analyzed additional journal entries for each of these five participants, we began to notice the positioning of insights relative to the other components. As the data are extensive (10–13 journal entries, averaging 250 words each, for five participants), Table 2 presents only selected exemplars to illustrate the diversity of ways in which insights are connected to the other four components. Even though we use colour coding to give readers a sense of how different COFRI components are positioned in relation to one another, we should note that the boundaries between components are frequently blurred. Aiming to distinguish between the components by drawing clear lines is, we believe, neither desirable nor productive.

Table 2. The relative positioning of Insights.

Challenge	Opportunity	Fear	Resistance	Insight

Participant	Sample Quote	Component
Cindy	Math really isn't just math anymore. Math is everywhere and I think we need to show students that. [The presenter] gave a good example of this when she brought up the topic of measurement. Students went home and were asked to find out who measures things, what are they measuring, how are they measuring, and then they had to draw a picture. This is something I immediately latched onto as this is something I would love to try in a classroom. I am interested to see what students come back with and to have some really authentic class discussions about their findings.	Insight → Opportunity
Olive	I have felt the content of the course to be "heavy" and felt guilt as an educator grasping at ways to apply this to my own classroom. I felt fear, of being culturally inappropriate within my own eurocentric bias. Our conversation with [presenter] created a real-life connection and put a face to the research. I understand that becoming a culturally responsive teacher is a process that will not happen overnight, and requires baby steps. My biggest take away from [presenter], aside from her sharing of her depth of knowledge, was her openness and passion for the subject matter. In saying this, I feel more confident in pushing myself away from my fears of making mistakes and embrace the learning challenges that appear.	Challenge → Fear → Insight → Opportunity
Felix	I think my concerns weigh heavily on me and make it difficult to get excited. I try to be reflective of my privilege and place in addressing social inequity as a white man in modern society. I have often struggled with my place in this narrative, and how I can appropriately uplift the voices of those who need to be heard without overstepping or filling the cliché "nice white person" role. Concisely, I am concerned about doing a poor job of bringing culture, responsiveness, and mathematics together and similarly concerned not doing ANY job out of fear of doing it poorly. My excitement is that there are many knowledgeable and caring voices out there who seem to have tread this ground with my same fears and have wisdom to share. I hope that I can learn actionable skills that are realistic within my context. I am excited to get to dig into the idea of "tokenism" and cultural appropriation and to hone my critical evaluation skills for what is or is not appropriate. I also hope to open new discussions to increase engagement and fulfillment for my students. When these ideas are brought together it does not seem easy to distinguish appropriate from appropriation as a cultural outsider. I have often struggled with this idea of being an outsider told to tell someone else's story—it feels wrong to me on a very personal level. These stories are not mine, and I do not follow or necessarily believe in them. I also believe they are important, worthwhile, and deserve full acknowledgement of their richness and importance to the people who do hold those cultural values and stories. When culture appears to be "tacked on" I tend to judge it appropriation. But today we also learned about the danger of mathemetizing/personalizing too. This sparked some riveting ideas in how to approach the role of western and other cultural math processes.	Fear → Opportunity → Challenge → Insight

Table 2. *Cont.*

Challenge	Opportunity	Fear	Resistance	Insight
Participant	**Sample Quote**			**Component**
Raymond	Specifically, as I hope (in my project) to help students explore the math behind wealth and income inequality, I first have to help them see that a) such inequality exists, and b) this existing inequality is somehow inherently bad. In my research, I've found that "socialists" (like me!) argue this inequality not only exists, but is growing, and that both the existence and the growth are inherently problematic. They use a wide range of data to make these points. But how do we define *income*? Or *wealth*? Employment, tax transfers, property, capital investments, debt, inheritance make these more difficult to define than I'd expected. And how much inequality should we expect in a "fair" country? Should a 21-year-old university student have already accumulated as much wealth as a 65-year-old who is preparing to retire?			Opportunity → Insight → Challenge
Iris	I certainly found some of the suggestions overwhelming. It's funny, a lot of the feedback from some of the seminars have been that people would like to see some sort of template or framework for helping develop us as CRP teachers. Yet, we've been provided a framework along with the reflective questions and here I am, feeling overwhelmed! I have this sense (not sure if it's grounded in any knowledge outside of my participation in this class), that CRP will not come as "naturally" to educators until the educators who experience CRP as students, are in the classroom in the teacher role. I see teachers teach exactly how they learned in school (which makes sense, you know what you know!). However, I do recognize that this cycle requires intervention. I think the best advice I've heard so far this course it to just start somewhere, to start small. I do believe there were some very [good] questions in the Tool for Reflection we saw tonight, that I would genuinely like to consider when teaching CRP. I valued questions encouraging me to ask if I genuinely connected to students' contexts and funds of knowledge, and assessing if students are engaged in higher-order thinking and critical analysis. Although I know this framework was considered "incomplete", however I do not devalue the types of questions I could be using to reflect after my introduction to CRP. Again, I need to start somewhere and I am no expert at CRP (yet!), so I think these reflective questions actually will only assist my progress.			Resistance → Insight → Opportunity → Insight → Challenge

In these examples, we see very different ways in which insights are linked to other components. The extract from Cindy's journal entry is, indeed, a case aligned with our initial working definition; an insight is expressed leading to a perceived opportunity. Yet, things appear to be a bit more complicated in other cases. In the journal entry of Felix, for instance, an expression of fear leads to the expression of an opportunity, leading to a challenge, which in turn points towards an insight. In other words, Felix had to pass through three other components in his reflection to reach, what was for him, an insightful moment. In the case of Iris, things become even more intertwined: An expression of resistance leads to an insight; yet that specific insight points to an opportunity, which in turn leads to another insight, only to conclude this chain with an expression of a challenge.

We wish to draw attention to several aspects of the data presented in Table 2. Firstly, while our original definition of an insight clearly pointed to relationships between insights and the other four components, this first conceptualization suggested a limited understanding of those relationships when we stated "an insight *might lead to* gaining new tools (challenge), moving forward with what one has (opportunity), halting/moving backwards (fear), or disbelief (resistance)" [emphasis not in original]. It is true that insights can be expressed first, and then lead into the expression of challenges, opportunities, fears, or resistance associated with that insight; however, it can be seen in the data above that, in some cases, the insights *emerge from* or are even introduced *between* PPTs' expressions of challenges, opportunities, fears, or resistance. Also, while insights are shown to be connected to at least one of the other components, in some instances, insights are connected to several components at once. This is an important finding as it can provide MTEs with key information about PPTs' progress/development/growth or, as the situation may entail, regression or feelings of being overwhelmed. For example, it might be a positive sign of growth in CRP for a PPT to express more fear or resistance early on in the course reflections while showing more expressions of opportunities and insights further into the course. For the purposes of this paper, however, we have not traced the developmental aspects of the journal entries; we leave that for another time.

4.2. Types of CRP-Related Insights

Our second research question asks: What types of CRP-related insights do PPTs have as a result of their engagement with the course?

Given our revised definition/description for Insight in our COFRI framework, we now return to our review of the literature on insights. We identified three different types of insights discussed in the research: those associated with mathematics concepts or the subject matter content as outlined in curriculum [45]; those associated with different ways of teaching, learning, and pedagogical approaches [47]; and those associated with ideological or sociopolitical realizations [48]. In this paper, we name these three types of insights mathematical, pedagogical, and ideological. We observed in the literature that most scholars focus primarily on only one of these types in their research. In our research, however, we have noticed the presence of all three types of insights in our data. Thus, in response to this research question, we note here examples within the five-participant case study data that serve to illustrate the presence of the three types of insights.

4.3. Mathematics Subject-Matter Insights

The five participants expressed mathematics subject-matter insights, along the lines of what Liljedahl [45] describes as mathematical realizations. Due to the focus of the course on CRP, the PPTs' subject-matter insightful comments underlined the links between mathematics and culture, acknowledging how mathematical knowledge "lies on the borderline between the history of mathematics and cultural anthropology" [49] (p. 44). One such example comes from Felix, who admitted that "I have rarely considered myself as part of any 'ethnic group' other than Canadian (which is [a] whole other discussion of identity for another time), so the idea of supporting people with a focus on promoting culture in, as, and for learning has so far felt fairly detached from my own experience". Likewise, Raymond commented on the underappreciated role of various forms of Indigenous mathematical knowledge, as opposed to "Western" mathematics: "My sense at the moment is that the 'western' math I teach is largely about developing general tools to apply in a variety of situations, while the intersections of culture and math that I hope to explore in my classes is very specialized (igloos, tipis, woven baskets, and so on)". Olive, in turn, expressed her personal disconnection between culture and mathematics as experienced during her own schooling. As she wrote, "[p]rior to these discussions, I can honestly admit, I had a disconnect between culture and math. I wondered how such a traditionally concrete subject such as math, could be connected to being culturally responsive". Olive later commented that her prior perceptions of mathematics as an inflexible subject with one and only right answer had a negative impact on her attitudes towards its learning:

> I traditionally have seen math as a 1 + 1 = 2, step-by-step, sequential, one-way-to-get-the-answer, kind of subject. I am excited to shift my mindset and open up my lens to a wider horizon. From experience as a learner, I frequently found math difficult and withdrew from the subject as it was deemed to be something, "I'm just not good at". I am curious to know if the approach had been shifted, if I may have found meaning and connection to deepen my understanding and approach.

In a similar vein, Iris presented her current "understanding of the word 'mathematize'" as "to treat activities or opportunities with a mathematical approach". In her reflections, she commented on how course readings provided her with opportunities to challenge her prior views of mathematics being a discipline independent of the real world:

> When reading the Culture-Based School Mathematics report, we were introduced to the very powerful effect of Indigenous mathematizing. Because this article was my first understanding of the word, and because it was discussed in such a positive light, I generally feel it conveys a favourable image for me. It brings to light the idea that we use activities or everyday opportunities in a way to conduct mathematical exploration. It also helps create a verb sense of the word "mathematics". (. . .) I was reading a book out loud and I paused for a think-aloud moment where I quickly did some math (I think I was calculating the difference in pay between white American males and non-white American females). After calculating verbally and on the board I returned to my book and said something along the lines of "so there is some math for ya". Immediately after that I was shaking my head, asking myself why I would say that. This again supports the notion that math is a subject, not a part of everyday thinking or opportunities (which it was, it was a natural moment where I thought out loud some information regarding important facts supporting my understanding of the text).

4.4. Mathematics Pedagogy Insights

Several journal entries from all five PPTs included insights about mathematics pedagogy. In line with Caniglia et al. [47], insights of this type mainly regarded the self as a mathematics teacher, the importance of knowing the students in one's class, knowing and being critical of the curriculum, as well as knowing appropriate teaching strategies, the same way Loewenberg Ball et al. [50] talk about pedagogical content knowledge. In one of his journal entries, Raymond, for instance, emphasized the importance of knowing one's students and utilizing their funds of knowledge. He reflected on some of the course readings, arguing that the articles "remind me to be looking for ways to draw on the funds of knowledge of not just my students, but of the families and communities they come from".

Like Raymond, Olive also highlighted the importance of teachers knowing their students. She expressed insights into the role of language in mathematics teaching.

> With that, it is important we are digging deeper into the importance of knowing all of our students and expanding our knowledge by developing connections between their world and the impacts it has on them when they come to school. I reflected on how the language I use in the classroom can have an impact on my students and my teaching of mathematics. In class, a discussion arose about the difference between more and less, does this look the same for every student? It appears we cannot assume students have specific language and vocabulary that is the same for each student. This helped me to critically reflect on my own teaching. . . By looking at my teaching practices through this critical lens, I can make adaptations and be aware of how the language I am using can impact my students learning.

For Felix, teachers need to become aware of personal bias and how specific teaching practices may be pushing some children to the margins:

> It is important to recognize and have awareness of our own personal biases and how this is reflected in the curriculum and many current teaching practices. One thing I noticed is the language being used at times is still "othering" a group or groups of people without using the appropriate language. I think it is important to observe the language we are using and to not further marginalize groups of people. It is important to be aware of our personal biases and consciously challenge this in our reflections and sharing. At times this is challenging as it is deeply ingrained into our societal westernized ideas.

From a different perspective, Cindy underlined the need for teachers to be critical of resources and tasks found in mathematics textbooks during lesson planning. She wrote the following:

> This course has made me aware of the different content that we should be incorporating into our mathematics lessons. The textbook activity with [presenter], in which we analyzed various questions in the textbooks, was just one eye-opening experience. A lot of those questions were very surface level and very tokenistic. In my opinion, they were just including them to say that they have checked off the box of including First Nations content. This same idea appeared in [classmates'] final project presentation, in the Big Math Book from the grade 2 Math Makes Sense resource. We need to be more aware of what we are teaching our students. Both of the above examples are very surface level and do not dive any deeper. In order to make these lessons deeper, we need to provide some more information. It may take more time to discuss this background knowledge but it is needed so students can not only appreciate the culture but understand the meaning and significance of the activity. However, this idea of lessons being surface level is something I have struggled with a tad and I probably still will in the future. [One class presenter] talked about how lessons need to dive deeper; however, [another class presenter] said that even a surface level lesson is better than not touching on the subject at all. So where do we draw the line? Surface level lessons can still be beneficial, but are students really getting that deeper understanding I mentioned earlier?

4.5. Ideological Insights

A third type of insight expressed by all five PPTs concerns ideological/sociopolitical realizations in relation to themselves as teachers [48], which are significant first steps towards supporting communities in vulnerable positions [51]. Olive wrote the following:

> I believe the first step to becoming a culturally responsive mathematics teacher is to make a conscious decision to become one. I do not believe it is something that is always instinctual based on the societal pressures and ideologies that are persistent in our communities. It takes a conscious step back from ethnonormative perspectives to open yourself to the understanding that this is not a reality for our learners in our classrooms.

For Felix, his sociopolitical realizations were expressed in the context of his learning about mathematizing, an action which he defines as "applying structured mathematics to a thing and/or the contexts surrounding it in an attempt to observe a fundamental principle of mathematics within that given 'thing'" He warned that this kind of general approach can "become a problem when we mathematize those contexts which are humanizing to people, especially marginalized groups". On reflecting, he added: "I don't see anything wrong with mathematizing a soda can, a public park space, the classroom itself, etc. But I do see the issues with mathematizing cultural artifacts, processes, and traditions".

Raymond, who described himself as a person with socialist dispositions, commented that his own project for the course aimed at helping students develop an awareness of social inequalities. With his course project, he aimed "to help them see that a) such inequality exists, and b) this existing inequality is somehow inherently bad" (see Table 2 for full quote from Raymond).

Another example comes from Iris, who raised some interesting points about when it is age-appropriate to introduce children to issues of social justice:

> I can understand why people would argue that critical consciousness and social agency would be more appropriately geared towards older grades. The subject matter often pertains to more mature learners, as often topics can require extensive knowledge of socio-economic issues which students may not full understand until they have more in-depth knowledge of economic relations. Having said that,

> I think the concepts of “right and wrong”, “standing up for what’s right”, and “fighting injustices” are concepts that can be taught, explored, and elaborated at any age. Now obviously fighting social inequalities and injustices are a little different than standing up to the playground bully in kindergarten, but these are still concepts that are understood from a very, very young age. I believe that, in order to build empathy in our future community learners, it is not only possible but critical to instill a sense of critical awareness at a young age. I often think about a quote I’ve seen pop up a few times in the past year; “If my child is old enough to experience racism, your child is old enough to learn about it”.

5. Discussion

In this paper, we set out to understand the ways in which insights are connected to challenges, opportunities, fears, and resistance in PPTs’ reflections throughout a semester-long CRP course. We also aimed to explore the types of CRP-related insights PPTs have as a result of their engagement with the course. Based on our analysis of all journal entries for five case study participants, we observed that our initial working definition of *insight* was limited, and missing a full portrayal of how the COFRI components can be related to each other. As noted earlier in this paper, in our initial data analysis and COFRI conceptualization, we noticed that the expression of an insight seemed to follow an expression of a challenge, opportunity, fear, or resistance. Although this "order" was reflective of our initial data analysis (the first two journal entries), as we delved into additional journal entries for each of these five participants, we began to notice the positioning of insights relative to the other components. Table 2 shows that there is no fixed positioning for expressing insights. PPTs, in their journal entries, may show how they first think through challenges or fears in their reflection before expressing a related insight, and vice versa. Given this, the first consideration in our revised definition was relative positioning of insights. Also, given that we were successful in categorizing PPTs insights into one of the three types of insights as found in the literature, we can see benefits in incorporating these three types into our revised definition.

With these two key findings in mind, we offer the following revised definition of insight:

> An understanding or realization of what is currently happening and/or how things could be. In addition to seeing what is currently happening, a person will generate new ideas for extending, adapting, and/or improving. In general, insights can be (1) connected to one or more of the other four components (challenge, opportunity, fear, or resistance) and (2) classified as a specific type (mathematical, pedagogical, or ideological). That is, an insight suggests a new direction which either emerges from or leads into a challenge, opportunity, or even fear or resistance depending on the "tools" one has. Consequently, an insight might connect to gaining new tools (challenge), moving forward with what one has (opportunity), halting/moving backwards (fear), or disbelief (resistance) and, in each case, typically highlights aspects of mathematics subject matter, mathematics pedagogy, or one’s ideological/sociopolitical perspectives on mathematics.

Having a tool to help clarify the connections between PPTs’ insights and the other four components, as well as the type of insight being expressed, carries significant implications for mathematics teacher education and for culturally responsive mathematics classrooms. First, it is important to note that the course, which served as the context for this study’s data collection, was designed and taught (by Nolan) prior to the existence of this COFRI framework and our explicit classification of insights; yet we can see that the course provided opportunities for participants to experience all three types of insights: mathematical, pedagogical, and ideological insights. We have already noted that previous studies generally focus on one type exclusively. Our work confirms that all these types can happen in the same course, one specifically focused on CRP.

A second significant implication of the findings presented here relates more generally to the practices of MTEs. In designing course experiences for PPTs, MTEs can draw on this

COFRI framework and our expanded conceptualization of insights to build a course that acknowledges the presence of each of these components and types of insights at various stages of the PPTs' learning while, at the same time, providing opportunities for growth and diversity of perspectives within. As MTEs ourselves who teach courses in CRP, we welcome this tool to help us explicitly aim for PPTs' development and expression of all three types of insights as they progress through the course.

In the course associated with this study, the instructor was not as intentional about these aspects of PPTs' learning about CRP. As a result, and as we observed through our analysis, PPTs tended heavily toward expressing the first two types of insights (mathematical and pedagogical) while ideological/sociopolitical insights were either absent or underdeveloped. Elsewhere, we (and others) have acknowledged an unfortunate lack of attention to the sociopolitical and critical dimensions of CRP [12,14,21]. We now believe, however, that the tool developed through this research will encourage us, and other MTEs, to be more deliberate in our course design so that richer, more diverse learning opportunities in/for CRP are made available for PPTs. In other words, it can be claimed that, through this research, *our* insight is that we can analyze *their* insights to understand what directions their learning takes them: Toward challenge, opportunity, fear, or resistance?

6. Concluding Thoughts

As analysis beyond the five-participant case study continues, we are paying closer attention to the timeline for PPTs' expression of challenges, opportunities, fears, resistance, and insights, with an aim to track both the connections and the types of insights being expressed as the course progresses. This information is valuable for MTEs to inform their course design. Furthermore, we claim that PPTs themselves would benefit greatly from reflecting on their own COFRI timeline. For example, understanding when an insight occurs, how (or if) it is connected to other components, and what type of insight it is holds promise for PPTs as they move forward to make a conscious decision to become a culturally responsive mathematics teacher. Unpacking her own advice about making "a conscious decision" to become a culturally responsive mathematics teacher, Olive offered some general direction for how to do this:

> This can be done by taking courses such as this, by doing topic-related readings, getting to know our learners' unique family backgrounds and worldviews; by doing these things [we] begin to open our perspectives and deepen our knowledge to become more culturally inclusive, and we can find ways to integrate these into our classrooms. I do not believe a person is one day just going to become culturally responsive, as discussed, like a checklist. This, like many things, is a conscious, ongoing journey that will have ups and downs and learnings disguised as failures. Through this journey, a teacher becomes culturally responsive.

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Integrating Societal Issues with Mathematical Modelling in Pre-Service Teacher Education

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Abstract: The complex societal phenomena occurring in our daily lives and the ongoing curricula demands of mathematics education imply the responsibility of teachers to discuss societal issues with their students in mathematics classrooms. Yet, the ways in which teachers respond to these demands are neither given nor straightforward. In this case study, we aim to understand how pre-service teachers are introduced to addressing societal issues during mathematical modelling activities through the examples utilised by a teacher educator. Theoretical perspectives from socio-critical modelling are used to investigate examples from a mathematics teacher education course where socio-critical perspectives of modelling activities were addressed. We found that the teacher educator included multiple activities with contexts relevant to pre-service teachers, such as littering, body images, and oil spills, and focused on problem posing. Also, the complexity of socio-critical modelling activities was illustrated by bringing various perspectives and alternatives, and a need for commitment to action and assuming responsibility was discussed. Our findings conclude that mathematical modelling can be one way of incorporating socio-critical issues in teacher education to prepare pre-service teachers to be, and become, critical and responsible citizens, yet, doing so requires the engagement of a community of teacher educators.

Keywords: mathematical modelling; socio-critical perspectives; mathematics teacher educator; teacher education

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1. Introduction

Society and citizens face numerous challenges, such as climate change issues, sustainability, pandemics, and refugee crises. The challenges are often multifaceted and require approaches that take into consideration political, economic, or ethical dimensions. To prepare students to deal with these challenges as sensitised and responsible citizens, educators, including teachers, pre-service teachers (PTs), and mathematics teacher educators (MTEs), also need to become sensitised and assume responsibility for their role in addressing complex socio–political issues in their respective fields of practice.

In mathematics education, approaching real-world problems often occurs through engaging in modelling. Mathematical modelling refers to the process of representing real-world phenomena or systems and translating them into a mathematical framework to analyse, predict, and understand a real-world problem. Previous studies have indicated that mathematical modelling competencies are central to educating responsible citizens [1,2] because they require making mathematical and non-mathematical decisions [3]. However, according to [4], modelling activities around challenging issues often focus on transforming real-world situations into mathematically solvable problems rather than addressing the complexities of the issues themselves or reflecting on the use of mathematical models. To avoid modelling activities becoming just a context to "dress up" mathematical problems, there is a need to move beyond mathematising problems and building students' experiences relevant to real-world situations, such as taking political action. Engaging students to use mathematical modelling as a tool to investigate issues of the world and society, challenge inequities, and take action has been discussed in previous research, such as [5–11]. In particular, ref. [11] highlighted that modelling activities overlap with mathematics for social justice by dealing with controversial real-world situations, reflecting on alternative solutions, and supporting students as "competent knowers and doers of mathematics" (p. 1).

Mathematical modelling is included in an increasing number of curricula worldwide [12], such as in Germany [13] and Norway. In Norway, where our study takes place, modelling recently became one of six core elements in the mathematics curricula reform for grades 1–10 [14]. According to the reform, students should learn about how models are used and become able to critically evaluate modelling in society. For instance, a competence aim for fourth-grade students entails learning to "model situations from one's own everyday life and explain one's own thought processes" (p. 8).

However, it can be challenging for teachers and students to respond to curriculum demands to do with mathematical modelling [15]. When there are additional expectations to address socio–political issues, more pedagogical challenges may emerge as teaching becomes more uncertain. Therefore, the role of teacher education in preparing future teachers to deal with the uncertainties and various demands of combining mathematical modelling activities with knowledge about society needs to be further examined. PTs need to be supported to engage in modelling activities regarding societal issues with their students [16,17]. However, little is empirically known about how MTEs support them during mathematics teacher education courses.

In this paper, we discuss a case study to understand how mathematical modelling and socio-critical issues can be integrated into teacher education. Based on the case study of one MTE, our focus is on understanding how mathematics teacher educators exemplify associations of modelling activities with societal issues. To investigate this, we use data from an MTE's planning and implementing of three introductory workshops on mathematical modelling to second-year PTs of grades 1–7 as part of a five-year teacher education program in Norway. Understanding how MTEs use examples of integrating societal issues in mathematical modelling activities can provide insights on supporting PTs to embrace the uncertainties and engage in those issues with their students.

2. Integrating Mathematical Modelling and Societal Issues in Teacher Education

Societal issues within mathematics education can involve teachers' and students' explorations of how mathematics intersects with broader social, economic, and political contexts. In particular, perspectives from critical mathematics education (CME) have focused on social justice issues, questioning the role and influence of mathematics in society, and how power dynamics and social inequalities can be reinforced or challenged through mathematical practice [18–21]. There are many forms of social justice and defining it is not straightforward, because what is just to one person might be unjust to someone else [22]. Social justice issues can include, for example, the fair distribution of wealth, resources, opportunities, and privileges or inequalities in society (Oxford Learner's Dictionaries [23]). According to [21], learning about issues of justice and injustice in mathematics school classrooms is not a matter of telling students what these issues are, but, rather, engaging them in developing experiences about what social justice is, and how it is formed. Teaching and learning mathematics with this perspective aims to support students in expressing what they consider as social and economic inequalities, oppression, and exclusion or address structural discrimination and power relations benefitting certain perspectives (e.g., valuing rapid economic growth at the expense of environmental problems and climate change). Although CME initially focussed on social justice problems, ref. [21] introduced the term environmental justice to include environmental issues as part of the concerns of CME. For example, ref. [24] analysed students' involvement in a modelling task to estimate the state of the Great Barrier Reef by 2050. They described that the students became aware of environmental injustices and their influence, not only on the lives of the Australasian people but also on their own. Engaging in this activity allowed students to combine their understanding of society with mathematics and develop a sense of agency and political change [24]. Although conducting and defining social justice is a complex and ongoing endeavour, rather than a single product, the aims of developing a social justice stance are not separate from mathematics learning goals, according to [25]. Ref. [25] discussed three specific mathematics learning goals (reading the world with mathematics, mathematical empowerment, and changing dispositions towards mathematics) as aligning with three goals of learning about social justice (socio–political consciousness, agency, and positive social and cultural identities). In this paper, we have a broad understanding of social justice, including societal issues, such as environmental justice, marginalisation, stereotypes, biases, power structures, and cultural and linguistic diversity.

Several research studies have investigated mathematical modelling within teacher education. Ref. [26] discussed experiences as MTEs from modelling courses and mentioned challenges they needed to deal with, such as balance theory and practice, appropriate teaching strategies, and content knowledge (e.g., modelling cycle, goals/perspectives, types of tasks, solving and creating modelling tasks, and planning and practising lessons). Ref. [16] described four dimensions of pedagogical content knowledge for teachers: a theoretical dimension, a task dimension, an instructional dimension, and a diagnostic dimension. Ref. [27] found that PTs could develop their competencies in these four dimensions when their experiences as learners of modelling were combined with their experiences as teachers of modelling. For example, PTs should develop competencies such as choosing appropriate modelling tasks and taking various approaches. Related to the findings of [27,28], the authors found that when PTs were provided with rich modelling tasks (e.g., good-quality tasks where PTs engaged in real-life problems where they made choices, developed representations, and engaged in decision making) and an emphasis on modelling, they demonstrated an understanding of mathematical modelling.

There are also some studies within teacher education focusing on socio-critical modelling perspectives. Ref. [10] explored how PTs connected mathematical modelling and social justice issues by investigating what kind of problems they posed. They found that the PTs posed problems of social justice issues involving both micro-level (e.g., individual or community issues) and macro-level (e.g., broader issues involving political and economic structures involving disadvantages for some groups). From the perspective of problem posers, ref. [10] proposed a conceptual framework for social justice-oriented mathematical modelling tasks consisting of social justice issues, realistic context, model development, and shareable processes. According to PTs' reflections in the study of [29], some problems were "too simplistic" for students, while other students had little background in posing open questions. Nevertheless, in her study, PTs were able to develop their open problem posing after observing their peers' problems and designing multiple variations of a potential task inspired by authentic pictures they took from their environment.

Ref. [30] integrated a critical mathematics education perspective with a socio-critical approach to modelling, encouraging PTs to reflect critically on the application of mathematical models in society. When the PTs developed modelling activities for students, ref. [30] found that community interactions and classical and critical knowledge were needed for developing authentic modelling tasks and fostering good modelling experiences for students. Ref. [9] suggested combining mathematical modelling with culturally responsive, social justice-oriented mathematics. In their study, an ongoing water crisis was used as an example to explore social and environmental justice issues. Their findings suggested that engaging in tasks addressing these issues increased PTs' and teachers' awareness of systemic injustice while strengthening their mathematical competencies. Refs. [31,32] described how MTEs stimulated critical mathematical discussions on the use of indices, such as the body mass index (BMI), inspired by modelling as critiqued by [7]. The primary school teachers in their course were shown a picture of a muscular athlete with a BMI of 35.8 kg/m^2 (which would put him in the obesity range). In indices such as BMI, mathematics is often implicit; still, it is crucial in defining the phenomena. Facilitating critical discussions amongst students could be one way of critically reflecting on the role of mathematical models in society.

Ref. [33] described mathematical modelling as one way of addressing socio–political issues related to injustice. He used the context of the Charlottesville rally (far-right groups' protests) with his PTs to explore issues of segregation in U.S. schools and racism in society. Drawing on research literature and his own experiences, he proposed a framework highlighting the importance of building a socio–politically oriented community. The knowledge about socio–political issues, curriculum, and mathematical teaching inform how to mathematise socio–political problems. He further highlighted that a key to building a socio–politically oriented community is having a network of colleagues committed to exploring societal issues, as well as resources (e.g., social media sites on equity and mathematics education). Ref. [34] described a modelling activity conducted by one group of PTs in their practicum. The PTs utilised pictures of refugees to encourage students to pose questions about the refugee crisis and supported them in posing examinable mathematical questions.

The choice of context can reflect the educational purposes of modelling as a purposeful activity. Ref. [35] exemplified contexts such as environmental problems, migration, and inequality, yet mentioned that identifying contexts that would work in classrooms worldwide is difficult. This is because different cultural experiences among students and teachers can have an impact on how one perceives things and lead to numerous variations of models. Related to this, ref. [16] explained that the aim and justification of modelling influence the choice of context or examples. For instance, a cultural justification, where relations to the extra-mathematical world are involved, authentic, real-world examples showing the role of mathematics more explicitly are suggested contexts. For example, ref. [36] discussed a modelling activity about ocean trash, where seventh-grade students investigated the Great Pacific garbage patch. This activity allowed students to engage both with mathematical thinking and knowledge about society since they posed questions about the origin of the trash, its effects on wildlife, and what actions could be taken in addition to designing a model to convey the size and density of the garbage patch and discussing various approaches.

Several examples of modelling contexts scale up or down items from real life, e.g., the Giant shoe problem and the Oldenburg's pick-axe with the Kassel Hercules [16,37,38]. Another variant of such proportional thinking is the Barbie doll. Ref. [39] asked students what Barbie would look like if she were the height of an average woman as an example of proportional reasoning. The differences between the doll and humans led them to discuss body images and eating disorders. Exposure to images of thin, idealised bodies, which dominate the social media, can contribute to body dissatisfaction. Ref. [40] described that Barbie has unrealistic measures and "promote(s) harmful weight attitudes including thin-ideal internalisation in young girls" and can "negatively influence children's feelings about their own bodies and their eating behaviours". Ref. [41] let her PTs read the article by [39] in a course about social justice in mathematics education. Her PTs' reactions to the paper were that it was "too feminist" and did not match their expectations of learning practical "tips and techniques" about how to integrate social justice issues in the mathematics classroom (p. 210). However, ref. [41] said the activity "represented tangible ways to both understand the marriage of mathematics and social justice and to feel good about doing what they could to address social justice issues in/through mathematics through more meaningful 'real-life' connections" (p. 210). Ref. [2] highlighted how the context of the COVID-19 pandemic has demonstrated that citizens need to understand how mathematical models function, help us to understand, predict, and overcome crises. It has also shown that citizens and decision-makers should learn to critically evaluate reports affecting how we act during a crisis and deal with the inherent uncertainty. They argued that these new demands had expanded the learning of modelling in education to not only include learning to apply mathematics to real-world contexts but also to include learning decision making, dealing with uncertainty, and adopting critical thinking. Issues of inequality were used as context when ref. [42] described a modelling task involving the distribution of wealth in the world as a context "to unveil social and political phenomena, and promote informed critical position" (p. 6). This summarises in many ways how context is particularly important in socio-critical modelling activities.

Several studies involving societal issues and mathematics have reported challenges. Ref. [9] described reluctance to mathematise controversial issues, such as racial profiling or discrimination. However, they highlighted that an increasing number of MTEs invited PTs to investigate fairness, environmental issues, and economic justice. Ref. [33] explained that colleagues, leaders, or others could question the use of socio–political context in mathematics courses from views about what mathematics courses should entail. He underlined that PTs might feel overwhelmed by the time it takes to develop relevant socio–political tasks, experience pushbacks from colleagues, or uncertainty to navigate challenging conversations. He suggested that PTs could find support elsewhere, such as in like-minded colleagues, various resources, starting with less controversial topics (environmental issues), and curricula stating the relevance of real-world problems or mathematical modelling where the context is essential. Ref. [8] reported that although PTs stated that they would include real-world situations, they were ambivalent about having controversial topics and issues of injustice in mathematics education. Ref. [43] described it as time-consuming to find contexts that work well with both developing responsible citizens and competencies in modelling. Some teachers in their courses stated it was difficult to identify and address the mathematics involved in the context (e.g., they considered the context of plastic waste to be more connected to biology). Others stated it took more time to understand the context than the benefits from the mathematical outcome. Ref. [43] highlighted that these challenges render the need to include controversial societal issues in mathematics classrooms more systematically, in order to make future citizens familiar with such issues and the inherent mathematics in them. Ref. [2] described that including societal problems requires competencies from many scientific fields. It can be challenging for teachers to combat subject-based approaches. However, real-world problems are interdisciplinary and challenging; still, as citizens, we need to find ways to deal with them.

3. Theoretical Perspectives

Socio-critical modelling perspectives focus on exploring societal problems with mathematics as a critical tool [7]. Ref. [11] highlighted that socio-critical modelling perspectives serve dual goals, one within mathematics (e.g., how mathematical models support decision making), while the other is within societal spaces (e.g., addressing inequality and injustice). They emphasised that when combining modelling and societal issues, one should connect each of the components of the modelling process to societal issues (see Figure 1), rather than only some components in isolation.

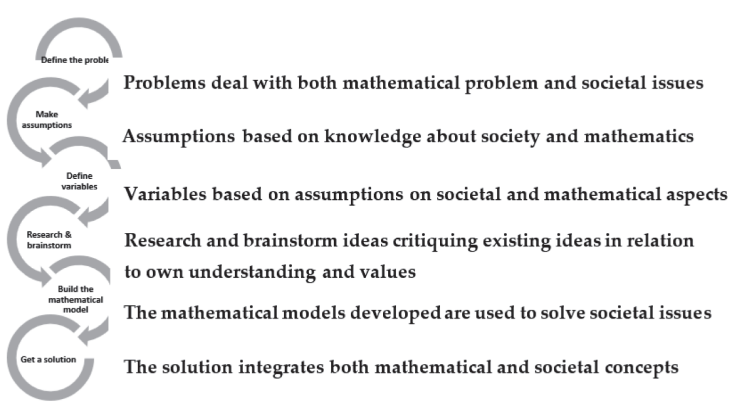

Figure 1. Combining mathematical modelling with societal issues [11].

Figure 1 illustrates the modelling process (on the left) and the process of combining mathematics and societal issues (on the right). Ref. [11] departed from the modelling process described by [44]. The modelling process is sometimes explained as different steps where students perform different activities, e.g., defining, structuring, and simplifying problems [16]. Ref. [11] suggested that all components in the modelling process should, ideally, include mathematical and societal considerations. That means it becomes insufficient to ensure that the problem deals with societal issues if the other steps of the process are not based on such factors as societal knowledge and understanding of mathematical data, or if the model developed does not aim to solve societal issues. Mathematical modelling and social justice, thus, become inherently intertwined through students' modelling processes. Integrating this duality in all steps of students' modelling processes is, thus, an aim when combining mathematical modelling and societal issues.

3.1. Modelling to Raise Awareness and Responsibility

Refs. [45,46] argued that mathematics education researchers, educators, and teachers have a responsibility to engage in societal issues. Mathematical modelling can be one way of assuming this. For instance, Ref. [47] highlighted that mathematical modelling could be used to raise awareness of critical issues in society. When investigating how socio-critical perspectives were present in a modelling project conducted by PTs in Argentina, one of the groups of PTs stated that their aim was "modelling to raise awareness" of their students rather than modelling "to obtain a super formula" (p. 573). The PTs focused on trash and recycling, and posed questions on the quantity and classification of trash. Mathematics became subordinate to the social aim, and mathematical modelling became a tool for understanding and reflecting on a phenomenon concerning the world outside the classroom. When introducing the modelling activity to the PTs, the teacher educators and researchers discussed that modelling activities should involve free choices of a real-world theme, interdisciplinarity, avoid pre-determined mathematical content, and offer reflection about mathematics, the model, and the societal role of mathematics.

Ref. [43] described an extended modelling cycle considering ethical, social, cultural, and economic aspects when controversial issues are in play. They emphasised that, to develop active and responsible citizens, mathematics education could include modelling activities where students could search for information about societal issues and contradicting discourses about scientific results or various reasonings grounded in ethical, social, or cultural considerations.

3.2. Modelling to Empower Students and Take Actions

Ref. [5] explained that mathematical modelling empowers citizens. It could provide them with the tools, the rights, and the responsibility to investigate critically and, if needed, reject mathematical arguments. They further highlighted that mathematical modelling could enable students to judge applications of mathematics used to describe and analyse aspects of our society. Refs. [5,6] described that the modelling activity with Barbie, which we mentioned earlier, is an example of using modelling as a critical tool for analysis. In a sense, it becomes a method of modelling for life and modelling with a purpose. Similar ideas are seen by [2,43], who argued that mathematical modelling is key to empowering students as responsible and active citizens. They problematised that mathematics education traditionally focused on concepts and competencies detached from societal implications. Therefore, they suggested that, alongside the teaching and learning of mathematical modelling, there are potential opportunities for developing understandings about socio–scientific issues, for inquiry-based learning, as well as numeracy, critical thinking, and 21st-century skills. When considering the professional development of teachers to support them realise these potential opportunities, modelling tasks involving controversial issues demanding ethical, moral and social reasoning and decision making could be discussed.

Ref. [9] visualised the modelling cycle next to a social justice cycle. In the latter cycle, it is emphasised to consider the broad social issue and build civic awareness and a sense of action. The modelling cycle is essential in modelling activities with students. If civic awareness and taking action are explicitly included in the modelling cycle, this can bring attention to the fact that these topics are relevant to consider when modelling. They described that, after the teachers in their study engaged in one example of a socio-critical modelling task, they began to imagine other relevant topics for modelling in their classroom, including immigration issues, recycling, and food insecurities.

Ref. [19] highlighted that teaching mathematics in critical ways is not an option in today's society, referring to current crises such as climate change and refugee situations. He stated that we, as teachers, have a responsibility towards future generations and to our planet, and what we do in the classroom matters. He used the phrase "reading and writing the world with mathematics" (p. 133) to describe a situation where students learn to use mathematics to investigate their society, understand forms of injustice, and enable them to act to change them accordingly.

Ref. [48] also emphasised action and stated that students could take action when critically investigating and reflecting upon real-world modelling problems. They described how dimensions of socio-critical modelling attempt to address multiple ways of working with real life, where students are supported to understand, explain, deal with, and suggest solutions to various problems. To take action on inequality issues, ref. [48] highlighted that students should be critically aware of mathematics as part of power structures in society.

Ref. [49], among other relevant studies, discussed that culturally relevant pedagogy can be strengthened through mathematical modelling, because students' backgrounds, knowledge, and experiences can be acknowledged to bridge home cultures and school. This could support students in developing critical consciousness, which refers to understanding societal systems and acting on them with a sense of agency [50]. Cultural issues are embedded in societal issues. For teacher education, they suggested that a focus on PTs posing open-ended modelling problems, making assumptions, and engaging in discussions and readings about culturally relevant pedagogy that could support understandings about the relations between societal and cultural issues in mathematics education.

4. Context of the Study

In our case study, we aimed to understand how an MTE integrated societal issues in mathematical modelling activities. The study took place in a teacher education institution in Norway as part of a larger design-based research project, "Learning about teaching argumentation for critical mathematics education" (LATACME). The project investigated what promotes or hinders PTs' learning about teaching argumentation for critical mathematics education in grades 1–7 multilingual classrooms. The MTE, the main subject of this case study, is the second author of this paper (Georgia) who was in her second year of working as a teacher educator when the study took place.

We investigated a course in the first semester of the second year of a five-year teacher education program, which included a period of school practicum. The course also included an obligatory assignment where PTs needed to describe and analyse their experiences of designing and implementing mathematical modelling in their practicum. The workshops were conducted before the practicum and lasted three hours each. Two out of three workshops of the first cycle and all three workshops of the second cycle took place digitally due to COVID-19 regulations. During the workshops, the main language spoken by the MTE was English, as her native language is not Norwegian. However, the MTE and PTs drew on Norwegian at times.

4.1. Data Collection

The data come from Georgia's two cycles of planning and implementing three workshops of mathematical modelling to three groups of PTs each year, as part of the mandatory mathematics education course. The empirical data included in this case study consist of the PowerPoint slides used in Georgia's workshops, her reflection notes before and after each workshop, transcripts from the audio-recorded workshops, and the PTs' written responses to tasks during the workshops, where available. Some sequences of the workshops were not audio-recorded because they were digital, and not all PTs had consented. The workshops aimed to introduce PTs to planning and implementing mathematical modelling activities in multilingual classrooms in grades 1–7. The theoretical framing of modelling in the workshops was based on socio-critical perspectives of modelling [7], where a modelling activity is understood as a problem (not an exercise) for the students, extracted from everyday life or sciences that are not pure mathematics (p. 294). Emphasis was given to problem-posing, framed within the ideas of activism and awareness of injustices in society [50,51]. To illustrate different perspectives of modelling activities and posing questions, Georgia used various examples from real-life situations, such as oil spills in the ocean and pollution caused by the accumulation of cigarette butts.

To support PTs in modelling activities, the group of MTEs in the teacher education program decided to use mathematics in three acts [52,53] as a teaching arrangement, or the adjusted approach; modelling in three acts [34,54]. Mathematics in three acts includes: identifying a problematic situation based on visual illustrations (e.g., pictures, films, graphs, and concretes) (Act 1), working in small groups to retrieve the necessary information to approach the problematic situation (Act 2), and solving and presenting solutions (Act 3). During Georgia's workshops, more emphasis was given to Act 1, connected to problem-posing. For example, she requested PTs to reflect on their role in supporting students to pose their own questions in a modelling activity, handling students' mathematical and non-mathematical questions, and taking action to change the situation identified as problematic.

As a beginning MTE, Georgia had been engaged in investigating her own teaching practice through action research [55,56]. However, her focus on investigating and improving her practice had been on integrating issues of language diversity rather than socio-critical modelling activities. The first author of this paper, Lisa, was also involved as an MTE during the LATACME project. Since we, as MTEs, shared similar genuine interests in mathematics education but came from different backgrounds and teaching experiences, we collaborated closely and decided to learn from each other's practices. For example, Lisa taught one workshop on mathematical modelling to one group of PTs before Georgia. Before and after our respective workshops, we reflected on our plans as MTEs and discussed the needs we perceived that PTs had. In a prior study we conducted based on a related dataset within the LATACME project [34], we became aware of the impact that our decisions and examples can have on PTs integrating societal issues into mathematical modelling. Even though the focus of that prior study was on PTs' practices after Georgia's modelling workshop, in the present study we extend our understanding of the topic by looking further into the same MTE's practices.

4.2. Data Analysis

To understand how the integration of societal issues in modelling activities was exemplified in the MTE's practice, we first categorised the data into broad themes based on the content of the workshops and the reflection notes. The emergent themes were: pedagogical approaches, choice of literature, curricula, modelling tasks and activities, challenges of modelling, modelling competencies and sub-competencies, problem posing, modelling cycle, teaching in language-diverse classrooms, and socio-critical modelling perspectives. Then, we identified similarities and differences across the themes based on the theoretical framework discussed in the previous section. We analysed the data according to two themes inspired by components of the modelling cycle, particularly posing problems involving both mathematical and societal aspects [11]. Other analytical lenses were: modelling to raise awareness and responsibility [43,47] and modelling to empower and take action [5,6,18,49,50].

Therefore, we grouped the MTE's ways of exemplifying connections between modelling and societal issues into the following foci: problem posing, raising awareness, responsibility, empowering, and taking action. In the next section, we analyse and discuss examples of how these foci emerged in the MTE's practice. The research study follows the ethical guidelines from [57].

5. Findings and Discussion

We structure the discussion of the data in two parts: (1) raising awareness and responsibility by posing problems focusing on societal issues; and (2) modelling to empower and take action. In the first part, we describe three modelling activities where the combination of mathematical and societal aspects was exemplified in relation to problem-posing processes. In the second part, we describe an example related to the modelling cycle. We chose this example because it provides opportunities to understand how MTEs can connect the theoretical framing of mathematical modelling and socio-critical perspectives, such as taking action. The examples we discuss are taken as potentialities in order to gain insights about MTEs' practices and decision making rather than as "ideal" examples of approaching the combination of mathematical modelling activities and societal issues.

5.1. Raising Awareness, Responsibility, and Problem Posing

We identified three examples of modelling activities related to problem posing that Georgia drew on in different workshops: littering, Barbie, and oil spill.

The littering activity was related to PTs' hiking experiences, a popular everyday activity in Norway. Therefore, this context gave the MTE the possibility of combining reflections on a micro-level (PTs and students can observe littering in their daily lives) and a macro-level (littering is a nation- and global-wide issue), as indicated by [10]. This example was introduced to PTs as a potential modelling activity in three acts [52,53], named "Don't let your hiking go to waste" and developed as part of LATACME. This activity was designed during the LATACME project, see https://prosjekt.hvl.no/latacme/wp-content/uploads/2020/09/Modelling-tasks-hiking-ice.pdf (accessed on 13 July 2023).

In Act 1, Georgia showed a slide with authentic pictures of a local hiking area where littering was observed (Figure 2). The pictures included different kinds of trash, such as plastic bags, glass bottles, cigarette butts, and cigarette packaging, between plants and in the lake by the hiking area.

Figure 2. Four of the pictures used to bring awareness about littering. Photo from a hiking trail in Kanadaskogen, Bergen, in June 2020. Pictures taken by Camilla Meidell and printed with permission.

She asked PTs to discuss what they noticed in these pictures, identify problematic situations, and pose questions in small groups in a shared document. Some of the questions that PTs shared were: "Why is it important not to litter in nature?" and "Which distance should there be between each bin for people to throw waste in, instead of in nature?" One group of PTs wrote:

> *Problem solving: How to reduce litter in nature? Reasoning and argumentation: Why is it important that people do not throw rubbish in nature? How much trash do we find? How many trash cans? Where are the trash cans? Near places to eat? If we increase the*

number of rubbish bins, will the amount of rubbish in nature decrease? Statistics: How much plastic, paper, residual waste, and food waste do we find in nature? Make forms and tables.

Problem posing can be challenging for students [12,16] and formulating questions in groups can be an opportunity to test out various questions they want to investigate. Engaging the PTs as learners of problem posing combined with being teachers preparing lessons for their students, in line with the ideas of [27], is one way of providing the PTs with the experience of posing questions when modelling. In this sense, PTs experienced problem posing around socio–environmental issues as learners, not only from their own perspectives but also from observing the problems their peers posed, similar to the PTs in the study of [29]. Later, PTs were requested to reflect as teachers and discuss how they could deal with possible challenges of problem posing processes in the classroom, such as dealing with the nature of students' questions, either mathematical ones or other questions that are (initially) non-mathematical (see [34] for related discussions). A focus on identifying mathematics can support students in seeing the mathematical aspects of real-world problems while bringing awareness of littering issues in society, similar to students who identified mathematics related to the refugee crisis in the study of [34].

Following the PTs' discussions and sharing of questions, Georgia included the following slide (see Figure 3):

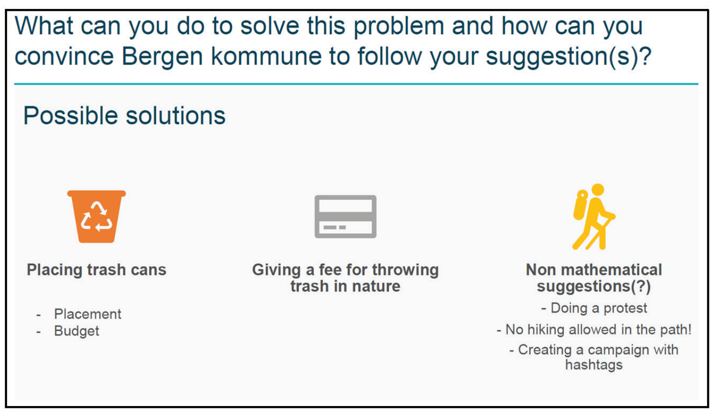

Figure 3. The PowerPoint slide from Georgia.

Georgia linked the littering activity with the idea of taking action by returning to PTs as learners and asking them to consider what they could do to "solve" the problem they had identified around littering and how they could convince the municipality where the hiking area belongs to follow their suggestions. Thus, the littering activity allowed Georgia to introduce a real-world problem combining mathematics and societal issues, highlighted as essential by [11]. Using the municipality as an audience in the modelling activity [58] can be considered a technique to work on the activity from the point of view of the agents who hold responsibility for the problematic situation or need to assume responsibility for its resolution. The first part of the MTE's question ("What can you do to solve this problem?") addressed PTs' personal responsibility as citizens, in line with the ideas of [2,43]. It also pointed towards potential actions, i.e., what the PTs can do, which can be seen as exemplifying the use of mathematics to change a problematic situation, following the ideas of [18]. In the second part of the question ("[...] and how can you convince Bergen municipality to follow your suggestions?"), PTs were asked to imagine the municipality as a "client" needing a solution. The idea of involving a client was highlighted by [11] as a fruitful approach to making the modelling task authentic. In this case, Georgia used the example of Bergen municipality as a client who needs to be convinced to pay attention to the problematic situation and, thus, broadened the littering problem from a societal to a political problem. However, the MTE did not ask PTs to work further with modelling the activity and the problems they had posed and, therefore, did not proceed to actually send suggestions to the municipality. Therefore, an opportunity was missed to exemplify making authenticity real and not imaginary, or, in [33]'s words, to use mathematics as a tool for socio–political change in society. As described by [58], the idea of an audience can change how PTs use mathematics in order to convince the audience about the significance of the problem at hand.

Following the question of convincing the municipality, Georgia presented information about the hiking area and more authentic pictures as different alternatives of what dimensions of the problem one could focus on. These different alternatives, or examples to explore, included: a geographical dimension (map and bins placement), a statistical dimension (personal opinions of the locals), an economic dimension (budget), an environmental–political dimension (taking action to protect the nature), and an ethical dimension (legitimacy). The diversity of dimensions reflects the complexity of considerations needing to be made when working on socio-critical modelling activities. The various dimensions could also support PTs' understanding of "modelling to raise awareness" rather than modelling for developing a "super formula", in alignment with the study of [47].

The geographical dimension included a map of the hiking area and its scale, which could be used as a potential model for placing regular trash bins or cigarette butt bins. Here, the issues of informing hikers about the placement of trash bins and prohibiting them from littering were brought up as possible solutions for reducing littering in the area. The economic dimension included some of the expenses that the municipality would have to make to place and maintain trash bins. For instance, Georgia exemplified questions in a PowerPoint slide such as:

> *How much do the cans cost? How big should they be? In what shape? Does it matter? How much money does the maintenance of the trash cans cost? How much does the pollution of not putting up trash cans cost? Picking up trash, environmental costs, wildlife, and ecosystem.*

This set of questions suggested how posing questions based on the economic dimension and the cost of placing trash cans can have extensions in environmental dimensions, such as air pollution. This seems to have allowed Georgia to compare the cost of taking action to place trash bins with the accumulated costs of *not* taking any action to do so (e.g., the impact on the environment, wildlife, and ecosystem of the hiking area). By bringing up this comparison between taking action and not taking action, PTs could develop an understanding of working on socio-critical issues and become aware that it involves a responsibility to make a decision that will have an impact on society, whether the decision leads to a change or not. Littered cigarette butts have environmental impacts, such as leaching toxic chemicals into the ground and water (e.g., lead and arsenic) and being a major plastic polluter as the filter consists mainly of plastic fibres. Georgia introduced the environmental–political dimension to PTs with a number of possible (initially) non-mathematical suggestions for the littering problem. These suggestions included organising actions, such as protests and online campaigns, or considering whether hiking in that area should no longer be permitted to protect the nature and ecosystem within the hiking path and the lake.

Littering is also an ethical issue and can lead to facing ethical dilemmas. For instance, do we have an ethical responsibility for other people littering? Also, littering cigarette butts is sometimes considered acceptable, which can make it challenging to act on the problem. In the workshop, Georgia included ethical dilemmas that could arise if it became illegal to litter in the hiking area. For instance, she posed questions such as:

> *If I am not going to place trash cans or trash bins, should I place cameras? Is that another solution? Is that ethical now? Should I be . . . Should I suggest to Bergen kommune*

> *(municipality in Norwegian) to do that? And who is going to be responsible for that? How much is going to be the fee for people who throw trash?* (workshop 1, p. 7)

These questions made available to PTs ethical concerns that may challenge everyday structures that are often taken for granted, such as the techniques used to identify individuals who litter, the settling of fees, and the municipality's role. Norwegian culture is often associated with being outdoors and protecting pristine nature from littering. While identifying people littering can be easily achieved by monitoring, people go to hiking areas to relax, so surveillance of citizens is very intrusive, and ethical considerations should be made.

In the modelling activity "Barbie in real life", Georgia showed a picture (picture: https://www.cbsnews.com/news/life-size-barbies-shocking-dimensions-photo-would-she-be-anorexic/ accessed on 13 July 2023) of a scaled-up Barbie next to a picture of a real woman, indicating that Barbies are often far from being realistic representations of women. Based on the pictures, a question was posed by the MTE, suggesting a potential modelling task: "What would Barbie look like if she was one of us?" Taking the starting point in Barbie and asking students to imagine how she would look in real life was also described by [5,6,39,41]. In the case of Georgia, according to her preparation notes, the choice of the Barbie context as an example of setting up socio-critical modelling activities was made in order to make links to PTs' obligatory assignment where the "Bungee Jump Barbie" task [59] was recommended by the team of MTEs. In the "Bungee Jump Barbie" task, students investigate how long the rope (rubber band) holding Barbie should be if she is dropped from a certain height to the point of just touching the ground's water surface. Georgia compared the reality factor of the two Barbie activities:

> *The use of modelling as content, in cases like this bungee jumping Barbie, (...) where the goal of the bungee jumping of Barbie is to learn math, like measuring distance and finding patterns. (...) That was a problem that was already set by a teacher. But from a more social and critical perspective, what would be interesting would be something like: Is this Barbie model realistic? Is it real? Which could also involve very interesting discussions around proportional thinking, different analogies, functional thinking, symmetry, measuring, and modelling. (...) My point here is that different modelling perspectives ask different questions, so not all questions start from a situation asking how much or how many there are* (workshop 2, pp. 10–11).

In this excerpt, Georgia referred to the "Barbie in real life" activity as an opportunity to extend the "Bungee Jump Barbie" and combine mathematics with socio-critical issues related to body image. Discussing issues of body image or eating disorders can be challenging for students and teachers because it can affect students in the classroom at a personal level. It can, therefore, be difficult for PTs to combine and balance the mathematical and societal aspects of this modelling problem. Georgia suggested various mathematical topics that could be relevant when exploring the "Barbie in real life" modelling task, similar to the task described by [16,37,38]. Thus, the "Barbie in real life" modelling activity could engage students in such as proportional reasoning, facing similar mathematical challenges described in their research. However, the PTs may find it challenging to recognise such problems as mathematical. For instance, while Georgia introduced various societal issues, the following dialogue took place during the first workshop (pp. 6–7):

> *PT: Could you please explain how you're playing this (referring to the societal context) into math or the modelling? How it comes together?*
>
> *Georgia: What is math for you? (pause) That does not look like math?*
>
> *PT: Doesn't really ... (pause) But if it's like you're taking focus on argumentation and reasoning, and then quite (inaudible) to mathematics ... then I'm in, then I can understand.*

The PT asked Georgia to connect the societal issues to mathematics explicitly. Instead of explaining, Georgia turned the question around and asked the PT what she considered as mathematics. The PT started pondering before she related the question to argumentation and reasoning, which are part of the mathematical competencies in the Norwegian mathematics curriculum [14]. "Barbie in real life" could encourage the students to investigate and engage in societal issues, such as body image stereotypes, societal expectations, and eating disorders. Bringing awareness of the matters of eating disorders could involve knowledge about the illness itself but also to socio–political dimensions of this illness. Students could investigate the social and economic cost of how this illness affects society [60] or how insurance companies treat this disorder and how this affects marginalised groups in society [61]. To extend a "feminist focus", as reported by some of the PTs in the study of [41], one could also focus on the occurrence of eating disorders in males, a historically under-researched area (males compromise about one in four cases in bulimia nervosa and anorexia nervosa [62].

The third modelling activity was the "Oil Spill". Georgia started by showing a map representing the span of an oil spill in the Gulf of Mexico in 2010 (Figure 4). This oil spill is historically one of the largest marine accidents, with more than 200 million gallons of oil being spilt [63]. Four questions accompanied the map (see Figure 4).

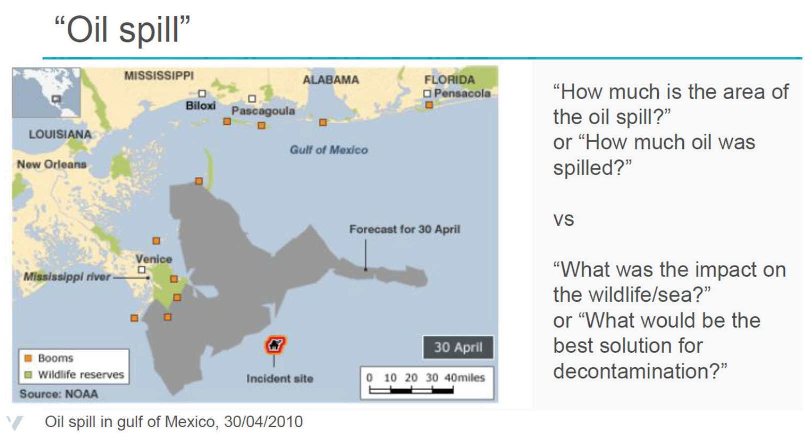

Figure 4. The slide in the fourth, fifth, and sixth workshops depicting a map of the oil spill in the Gulf of Mexico and the four questions. Map from "Oil spill in Gulf of Mexico in maps and graphics" by BBC (NOAA) [64] (http://news.bbc.co.uk/2/hi/americas/8651333.stm, accessed on 11 July 2023).

Each question in Figure 4 exemplifies posing problems for potential modelling activities. The first two questions suggest a mathematical focus on quantifying and estimating. Estimating the area or the amount of spilt oil is not a straightforward task. It involves (mathematical) challenges, such as deciding the surface area and the average thickness or risk analysis. It also could include political implications, such as whether oil drilling is still worth it and who should bear the costs. It resembles the modelling task described by [36], where students investigated the Great Pacific garbage patch and posed questions involving the extent, environmental effects, and potential actions. Estimations of geographical areas or crowds from a socio–political perspective are described by researchers such as [65–67]. While [65] discussed various estimations on casualties in the war in Iraq, the two latter researchers investigated how students estimated attendees in protests in Chile and Greece, respectively. Combining mathematical estimations with the political implications of oil spills can be fruitful in developing an integrated understanding that mathematics is not just a tool separated from decisions in society. Making PTs familiar with such decision- making through modelling tasks has the potential to develop responsible citizenship, according to [3].

In her reflection notes, Georgia wrote about the "Oil Spill" modelling activity:

> *I think it's interesting and critical. (. . .) It is again a modelling task with Critical Mathematics Education (CME) concerns, and it is much more realistic than the spilled*

ink task, even if they share the same (mathematical) ideas. One option could be to show both tasks and ask about (PTs') opinion and compare the tasks since they are similar and have similar (mathematical) ideas and aim (workshop 4, p. 4).

She referred to another mathematical task she had used earlier in her teaching, where students/PTs were expected to estimate the area of ink spilt into a paper surface. Similar to comparing the "Barbie in real life" with the "Bungee Jump Barbie", the comparisons made between the "spilt ink" task and the "Oil Spill" activity allow the MTE to exemplify reflections on modelling activities that extend mathematical skills to knowledge around socio-critical issues, in this case, environmental phenomena. Although the "Oil Spill" modelling activity occurs in the Gulf of Mexico, it is transferable to other places worldwide, as well as in the context of Norway. In Norway, where the PTs live, a potential oil spill could have catastrophic impacts on the wildlife in areas such as the Lofoten islands and the Arctic Ocean. To prevent this from happening, the Norwegian government has temporarily suspended all oil exploration of Lofoten. As a nation profiting from oil and gas exploitation and having a long coastal line at risk of being affected by oil spills, it is imperative to consider risks and impact assessments when deciding whether to drill or not. However, estimating and calculating the potential damage to wildlife can be challenging and economic considerations are often prioritised. Therefore, introducing questions with a mathematical and with socio–environmental focus could exemplify to PTs that a modelling activity could include considering more quantifiable dimensions (e.g., income from oil and gas) and less quantifiable dimensions (e.g., value of wildlife and pristine nature). This example could provide opportunities for PTs to support their students to reflect on what matters in a model, what kind of assumptions should be included, and how to deal with assumptions not included.

In Norway, the industry is part of funding their welfare state and is part of the Norwegian culture [68]. Integrating cultural knowledge into mathematical problem solving is in line with the ideas from [49]. The students' cultural knowledge about oil is not a problem selected for a particular group of students but relates to everyone living in Norway. It concerns those working in the industry (or who have parents working) and those benefitting from that industry (e.g., through free health services and school). However, although the petroleum industry has various positive impacts on Norwegian society, it is crucial to investigate problematic issues of the industry critically. Living in an oil-rich nation should bring about responsibility, as described by [2,69], and awareness of the potential consequences of accidents or the long-term impacts of climate change. Students could develop their competencies to understand and act on societal systems as well as their critical consciousness [50]. There are ongoing public and political debates in Norway, such as stopping to search for more oil and gas and preventing explorations in certain areas (such as Lofoten Islands and the Arctic).

5.2. Modelling to Empower and Take Actions

Georgia included theoretical dimensions of modelling as described by [16,38]. She used several visualisations of this process when introducing the modelling cycle. In one of the workshops, she added the word "Action!!" in the cycle described by [70] (see Figure 5).

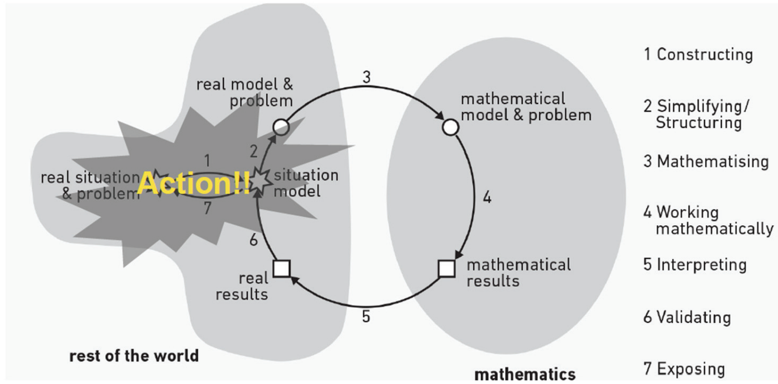

Figure 5. The slide in the second workshop showing the word "Action!!" added in the modelling cycle by [70].

While showing the modelling cycle in Figure 5, she said:

> *But (. . .) after working with a critical perspective, what I'm always interested in as a problem solver, and a modeller, is doing something in the end that can change something in a situation that is really problematic. I have worked with all these numbers and problems that concern society, the community, the school, and my home. I have a responsibility as an individual and with others that I have worked with also. To take action, to do something with the power of math, the numbers, the power of the math that math has given me. Because that is another aspect of math, it empowers the problem-solver* (workshop 2, page 12).

She started by emphasising a critical mathematics perspective connected to her own role as a problem solver and modeller in changing a societal problem. She continued by connecting her responsibility as an individual with taking action with the help of mathematics, ideas which have been introduced by [19,50]. By forwarding these ideas to PTs, Georgia exemplified taking a form of action. Further, she took an opportunity to exemplify that mathematics is not neutral but has the potential to empower, in line with what [2,5,43] described. When Georgia added and highlighted the word "Action!!" into the modelling cycle, as shown in Figure 5, she extended the modelling cycle to include socio-critical perspectives. Similar extensions have been described by researchers such as [9,43,48], who have all presented various versions of the modelling cycle to combine social-critical aspects with the traditional cycle. Based on these studies, as well as on [11], who emphasised that socio-critical perspectives should be integrated in all parts of the modelling process, Georgia could have used this opportunity to be more explicit in exemplifying that the need for "Action!!" concerns all parts of the cycle, rather than just the end.

Later, the MTE further elaborated on how this responsibility and action could manifest within aspects of one's everyday life:

> *For example, if I work with a problem that is about plastic, as in plastic packaging, trash in any way, or cigarette butts, what will I do in the end? Will I actually tell my parents and my family when I see someone throwing plastic packaging on the ground or cigarette butts? How will I convince them that this is not rational behaviour, not responsible, because I have done the math? (. . .) That is why I am adding this action in this part of the modelling cycle. It is important that you take action. When you come to work with a problem, it is a social problem, and I have some responsibility. I have "ansvar" ("responsibility", in Norwegian) to tell others and make my voice heard* (workshop 2, pp. 12–13).

With indirect questions to PTs, she brought examples from everyday life, plastic packaging, trash, and cigarette butts, and connected them to the importance of taking action and assuming responsibility. By doing so, the MTE could exemplify that action is not a single, limited, and finite doing but that working on a modelling activity with socio-critical concerns comes with a commitment to action. For example, it involves a kind of societal commitment to being a responsible citizen and spreading awareness about the problem. Dealing with societal issues such as trash can include a focus on systemic and global challenges, e.g., as described by [36] in the garbage patch activity or by [47] in the trash and recyclable collection in Cordoba city. It can address systemic injustice, as described by [9] or [33], since those dealing with consequences of littering may not be the ones causing it. However, it can also have an individual focus. Reflecting on our behaviour can bring awareness of what we can do as individuals, which is important to consider for taking action. For example, when referring to dealing with other people littering, the MTE connected the commitment to action with a personal dilemma one might face in everyday life. That is because, on the one hand, it can be uncomfortable to steer others' behaviour and be perceived as moral policing, but, on the other hand, having "done the math" leaves one having to make decisions based on what is mathematically right and ethically appropriate. Therefore, according to Georgia, it could still be challenging to "convince" people to behave rationally.

Encouraging the ideas of responsibility is a critical part of mathematics teachers' and teacher educators' roles, according to several studies, such as those of [45,46]. Yet we can start questioning how much of that responsibility is shared and should be "transferred" to students. Students are often innocent bystanders, and it is unfair to burden them with the responsibility of socio-critical issues, such as climate change. However, many students are already aware of these problematic issues, and having spaces to make their voices heard and take action has implications both within and outside the classroom borders.

6. Conclusions

In this article, we discussed integrating societal issues in mathematical modelling in teacher education. Based on the case of one MTE, we identified ways by which this integration was exemplified and the potentialities for PTs' and students' learning.

For example, we found that the MTE included three modelling activities, "Littering", "Barbie in real life", and "Oil Spill", with a focus on problem posing. Focusing on posing problems and identifying mathematics can support PTs and students in recognising the mathematical aspects of real-world issues while simultaneously bringing awareness of problematic issues in society. The MTE's choice of these contexts was also an implicit exemplification because they were relevant to PTs' everyday life (hiking, oil spill) and their coursework (Barbie). As well, adding the municipality as an audience in the modelling activity of "Littering" as an imagined client [11,58] was also a practice that could contribute to PTs' understandings about the purpose and implications of working on a modelling activity that concerns society. Another practice of the MTE was bringing in various problem dimensions, such as geographical, statistical, economic, and environmental–political. Showing a multiplicity of dimensions within the same modelling activity, thus, could support PTs in understanding the complexity of such activities and the different alternatives that are available. For instance, in the economic dimension of the littering activity, Georgia compared two alternatives where critical choices were required: that of placing and not placing trash bins.

Further, we identified that Georgia compared the "Barbie in real life" and the "Oil Spill" activities to tasks not focusing on socio-critical perspectives [5,6,39,41]. Considering that PTs were at the beginning of their second year of teacher education, showing such comparisons could support them in designing socio-critical modelling activities by challenging and broadening the tasks they are more likely to have encountered as mathematics learners and teachers. Knowing that many PTs have difficulties with developing socio-critical modelling activities, starting from challenging what they can already access might seem less overwhelming for them than dealing with the uncertainty of constantly developing new activities.

Lastly, the MTE focused on taking action and assuming responsibility [19,50]. This was connected to: theoretical perspectives of teaching and learning modelling, such as when adding the word "Action!!" in the modelling cycle, and to self-reflections on being committed to taking action and being a responsible citizen empowered by mathematics [2,5,43].

Even though we identified multiple situations where the MTE combined mathematics and socio-critical perspectives, there is still room for improvement both in the individual work of MTEs and within the teacher education community. Limitations of this case study include risks of bias. Also, the study took place during COVID-19 restrictions, and although the PTs interacted in online groups, they were not physically in the same room, and their communication was influenced by the boundaries of digital tools (e.g., some PTs did not have cameras).

Implications of this study are that socio-critical issues should not be separated from mathematics education courses about modelling; rather, the focus should be given to both mathematics and societal issues in all parts of the modelling process [11], if PTs and students are to be empowered as responsible citizens. Further research could investigate more holistically the modelling process when combining mathematical aspects and societal issues. Future research could also concern the support MTEs have when including mathematical modelling and societal issues. In our case study, the MTE (Georgia) was supported by her Ph.D. supervisors, both experienced researchers in research fields such as critical mathematics education, and some of her colleagues, including the first author (Lisa). Thus, a small informal community supported the MTE in discussing and reflecting on her teaching. However, as [33] highlighted, combining socio–political issues with mathematics can be challenging and a well-established community of MTEs engaged in socio-critical issues of modelling activities is needed. Future research could explore how MTEs can be supported to include societal issues, particularly novel teacher educators or educators who have not previously included socio–political issues in their mathematics teaching.

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Institutional Review Board Statement: The study was conducted in accordance with the Declaration of Helsinki, and approved by the Institutional Review Board of Sikt Norwegian Agency for Shared Services in Education and Research (protocol code 950095 and from 1 September 2018 until 31 December 2023).

Informed Consent Statement: Informed consent was obtained from all subjects involved in the study.

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Article

Bringing Critical Mathematics Education and Actor–Network Theory to a Statistics Course in Mathematics Teacher Education: Actants for Articulating Complexity in Student Teachers' Foregrounds

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Abstract: In this paper, we discuss how critical mathematics education (CME) and actor–network theory (ANT) come together in a mathematics teacher education course that focuses on the thematic context of climate change to study statistics. Acknowledging the complexity that student teachers encounter when asked to move from a mainly instrumental treatment of statistics toward a critical foreground of data in society, we turn to explore the actant networks, as theorized by ANT, utilized by student teachers when asked to imagine teaching from a CME perspective. For this, our study is based on a series of interviews with student teachers who participated in a statistics course where pollution data graphs were discussed, inquiring about their role as future critical mathematics teachers. The transcribed interviews, analyzed through ANT, inform us as to how student teachers' foregrounds are being shaped by actants such as the curriculum, social justice, democracy, and source critique, among others. Based on the above, we recommend that teacher education should invite active discussion of the complexity created when a CME perspective is required. This move would allow for a critical approach to critical mathematics education itself that could prepare student teachers to navigate, instead of ignoring or opposing, such complexity.

Keywords: statistics education; critical mathematics education (CME); climate change; actor–network theory (ANT); teacher education

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1. Introduction

Mathematics education runs the risk of producing banal mathematical expertise, as argued by [1], and this is especially true for courses in statistics education, which tend to emphasize an instrumental use of data, disconnected from the sociopolitical and cultural contexts of their production. Moreover, this is also the effect of not considering the societal implications of certain choices when dealing with mathematics, or of referring to mathematics as something that is always neutral and objectively true. Such views and practices can be potentially dangerous in a democratic society that aims for citizens' agency in relation to societal challenges such as climate change. For this, the goal of critical mathematics education (CME) is both to empower students to become critical thinkers with respect to how mathematics is used in action, and to create awareness of such dangers [2]. It is with these thoughts in mind that we created a statistics course inspired by the principles of critical mathematics education at a teacher education program in Sweden [3]. This move aligns with the aims stated by the Swedish curriculum when it argues that school is responsible for ensuring that "each pupil on completing compulsory school can make use of critical thinking and independently formulate standpoints based on knowledge and ethical considerations" [4]. However, although critical thinking is mentioned in the Swedish curriculum, there is not much guidance as to how this should be covered. The curriculum focuses mainly on disconnected mathematical content, without considering the need for employing thematic contexts (such as climate change) in which students' critical competencies could develop alongside mathematics. As such, the choice to cultivate a critical perspective on mathematics teaching based on climate change remains unsupported, as we have noted in a prior study on mathematics teacher education [3].

Researchers who are active in the field have investigated critical mathematics and climate change in various ways and using different approaches. For example, Steffensen et al. (2021) [5] discussed classroom lessons designed by teachers to develop students' critical mathematical competencies in a climate change context and analyzed the outcomes of these lessons. The study suggests that complex issues such as climate change bring forth an awareness of the formatting powers of mathematics. The "formatting power of mathematics", a concept developed by Skovsmose (1994) [6], signifies that mathematics potentially changes the ways in which we act, think, and experience our reality. In a related area of her research, Steffensen (2020) [7] identified how students' critical mathematical competencies appear in their attempts to enact argumentation when they discuss the themes of climate change. In the study, students participated in dialogues that involved mathematical, technological, and reflective argumentation and were based on multiple perspectives, such as environmental, economic, and ethical concerns. Her conclusion was that critical competencies are important to enable students to become critical citizens. The two studies mentioned above were empirical ones; in contrast, Hauge et al. (2017) [8] took a theoretical approach when they developed a framework that brings forward three categories that support critical reflection when mathematics is employed to discuss climate change: climate change as a vehicle, climate change as critique, and climate change as content. These categories help visualize different educational perspectives on climate change, and they can be seen as one way of grasping, and even narrowing down, the complexity when climate change is introduced into critical mathematics education. Weiland (2019) [9] took a more envisioning approach as he discussed how ideas from critical mathematics education (CME) could be used to transform the type of experience that students face with statistics in the school mathematics curriculum, and he then discussed what critical statistics education could be, using key ideas from critical mathematics literature, such as "critical thinking". Critical thinking, in the case of statistics, involves the idea of using statistics to critically examine the underlying structures and hidden assumptions present in society through specific data and, furthermore, to critique and understand these hidden assumptions. He further discussed the potential challenges faced when educators strive for critical statistics in the pedagogical context, since these often bring to the foreground sociopolitical controversies around race, sexuality, and/or ethnicity voiced within political campaigns, requiring the existing rules or norms of institutional administration and policy to be confronted.

Our way of working with climate change in this study could be conceptualized as working with a "thematic context", which offers opportunities to appreciate the potential of critical thinking in mathematics and a critical reflection on the significance of mathematics in real-life situations. The focus on "thematic contexts" was a core method for Ole Skovsmose (1994) [6] when he introduced the philosophy of critical mathematics education, with the aim of raising awareness of democratic citizenship as active participation in social practices enacted in the mathematics classroom. This was utilized by Chronaki (2000) [10] to inquire as to how mathematics teachers encounter the complexity of coordinating linkages across disciplinary areas concerning constructions of both the theme or the embedded mathematics, and the author notes the challenges encountered by teachers. Since then, several studies, such as the ones mentioned above, trying to introduce critical mathematics education in the field of institutional mathematics teaching and teacher education, have identified difficulties in the form of risks or dilemmas faced by teachers and students when they attempt to implement it within their local educational institutional settings. Moreover, they seem to agree that the teaching situation becomes even more complex when a controversial thematic context such as climate change is introduced through a

CME perspective. At this point, we can conjecture that complexity increases when new issues and potential connections are introduced into a teaching course that tries to move beyond instrumental learning of statistics. Therefore, we are interested in exploring how this complexity, an inevitable component of CME, is experienced by the student teachers themselves. For this, we employ actor–network theory (ANT) which provides a method to investigate complexity as a network of actants including the student teacher (as described in more detail below).

As such, our approach in this study is twofold: on the one hand, the philosophical standpoint of CME brings the assumption that both mathematics and mathematics education are not neutral. Skovsmose (1994) [6] discussed the concept of "the formatting power of mathematics" when he asserted that mathematics plays a role in how we see and act in the world. In other words, it produces a social and physical world after its own image. This power of mathematics is double-edged. Many great achievements in science and technology have been made possible by mathematics, but mathematics is also involved in technological catastrophes such as wars and mass destruction [11]. Mathematics not only presents the world as it is, but also formats how we act—it changes the way we think and how we perceive our physical reality. The goal of critical mathematics education is to understand this formatting power of mathematics and to empower people to examine it so that they will not be controlled by it [2]. Driven by these core ideas of CME, mathematics has been conceived as a formatting power for articulating issues of climate change [12]. Mathematics can potentially influence how climate change is perceived and formatted as solvable, predictable, etc. Coles et al. (2013) [12] presented examples of how this could be illustrated in a practice setting, and it is with these thoughts in mind that the teacher educator set up the course [3].

On the other hand, ANT states that a given social situation is made up of actants and connections comprising a network [13]. This network concept allows us to search for the actants and their connections when the student teachers enter the field of CME through the thematic context of climate change. ANT is used in this study as a methodology and philosophy to shed light on the student teachers' situation. There are several other possible theoretical frameworks that could have been used for our purpose—for instance, Foucauldian discourse analysis [14,15] or discourse theory used in mathematics education [16]. Here, we are intrigued by ANT, in that it includes an open starting point when analyzing a social situation, in the sense of not taking a certain concept of "social" for granted [13]. Rather, ANT starts from the material data, including the utterances, gestures, and objects utilized, trying to avoid fixed notions and defined concepts of the "social". Using ANT's abstract framework allows us to capture things that we otherwise might not see as they tend to remain invisible. For instance, if we were to only use CME, the utterances produced through the interviews would end up in concepts predefined by CME. However, by analyzing our interview data through the ANT approach, we do not have to rely on a CME-based conceptualization but, instead, can expand through emergent and unexpected notions.

We will describe our use of CME and ANT in more detail below, but we wish to state the purpose of this paper here, adopting concepts from these theories. The purpose of this paper was to search for complexity by identifying the actants (i.e., both human and nonhuman actors) that allow student teachers to articulate their foregrounds (i.e., expectations, aspirations, and hopes for their future) in how climate change and critical mathematics education could come together for teaching statistics. This study was organized through an obligatory statistics course and was part of a four-year teacher education program. In doing so, we were able to observe the possible tentative networks around specific actants mentioned by each of the student teachers, which reveal how they experience complexity. Complexity in our case refers to all the old and new elements, parts and connections, which are being introduced to the current and imagined teaching and learning situation, and it is this complexity (i.e., parts and connections) that we examine here.

This paper starts with this introduction in Section 1 and then continues with the theoretical considerations, where both CME and ANT are discussed in relation to the study (Section 2). The methodology section follows, describing the study's setting, including the specific methods used for data collection and data analysis (Section 3). Then, the analysis and findings are discussed in five cases of student teachers (Section 4) and, finally, the conclusions of the study are noted (Section 5).

2. Theoretical Considerations: CME and ANT

As mentioned above, this section discusses how the two theories (i.e., critical mathematics education (CME) and actor–network theory (ANT)) provide concepts that could support our efforts in the context of this study to delve deeper into the complexity faced by student teachers when they are asked to move from a statistics course that fulfills the curriculum to a course that utilizes the thematic context of "climate change" to approach mathematics and statistics from a critical perspective.

2.1. CME and Foregrounds

What is often taken as the background of a person cannot be the only factor that determines what their behavior and performance could be at a given time. Skovsmose (2007) [17] interrogated the idea of a fixed horizon of opportunities and employed the concept of foreground to refer to what new social, political, and cultural contexts might provide by arguing: "However, not the opportunities as they might exist in any socially well-defined or 'objective' form, but the opportunities as perceived by a person. Nor does the background of a person exist in any 'objective' way" (p. 6). Although the background refers to what a person has already done and experienced, such as the situations in which the person has been involved, the cultural context, and the sociopolitical context, as well as their family traditions, the person can still interpret their background in diverse ways. The foreground and the background, taken together, generate practices, perceptions, and attitudes that are regular without being consciously coordinated or governed by any rule or ritual. Moreover, the person's foreground and background need not always be in harmony with one another; they can incorporate conflicts and contradictions. A person can conceptualize different sets of foregrounds that contrast and do not align with their organized background. As such, foreground and background are continuously reworked and remolded in dynamic and relational ways with diverse characteristics. Skovsmose (2007) [17] gave a concrete example of how a person's foreground can be visualized: "an airplane passing by up there high in the sky making a fine white line, signifying that there are many different places to go" (p. 8).

Skovsmose (2007) [17] argued for the importance of grasping the specificity of an action through its intentionality. The intentions of a person are not simply grounded in their background but, equally, emerge from the way(s) in which the person revisits possibilities through action. Intentions express expectations, aspirations, and hopes. Intentions make up a constitutive part of any action. Actions become not simply caused by the past but represent forms of grasping the future. When we want to try to understand how and why a person is acting, it is important to obtain an understanding of the person's foreground and background. In this study, the concept of a person's foreground allowed us to approach student teachers, not with pre-given backgrounds that shape and sometimes fix (with stereotypes) how they act as future teachers, but with attention to how student teachers can imagine their mathematics teaching in action. As such, we turned toward investigating student teachers' visions of their roles as mathematics teachers, and focusing on their foregrounds seems useful in this respect.

Recently, Skovsmose (2023) [18] went into detail about what the core of CME is, explaining that it is about concern and hope, and arguing that with concern comes hope for change. The main concern for CME is how society is influenced by what takes place in the classroom; critical mathematics education is concerned about the students, their learning in the classroom, and their roles as future citizens. Education for citizenship could mean preparing students to fit into the given social order, but it could also be education for autonomy, enabling students to become critical citizens. Through this, critical mathematics education is concerned about how mathematics teaching and learning addresses societal challenges, including questions about the environment [18]. Skovsmose (2023) [18] explained that traditional critical positions did not express concern about the natural environment, but only for humans' critical competence. Nature has been considered as an infinite resource that needed to be used for creating welfare. A deep concern about our environment has been broadly expressed since then. Nature has now been recognized as limited and fragile. Concern about our environment has become part of critical mathematics education. While the environment is one relevant focus for critical mathematics education, the focus on social justice is always present in CME. Environmental issues, which are the focus of this study, could also be thought of in terms of social justice. For instance, natural resources are not equally distributed around the world. Some groups of people benefit much more than others. Some nations use far more resources per citizen than other nations. Pollution is a significant problem, but it does not affect everybody in the same ways. Skovsmose (2023) [18] argued for a mathematics education for social justice in terms of processes that engage students in the very formulation of what social justice could mean, and not as an education informing students about justices and injustices. A principal step is to engage students and student teachers in a pedagogical process of identifying and articulating for themselves and their community what they find to be just or unjust, and to be ready to critically challenge fixed opinions. We have fully adopted this standpoint in our study and implemented it in workshops with student teachers, where we aimed not to lecture about what justice and injustice are, but to invite the student teachers to relate with specific examples where they could foreground their potential actions. We also made use of the model presented by Skovsmose and Borba (2004) [19], which conceptualizes the interplay amongst the current situation (CS), the imagined situation (IS), and the arranged situation (AS), as exemplified by the corners of the triangle in Figure 1.

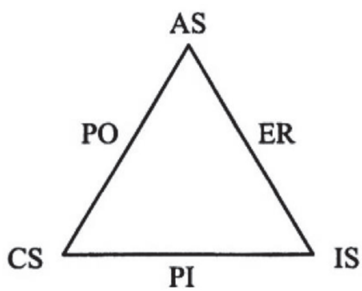


Figure 1. Model of researching critical mathematics education in action.

This study focuses on the imagined situation (IS) as the desired pedagogical setting, where student teachers can articulate their foregrounds as critical mathematics teachers through a statistics course that utilizes climate change as a thematic context. To move toward the imagined situation, the student teachers must experience a process of change from their current situation (Figure 1). For this process, developing spaces for pedagogical imagination (PI) is important to foster the relationship between the current situation and the imagined situation. In this, the relationship between the current situation and the arranged situation is established by practical organization (PO). Practical organization consists of planning activities perceived as necessary for establishing an arranged situation. Finally, explorative reasoning (ER) refers to the critical and analytical process of reconsidering the pros and cons of the imagined situation considering participants' experiences and their roles in the arranged situation. Thus, explorative reasoning is a process through which the feasibility of pedagogical imagination is discussed, along with the innovative elements that allow for its practical organization [19] (p. 216). We used this model to construct our questions in the interview guide (Appendix A): one question that deals with the arranged situation in the course, two questions that concern the imagined situation, and two open questions that allow the student teachers to refer to the current situation. More on the interview questions will follow in the Methodology section below.

2.2. ANT and Actants

CME and foregrounds do not seem sufficient to understand what enables and prevents student teachers from changing the current situation to an imagined situation that foregrounds the critical mathematics education vision. Vithal (2000) [20] argued that CME in fact introduces a serious contradiction when exploring a theory that attempts to introduce a critical democratic perspective to an educational setting without it being an imposition. However, that imposition must be made "…in the hypothetical situation precisely in order to make such ideas more widely available and to understand what they can mean in reality" [20] (p. 6). This contradiction could be one reason why student teachers are prevented from changing the current situation to an imagined situation; in order to find other issues like this one, we need to ask what enables the student teachers to act or not to act. Actor–network theory (ANT) provides a way of describing this affordance through a network [13,21]. ANT tries to understand the situation based on relationships between what are called actors or actants. The difference between actors and actants is that the term actant refers to an abstract structure, whereas the term actor is a concrete one, as described by Latour "…going from abstract structure -actants- to concrete ones -actors" [22] (p. 8). The term actant is used to suggest that agency is assigned not only to humans but also to nonhumans such as animals, physical things, and ideas. In this study, we adopted the term actant throughout the investigation.

As such, the actant represents anything that has the possibility of producing a particular effect and, thus, has agency (Smelser and Baltes, 2001) [23], or in Latour's words "An actant can literally be anything provided it is granted to be the source of an action" [23] (p. 7). Actants can also be socially constructed ideas, such as legal codes and ideologies [24]. The relationships between actants that we iterate as a collective over time represent a way of thinking about how things are also in a process of re-creation in critical mathematics education [25]. Furthermore, there are multiple connections amongst actants that constitute a network, and zooming into a particular actant would reveal yet another network [13]. Specifically, Latour argues "A network, in this second meaning of the word, is more like what you record through a Geiger counter that clicks every time a new element, invisible before, has been made visible to the inquirer" [26] (p. 799). To give a concrete example, one could consider a classroom with a projector. During a lecture, that projector would constitute an actant, since it is a source of actions in how the social situation plays out. With actor–network theory, when one tries to describe the situation, one would have to include the projector as an actant, along with other sources that allow action. However, if the projector breaks down, then there might be a need to zoom into "the projector" and reveal the network that describes the situation, (its parts, wires, etc.), but as long as the projector is working, for the purposes of describing the situation, the actant "the projector" is sufficient [27].

In this, agency can be described as the ability of an actant to mediate another actant. In ANT, there are two main concepts: mediators and intermediaries. Mediators "…transform, translate, distort, and modify the meaning of the elements they are supposed to carry" [13] (p. 39). Intermediaries, on the other hand, are what transport meaning without transformation: "…defining its inputs is enough to define its outputs" (p. 39). An example of an intermediary can be an administrative instance that handles some applications and simply transfers them to the next instance. Mediators, on the other hand, transform meaning and content. However, how do we distinguish between mediators and intermediaries? Latour mentions that to use ANT is nothing more than to "…become sensitive to the differences in the literary, scientific, moral, political, and empirical dimensions of the two types of accounts" (p. 109). This means that, in our case, when inquiring into how student teachers talk about their foregrounds as future critical teachers of statistics, we must be sensitive to when something is introduced that transforms meaning. This means that we encounter entities that, for some other investigations, could be considered actants, but not for a particular one and, therefore, can be ignored. ANT emphasizes language displacement from one frame of reference to the next that Latour (2005) [13] calls infralanguage and argues "In my experience, this is a better way for the vocabulary of the actors to be heard loud and clear" (p. 30). An example of how we use this can be found in the Methodology section.

The notions of actant and network allow us to conceptualize a picture of the complexity that a student teacher in teacher education moves with when engaging with critical mathematics education and climate change as a process and not as a fixed outcome. This allows us to revisit critical mathematics education as a process where mathematics, as a nonhuman actant, also acts and ask questions such as "What is mathematics? What is it that we do when we do mathematics? Who acts? And with whom? Is it us? Only us? Us alone?" [25] (p. 31). Such questions allow us to move beyond the immediate concern of mathematics as a human construct and explore the potential network of relations afforded across diverse actants. Latour (2011) [26] explained how both actants and networks suggest fragility, and the empty spaces between the arrays imply possibilities, as they could possibly be inhabited by another actant or connections between present actants. Especially important is what the network does to universality; any part of the network is accessible from anywhere in the network, just as there are enough "...antennas, relays, repeaters, and so on," [26] (p. 802) to sustain the network. Latour further argued that "In network, it's the work that is becoming foregrounded, and this is why some suggest using the word worknet instead" [26] (p. 802). In other words, by using the concept of a network, it is possible to localize where and through which other actants a given actant is influenced. Rather than an existing stable entity, the network can better assign a mode of inquiry that "...learns to list, at the occasion of a trial, the unexpected beings necessary for any entity to exist" [26] (p. 799). Networks make visible the configuration in which actants—human and nonhuman—are entangled and, in different ways, emerge as significant and powerful. Latour wants us to move away from accepted concepts that hinder deeper understanding, for example "...nature, society, or power, notions that before were able to expand mysteriously everywhere at no cost" [26] (p. 802). In so doing, he writes, the forces that affect people can be seen in a clearer way. These ways of working with actants and their interconnections enable us to articulate networks of specific situated relations.

With these ideas in mind, we can now formulate our research question: what might be the networks of relations that transform the mathematics student teachers' attempts to foreground classroom teaching within the milieu of CME through the thematic context of climate change? To explore this complex question, we undertook an empirical investigation in which we inquired as to the actants and their interconnections and relationships, as presented in a series of interviews with student teachers after completing a statistics course for critical mathematics education and climate change. ANT does not provide much explanation about *why* the actants and their interconnections are there, or about how they contribute towards creating a relational network that affords student teachers a critical stance for both statistics and climate change. In this, specific concepts of CME, such as foreground, imagined situation, and pedagogical imagination, could allow us to answer such questions.

The present study can be viewed as being situated in the above model (Figure 1), where the student teachers participate as co-researchers in this complex process, offering vital ideas about how their foregrounds as critical mathematics teachers of statistics could be materialized (or not) through the thematic context of climate change. Since the student teachers' ideas are heavily grounded in both human and nonhuman resources related to their living experiences, our study turns to the concept of actants as discussed in actor–network theory (ANT) as a way to inquire as to the types of knowledge, and elements that act with them, to reach their imagined situation.

3. Methodology

In this section, we outline the methodology of this study by recounting how the theories have contributed to its organization around the research question. We also reflect on the challenges when these theories are utilized with empirical data and discuss how these could be overcome. The present section comprises four subsections: the first discusses the study setting by providing details of the statistics course in teacher education around the vision of critical mathematics education through the thematic context of climate change; the second outlines methods for data collection; the third outlines the methods used for data analysis; and the fourth discusses the ethics of this study.

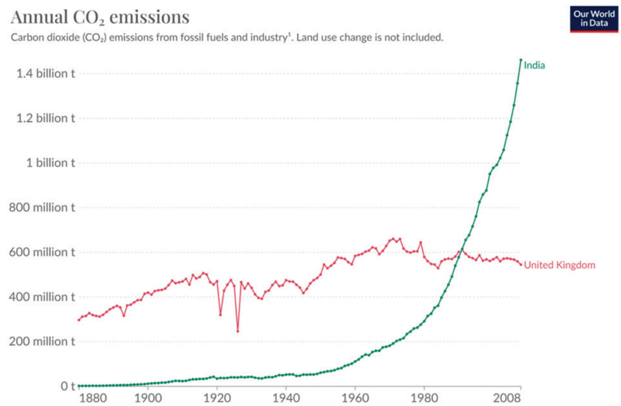
3.1. Study Setting: A Statistics Course for Critical Mathematics Teacher Education

As mentioned in the introduction, this study took place at a teacher education program at a university in Sweden, focusing on a compulsory statistics course for student teachers. It contained five cycles of two-hour lectures and two-hour workshops. The first author of this paper was the teacher educator for the course in this study, whilst the overall design and analysis were the collaborative work of all of the authors. In this sense, this paper could be seen as the extension of a prior self-study (see also: [3]) where the second and third authors, as critical collaborators, participated actively in both studies.

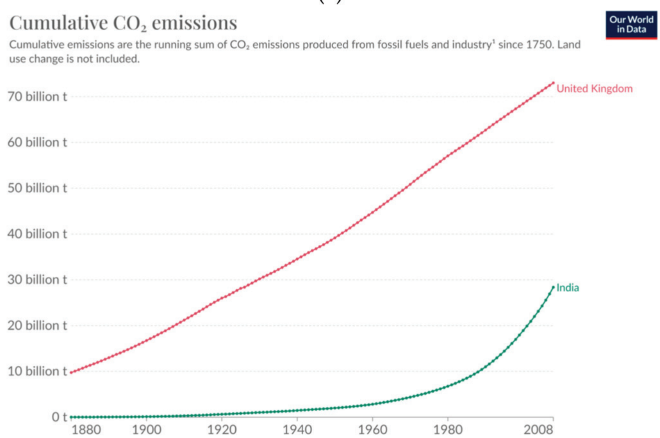
The course typically comprises 20 to 40 students studying to become teachers for pupils in the age group of 10 to 12 years old. The statistics course deals with fundamental concepts such as mean and median values, as well as methods for data visualization such as diagrams, graphs, histograms, table charts, etc. The course also examines some didactic ideas that can be used in a school setting. Being sensitized to the need to revisit mathematics teacher education in the light of current societal and environmental urgencies, the present study considers both climate change and critical mathematics education as key axes for redesigning the statistics course (removed for peer review). This is exemplified during the course by a short introduction to what CME is, including related activities provided during the workshops. In these activities, we asked the student teachers to reflect on and discuss how different types of graphs could change the perception of climate change. For this, the graphs identified already by Coles et al. (2013) [12] were employed (Figure 2). The choice of these graphs was made on the basis that, in our opinion, they clearly bring forth the formatting properties of mathematics and are suitable since they deal with climate change, our thematic context of choice. Although Coles et al. (2013) [12] proposed these graphs for use in secondary school classrooms, in this study we employed them with student teachers enrolled on a statistics course. The first two graphs below show the amounts of carbon dioxide emitted from the countries Great Britain and India from the year 1880 until 2008. The first graph shows the year-to-year emissions from both countries. The second is a cumulative graph, where the previous years' emissions are added to the coming year's emissions. As an example, reading the data point from the year 2008 shows the total amount of emissions since 1880 for that country. The third graph shows the emissions per capita from 1950 to 2008 for the two countries.

3.2. The Interview as a Method for Eliciting Student Teachers' Foregrounds

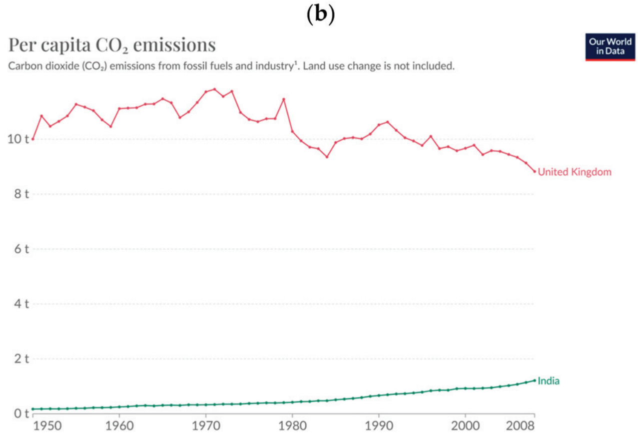
Our research question, as mentioned above, was "What might be the networks of relations that transform the mathematics student teachers' attempts to foreground classroom teaching within the milieu of CME through the thematic context of climate change?" To explore this question, we conducted an inquiry study. Having organized the statistics course around the two key axes of climate change and critical mathematics, we moved toward organizing our inquiry based on interviews with a small number of student teachers who volunteered to participate. The data for the present study were derived from a series of semi-structured interviews with five student teachers.



(**a**)



(**b**)



(**c**)

Figure 2. Graphs used for the CME statistics course (graphs created at https://ourworldindata.org/, source: Coles et al., 2013 [12], pp. 45–46). (**a**) Annual fossil fuel emissions for the UK and India (1880–2008). (**b**) Cumulative fossil fuel emissions for the UK and India (1880–2008). (**c**) Annual fossil fuel emissions per capita for UK and India (1950–2008).

An important principle from Brinkmann and Kvale (2018) [28] used in the interviews is to allow silence. By allowing for pauses in the conversation, the student teachers are given time to associate and reflect, and then they can break the silence themselves with significant information. These pauses are denoted as # for shorter pauses and ## for longer pauses in the transcripts. The interviews were approximately 30 min long, and an interview guide (Appendix A) was used, containing five main axes of questions around which the conversation could take place. The questions were as follows: Q1: What did you think of the statistics course? (A simple, open question to start the conversation.) Q2: In the lectures I mentioned the following examples: How did your thoughts go during the lecture and afterwards, around this example? (Appendix A) Q3: How could you as a teacher work with this in your teaching in a class, grades 4–6? Q4: How would a class in grades 4–6 benefit from this in their teaching? Q5: Is there anything else you want to add?

For each question, there were supplementary or follow-up questions addressing the guidelines by Brinkmann and Kvale (2018) [28] for how a semi-structured interview should be conducted. According to them, the first question should be an open question that aims at producing spontaneous, rich descriptions where the student teachers themselves describe what they experience as the main aspects of what is being investigated. The other four questions aim at being easy to understand, short, and without academic language. The questions were also evaluated with respect to both thematic and dynamic dimensions: thematically about producing knowledge, and dynamically regarding the interpersonal relationship in the interview. We aimed at formulating the questions in such a way that they would contribute both thematically to knowledge production and dynamically by promoting good interview interactions. The follow-up questions aimed at prolonging the student teachers' answers and maintaining a curious, persistent, and critical attitude during the interviews [28]. "Could you tell me more about that?" is an example of a follow-up question that is used—a kind of a probing question with the aim of pursuing the answers and probing their contents, but without stating which dimensions are to be taken into consideration. Structuring questions is another example.

The aim of this type of question is to steer the course of the interview in order to break off long answers that are not the focus to the investigation—for example, by briefly stating the understanding of an answer, and then saying "I feel that you see many benefits regarding the use of climate change as a subject, but how do you think it affects the learning of mathematics?". A concrete example of this used during the interviews is the question "How do you think it affects the learning of mathematics or the understanding of mathematics?", used when the course of the interview tended to go in an unwanted direction. Another example is "The way we're talking about these diagrams now, do you think that talking about it in this way is something that benefits students' math learning?". Further, the questions were produced using the research methodology (Figure 1) presented in the Theoretical Considerations section. Question 1 (Appendix A) is an open question that allows the student teachers to reflect on all parts of the model (Figure 1), i.e., current situation, arranged situation, and imagined situation. Question 2 revolves around the arranged situation, starting with a description and a reiteration of what was worked on during the workshops. Questions 3 and 4 focus on the imagined situation, and a final question asks whether the student teacher wishes to add anything.

Järvinen and Mik-Meyer (2020) [29] discussed eight different approaches to qualitative analysis, of which ANT was one. They argued that the main objection to using qualitative interviews together with ANT is that the interview contradicts the fundamental premise of ANT, i.e., that human actors should not be privileged over nonhuman actors. Interviews may vary in how much focus they place on the key human actors, i.e., the interviewee and interviewer. However, over the last decade or so, several studies have applied ANT concepts to interview materials, with a focus on mapping the interactions between human and nonhuman actors [30–35]. Specifically, Järvinen and Mik-Meyer (2020) [29] elaborated on challenges in ANT-inspired analyses of interview materials. Firstly, it is important that the interview opens a space where the interviewees are given the opportunity to articulate those elements in the network that are important to them. The interview therefore needs to adopt an open and exploratory approach. This was achieved in our study by making sure that the questions and the follow-up questions did not close this space but promoted further elaborations on the same topic. One of sociology's core tasks is to show how social structures operate in relation to human actors, and since ANT has a different starting point it leads us to another challenge in ANT analyses, concerning how we view the data material and what the informants tell us. When we use the principle of "following the actor", we cannot presume that social structures exist and exert influence on the actors. This must remain an open, empirical question. We must take the informants' statements at face value instead of explaining their statements with reference to factors outside of the data. We should not assume that social categories such as "ethnic minority", "working class", or "gender" are necessarily suitable for understanding a given situation. An ANT analysis requires that the relevance of these and similar categories is shown through the data—that these categories "act" in the data—before they can be assigned relevance in the analysis.

All interviews were performed in the Swedish language, recorded, transcribed in Swedish, and then translated from Swedish to English. The analysis involved carefully reading all interviews as data produced by student teachers as they responded to the interview questions. The data were also analyzed with respect to our research question, which means that we focused our analysis on whether or how the student teachers related to their imagined future teaching and classroom, i.e., their "foreground". This occurred mostly in their responses to questions 3 and 4 from our interview guide (Appendix A). These questions are directly related to the concept of foreground and the imagined situation. However, there were instances based on questions 1 and 2 where this also occurred. This dataset is what we analyze in Section 3.3.

3.3. Analyzing the Interviews: Inquiring for Actants in Student Teachers' Foregrounds

The analysis here was also inspired by the study of Boistrup and Valero (in press) [36], especially with regards to how the data were handled. They embraced some of the notions and analytical strategies of Bruno Latour to think about the narratives of mathematics education as a field of research. This allowed them to conceptualize a network in which mathematics education forms a part. Working with Latourian tools, they performed a limited empirical investigation of how mathematics education research texts from 2004 to 2020 establish relationships to PISA, as well as which controversies are noticeable in the research. A noticeable difference is that while they analyzed documents, we analyzed interviews. However, the methodology of how actants can be located is similar, using one actant as the entry point, finding connections to other actants, and then organizing these actants in a spreadsheet. In addition, the frequencies of actants were located in the interviews for each of the student teachers (Appendix C).

Finally, the digital tool Visio was used to create the tentative actant networks (as seen in Section 4), allowing us to move the actants around without losing the connections. The networks were created by first drawing the actant student teacher, since that was our entry point into the investigation, followed by the generalized actants and the connections. We continued this process until we had covered the identified generalized actants and connections from the student teachers' utterances. The layout of the different actants was made on a practical basis to allow all of the connections to be seen as clearly as possible. This layout can also be achieved in other ways; what is constant is the connections between the actants.

We now apply our concept of actants and connections from ANT, as described in the theoretical section. In the dataset, key actants were identified and systematically organized in a spreadsheet using the categories of actants, generalized actants, description of generalized actants, and connections. This spreadsheet is not included in the article, due to its large size. First, by carefully reading through the student teachers' statements, we noticed when things were introduced in their utterances that transformed meaning. The word or words used to describe that "something" were then transferred to the spreadsheet in the actant column. As an example, one of the student teachers talked about CME and then related it to the idea of source criticism; we noticed from their statement that the student teacher then started to talk about imagining CME exercises as going through newspapers to search for statistical content and critically examine them. The words "source criticism" were then transferred to the actant column.

Second, the generalized actant denotes when similar terms are used in the data for the same thing. For instance, if the student teachers talked about "being fooled" or "being tricked", we thematically grouped this under the generalized actant "manipulation". In this way, we used ANT's notion of an infralanguage [13], making sure that as little transformation as possible of the intended meaning had taken place, to allow us to see the actant more clearly. Third, the description of generalized actants gives a more in-depth description of the generalized actant than the name provides. Fourth, "connections" specify what inter-actant relations the student teacher creates through their utterances. For instance, in the example above, where the student teacher connected CME with source criticism, the connection is denoted in the column "connections" as "CME-source criticism". Overall, our inquiry envisions student teachers as potential actants (i.e., future critical math teachers) through a network of other actants and interconnections. This means that when trying to map the network, i.e., the actants and their connections, we imagined each of the student teachers as entry points. In Appendix B, we list the generalized actants for all five student teachers.

These generalized actants represent the actants that were mentioned in the interviews in relation to how the student teachers foreground themselves as critical statistics teachers. The generalized actants are illustrated by quotes from the interviews, and descriptions of the generalized actants are provided. An example from the list in Appendix B is the generalized actant "Source criticism". In the description, we can see that this is about the "The idea that facts need to be checked for accuracy". The example quote is from the interview with Nadir when she said: "There is so much information online, but it must be from sources that are close to our time and so that you have discussions like this about source criticism". Utilizing the generalized actants and connections from our spreadsheet, we can now create a network for each of the student teachers based on their utterances. This allows us to easily see which generalized actants each student teacher connects to, as well as which generalized actants they make connections between.

Our identified generalized actants allowed us to discern the networks described by the student teachers when entering the field of CME in the thematic context of climate change, in relation to their foreground. We argue that these generalized actants influence the student teachers' foregrounds, i.e., their hopes and aspirations for their future teaching in statistics using CME and climate change. The generalized actants provide constraints and affordances (Appendix B): constraints in the sense that the actant suggests certain ways of doing things and not others, and affordances as the actants provide sources of actions.

3.4. Ethical Considerations

The interviews were conducted by the first author of this article and took place after the statistics course was already completed and assessed. This was to reduce the risk of any ethical implications or bias from the fact that the interviewer was also the teacher of the course. Otherwise, there could be a risk of student teachers thinking that their participation would somehow influence their grading on the course. Although all the student teachers who participated in the course were given the chance to participate in the follow-up interview, five of them agreed, and they were all female. This might be partially because females are over-represented in this statistics course (i.e., at time of this study there were 19 females out of the 26 students enrolled), but this certainly needs further analysis considering the increased feminization of the teaching profession. The fact that the teacher was also the interviewer might have affected who and how many participated in the interview. It is hard to say exactly how and in what way(s) this influenced participation. As we see it, it could go both ways; it could mean fewer participants, but it could also mean more participants than would otherwise have been the case. For instance, having some general ideas about who the interviewer is and what they stand for could remove some uncertainty that might cause some student teachers not to participate. On the other hand, a bad experience with the course and the teacher in general might cause reluctance to participate. The teacher being the interviewer could also potentially influence the ways in which student teachers answer the questions, consciously or unconsciously giving the answers that they think they know that the teachers want to hear. To deal with this, it was important to emphasize that the course was completed, and that the interviewer no longer had the role of the teacher as evaluator of the learning process. In this case, it could also go either way, because now the student teachers had the chance to say exactly what they thought about critical mathematics, without the risk of being judged or graded for it. Based on the interviews, there were some cases where this happened, which we interpreted as a sign of the participants feeling free to express their thoughts.

All the interview participants were ensured that they would remain anonymous and that none of their personal information would be accessed. As such, in our interview setting, we stayed away from personal questions that could make any direct reference to their background or personal history. This was for ethical reasons, making sure that it would not be possible to use the information provided by the informants to figure out their identity. All the participants have been given pseudonyms in this text (i.e., Nicole, Sophie, Nadir, Iman, and Estelle). Conversation was initiated with the local ethical authorities at the university where we are active, and we received a letter of confirmation stating that no formal vetting application to the "Swedish ethical review authority" was needed due to the nature of the study.

4. Actant Networks for Student Teachers' Foregrounds

The actants and actant networks created by the student teachers as described below should be seen as a tentative answer to our inquiry addressing the following question: What might be the networks of actants and their connections that articulate the mathematics student teachers' attempts to foreground classroom teaching within the milieu of CME through the thematic context of climate change? Along each of the visualized actant networks, we give a short summary of the interview, followed by some example quotes. We investigated these quotes using concepts from CME and ANT and situated how the student teachers brought the actants into relation with the core elements of bringing CME into action (see the model in Figure 1, Section 2). Many of their utterances, produced through the interviews, can be situated in how the student teachers' imagined situation is foregrounded as they try to express and articulate specific ways of getting there that involve them in explorative reasoning (ER) and pedagogical imagination (PI). This is partly due to the nature of the questions asked in the interviews, which place an emphasis on their future classroom and teaching. The imagined situation, we argue, is a part of the student teacher's foreground, as it grounds their hopes and aspirations in relation to specific materializations concerning the teaching and learning situation. The imagined situation is often set in contrast, in the student teachers' utterances, to the current situation (CS), where mathematics is of a more instrumental type. With this in mind, we present below the cases of all five student teachers and discuss the actant networks that interpret each of their foregrounds as critical mathematics teachers employing the thematic context of climate change in their teaching of statistics. For clarity, for the first student teacher, Nicole, we have put the words stemming from the theoretical framework in italics.

4.1. The Case of Nicole: Mathematics as a Social Activity and Pupils as Not Manipulated

As we can see in Figure 3, the actants comprising Nicole's network are "manipulation", "climate change", "real world", "math as social activity", and "fear and hope".

All of the actants in Nicole's actant network (Figure 3) are directly connected to CME, but not to other actants. Nicole talked very much about CME as a social activity, and she referred to her own experiences of acting as playing and making something physical during the class. She recollected "*I think you learn more when you sort of get to do something physically as well. For me, if I think back to my own schooling—I remember those things where we kind of had plays. But I don't remember all the written tests I had*", and she added "*Then I also think it will be a lot of fun, so that's one reason they'll remember it more*". Nicole connected CME (*the arranged situation*) with the *actant* "math as social activity", resorting to her past experiences for conceptualizing what CME might be. Then, by using *pedagogical imagination*, she tried to envision what the benefits would be by using CME and "math as social activity" together in that *imagined situation*. She also connected the *actant* "math as social activity" with the actant "fear and hope", in that she hoped, based on her past experiences, that it could enhance the ability of students to remember what was said during the teaching and learning situation.

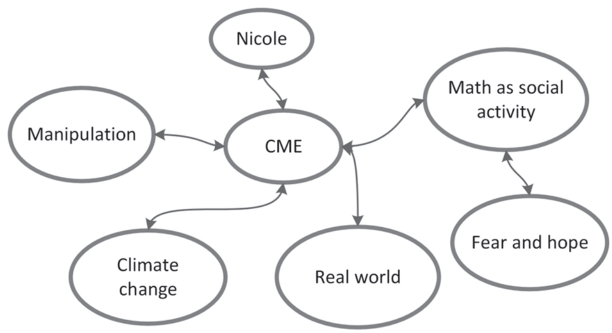


Figure 3. Nicole's actant network.

The *connection* she made between CME and "math as social activity", we argue, allowed her to make the next step in her *foregrounding* of the *imagined situation*. As she also said, "*You can manipulate quite a lot with diagrams depending on what you choose to focus on and what purpose you have with the various diagrams, sort of*". Nicole connected CME with the actant "manipulation". Nicole wanted to make sure that the pupils are not tricked or manipulated when it comes to the content of statistics. We can see that Nicole's *foreground* and role as a future teacher in relation to CME and climate change is to *facilitate social activity and make sure the pupils are not manipulated*.

4.2. The Case of Sophie: The Ethics and Source Criticism

As we can see in Figure 4, the actants that comprised Sophie's network around CME are "pupils' age", "source criticism", "math as social activity", "climate change", "politics", and "social justice".

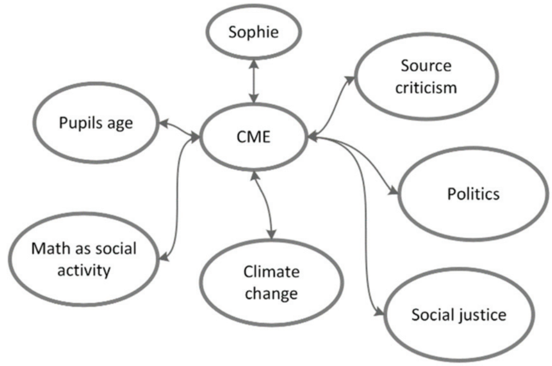


Figure 4. Sophie's actant network.

All of the actants in Sophie's actant network (Figure 4) are connected directly to CME, but not amongst themselves. Sophie talked about CME and climate change as she reasoned about how children are the future and, therefore, climate change should be especially important to them. Sophie embraced the idea of classroom argumentation and reasoning in the classroom to discuss the offered diagrams concerning India and Great Britain, so to highlight their underlying meanings she said "*Two different diagrams, the same countries but different diagrams, it can be a bit of source criticism too. What is actually right?*" Sophie connected CME in the arranged situation with the actant "source criticism" in her way of conceptualizing CME. In the interview, she said:

> *"…and it's good if you had seen this in a newspaper that you would have been able to understand it, because we are discussing if two students had discussed then they would have been able to learn about diagrams together and then be able to understand it themselves, if they had seen it in a newspaper or elsewhere".*

By using pedagogical imagination, she gave a concrete example of when this would be useful for understanding diagrams when reading newspapers. In response to the question "Do you have any more situations like this where it might be useful to have that understanding?", she responded "*So also on social media, if there is, so they take up statistics on how, who has the most followers. I don't know, even the weather*". Unfolding her pedagogical imagination further, she gave another concrete example that could be used in her imagined situation: students discussing statistics on social media or concerning weather. We argue that the connection she made between CME and the actant "source criticism" is what enabled her, in her pedagogical imagination, to come up with the idea of investigating statistics on social media or weather data. All in all, Sophie's foreground and role as a future teacher in relation to CME and climate change is to *go into ethical discussion in the classroom and see CME as an instance of source criticism*.

4.3. The Case of Nadir: A Political Stance for Democracy through Discussion

In Figure 5 we can see that the actants encountered by Nadir are "democracy", "politics", "interdisciplinary", "real world", "politics", "CME", "instrumental math", "social justice", "climate change", "math as social activity", "UN", "pupils' age'", "fear and hope", and "source criticism".

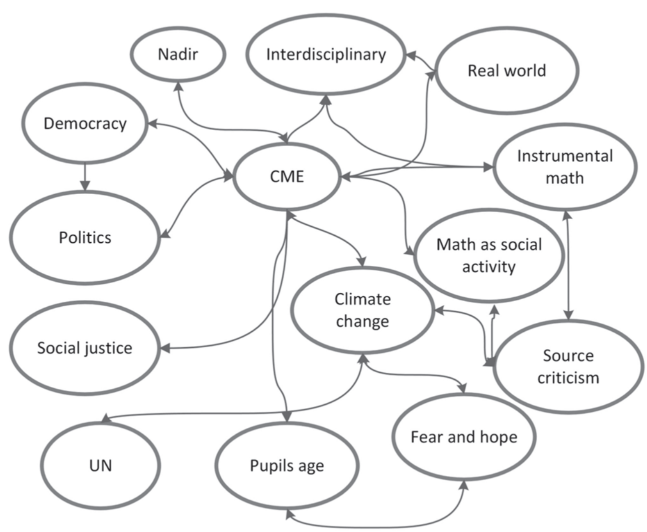


Figure 5. Nadir's actant network.

In the actant network in Figure 5, one can see many more actants as compared to the previous two cases of Nicole and Sophie, with many more complex connections across and between them. Nadir went into lengthy and complex thoughts in relation to CME and climate change. She took a more political perspective and connected democracy and social justice to CME, even relating these ideas to how the United Nations addresses climate change. Nadir related to our current reality of climate crisis and emphasized the importance of facts and source criticism. She compared how democratic countries respond to issues of climate change and expressed a desire to see serious discussions around climate change in countries that are assumed to be nondemocratic. For instance, she said:

> *"Politicians have a lot, really decide like this, this international, or this national, within our country, what shall apply, what is done. That's not how it looks in all countries, really. Especially if it's not a democracy."*

Nadir connected CME (the arranged situation) with the actant "democracy". She foregrounded herself as investigating what CME is in relation to democracy. Further on in the interview, by using pedagogical imagination, she came up with the idea of using emissions data (for example, from China) to discuss whether emission levels are related to whether the country is a democracy. The connection Nadir made between the actants "CME" and "democracy", we argue, enabled her to construct a pedagogical imagination to reach a concrete example to examine. Nadir further connected CME with interdisciplinary study, and she argued that since subject areas are not separated in the real world, the school curriculum should not separate them either. She said:

> *"They [pupils] must be able to make logical arguments like this and be able to respond to arguments, there are certain, so like this, knowledge requirements, it's not just that they should know exactly what a table is, but it will be, so it's [real-world] not as separated as it [school] is."*

In relation to this connection to the real world, Nadir was also concerned about the pupils' feelings in relation to climate change, as exemplified in this statement:

> *"It should not be too negative so that, that the students get too negative a view, there is hope there is like, present it that way, but at the same time they need to be aware of this stuff."*

CME and "real world" were connected actants for Nadir, who also connected the actant "fear and hope" with the real world, in the sense that she was concerned that CME using climate change as a thematic context might inflict negative feelings. We argue that the actant "real world" both improves and restricts her ability to foreground herself as a critical mathematics teacher. The actant "real world" presents her with a dilemma. Here, we can see teacher education playing a vital role in resolving and clarifying these types of dilemmas. Nadir's foreground/role as a future teacher in relation to CME and climate change is mostly a *political one, emphasizing the democratic importance of discussions*. She also brought up the UN as a part of setting up her future teaching, as the UN provides guidelines in relation to education and climate change. Nadir was also concerned with the emotional impact of introducing climate change and how best to deal with this.

4.4. The Case of Iman: Grounded in the Curriculum for Pupils

In Figure 6 we can see that the actants involved in Iman's network are "instrumental statistics", "manipulation", "interdisciplinary", "math as social activity", "CME", "teacher traditions", "curriculum", "climate change", "social justice", "fear and hope", "real world", and "democracy".

Again, we can see a complex actant network (Figure 6) with many interconnections established. Iman also expressed complex thoughts about CME through the thematic context of climate change, much like Nadir did, but Iman tried to turn to the curriculum to resolve her issues in relation to CME. Iman connected social justice with CME and connected climate change with the curriculum. She also reasoned around how the curriculum could be interpreted in many ways, and she suggested choosing based on one's own knowledge. For instance, she said:

> *"The curriculum can be twisted and turned so that you align with it, if you take up the broader issues, I think. So, you can always find something that matches more or less well. I think that you yourself have to be committed or knowledgeable about the field, otherwise it will just be a mess if you try to lecture about something you don't know."*

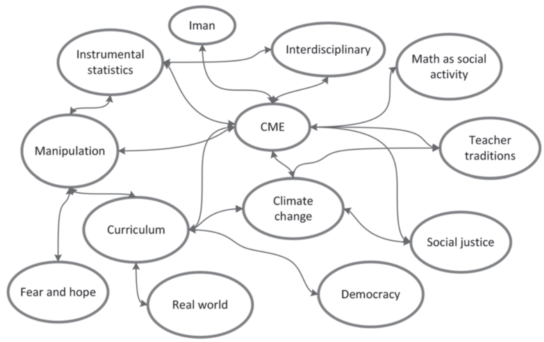


Figure 6. Iman's actant network.

Iman connected the actants "CME" and "curriculum" in her efforts to foreground her imagined situation as a critical mathematics teacher. Her statement can be seen as a part of the process of explorative reasoning (Figure 1) towards the imagined situation. We argue that the actant "curriculum" both affords her opportunities and restricts her in her quest to reach the imagined situation. While the curriculum states that critical thinking should be practiced, it does not say how and to what extent it should be covered; this could potentially be a source of dilemma for the teacher, as mentioned earlier. In her exploratory reasoning, she also found a way of overcoming this obstacle, arguing that parts of the curriculum could be seen as covered in a broader sense. Whether this argument should be used to introduce critical thinking is an open question. On the one hand, she based her choice of using CME and climate change on the curriculum, but she preferred to look at it from the pupil's perspective. She argued that she and her future pupils should not be easily manipulated. She also said:

> *"And when they usually ask why we should learn this, well it's so you won't be fooled by someone else or something else. It's something they can understand, they don't want to feel stupid, nobody wants to feel stupid."*

Iman connected the actants "CME" and "manipulation" when she tried to foreground her imagined situation, and she also connected the actants "manipulation" and "fear and hope". The connection that she made between "CME" and "manipulation" allowed her to use her pedagogical imagination to find the argument that she would use with the pupils to persuade them to participate, at the same time presenting hope that they would not feel stupid, but also fear that not participating could result in them being perceived as stupid. Iman also reasoned about teaching traditions in relation to CME, considering whether to push issues or to allow everything in the discussions—for instance, allowing climate change denial. Her foreground and role as a future teacher in relation to CME and climate change is *grounded in the curriculum and concerned with pupils not being manipulated*, and she sees opportunities to talk both about climate change and social justice.

4.5. The Case of Estelle: Caring for Young Pupils' Thinking and Feeling

In Figure 7 we can see that the actants in Estelle's network are "pupils' age", "CME", "math as social activity", "climate change", "real world", "instrumental statistics", "fear and hope", and "manipulation".

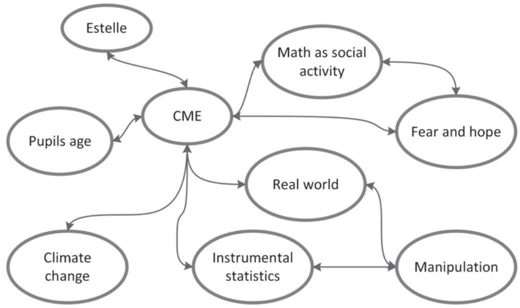


Figure 7. Estelle's actant network.

At first glance of the actants in the network (Figure 7), it looks as though they are mostly connected to the central actant "CME", as in the cases of Nicole and Sophie, with only two subnetworks created: one between "math as social activity" and "fear and hope", and the other amongst "instrumental statistics", "real world", and "manipulation". Estelle explicitly stated that she actively considers her role as a future teacher in statistics, along with how she could incorporate CME and climate change. She said:

"I think that at that age there are so many other things that are more important, and math has generally been a subject that many people find difficult. That oh, and then it becomes extra important that, well, they should be able to relate to it and see what is around them in order to show this with carbon dioxide, you can certainly find ways to show the emissions but perhaps on a smaller level, where, for example, ok, what does it look like in our area, or how does it look maybe in Sweden or in Malmö, or in different areas in Malmö, but that it's like, ok if you live in a certain area, maybe you can compare ok how do the emissions look there, compared to there. So that it's something they know that they can relate to."

Estelle connected the actants "CME" and "real world". The connection between CME and the real world enabled her to use her pedagogical imagination to come up with a concrete example of what this would look like, i.e., examining emissions locally to connect to pupils' own real world. She also connected the actants "CME" and "pupils' age". This, we argue, imposes some constraints on her imagined situation in her pedagogical imagination process, in that she needs to find a way of working that suits the particular age group of the pupils.

Estelle continued by saying:

"So I think it is, that everything can be learned but it has to happen early. It has to happen already at the first level, to bring in that thinking to see this abstract, to kind of see this critical and then it builds up, and you train."

In her quest to achieve this imaginary situation, she addressed this potential age constraint by arguing that critical thinking must start at an early age, in order for pupils to learn what it is. Once again, we can see teacher education playing a vital role in addressing and clarifying such issues—in general terms, supporting affordance and removing constraints, if possible. She thought about how the diagrams look visually to the pupils, and she stated the following when comparing diagram 1 and diagram 2 (Figure 2):

"The other feels a bit like -where are we. This is going to be ok. I think it will be clearer. It affects a lot. You want to capture interest, not lose interest in it."

Furthermore, Estelle connected the actants "CME" and "fear and hope". The actant "fear and hope" imposes some constraints on Estelle's quest towards the imagined situation and constrains her in her pedagogical imagination because she is afraid that the pupils will lose interest if the diagrams are not clear enough. Her way of foregrounding her role as future teacher in relation to CME and climate change shows more of a *caring role*. She thinks about the feelings that CME and climate change might stimulate, as well as which age group this teaching would work for and how it could be visually displayed so that it becomes easy for young pupils to understand.

Summarizing the discussion of the above cases of student teachers who participated in these interviews, we noted three points as we cross-examined them: (a) some actants are more frequently used than others in student teachers' attempts to foreground themselves as critical mathematics educators, (b) the quality of the actant networks created varies across the five cases, and (c) student teachers' sense of the complexity differs. First, as we can see in Table A2 in Appendix C, the two actants "CME" and "climate change" were the ones used most frequently throughout all five interviews. This has to do with the nature of the study, as CME and climate change were introduced in the statistics course and became a core part of the discussions. The student teachers referred to them and elaborated on them further, as we can see from their transcribed utterances. These elaborations can be seen in the actants created by the student teachers (i.e., "real world", "math as social activity", "instrumental statistics", or "manipulation") that show a relatively high frequency and can be perceived as crucial actants for the participants to foreground themselves as critical statistics teachers.

Second, the quality of the actant networks created by student teachers is also a significant outcome of this study. In particular, Nicole and Sophie created very simple networks without intermediate connections, while Nadir and Iman constructed much more complex networks with subnetwork relations, and Estelle's could be categorized as being of moderate complexity. An overview of the above case studies shows how the actant networks presented above illustrate how differently the student teachers conceptualize CME in the thematic context of climate change and how they foreground themselves as critical mathematics teachers. Specifically, all five were interested in CME in the thematic context of climate change, but two of them (Nadir and Iman) talked intensely and made intricate elaborations, while the rest were more reluctant to speak. The differences observed, of course, could be partially based on the personal histories of these five student teachers who volunteered to participate in the interviews. Despite their volunteer status, some were more reluctant to speak about what they were not sure about, while others gladly shared their thoughts. We could also add that there is more to explore with these interviewees than the actant networks presented here can show. Specifically, it was possible to see that giving a broader picture of each of the interviewees could have opened more dimensions in this discussion, such as the roles that could be played by other identity markers such as gender, race, or ethnicity. However, we also needed to follow through with our guarantee of total anonymity. This anonymity could also mean that the participants showed more willingness to speak more openly about their experiences on the course.

Third, the above allows us to more deeply grasp the complexity embedded in attempts to bring a CME perspective into school classrooms for discussing statistics through thematic contexts that deal with societal challenges such as climate change. We argue that the complexity increases in the sense that new actants and connections need to be introduced to the practicalities of designing teaching and learning in a CME context. The student teachers in this study indicated that this dimension cannot be ignored, but awareness is required. Iman's statement is indicative: "It is good to use statistics in different ways in teaching, you just have to think about not using everything at the same time, I think." There were also several occasions when the student teachers paused and made comments such as "I don't know what I am trying to say" and similar formulations.

Overall, all of the student teachers talked in positive terms with regard to using CME in the thematic context of climate change. The actant networks discussed here show similarities in that all of the participants brought in CME and climate change through their utterances as very positive things to incorporate. The noted differences across the actant networks produced concerns with respect to the number of actants that the student teachers connect with their foregrounds as critical statistics teachers. This could be due to either a reluctance to speak of what is not clear or the fact that they simply did not see any other actants, i.e., ways forward. Since this study constitutes an introduction to CME in the thematic context of climate change, we would like to add that more actants would probably come into play as the student teachers progress in their understanding of and familiarity with CME. However, the student teachers' own hesitation is noted, either through articulating their opinions regarding the actants through the interviews and making specific actant connections, or through body language, silences, and pauses as the interview was carried out. Certainly, these issues need further research in the context of future studies.

5. Concluding Remarks

We agree with other colleagues in this Special Issue that critical mathematics education is an important opening for mathematics teaching and teacher education, for reasons already mentioned in the introductory section of this paper, and that its role is crucial for preparing students for a future as democratic citizens [37–40]. For this to happen in the context of teaching practice, teacher educators and student teachers must envisage themselves as future critical mathematics educators who are able to bring an active discussion of the formatting power of mathematics in society into the school classroom. In particular, the use of societal challenges such as climate change can be a thematic context for student teachers to enact critical pedagogical imagination, awareness, and sensitivity with respect to the embedded complexity. Moreover, critical mathematics education provides ways of enacting mathematical thinking grounded in cultural complexity [41] and becomes exemplified in cases where thematic contexts and contents reflect the desired culture of the curriculum, which risks ignoring the cultural diversity of the teaching and learning situation. The importance of CME when the thematic context of climate change is emphasized in teacher education was embraced by the student teachers in different ways. All student teachers foregrounded themselves as aiming to become critical mathematics teachers, but they expressed varied degrees of interest, doubt and hesitancy, and, sometimes, refusal. This became evident in the ways in which they resorted to different actants for articulating their foregrounds as critical mathematics teachers.

Bringing a CME perspective to how a statistics course embraces data related to questions of climate change inevitably increases the complexity of the teaching and learning process. We found that this increased complexity is perceived when new actants and connections are introduced by the student teachers themselves to justify their future choices. The actants introduced will, in turn, produce new actants that are grounded in the ways in which each of the student teachers conceptualizes CME in the thematic context of climate change, as we have seen from analyzing their interviews. At the same time, these new actants provide both constraints and affordances—for instance, when the student teachers started to talk about "source criticism" and then suggested ways of working with CME, such as looking for graphs in newspaper articles. Source criticism is a research field in itself and brings with it certain ways of doing things with data, truth, reliability, fake news [42], etc. For instance, investigating a homepage's network addresses the credibility of information [43]. As an example, looking at the homepage's top-level domain (i.e., .com, .gov, .edu) can give some indication of the credibility of the homepage [43].

We can see many similarities between our study and, for example, the studies by Steffensen [5,7] examining how participants articulated CME in the thematic context of climate change, but while she examined their attempts to articulate in relation to critical competencies, the ANT approach allows actants to reveal themselves through the student teachers' utterances. There are pros and cons to each approach, but we argue that an advantage of our approach is that it allows the actants themselves to reveal what was not yet known in advance, and this could be beneficial for teacher educators to tap into. For instance, the actant "teacher traditions" was an unexpected actant for us in this investigation. Here it is important to ask how student teachers relate to a normative position such as 'teacher traditions' in their attempts to open up for lessons that incorporate critical mathematics teaching and learning? This might include taking a political activist stance that foregrounds a specific ideology or embracing a pluralistic approach allowing all types of opinions—even climate change denial. In turn, this dilemma might also suggest specific modes of doing things; for instance, what data to choose, what graphs to select and, overall, creating spaces in which student teachers foreground themselves as critical statistics teachers. We would like to add that we, in this text, ourselves as researchers did not try to teach the student teachers what to do, but instead, tried to learn with the student teachers in this complex process. When facing urgent societal challenges like climate change, some may opt toward an activist stance that strives to apply solutions for resolving the problems by avoiding engagement with a time-consuming democratic process. However, we argue that there is still a need to learn with the student teachers by opening up such complex discussions, despite the risks of not being able to produce always viable answers.

Creating more awareness of how such actants could potentially work so that teacher educators could grasp the complexity of bringing CME into their mathematics teacher education courses could be a proposal derived from this study. In relation to grasping the complexity, such actants could help them not only to reduce the constraints, but also to build and work with them as potential affordances. At the same time, this will also put more focus on the process of stimulating pedagogical imagination and bridging the current and the imagined situations, which we argue should be an important part of teacher education and is also emphasized by Skovsmose et al. (2023) [37] in this Special Issue.

The actant networks presented here, both in the five cases discussed in Section 4 and as generalized actants in Appendix B, can be used as exemplary vignettes for creating reflexive dialogues in mathematics teacher education courses or workshops. These vignette-based dialogues could invite student teachers to try to make their own actant networks with respect to how they foreground themselves as critical statistics teachers. This process would highlight the importance of pedagogical imagination to the student teachers. A similar proposal has been made by Rubel et al. (2021) [44], who discussed the critical reading of data visualizations by bringing CME into the thematic context of the coronavirus pandemic. They employed the triplet of *renarrating* (i.e., talking about what stories can be told in these pictures), *reframing* (i.e., what relationships should be highlighted), and *reformatting* (i.e., what has been left out).

We see our work as adding to such attempts to bring CME into mathematics education. Specifically, our actant network analysis brings an additional critical approach to critical mathematics education itself. Table A1 in Appendix B can act as a guide to potential actants coming into play during the design of teaching and learning that aim to use CME to discuss climate change. For instance, teachers could discuss how the actant “source criticism” is related to CME. Teachers could discuss the differences between “instrumental statistics” and “critical statistics” or consider how CME could be beneficial for democratic purposes, and even to describe what the United Nations’ documents report about climate change. Moreover, to discuss how a student teacher should navigate the curriculum in relation to CME and climate change, one could give examples of how CME could become a social activity that brings joy and hope, rather than being frightening, and discuss what age group(s) this is suitable for and what different teaching traditions a student teacher could apply—being normative in driving an issue or taking a more “pluralist” approach, or even allowing the expression of opinions that come close to climate change denial [42] (p. 76). This approach requires an increased focus on the imagined situation, and that the mathematics teacher is prepared to consider other ideas than those described by the curriculum [37]. The complexity of some of the actant networks presented here might seem daunting and could cause reluctance to introduce CME among both teachers and student teachers. We see this study as an attempt to alleviate such hesitation. Having teacher educators who are somewhat familiar with this complexity might help reduce such hesitation. Another implication of the approach used in this study is that the teaching and learning situation will be co-constructed with the student teacher perspective in mind by the teacher educator. This would require that teacher educators learn to listen actively to how student teachers engage with critical mathematics teaching and recognize the value of informal understanding as it is also highlighted by Hough and Solomon (2023) [45].

This study inquired into student teachers' foregrounds when introduced to CME through the climate change thematic context. We believe that further studies with student teachers who are much more familiar or knowledgeable with statistics, CME and climate change would be of equal interest. What would their networks look like and how would these compare with the ones created by the novices of this study? How do they envision themselves as critical mathematics teachers and how do their familiarity, knowledge, or experience with diverse concepts support their foregrounds? What different actants would this bring that are beneficial to teacher education? In a broader sense, this study shows how ANT can be used as a theoretical and methodological approach for studying how a particular teaching and learning setting (i.e., CME and climate change) could be enacted by student teachers as a process of pedagogical imagination in mathematics teacher education by appreciating its embedded complexity.

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Institutional Review Board Statement: The leader of the Ethical Board of the University of Malmö considers that this study involves adults and since no obvious sensitive personal information will be handled, it is no need for an ethical application to the National Ethics Committee.

Informed Consent Statement: Informed consent was obtained from all subjects involved in the study.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Interview guide:

Advance information:

- Welcome and brief about myself (if necessary)
- Description of the interview
- Agreement on the planned duration of the interview
- Information on the use and publication of data
- Explanation of consent

Getting Started

A short introduction with a simple open-ended question to start the conversation.

Introduction with a brief description of the research and why I need the informant's help:

The research is about examining in practice how mathematics learning is affected by working with critical mathematics in mathematics teaching, with the help of examples regarding climate change. Critical math is illustrated in some examples that show that it is possible to create narratives with the help of data selection and choice of diagrams in statistics. The research aims to gain knowledge about this when teaching in teacher education (grades 4–6) in the subject of statistics. I need your help with experience from the course to be able to see what knowledge can be gained from this type of teaching.

Main part

Questions and follow-up questions:

Question 1: What did you think of the statistics course? (Simple, open question to start the conversation)

Supplementary questions:

Can you develop that?

If the discussion drives away from mathematics learning, some question related to this may be necessary. For example: I notice that you are very interested in the environmental issue, but how does it affect mathematics teaching?

If it comes to working methods, amount of work: How did you feel that the working method and the amount of work were in the work around critical math?

What do you see as other advantages and disadvantages of this way of working in mathematics teaching?

Question 2: In the lectures I mentioned the following examples. How did your thoughts go during the lecture and afterwards, around this example?

These statistics suggest that climate change must be addressed at country level.

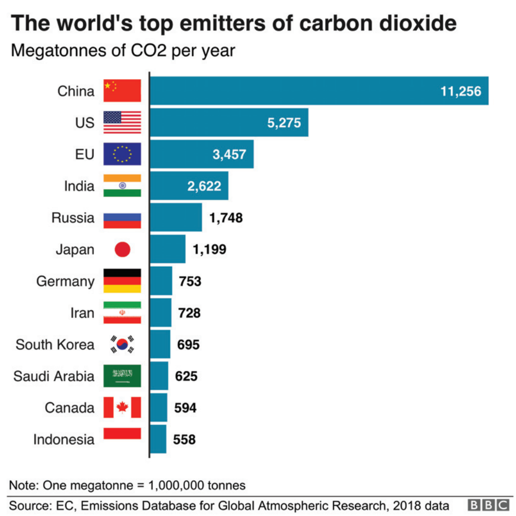

Figure A1. The world's to emitters of carbon dioxid.

When carbon dioxide emissions are presented below, it suggests that the discourse should be conducted as a global issue.

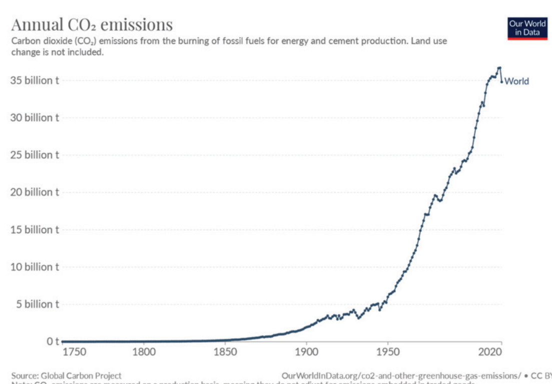

Figure A2. Annual carbon dioxide emissions in the world.

Supplementary questions:

If interest is shown in using this in their own teaching, appropriate questions may be: Do you see that the pupils should work with this themselves or is it the teacher who should teach about it? Is it a collaboration between the pupils and the teacher or should the teacher inform about this? How do you see that the pupils can work with this themselves? What is the benefit of the teacher being aware of this (critical math)? Is there any benefit to mathematics teaching in that teachers and pupils work with statistics in this way? If so, which is it? How do you think it affects learning?

Question 3: Another example that illustrates how the choice of data and diagrams affects the discourse (Figure 2).

The diagrams shows carbon dioxide emissions for the United Kingdom and India, respectively. The first diagram shows how emissions for the two countries have changed since 1880.

The second is a commutative or accumulative diagram, i.e., the sum of the emissions can be seen on the right of the diagram.

The third diagram shows the emissions per person for the two countries between 1950 and 2008.

As can be seen, the different diagrams create two different images or narratives of the situation.

How could you as a teacher work with this in your teaching in a class, grades 4–6?

Follow-up questions: If the discussion turns to the issue of workload: Is there any way to weave this into the regular teaching so that the mathematical and critical math are treated simultaneously?

If the discussion drives too much to the climate issue, ask a question that brings you back to mathematics. (However, some thoughts on the climate issue are interesting to include). One question may be: I feel that you see many benefits regarding climate change as a subject, but how do you think it affects the learning of mathematics?

If the discussion is about concrete ways of working with this: Do you want to develop that? Again, if the answer is more about climate change try to link back to mathematics teaching.

Question (to show that I connect to what was mentioned previously): You mentioned before that... Do you see any connection between this and what you just mentioned?

Question 4: How would a class in grades 4–6 benefit from this in their teaching?

Follow-up questions: How do you think it affects mathematics teaching that awareness increases around these issues (critical math)? How did you, as a student-teacher, experience that it affected your own motivation and learning about working with statistics. How do you think it affects your motivation as a teacher?

Question 5: Is there anything else you want to add?

Review

Brief summary of the interview. In this way, misunderstandings can be avoided. Thanks for the interview.

Going forward

Description of what will happen next with the answers. Whether the interviewee will be informed about the continuation of the study and the results of the research.

Appendix B

Table A1. Description of the thematically grouped actants with example quotes from the interviews.

Generalized Actant	Description	Example of Data Quotes
CME	This actant introduces the idea of critical mathematics education. The diagrams (Figure 2) format how the lectures will be carried out and imply certain ways of thinking. Opens new ways of thinking.	Sophie: "Yes, I think so, you get a more basic understanding of both the subject and the diagrams. If we hadn't had this discussion, we wouldn't have really understood what the diagrams are about. That is, what their underlying meaning really is. And that benefits learning, I think."
Democracy	Democracy as an actant refers to ideas about preparing citizens to be well informed and critical, e.g., valuable in democratic elections. Some STs referred to this as a reason for using CME.	Nadir: "Because if you present a table of that country and then it gets into perhaps society as well as within the social sciences. For example in China, it's not a, well, democracy in that way. Could that be the reason it looks, well, things like this. There are many reasons why for example."
Politics	The actant politics, in our case, refers to states of affairs that the ST cannot change or influence. Politics or similar words were used by some STs to talk about things that they felt that they had no way of changing or influencing.	Sophie: "Then you also know the level of politics that politics is different in the different countries. It can also affect emissions." Nadir: "it's not that easy to be able to link, we live in a political world, a lot lies in this kind of politics" Nadir: "Yes, well, politicians have a lot, really decide like this, this international, or this national, within our country, what shall apply, what is done."
Math as social activity	Involves discussion and acting out what is being discussed.	Nicole: " yes, that you think that mathematics belongs at school, it's not something I need to use at home. It's just something unnecessary that you have to do at school, but if you do it more socially, you might think that—well, it actually belongs outside as well. There is a reason why we study in school, it is so that we can use it in real life."
Instrumental statistics	Suggests a more traditional and instrumental way of looking at mathematics, i.e., technical, abstract, and disassociated from the social context of the data used. Some STs referred to previous experience of mathematics without CME.	Nicole: "You might get a different picture of mathematics, instead of that, this traditional mathematics teaching, that you just have to sit quietly and work in a book, that you get the whole thing, well mathematics can be a social thing, that you put in values, you can discuss, yes that you get a more enjoyable picture of mathematics as well."
Curriculum	Sets part of the stage for what should be done and what can be done during the lessons. Some STs referred to this in the interview as a constraint on what they are supposed to do in mathematics education.	Iman: "For one thing, the curriculum says that we must teach about the environment and the outside world and be able to reason and become a good citizen of society and make oneself understood."

Table A1. *Cont.*

Generalized Actant	Description	Example of Data Quotes
Pupils' age	This actant refers to ideas about how pupils' age puts some constraints on what can be taught at a specific age or maturity level.	Estelle: "It has to happen already at the first level, to bring in that thinking to see this abstract, to kind of see this critical and then it builds up, and you train, so that's why it's very late. It depends on who and what kind of prior knowledge they have with them."
Manipulation	The idea of not being told the truth. This can occur in a conscious or unconscious way. The idea that there is an objective truth, and that the teacher should tell and show that truth.	Nicole: "like yes you can manipulate quite a lot with diagrams depending on what you choose to focus on and what purpose you have with the various diagrams, sort of." Iman: "if you once show this and want to discuss how statistics can deceive the eye depending on how the bars are set"
United Nations (UN)	The UN provides guiding documents that the teacher relates to. This can be seen as one of the actants that build up the actant climate change.	Sophie: "I think the UN and their goals are very clear, [eum], then they have done this, that is, based on facts and research and all that kind of stuff. And then you can start from their ages, what do the students find interesting, what do they want to know."
Teaching traditions	Actant that refers to the idea about the teacher's role or what teaching tradition to follow—for instance, being normative and pushing a certain standpoint, or taking a "pluralistic" stance allowing all kinds of discussions. "Every tradition has its shortcomings and strengths." [46] (p. 76).	Iman: "Try not to put the words in their mouths, because then I only think certain will back off—"no, I don't want to" Iman: "Overcommitted if you stand very clearly on a position, politically for example, then it can be easy to push the issue."
Climate change	Actant that contributes as an important subject to illustrate CME and, at the same time, creates an awareness of the subject.	Iman: "The climate issue doesn't feel as problematic, if you do, because there you can stick to the facts quite a bit, I think. But that's because I think this is a big problem, but so do the majority of world leaders, even if not everyone agrees." Nicole: "right now we have a big climate crisis so we all really have to try to lower our emissions as much as possible so that it doesn't increase the greenhouse effect."
Source criticism	The idea that facts need to be checked for accuracy.	Nadir: "There is so much information online, but it must be from sources that are close to our time and so that you have discussions like this about source criticism."

Table A1. *Cont.*

Generalized Actant	Description	Example of Data Quotes
Real world	The idea that school is separate from the real world. Some STs mentioned that school activities need to connect to the real world, or words to that effect.	Sophie: "It is very easy to believe that it is only like this. But it is not, there are many reasons why it is represented in this way. So their feeling towards mathematics, yes it gives them a more realistic and reality based picture. Can I imagine because math is abstract it needs to be more concrete that way." Nadir: "but how to make it clear to students and that they shall find this connection between then, as I said, reality and these tables and diagrams." Nicole: "well, it actually belongs outside as well. There is a reason why we study in school, it is so that we can use it in real life." Nicole: "It is quite often that you get the question, well, why should we learn this here? Why should we learn to calculate the root of something, we don't need to be able to do that just like in reality. But sometimes you actually need to be able to do that, as well"
Fear and hope	This actant has to do with STs' concerns about pupils' feelings. Climate change can be frightening. Some STs expressed concern about the feelings that a certain subject might inflict on their pupils.	Sophie: "a little like this that the students get this type of negative image like this, like that the topic that you need to touch on and, especially for the younger ages, because we have talked about that in other courses that it should not be too negative so that, that the students get too negative a view, there is hope there is like, present it that way, but at the same time they need to be aware of this stuff."
Interdisciplinary	The idea of introducing a combination of multiple academic disciplines into one activity—for instance, mathematics and geography.	Iman: "I am very much in favor of doing it at the same time. I like the integrated subject. So I absolutely think that this could be brought up in both the NO lesson and in the math lesson."
Social justice	Actant that contributes as an important subject to illustrate CME and, at the same time, creates an awareness of the subject.	Sophie: "Then equality and the social problems are also a big issue. But I would probably have chosen the climate issue first."

Appendix C

Table A2. Frequency of each of the generalized actants used in all interviews; the top row shows generalized actants, while the first column shows the participants.

	CME	Democracy	Politics	Math as Social Activity	Instrumental Statistics	Curriculum	Pupils Age	Manipulation	United Nations (UN)	Teaching Tradi-tions	Climate Change	Source Criti-cism	Real World	Fear and Hope	Interdiciplinary	Social Justice
Nicole	5			4				2			1		3	1		
Sophie	11		3	2			1				11	2				3
Nadir	11	2	2	3	4		2		2		6	6	5	1	2	1
Iman	10			2	3	2		4		2	4		2	1	1	2
Estelle	14			3	4		5	4			1		5	3		
Frequency	51	2	5	14	11	2	8	10	2	2	23	8	15	6	3	6

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education sciences

Article

Storylines in Voices of Frustration: Implications for Mathematics Teacher Education in Changing Times

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Abstract: We have interviewed becoming mathematics teachers, in the last semester of their education, asking how they experience their time as teacher students with the focus on inclusive teaching. In their forthcoming daily work, they will be responsible for arranging for inclusive teaching that addresses all the learners' needs in mathematics. We believe the voices of future teachers are important to include in conversations about how programs prepare future mathematics teachers for the work of teaching in today's schools and classrooms. We used storylines as a theoretical construct to discuss the socio-political aspects of mathematics teacher education through the lens of two research questions: What storylines emerged in interviews with becoming mathematics teachers in their last semester of teacher education when they talked about teaching in diverse classrooms? What implications might these storylines have on mathematics teacher education? Our analysis made us aware of three important storylines: (1) storylines about the importance of language in mathematics education; (2) storylines about the importance of accepting diverse methods when doing mathematics; and (3) storylines about issues of invisibility at play in mathematics classrooms. In this paper, we discuss the importance of creating space for discussions in teacher education about issues that may challenge inclusive practices in mathematics classrooms.

Keywords: diversity; teacher education; minoritized students; mathematics; storylines; language

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1. Introduction

Teacher education and schools need to lead the way in the changing times. Over the last decades, Norwegian society, like that of many other countries, has experienced demographic changes. Diversity in the population has become much more prominent and visual [1]. Norwegian schools and classrooms mirror the demographic structure of todays' society. Newly educated teachers in their forthcoming professional years will meet multilingual and multicultural classrooms, which we will refer to as diverse classrooms. They will be responsible for arranging daily inclusive teaching that addresses all learners' needs in mathematics.

We argue that the development of knowledge and awareness regarding inclusion in mathematics for all learners must be included in what are commonly called 21st century skills, the new professional skills that the social and technological changes in modern society pave the way for. There are broad suggestions as to what these skills should include (e.g., [2]). The official Norwegian report, The School of the Future: Renewal of subjects and competences, highlights the following as key competences for current and future learners: "creativity, innovation, critical thinking and problem-solving" [3] (p. 10). Often, 21st century skills are connected to the skills that students need in order "to enter the era of globalization, anticipate the fast advancement of science and technology, and utilize information technology in various activities" [4] (p. 61). There is no doubt that teacher education must educate teachers who can prepare learners for the demands of the 21st century. Because it is impossible to ignore the social changes in our society, we consider the teachers' competence in and strategies for promoting inclusion in mathematics for all students, regardless of social background, as 21st century skills for teaching. We regard this as an important topic that new mathematics teachers should be introduced to and prepared for during their education.

As mathematics teacher educators and researchers working within the framing of the research project "Mathematics Education in Indigenous and Migrational contexts: Storylines, Cultures and Strength-based Pedagogies" (MIM project) and focusing on minoritized learners in mathematics education contexts, we are interested in understanding how mathematics teacher education in Norway prepares becoming teachers to teach mathematics in an inclusive way that meets the needs of the diverse learners, including the minoritized learners in particular, in the classrooms in Norway. We interviewed students who were becoming mathematics teachers (in this paper we use *becoming teachers* for the master's students who attended mathematics teacher education classes) and were in the very last semester of their education about how they experienced their time as teacher students; the focus was on inclusive teaching for all. We believe that their voices are important to include in the debate about how teacher education programs in mathematics prepare becoming teachers for the work of teaching in the schools and classrooms of today. Not only are the becoming teachers experienced students in the last years of their education, they also have rich experiences from teacher placement. Hence, these students' voices are a unique means of gaining insight into the current situation of teacher education in mathematics and the relations between the institution's focus and the present situation in schools.

We found that these becoming teachers expressed frustration about the differences in perspectives on teaching between their university courses and what they experience in practicums. Our interest in how mathematics teacher education prepares future teachers for their daily work aligns with the work by Raaen and Thorsen [5] who say that "the legitimacy of teacher education depends on its ability to offer professional learning that will enable student teachers to meet formal requirements as well as taking good care of the actual academic and social needs they are to face in school" (p. 1). This legitimacy is addressed through an analysis of what becoming teachers are expressing about the discourses in schools as compared to those they encountered at university or in societal conversations. The following research questions guide our analysis:

1. What storylines emerged in interviews with becoming mathematics teachers in their last semester of teacher education when they talked about teaching in diverse classrooms?
2. What implications might these storylines have on mathematics teacher education?

1.1. The Socio-Political Context for Teacher Education in Norway

We will start by describing the socio-political context for mathematics teaching in Norway before we explore the already-known tensions in teacher education. As they are in the transition from becoming teachers to full-fledged teachers, the students in our study have been actors in two connecting worlds, in school placements and in teacher education institutions, and they have experienced tensions and frustrations in both settings. Hence, these two contexts work as a backdrop for our conversations with the becoming teachers.

The Norwegian Grunnskole (age 6–16) is one of the core elements of the Norwegian welfare state for the promotion of social mobility. In fact, a goal of schooling in Norway is to iron out social differences and give all students equal opportunities [6]. However, there have been concerns raised about whether the Grunnskole manages to fulfil the goal of promoting social mobility. Social differences are rather exaggerated during the years students spend as learners in Grunnskole [7–9]. Reisel et al. [10] state:

> *The Norwegian unitary school system is not particularly well equipped to handle student diversity, and that this can make it difficult for students with minority back-grounds to fit in. The tradition seems to identify a contested institutional field, where teachers attempt to handle a diverse student body, without adequate curricular tools to do so.*

So, even though the Norwegian society is well known for its high level of equity and social mobility, there are extra challenges for minoritized students in the educational system.

To recognize all students for who they are and to adjust for what they need as learners is an ideal for every teacher. This has had a prominent position within the Norwegian curriculum over the years and has received the label "tilpasset opplæring" (TPO, "differentiated instruction"). It is a strong guideline which emphasizes that every student must be given opportunities for learning and growth regardless of their unique situation. Like TPO, inclusion has a strong position in the curriculum: "It is important to recognize and value the diversity of students, and to see diversity as a contribution and resource" (Translated from: "Det er viktig for å anerkjenne og verdsette mangfoldet av elever, og for å se på mangfoldet som en berikelse og ressurs.") [11] (p. 3).

In addition, mathematics as a school subject is important in developing learners' competence in critical thinking and for democratic citizenship. Also, mathematics has important connections to learners' future opportunities in their social lives. Within the research field of mathematics teaching and learning, these connections are well known; for instance, Williams [12] argues that mathematics has major exchange value in our society: the high status of mathematics qualifications positions mathematics as a ticket to a future or to a good life [13]. Hence, it is of huge importance that all learners get the opportunity to develop mathematics skills and knowledge in the spirit of TPO and inclusion, regardless of their background or language skills.

The becoming mathematics teachers need to be prepared for meeting diversity in all aspects, with consciousness and care for all students (in this paper we use the word *learners* for pupils in schools). In this sense, teacher education plays an important role. In particular, the becoming teachers need to know about the importance of diversity and inclusion in mathematics [14] and to arrange for every student to have the space to build on their own thinking [15]. Furthermore, they need to know about how the dynamics of exclusion may play out. To not be aware of such issues may have serious consequences; exclusion may be hidden in plain sight. To be excluded as a mathematics learner could exclude young people from further education and life in general [16,17].

1.2. Tensions in Teacher Education in Norway in the 21st Century

How universities and schools are in touch with the changing needs in society has been a debated topic in different research contexts over the years. In the North American context, where diversity has been a considerable force for educational reforms and challenges for over a century, Bascia and Jacka [18] show how the way that educational institutions respond to societal change is problematic. While educational systems should play a leading role in promoting the emerging and changing needs of youth, there are challenges based on how institutions respond to changes in the social society. We believe that these challenges apply in the Scandinavian context too. While societal changes are rapid, change in educational institutions is slow-moving, piecemeal, and fragmented. Hence, there seems to be a time lag or reaction difference between societal changes, responses in schools, and mathematics teacher preparation.

The discipline of educating teachers goes back several centuries. There has been ongoing discussion about what kind of knowledge teachers need; this discussion goes back to the beginning of the 20th century and Dewey's [19] work. He identified two forms of knowledge that are important for teachers' work: "the skill and proficiency in the work of teaching" and the use of "practice work as an instrument in making real and vital theoretical instruction; the knowledge of subject-matter and of principles of education" [19] (p. 9). He considered possible ways to bridge the gap between these forms of knowledge. Since then, the discussion on how to bridge the gap has continued both in the research literature and in public debates [20]. Questions like "What counts as legitimate knowledge in education?" and "How can such knowledge be obtained?" have been of interest to several scholars [21]. This is also the case in Norway [5]. A national report, the NOKUT report [22], shows that teacher education is fragmented by a divide between theory and practice, without a clear line of communication between the two. A future discussion on how to bridge the gap seems to be important. A more recent report, from 2016, about strategies for teacher education in the future, emphasizes that the "gap between campus and the world of work generally remains too wide" [23]. As we show, this gap appeared implicitly and explicitly in the interviews. The becoming teachers expressed frustrations. Hence, the discussion of the gap in teacher education is not just a discussion among researchers. It is an important part of everyday life for the becoming teachers.

Another tension that becoming teachers meet relates to the move they have to make between research-based learning (what they learn at university) and practice (what they learn in schools). Jenset and Blikstad-Balas [24] underline the fact that, in general, becoming teachers in Norway emphasize that what they learn in practice, in teacher placement, is for them the most valuable aspect of their studies. Prior research has used a number of metaphors for this move, such as bridging a gap (e.g., [20]), border crossing (e.g., [25]), or medical metaphors like those in the political discourses about "evidence-based approaches" in education [25]. The border crossing metaphor suggests a currency exchange between the theory (valued in research) and the practical applications that are highly valued in educational practices. "As progress continues in efforts to promote border crossing, those who dwell on each side should seek to respect the world across the border" [25] (p. 183). An alternative approach is presented by Bjerke and Nolan [26]. They draw on Pereira [27] and argue for the need for a post-field third space focusing on the practice–theory transition, which includes the voices of teacher educators, teacher mentors, and teacher students. They show how disruptive pedagogies in teacher education can "invite awareness and action toward disrupting and challenging dominant discourses in mathematics classrooms" [26] (p. 11). We argue that becoming teachers are primarily the ones who actively cross this border a number of times during their five years in teacher education. Hence, their voices are important.

However, an interview study conducted at four Swedish universities showed that becoming teachers do not feel properly prepared for teaching in linguistically diverse classrooms, indicating a deficiency in teacher education programs [28]. (We believe that the same feelings are present in Norwegian universities.) Teacher educators in turn express an awareness of the need to address multilingualism, but they also express uncertainty and a lack of consensus about where measures should be included in teacher education. Usually, the responsibility falls on teacher educators in the language subjects, especially becoming teachers in the "Swedish as a second language" subject [28,29]. In Sweden, "a Swedish linguistic norm is taken for granted" in teacher education [30] (p. 46) and the case is similar in Norway [31]. However, Lundberg [32] maintains that teachers' attitudes and beliefs are rather positive towards multilingualism and multilingual pupils, although "sceptical views, often based on monolingual ideologies, are present and are likely to pose challenges for the implementation of pluralistic policies" (p. 266) in schools. According to Dewilde [33], Norwegian classrooms are also subjected to a language ideology that promotes Norwegian language as the only language of instruction.

As we can see, there are several tensions in teacher education, and the becoming teachers in our study were exposed to and were a part of these tensions during their education. Thus, their stories have huge importance in the endeavor to understand the dynamics between what the teacher students learn and what they develop knowledge of during their education and to understand what will be important knowledge and skills for becoming teachers.

2. Theoretical Framework; Storylines

We chose storylines as our conceptual lens for investigating the becoming teachers' accounts of their experiences from their time in teacher education. This choice enables us to discuss mathematics teacher education from a critical and socio-political perspective. Positioning theory puts communication acts, positioning, and storyline in a triad: any communication or interpretation of communication needs to envision a known storyline and to position the current interlocutors within the storyline [34]. Storylines, as conceptualized in position theory, can be described as peoples' worldview or the big stories influencing how people are positioned in interaction [35]. By worldview, we do not mean something static; storylines are dynamic and full of potentiality. This means that (1) storylines influence peoples' interactions; (2) interlocutors influence each other; and (3) the actual interactions influence the coming interactions. Hence, storylines are influenced by the actual interactions and the interlocutors' histories from previous interactions, and they will also have an impact on future interactions. Davies [36] explained this complexity by saying that "we are imbricated in our relations with each other, in our workplaces, in our writing and our thinking. We are the thick tangles of our relationality and the assemblages of power/knowledge that make up our lives" (p. 474). When we chose storylines as a theoretical construct to frame our discussion of the becoming teachers' stories, we were motivated to focus on the complexity described by Davies.

Herbel-Eisenmann et al. [37] describe how "a storyline tends to be a broad, culturally shared narrative that acts as the backdrop of the enacted positionings. The storyline that is invoked or called forth by the participants shapes and constrains the kinds of positions that can be enacted" (p. 104). Importantly, it is naïve to claim that there is one single answer as to which storylines are at play in any human interaction. As Herbel-Eisenmann et al. [37] note: "Because there are multiple storylines and positionings at play in any interaction, the same communication actions can be interpreted in more than one way" (p. 104).

Moreover, storylines are contestable and contingent [34]—contestable because people may either accept or resist a storyline and act within it or within a competing storyline, even when they are involved in the same interactions. The contingency of storylines means that when people are connected in an interaction, they may not be acting or talking within the exact same storyline. In other words, our interpretation of storylines will be colored by who we are and our experiences. Any situation can be interpreted with different storylines, and there is no correct storyline or positioning in any given situation [34].

Using storylines as our theoretical lens means that we recognize that culturally shared repertoires are important in all storylines; they are perhaps especially important in storylines pictured from mathematics education perspectives in indigenous and migrational contexts. Indeed, mathematics education itself has a cultural repertoire. Andersson et al. [38] found three storylines about culturally shared repertoires in public news media about mathematics education and minoritized groups: (1) the majority language and culture are keys to learning and knowing mathematics; (2) mathematics is language- and culture-neutral; and (3) minoritized students' mathematics achievements are linked to culture and gender. Undoubtedly, these storylines of the general public would also be known to becoming teachers.

Previous research on storylines from work by mathematics education researchers [37] and in the public news media [38] are powerful voices that can (re)produce 231 "common sense" about mathematics education [39]. However, because storylines are reciprocal—human interaction goes both ways [34]—the "consumer" voices of mathematics education are also important in understanding mathematics teacher education.

Importantly for this study, becoming teachers meet storylines in university, in practicums, and in society, and these storylines are contestable and contingent. Hence, a way to rephrase our research questions could be to analyze what happens when these becoming teachers meet conflicting storylines and how they handle these tensions or frustrations that they talk about.

3. Methods

This research is part of the Norwegian Research Council's FINNUT-granted project MIM: Mathematics Education in Indigenous and Migrational contexts: Storylines, Cultures and Strength-Based Pedagogies (see https://www.usn.no/mim (accessed on 8 July 2023)); it is a collaboration between Canadian, American, and Norwegian researchers, drawing on participatory approaches to investigate educational possibilities and desires in times of societal changes and movements. Although we focus here on the Norwegian context, we recognize that these kinds of societal changes and movements impact many countries throughout the world. With these changes, language diversity may be the most obvious challenge in mathematics classrooms, but this reality also connects to cultural differences and the conventional characteristics of the mathematics subject and discipline [40]. Indigenous communities have experienced linguistic and other challenges for decades as a result of colonization.

The data for the research reported here come from in-depth, semi-structured interviews with nine teacher students who completed their five-year mathematics teacher education while writing master's theses that were connected to the MIM project. The nine students, from two universities, all had experience of working in schools as teachers in addition to their five-year university education. Some of them drew on teaching incidents when reflecting on their teacher education experiences. Three of the students had experiences from working in indigenous school contexts in northern Norway, and all of them had experiences from multilingual or multicultural primary school classrooms in urban schools. One student had experience from introductory classrooms where newly arrived students aged 16–21 years old were introduced to the Norwegian school system while learning Norwegian. Mathematics teacher students were informed about the MIM project during their first years in teacher education. They were students at institutions where MIM participants give lectures in mathematics classes. After learning about the MIM project, some of the master's students expressed interest in migrational and indigenous aspects of mathematics classrooms. Some of the students chose this as the topic for their master's degree studies. The interviewees were recruited from this last pool of students.

All the interviews except one took place in the student's last semester while they were writing up their master's theses. Each interview used Zoom and involved two people from our research team and one student. The interviews were recorded, anonymized, and transcribed in Norwegian. In the semi-structured interviews, the students were invited to share their experiences and reflections freely. Examples of the interview questions include: Can you briefly say why you chose teacher education and not least why you chose to become a mathematics teacher? Can you describe one successful lesson where you organized mathematical learning opportunities? What have you learnt from teacher education about mathematics instruction for minoritized and indigenous students? Our ethical protocol was an important part of ensuring open communication from the students. The participants were informed in advance about the study, through a written informed consent process before entering into the research. Participation was voluntarily and the participants were informed that they could withdraw from the research without giving a reason and without negative consequences.

The analysis process was iterative. We started out by independently reading each interview multiple times, trying to understand the student's stories. We took notes of what the becoming teachers talked about regarding their teacher education experiences; what they said they learnt (or not) during teacher education; and what struck us as important in their interviews. When we met as a group, we critically reflected on and discussed the different themes and topics as potential storylines. The first selection process involved asking whether the themes or topics were relevant for teacher education. The next process involved discussing how the remaining potential storylines were connected to the storylines that were already known (by us) from the relevant literature in mathematics teaching and learning. This made us realize how some topics were interconnected, and the list of storylines was once again reduced. We considered whether the storylines we had identified in the becoming teacher interviews were overlapping or contrasting, whether some of them were overarching, or whether some of them were less prominent than others. This made us reflect on our findings and on this article, and we decided upon the following three storylines: (1) storylines about the importance of language in mathematics education; (2) storylines about the importance of accepting diverse methods when doing mathematics; and (3) storylines about issues of invisibility at play in mathematics classrooms.

The participants in this study were master's students in mathematics teaching and learning in their very last semester; the students wrote their master's theses as part of the MIM research project. This means that they had a special interest in equity and social justice questions and wanted to develop strength-based pedagogies in diverse mathematics classrooms. The becoming teachers in our study were involved in the MIM project, even if their theses explored different topics. Also, we as researchers are involved in the MIM project. We are aware that the becoming teachers' interpretations and reflections of experiences from their educational pathways are colored by who they are, as the analysis is colored by who we are. Moreover, we are aware that to name storylines through a research process and to present them to a reader may reduce their complexity. This resonates with what Gerbrandt and Foyn [41] noticed in their discussion on how to hunt storylines; to name a storyline and present it to an outsider could be compared to freezing a picture of a movie. The frozen picture would in this case include us as researchers, the transcripts, and the relationships between them. As Gerbrandt and Foyn [41] state, "It is like three threads coming together, and it is within these threads and in the spaces between, that we are trying to capture the surrounding narratives" (p. 329). We are aware that other researchers and other becoming teachers could have enabled other storylines to emerge. However, this does not mean that we regard our findings as unimportant. Rather, our interpretations are possible meanings that are important to highlight—they promote a discussion on teacher education in changing times from the perspective of the MIM context.

4. Results

Even though the becoming teachers in our study gave an account of their own reflections, thoughts, and experiences from their specific and unique contexts, our analysis show how they talk about similar topics that cross-cut the data. This enabled us to become aware of the storylines about mathematics teaching and learning which surround the becoming teachers in their educational pathway towards becoming full-fledged mathematics teachers. The structure of our findings is presented as follows: for each storyline, we first introduce how the given storyline is presented in the research literature; then, we turn to how the storyline was expressed in the becoming students' narrative.

4.1. Storylines about the Importance of Language in Mathematics Education

4.1.1. Storylines of Mathematics Education and Language in Research Literature

At university, becoming teachers usually learn that languages are to be seen as a resource in diverse multilingual classrooms; this is in line with what the research has shown in, for example, Canada [42], India [43], New Zealand [44], Sweden [45], and other countries. They may learn that code switching—the practice of switching among languages within a conversation—supports multilingual students when learning mathematics [46]. In a South African context, Chikiwa and Schäfer [47] conclude "that consensual understanding of best practices for code switching is required to promote code switching that is precise, consistent, transparent and thus supportive of teaching for conceptual understanding of mathematics" (p. 244). Becoming teachers may also learn that translanguaging may support mathematics learning, as was shown in a Hong Kong context [48] and in Spain [49]. The theory of translanguaging challenges simplistic divisions between languages and describes "translanguaging" practices that are more nuanced than code switching, including non-standard languages and words and complex and shifting linguistic identity.

However, in Norwegian schools there seems to be a *Norwegian language only* storyline, a storyline about monolingual ideologies [33]. We decided to write about it as a storyline as we, and others, cannot find evidence that "Norwegian only is best practice" is required in steering documents. There seems to be an established truth that says that you learn Norwegian better or faster if the language of instruction, Norwegian, is the only language allowed in the school classrooms (as, we note, is the case for English in a number of other countries) [44,50]. Nortvedt and Wiese [51] interviewed Norwegian mathematics teachers about "their classroom practices and on how they adapt teaching and assessment situations to migrant students" (p. 527). These teachers addressed language issues, explaining how a common language is considered crucial for mathematics communication and student-centered teaching. The importance of language is also addressed in Norwegian policy documents. These documents argue that knowing the majority language is crucial in processes of inclusion [52].

Choosing the language of instruction in mathematics classrooms is not simple. There are, among other aspects, the language(s) spoken by the teacher; the language(s) spoken by a particular learner; and the language spoken by the other learners. Moreover, there is the language advocated by policy documents, which tends to be the majority language [53]. On the one hand, it is helpful for the learning opportunities if there is a common language in the classroom. On the other hand, it is helpful for the particular learner if the language of instruction is adapted to a language familiar to the learner. Hilt [52] explains part of this complexity: "Both inclusion and exclusion processes are necessary in order to draw distinctions between a system and its environment, in the case of politics and education as well as other systems. After all, it is not the minority language in itself that is the problem, but how the lack of a common language challenges the extensiveness of educational communication" (p. 108). This explanation intersects with the Norwegian teachers' view of language as both a key and an obstacle to learning [51].

The becoming mathematics teachers might have first-hand experiences with language-related tensions when moving between the theory–practice transitions and the practice–theory transitions [26]. While acknowledging the language-related challenges, we suggest that it is important that mathematics teacher educators facilitate the becoming teachers' opportunities to critically reflect on these kinds of experiences. Teacher educators need to help becoming teachers enact critical emotional praxis so that they can disrupt the taken-for-granted language-related storylines in mathematics education. This can provide becoming teachers with the opportunities to understand and challenge how they "address questions of otherness, difference, and power" [54] (p. 307).

4.1.2. The Expressed Storyline about Language as Important in Mathematics Education

All the becoming teachers express and underline the importance of the connections between the language of instruction, the languages allowed in the classroom, and mathematics education, and they wish they had learnt more at university. We start by sharing one of Vilde's practicum experiences after meeting a 10-year-old immigrant boy in school:

> *I met a boy who had come to Norway two years ago. And he was a real resource in maths [. . .] because here he contributed to the classroom environment, knowing that somehow "here you are allowed to answer incorrectly". He helped to ask a lot of questions, i.e., "stupid" questions and, as it were, big questions and everything he wondered about, he asked about. And it also contributed to the fact that the others "here we can. Here it is allowed to . . . ask what we wonder about". And he was good at explaining to the others and putting into words how [. . .] he was very good at explaining things in a very simple way to other students who found the things we worked on difficult. He was a supporter for me in my very first maths lessons, he was a supporter because he understood what the others didn't understand and helped me explain them [. . .] I experienced this as a real resource in the classroom [. . .] but of course he lacks many terms which we cannot uncover in such a conversation.* (A:1) (A is short for "Appendix A" in the Supplementary Material, and 1 is the ordinality of the utterance within the Supplementary Material)

In alignment with the university scholarship, Vilde considered him as mathematically knowledgeable, and as a resource in the classroom because he explained and because he supported the other learners in his translanguaging way. Vilde experienced the way in which his questions opened the mathematical conversations because his questions and explanations enabled him and her to explain things for the other learners; but also by making it appropriate to ask questions in any language or a mixture of languages. However, the ordinary class teacher considered this boy as low achieving in mathematics because he did not understand the questions during test situations:

> *But then his teacher asked me about what I thought of him, like what grade I would think he got, I said he must have a 5 or 6 [6 is the top mark in the Norwegian grading system] because he has a lot of competence in maths. And then the teacher says no, he is between 2 and 3 because he does not understand the questions in the test situations.* (A:2)

Contrary to Vilde's view of the boy's mathematical competencies, the teacher was more oriented towards the boy's test results. Vilde reflected further on how test situations could be adapted to the learners' language:

> *And then there is a test, then he [the same boy] asks: "What does increase mean?" Then there is that word that sort of. . . then of course I helped with that, but I had actually been told that I shouldn't help with questions, but then it was in a way the language that was a challenge for his performance, because it was quite clear that he had a high level of competence in maths.* (A:3)

Vilde's story is not a solitary voice about an awareness of language challenges and how to meet them. This situation was challenging for the becoming teachers. They expressed a wish to adapt the tests so the language would become familiar to the learners, but they were not allowed to make these kinds of adaptions. Marja said, when commenting on the Norwegian language only storyline, that:

> *[. . .] then you can see that a storyline like this is more real with language being the key to everything, language and culture being the key to everything.* (A:4)

Beth described a similar experience:

> *I've been in practice and, as I said, I've met students who have Norwegian as a second language and I've come across problems, but I haven't thought about how to fix those problems until we've had those subjects [at the university], that how do you manage to get into them, how will they manage to understand us?* (A:5)

Kim raised a similar concern related to the context of superdiverse classrooms (see, e.g., [55,56]):

> *Yes, a few have been able to discuss in other languages. But the problem here in Oslo is that there are so many groups that meet in the same classroom. So, it is quite rare that you have two students who speak the same language. Then that opportunity [to use another language than Norwegian] doesn't really exist. Even if someone who is a little better in Norwegian can tell you that we say it this way and that way, in Arabic. But then it stops a bit there. Quite simply. Also, because many have only learned the concept we use in Norwegian, since they have attended children's and youth schools in Norway. So often they can't do it in their mother tongue either.* (A:6)

Part of the becoming teachers' expressed frustration relates to the need for a common language and language rigidity within the mathematics classroom—the sense that everyone should be using the same language. Another part of their experiences is related to the learners' wishes and reflections. The becoming teachers' experience was that the learners really wanted to learn mathematics in Norwegian, as Henning says:

> *They point out that they would like to just learn it in Norwegian in one way, with Norwegian language and terms, and that it becomes almost confusing to mix their own mother tongue into it. The way that they don't mix up the terms is special. Like the boy*

> *who then speaks both Italian and Tigrinya, he in a way very much wants to learn it in Norwegian and speak Norwegian and to use the Norwegian language in a way. So that he can join the class. [. . .] He is afraid of misunderstanding terms, so he would very much like to memorize it. Learn it in Norwegian.* (A:7)

This storyline focused on the becoming teachers' reflections on how to communicate about mathematics with learners that have language and culture backgrounds that differ from those of the dominant context. They were frustrated and explained how they experienced struggles in the "Norwegian language only" classrooms. As the last utterance illustrates, however, we should not forget the learners' wishes and concerns about learning mathematics in diverse classrooms.

4.2. Storylines about the Importance of Accepting Diverse Methods when Doing Mathematics

4.2.1. Storylines of Mathematics Education and Method Rigidity in Research Literature

The pace of the world is rapidly changing, and we (the society) can only imagine or predict the skills needed for the 21st century. This increased pace and uncertainty has implications for schools and higher education. Schleicher [57], Director for the Directorate of Education and Skills—OECD, explained that:

> *Education today is much more about ways of thinking which involve creative and critical approaches to problem-solving and decision-making. It is also about ways of working, including communication and collaboration, as well as the tools they require, such as the capacity to recognize and exploit the potential of new technologies, or indeed, to avert their risks. And last but not least, education is about the capacity to live in a multi-faceted world as an active and engaged citizen. These citizens influence what they want to learn and how they want to learn it, and it is this that shapes the role of educators.*

These changes have implications for teacher education, and one of the current method debates is about what should be considered as mathematics knowledge. Using the words of Skemp [58], this debate is about tensions between instrumental and relational knowledge. Others have addressed what was almost the same debate by using the words procedural and conceptual knowledge [59]. More recently, remnants from this debate were given topicality and were discussed at a culturally oriented level in discussions about culturally responsive pedagogy (CRP) and mathematics teacher education. Nolan and Keazer [60] describe CRP "as a critical component of teaching in ways that value and incorporate children's diverse cultural and community knowledge resources" (p. 151). This approach can be considered a response to the beliefs about mathematics as non-complex, value-free and culturally neutral [61]. When mathematics classrooms are framed within these beliefs, the learning situations tend to be oriented around direct instructions that promote method rigidity [62]. These debates can be pictured using the border crossing metaphor [25]. The non-complex and rigidity approaches are on one side of the border, and the complex, relational, and diversity approaches are on the other side.

Returning to the method debates, we recognize teacher education as being crucial for teachers' knowledge about the importance of allowing and encouraging all learners to use diverse methods. One concern raised in the method debates is whether diverse and multiple methods are beneficial for all learners. Lynch and Star [63] explained some of this complexity: "Despite apparent professional consensus, debate continues about whether instruction with multiple strategies is beneficial to all students or only to high-achieving students" (p. 7). Another part of the complexity within this debate is that the changing times are influencing what is considered useful to learn and whether it is appropriate to learn it. Cultivating learning in the 21st century can be demanding for teachers, and the challenges and fears might be more visible than the opportunities and insights [64]. This is part of the backdrop for our wish to learn more about how becoming teachers reflect on the methods available for learners in mathematics classrooms.

4.2.2. The Expressed Storyline about the Importance of Accepting Diverse Mathematical Methods

The importance of accepting diverse mathematical methods is a storyline at play in the tensions between the freedom to use multiple methods and the expectations related to method rigidity. This storyline can be considered a response to the storyline about method rigidity (e.g., [63]). This storyline was visible in the interviews when the becoming teachers shared reflections and experiences related to methods that differed from the ones addressed by a particular teacher. One example is from a student's perspective when the becoming teacher, Vilde, described a situation where a foreign student explained her experiences related to a method from her home country:

> *The student said: 'in my country, I have learnt to calculate a percentage like this.' She showed me an algorithm which was brilliant! She then told me that she was not allowed to use this method on tests. She explained 'because here I am supposed to do mathematics as we do it here.'* (A:8)

This utterance exemplifies that migrant learners might experience that they are not allowed to use methods in Norwegian classrooms that they learnt and used before moving to Norway. One consequence of method rigidity might be that students are not given agency and authority to think mathematically. Instead, they are positioned as non-competent, with the duty to follow the rules and procedures given by the teacher. The becoming teachers explained how this positioning might influence how students are experiencing mathematical opportunities, or the lack of these opportunities. As Vilde explained:

> *it is related to students not considering themselves as successful in mathematics. They are afraid of trying (. . .) they are not trusting themselves, like: 'No, I don't think I am right'.* (A:9)

However, if learners are not allowed to use the methods that are familiar to them, it becomes challenging for the learners who must relearn and also understand why. Understanding is another method-related aspect that came up in the interviews. Jørn described:

> *the ideal teacher as a person that gives students the opportunity to understand.* (A:10)

He had many experiences during placement periods with teachers that emphasized drill over understanding:

> *you have to practise, practise, practice* (A:11)

Jørn continued to explain that he saw understanding as more helpful for mathematical learning opportunities than practicing algorithms without understanding. The becoming teachers shared experiences related to method rigidity:

> *So, among the mathematics teachers working at that school, there are expectations about which methods and strategies to use.* (Vilde, A:12)

> *They [migrant learners] might have other methods that are incorrect, or they are not incorrect, but they can be interpreted by the teacher as incorrect (. . .) for example, if they [learners] are from South-Africa and their father who is an engineering teaches them equations, their teacher might reject it because the method is not familiar to the teacher.* (Emma, A:13)

The becoming teachers expressed frustration related to method rigidity. On the one hand, they experienced situations in which teachers were limiting the available and allowed methods in mathematics. On the other hand, they experienced multiple and diverse methods as a great source. In the words of Vilde:

> *So, then among these teachers who worked at this school and have mathematics, they have an idea about which method of procedure should be followed. Then there was a group in the classroom with many different ways of solving percentage calculations, which was absolutely brilliant for me. Which gave such a huge bang for the buck in other ways of doing it. I came back to university and just "okay, look here now! In this country they do*

> *it like that and in this country they do it like that!" And this is a great resource! Sitting in a classroom, you suddenly have ten ways to do it instead of one way to do things.* (A:14)

The way this storyline was at play in the interviews illustrates that the becoming teachers move in the intersection between method rigidity and the freedom to use multiple methods. On the one hand, they shared experiences with students and teachers operating within the frames of method rigidity. On the other hand, they advocated the freedom to use multiple methods. Henning explains how the master's courses in teacher education changed his understanding of the importance of accepting diverse mathematical methods and led him to challenge experienced mathematics teachers:

> *I have noticed that I as a becoming teacher educated at a master level, am challenging some of those around me. It relates to how things are done in the way of thinking I have with me from the university. It challenges them.* (Henning, A:15)

The becoming teachers described another way that they handled the frustrations related to method rigidity vs. method freedom; this was to go behind the teachers' backs:

> *So, we went behind the backs of these teachers and did things the way they [learners] did things because then you could transfer it to the way these teachers wanted it then [. . .] I thought it was strange.* (Vilde, A:16)

This storyline focuses on the becoming teachers' frustrations related to method rigidity. They described a tendency towards method rigidity in schools, where the methods familiar to the learners were not necessarily accepted as valid in the classroom. The becoming teachers recognized the teachers to be the source of the method rigidity; for example, the becoming teachers were not allowed to translate words to help learners during test situations. In other situations, it was the learner that explained that their methods were not considered valid or allowed in the classroom. The frustration originated from the experiences with method rigidity and the becoming teachers' expectations about mathematics as a way of thinking rather than a count of test scores.

4.3. Storylines about Issues of Invisibility at Play in Mathematics Classrooms

4.3.1. Storylines of Mathematics Education and Issues of Invisibility in Research Literature

To be aware of students' social background and how that affects how they are positioned as mathematics learners has been explored over the years by several researchers [65–68]. Their work explores how a white, male, middleclass background position students in a privileged position as mathematics learners. These students are easily noticed and given attention by the teachers while other groups of students and their needs are more invisible. Stepping outside of the mathematics classroom, the occurrence of visibility and invisibility for students that are different from the majority groups continues. Wing [69] focuses on race and discusses how Asian students expressed feelings of invisibility and insignificance and notes how this group of students was rendered invisible by widespread acceptance of the "Model Minority Myth", that Asian students are different from other racial minorities within the American context. Moreover, language issues are also part of the dynamics of invisibility, such as in the way that minority languages become invisible because of the dominance of majority languages. Major [44] notes, from a study conducted in New Zealand, how the use of English in schools is taken for granted as the norm and is used as the dominant language. This resonates with studies within the Norwegian setting. Schipor [70] and Krulatz et al. [71] claim that minority languages in Norwegian schools tend to be regarded as less valuable than Norwegian and English.

From studies within the field of gender and mathematics, we know that invisibility is a mechanism to hide being different and exposed. Walls [72] explores girls' invisibility and argues how female students in mathematics "are required to don a cloak of invisibility that affords them temporary status as honorary males in a male domain" (p. 47). However, invisibility is not just about the students' wish to not be seen, it is also a question of being noticed by others. The combination of students' invisibility and teachers' failure to see is discussed by Foyn and Solomon [73]. They explore these dynamics in the case of

Sarah, a female high-achieving student who was overlooked and not recognized for her mathematical competence. They argue that there was a double bind; Sarah was caught between the others' 'failure to see' and her invisibility in that she was unable or unwilling to perform smartness.

Even though teacher education has a focus on TPO and inclusion for all students and learners in diverse classrooms as an interdisciplinary topic, the dynamics of seeing, noticing, and recognizing all students for their uniqueness as mathematics learners, regardless of their social background, are debated and will be a challenge that newly educated teachers will face when they start their professional work.

4.3.2. The Expressed Storyline about Invisibility in Diverse Classrooms

The storyline about issues of invisibility at play in mathematics classrooms relates to learners having a mother tongue that differs from the dominant language (in our study the dominant language is Norwegian).

> *And I see nothing! I see nothing at all! I don't think it is any of them who talk about it. So, it is totally invisible.* (A:17)

These are Vilde's words describing some of her experiences with teaching in diverse classrooms and with students that have a mother tongue other than Norwegian, and we noticed a sense of frustration in her words. Vilde's teaching experience was in a school that had a small proportion of students that did not have Norwegian as their first language, and she explained that it was hard to know or notice whether the learners met the challenges of having Norwegian as the instruction language in schools or which strategies the teacher used to help the learners.

> *And of course, it might be that things are done that I am not aware of. But I haven't seen anything other than a comment in the hallway: 'I do struggle a bit about the language and. . .' And that's that!* (A:18)

She elaborated on the reasons why the students' challenges with regard to language did not get attention:

> *I think it is a bit unfortunate for the students, because there are too few of them. That it's not such a big problem in a way. Then you rather go under the category that you have slightly greater challenges in the subjects.* (A:19)

It seems like it passed under the radar. The low proportion of students with a mother tongue other than Norwegian made it hard to notice them, or it made them invisible.

The becoming teacher Kim, on the other hand, describes the opposite case, when having a mother tongue other than Norwegian is normal and is the case for the majority of the learners. She elaborated on she experiences the support teachers have when they teach diverse groups of learners in mathematics. Kim said:

> *For the subject's concerns, it has been ok, but it is like the culture part is invisible. Here [at my school] everybody is different, that is how it is. So that's not much, I think the knowledge of this is scarce, at schools. Especially here. It has not been a topic. Quite simply.* (A:20)

To have a large group of students that have another mother tongue does not catch the attention or become a case discussed among the teachers, at least not to the knowledge of the becoming teacher Kim.

While Vilde and Kim talked about how the students' situation of having a mother tongue other than that of the instruction language was passing under the radar and not given attention, Beth raised the issue, based on her experiences, that some students were trying to hide being different. She said:

> *In that class there were some who had Norwegian as a second language. Then we had two girls in the class who hadn't been in Norway that long. So, they didn't speak Norwegian very well, [. . .] the two girls were very quiet and there I noticed that I was having*

> *problems and felt that they didn't understand me. Then I asked if they understood the task, and they said yes. At least as I have experienced that they often say. Yes! It's going well. They also don't understand anything I say, but they do the math problems. They didn't really need any help. I didn't get into them. They didn't understand what I said.* (A:21)

Beth's words illustrate a frustration about not getting into a position to help the students with the math problems because something made them not talk about the problems they encountered. She described how the students were apparently not comfortable about having their problems exposed or seen.

In these three becoming teachers' stories, they shared how language issues in diverse classrooms become invisible, either because the students' problem is hard to see or because the students hide their problems. Going back to Vilde, she described an important consequence of this invisibility: the teachers talk about these learners as groups (they) and not as individual humans. She reflected on her way of describing the students:

> *I am now aware that I/we say "they". That is really uncomfortable, [. . .] that I distance myself from someone, and that is very uncomfortable.* (A:22)

This storyline focuses on the becoming teachers' frustration about the complexity of the issues connected to students' invisibility in diverse classrooms. They describe how they became aware of how language issues, which they were conscious of as a challenge from their time in teacher education, were hard to notice within the dynamics of the mathematics classrooms, despite the fact that they brought this awareness with them. Moreover, the becoming teachers expressed a frustration about how the learners' challenges connected to having a mother tongue other than Norwegian were reduced to more general challenges in learning and that there was little attention paid to these issues in the daily work in schools.

5. Discussion

Through our analysis, we found that the becoming teachers expressed frustration about mathematics education in diverse classrooms and that the knowledge they get in university does not always match what they experience in practicum. In most cases, the becoming teachers expressed their frustrations by sharing stories from their practicums and from their time in mathematics teacher education. We consider these storylines as interesting contributions to our first research question: What storylines emerged in interviews with becoming mathematics teachers in their last semester of teacher education, when they talked about teaching in diverse classrooms? Here we discuss the three storylines presented in our results: (1) storylines about the importance of language in mathematics education; (2) storylines about the importance of accepting diverse methods when doing mathematics; and (3) storylines about issues of invisibility at play in mathematics classrooms.

The storyline about the importance of language in mathematics education occurs frequently in our data from the interviews with the becoming teachers. We identified this storyline in two different arguments. First, a common language is important to facilitate communication in mathematics classrooms. Second, language is important to be positioned as knowable in mathematics. The first argument addresses the challenges of diverse classrooms in changing times [1]. It relates to the international storyline about language as a resource in diverse multilingual classrooms (e.g., [42,44,47]). The second argument addresses issues of mathematical competencies. The becoming teachers in our study described how learners who can express sophisticated mathematical knowledge when allowed to express this knowledge in familiar ways could be considered as low-achieving learners in mathematics when they were limited to using the majority language only. This storyline shares similarities and intersects with the storyline the majority language and culture are keys to learning and knowing mathematics from a media analysis conducted by Andersson et al. [38]. Interestingly, the becoming teachers also shared experiences from learners favoring the majority language over their home language. In a world of superdiversity, some learners have several languages, e.g., a home language, the previous language of instruction, the current language of instruction, and the majority language. The becoming teachers' expressed frustrations about this storyline, which illustrates the importance of addressing language-related issues in mathematics teacher education in the 21st century.

The storyline about the importance of accepting diverse methods when doing mathematics is recognized in the field of mathematics education and overlaps with the skills needed in changing times. The importance of diverse methods is visible in mathematics education literature [59,63], as well as in official documents [74]. Nolan and Keazer [60] emphasize the importance of opening spaces where learners can build on their "diverse cultural and community knowledge resources" (p. 151). In our analysis within this storyline, we noted that the becoming teachers expressed frustrations about minoritsed learners not being allowed to use methods familiar to the learners from earlier classrooms. The becoming teachers described diverse multiple methods as a resource, and their frustration was directed at their experiences with teachers favoring method rigidity. We see this frustration in relation to the tensions that becoming teachers encounter when they move between practice (what they learn in schools) and theory (what they learn at university) [24,26]. We identified two different approaches in our analysis connected to how the becoming teachers navigate within these tensions: (1) challenging the taken-for-granted situation in schools and (2) going behind the teachers' backs. Interestingly, the becoming teachers could have ignored these issues within the tension of research-based learning and practice. Contrary to the findings of Jenset and Blikstad-Balas [24], our findings are not connected to what becoming teachers learn in teacher placement is most valued; the becoming students in this study drew heavily on issues that had been discussed at university. Both approaches are motivated by taking the learners' perspectives and cultivating spaces for the learners to learn from strength-based pedagogies anchored in the learner's experiences.

The storylines about issues of invisibility at play in mathematics classrooms draws attention to the becoming teachers' frustrations when they realized that they were categorizing minoritized learners rather than seeing them as individuals. We noted that this invisibility might be about language. The focus tended to be more on how to learn Norwegian and less on how to use the learner's home language to strengthen the mathematical learning opportunities. We also identified that the becoming teachers addressed issues of invisibility related to the learners' cultural background. The issue of invisibility in our data is mostly about becoming teachers not being able to see, even though they expressed awareness of the importance of language in the students' learning of mathematics. The becoming teachers in our study expressed frustration when they realized that they were being captured in the dynamics of how the learners' cultural background becomes invisible. Foyn and Solomon [73] have discussed how invisibility can be reciprocal. In other words, learners can be positioned as less competent by their surroundings (e.g., teachers) and in order to meet their duties following such positioning, they might conceal their learning potential. We recognize the becoming teachers' frustration as one of the first steps toward their awareness of learners' social backgrounds in diverse classrooms [65–68]. The fact that the becoming teachers were sharing stories about invisibility, describing frustration about how they could not see the different learners' languages and backgrounds, indicates that they were drawing on alternative storylines that may challenge the education system. They know the diversity is there, but they cannot see it because the existing storylines are powerful and define what is possible to see. In addition to this, during the analysis connected to storylines about language and method rigidity, we identified issues of invisibility related to how learners' mathematical knowledge can be invisible when schools do not have the resources to use diversity as a starting point for strength-based pedagogies.

The three storylines that emerged during the research process of this study are all connected to inclusion in mathematics, particularly for the students with minoritized backgrounds. Within the debate about how schools can arrange for TPO and inclusive practices for all students, recognizing the issues related to the tendency in Norway to consider minority languages as less valuable than Norwegian [70,71] will be of importance.

Also, to open up the use of diverse methods in mathematics classrooms, to meet the needs of a diverse group of mathematics students, to direct actions for inclusion, and to act with an awareness of how being different may be disguised with invisibility will be important in order to avoid that exclusion from mathematics happens without realizing it.

The second research question guiding this study is: *What implications might these storylines have on mathematics teacher education?*

The presented storylines focus on the becoming teachers' reflections on mathematics education in diverse classrooms. We have seen how the becoming teachers were frustrated and struggled in the "Norwegian only" classrooms. How can we in teacher education support becoming teachers to be more prepared to meet 21st century learners in regular teaching situations? How can becoming teachers facilitate learners' opportunities to express their actual mathematical understanding in classroom communications and test situations? The frustration we sense in the transcripts needs to be met by mathematics teacher educators, and the practice–theory transition [26] might be a catalyst for challenging the practices behind this frustration. We are aware that the educational institutions' way of change is slow-moving, but from our point of view, to raise discussions on the existing storylines that challenge inclusive practices in mathematics classrooms will be a strong entrance point for these changes.

We consider 21st century skills as an important backdrop when identifying implications for mathematics teacher education [4]. Some of the key competencies in 21st century skills are creativity, innovation, critical thinking, and problem solving. Preparing becoming teachers to be "change agents" [74] (p. 4) is crucial for them to be responsive to changing times and tomorrow's society. We therefore propose that teacher education should and must initiate disruptive and transformative practices. The becoming teachers said that critical mathematics education [75] in teacher education had changed their way of thinking. One consequence of this change, the becoming teachers explained, was that they challenged established mindsets in schools during practicums. We do not know how these challenges were met by the teachers. Inspired by Bjerke and Nolan [26], we suggest that teacher education is facilitating post-fields that invite critical conversations about tensions related to these kinds of challenges.

Our findings show that the becoming teachers expressed a strong awareness of the learners' mathematical knowledge rather than their test scores. According to the becoming teachers, this did not align with the teachers who followed storylines of rigidity for both language and methods. The becoming teachers did not always challenge the (mentor) teachers; sometimes, they went behind the back of the teacher and used methods familiar to the learner as a starting point. The intention was to use the familiar method as a gateway to the method(s) favored by the teacher. We wonder whether the lack of direct communication about different ways of understanding mathematical knowledge relates to issues of power. We suggest that teacher education invites becoming teachers to reflect on how they can contribute to dismantling structures of power in their communication with teachers. Darling-Hammond (2004), in Chubbuck and Zembylas [54], suggested something similar: "the profession would indeed be well served if preservice and in-service teachers were given opportunities in their teacher education classes to engage in intrapersonal reflection on their emotional understanding of justice related issues" (p. 307).

To summarize, the implications of the three storylines identified in the becoming teachers' stories of their time in teacher education are connected to the importance of creating space for discussions about issues that may challenge inclusive practices in mathematics classrooms. Even though the issues in this paper are discussed within the framing of the MIM context, we consider it to be important for all learners in mathematics classrooms in the 21st century, regardless of social background. Discussions of inclusive practices should take place so that participants in both worlds can share and connect their experiences from mathematics teaching from different perspectives. We suggest a need to add a post-field space where becoming teachers, teacher educators, and mentor teachers can meet.

Within the frames of mathematics teacher education, positioning theory, inclusive principles, and critical thinking, we identified three storylines and discussed the implications they have for mathematics teacher education. Looking back on the text in retrospect, we realize that the inclusive principles were about to become invisible in the last part of the text. This illustrates the importance of being aware of how the dynamics of exclusion may play out. In our case, we had the opportunity to go back and to try to give more attention to what hindered the attention to inclusive practices. For learners and teachers in mathematics classrooms, this opportunity may not be the same; awareness is crucial for navigating within the dynamics of invisibility. To not be aware of storylines that may hinder inclusive practices may pave the way for exclusion to happen in plain sight.

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Article

Addressing Language Diversity in Early Years Mathematics: Proposed Classroom Practices through a Live Brief Assessment

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Abstract: There is a growing emphasis on the role of language in teaching and learning mathematics, most significantly in classrooms with increased language diversity. Consequently, teachers face considerable challenges in accommodating diverse needs and must employ strategies to support all students. It is, therefore, crucial to provide prospective teachers with opportunities to enhance their pedagogical approaches while raising their awareness of the relationship between language and mathematics. In this respect, Live Brief assessments in Higher Education, which involve students working on authentic projects/tasks from a school, may be a promising avenue. This research draws on the 19 Live Brief group presentations prepared by a total of 118 Year 1 prospective primary school teachers, specifically focusing on the language-related challenges faced by a local school in early years mathematics. The data encompassed prospective teachers' proposed practices, including one-to-one, small group and whole class activities, that aimed to address language diversity. Data analysis was informed by Moschkovich's three perspectives on the relation between language and teaching and learning mathematics, namely lexicon, register and situated-sociocultural perspectives. While a lexicon perspective was commonly evident in the activities, the manifestation of a situated socio-cultural perspective mainly in the one–to-one activities is noteworthy, given its social and discursive nature. Three themes encapsulated a range of practices suggested in the findings: explicit vocabulary teaching, different strategies of scaffolding and utilising multi-sensory approaches. While the lexicon and register perspectives were commonly evident, the situated socio-cultural perspective was much less commonly manifested in the practices. We offer implications to initial teacher education curriculum, future research and policies about teaching and learning mathematics.

Keywords: language; mathematics; England; Live Brief assessment; language diversity; early years

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1. Introduction

The importance of language in teaching and learning mathematics has gained increasing attention, especially within diverse language contexts to promote equitable practices [1–8]. In Sfard and Kieran's (2001) [9] view, mathematical thinking and doing cannot be differentiated from the act of mathematical communication, which occurs through different channels, including multiple languages, different variations of the same language, gestures, diagrams, symbols, etc. It is, therefore, crucial that all pupils, especially if the language of instruction is different than their home language, need to be supported to be able to learn to communicate mathematically [3,10]. In this respect, the early years context is particularly significant. Research suggests that many socially disadvantaged children in early years are less likely to develop their spoken language and vocabulary [11], which makes early interventions essential to address issues around language skills and mathematics knowledge [12,13]. Nevertheless, there appears to be a research and practice gap in terms of how to prepare teachers to address language diversity in classrooms [14–17] to develop meaningful interventions. Although the conceptualisation of this support, especially in initial teacher education, is a complex endeavour due to limited access to schools' socio-cultural contexts, it is vital to explore prospective teachers' perspectives and proposed practices through authentic assessments.

In Higher Education, Live Brief assessments (real-world projects) are one possible avenue for providing access to schools' realities. Typically, Live Brief assessments aim to provide authentic tasks, which are within the scope of learning outcomes of modules, and are presented by external organisations to university students. University students collaborate to suggest development ideas and/or solutions to address the task at hand. Subsequently, university tutors and external organisations provide feedback to students' solutions/ideas. Arguably, such authentic assessments may offer enhanced access to the complexities of schools' practices with noteworthy caveats that we will examine in the following sections. Addressing those caveats, Live Brief assessments may provide an opportunity for prospective teachers to influence schools by bridging theory and practice. Within the context of this research, Live Brief assessments are utilised in an undergraduate course focused on subject knowledge in mathematics. Students were tasked with addressing a local school's development idea, titled 'How to Close the Language Gap in Early Years?'

This research addressed the following questions to address the aforementioned gaps:

- What are teaching practices that prospective teachers propose through Live Brief presentations to address language diversity in early years mathematics? (or as the school stated, to address 'the language gap' in early years mathematics).

We the authors of this paper, would like to position ourselves within this research before we outline some key literature on this field. Primarily, we believe that our identities have influenced how we approached this particular research, including the types of questions we asked and how we interpreted the data. Firstly, we possess language diversity ourselves, in the sense that our home language, Turkish, is accompanied by English in our language repertoires. Both of us have experience of teaching mathematics in Turkish and English, in contexts where languages other than those were the dominant languages of instruction. For instance, the first author taught mathematics in a primary school in Finland, while the second author taught mathematics in Denmark. Moreover, we are both passionate about teaching mathematics for social justice and reject the discourses of 'innate ability' and deficit approaches. In contrast, we firmly believe that everybody can and should learn mathematics as a human activity because it holds the power to enable asking the right questions to understand and change the world for the better.

With these reflexive notes in mind, we begin by defining language diversity and scoping the literature on the complex relationship between language diversity and mathematics. We then offer details about the context and design of this research before presenting the findings. We conclude with revisiting the related literature for final remarks and offer implications for teacher education, future research and mathematics education policy.

1.1. Language Diversity and Mathematics

1.1.1. Language Diversity

Considering current world events, including migration and technological advancements, 'mathematics education is always happening in the context of language diversity' [18], p. 4. Drawing from Planas et al. (2018) [5] and Barwell et al. (2016) [18], we understand language diversity as 'the languages of the learners as they interact with mathematics but also to the languages for communication: official languages of instruction, languages of teaching, and languages of thinking and learning' [5], para. 17. The use of the term language diversity is, therefore, not solely connected to the concepts of 'multilingualism' or having 'English as an additional language (EAL)'. Instead, it accounts for a broad range of communication mediums such as dialects, sign languages, diagrams, symbols, etc. In fact, these very concepts can be problematic considering teaching mathematics for social justice.

One reason is that the aforementioned concepts marginalize some groups of students as they are not part of the norm. This may result in deficit approaches in teaching and learning mathematics, such as simplifying content to make it 'more accessible' [19]. Additionally, as Barwell et al. (2016) [18] discussed, the concepts 'multilinguism' and 'EAL' imply static forms of communication and a particular focus on languages, rather than learners. Barwell et al. (2016) [18] also state that it is often difficult to have neat boundaries between languages, hence these concepts are not particularly helpful for designing equitable mathematics teaching practices. García (2017) [20] agrees with this stand, by advocating for 'translanguaging', which involves leveraging all the language repertoires learners possess, transcending traditional language boundaries and shifting the focus from languages to learners. Similarly, Mazzatti (2022) [21] p. 4, defines translanguaging as 'complex ways of using languages to communicate, to understand, and to transform'. Nevertheless, the ways in which these languages are used in mathematics classroom by teachers can be politically and socially constructed under the dominance of monolingual curricular standards [22]. Chronaki et al. (2022) [22] suggest that considering the relational and interactive act of dialogicality in translanguaging is important to prevent the marginalization of some students. Even within the 'monolingual' teaching context, the term 'heteroglossia' is helpful to understand the complexity of learners' language repertoires. Bakhtin's (1981) [23] notion of heteroglossia refers to various forms of speech types, due to, for example, socio-cultural differences. Language diversity includes and goes beyond the concepts of 'multilinguism', 'EAL, 'translanguaging', and 'heteroglossia' and draws attention to 'language as a resource' rather than an obstacle. To account for such variation, while policy rhetoric commonly utilises the term 'English as Additional Language' (EAL–e.g., [24]) in the context of this paper. (While offering an extensive theoretical account on the aforementioned concepts in this section is beyond the scope of our paper, we believe the descriptions provided above serve to conceptually situate our research).

1.1.2. The Nature and Purpose of Mathematics

The notion that mathematics is a universal language and/or a culture-free subject, predominantly emphasizing cognitive aspects of learning, has been challenged. The "social turn" [25] and subsequently "socio-political turn" as termed by Gutiérrez (2013) [26], have been influential in reconsidering the nature and purpose of mathematics education. This change in perspective has led to an increased recognition of the interdependence between mathematics education and the sociocultural contexts in which teaching and learning take place. Particularly, it highlights how this context influences the languages used in mathematics. Concepts such as 'ethnomathematics' [27], 'speaking mathematically' [28], and more recently, the 'situated sociocultural perspective' [4], and the 'culturalist perspective' [29] reflect this shift. Furthermore, Gutierrez's [30,31] four dimensions of equity is another helpful framework to consider the nature and purpose of mathematics. She discusses how teaching mathematics often aims to provide 'access' (e.g., teaching vocabulary in English) and 'achievement' (e.g., high scores in tests), but oversights 'identity' (e.g., how students develop their mathematical identity within language diversity, whether they have opportunities to use their cultural and language resources) and 'power' (e.g., developing sense of agency and consciousness over mathematical learning to understand and change the world). All these perspectives underscore that mathematics is increasingly considered as far from being a language and culture free subject, in fact the opposite holds true.

The research on mathematics education and language has also undergone a significant shift in perspective since its establishment in the 1970s. As such, mathematics teaching and learning in a language-diverse context has attracted much attention, as evidenced by the increase in research in the field (e.g., [3,10,18,30,32–34]). Initially, the focus was on a deficit perspective, which emphasized the challenges and achievement gaps faced in language-diverse contexts (e.g., [32]). However, there has been a transition towards perspectives that recognize the socio-cultural and political dimensions of language and the wide range of language repertoires learners bring to the classroom [20]. Recent changes account for sociocultural aspects of teaching and learning mathematics and view language as a valuable resource in mathematics education (e.g., [3,10,34]).

1.1.3. Addressing Language Diversity through Pedagogy

The role of teachers and pedagogy in mediating language diversity in mathematics teaching has been widely addressed in the literature (e.g., [8,22,35–38]. In line with much research in this field, Schleppegrell (2007) [39] concluded that the role of teachers for communicating mathematically was imperative. Previous research suggested a range of perspectives, concepts and potential support mechanisms to characterise teachers' role and pedagogies. For instance, Lucas et al.'s (2008) [35] concept, *linguistically responsive mathematics teaching,* underscores the importance of three pedagogical practices. These practices are acquiring knowledge of the learners' linguistic backgrounds; identification of potential linguistic challenges/demands that exist in the mathematics tasks; and scaffolding strategies to enable learners to participate and succeed in mathematics as key pedagogical approaches to address language diversity. Scaffolding, as conceptualized by Bruner (1975) [40], involves learning with the support of a 'more knowledgeable other'. Vygotsky's (1978) [41] ideas typically complement discussions on scaffolding, as he posits that learning occurs through social interaction in which language plays a crucial role. In regard to specific scaffolding practices, Zahner and Sterling (2022) [42] suggested note-taking, highlighting mathematical words, elaborating on technical terms, reasoning from context, correcting pronunciation, using home languages, and reading textbooks, which are captured as 'language access strategies'. On the other hand, García (2017) [20] argued that pedagogies should go beyond merely scaffolding and facilitate the transformation of learners as unique subjects in language-diverse classrooms, similar to to Gutierrez's (2007; 2012) [30,31] dimension of identity.

Home languages, as an essential component of one's identity, have been suggested as an indispensable resource for learning mathematics [3,8,43]. In their longitudinal research in Germany, Peter-Koop (2010) [44] found that kindergarten children (age 5) who were identified as migrants, demonstrated significantly better performance when they were offered mathematical tasks in their home language. Moreover, Chronaki et al.'s (2015) [45] found that the use of multiple languages to teach number words to children aged 4 to 6, did not only benefit the students whose home language is different than the language of instruction; it also benefitted others. Such practices can enhance all students' self-confidence while challenging the assumptions around the deficit view of students with diverse language needs. Chronaki et al.'s (2015) [45] research also underlines the importance of collective efforts to challenge such deficit notions by involving not only children, but also parents and teachers. Additionally, Clarkson (2009) [46] suggested a number of effective practices in terms of the ways in which pupils' informal language/home language can be used to progress to a more structured and academic mathematical language. There is a strong agreement in literature that limiting the use of home languages and in fact, 'simplifying' mathematical language in classrooms might inhibit students' mathematical understanding [4,19,47]. Such practices limit pupils' agency [48] and in turn might affect how they develop a positive mathematical identity [49].

Another important theme in the literature is using multimodality as a pedagogical lens to address language diversity in mathematics. Multimodality can also be seen as an important part of translanguaging as it refers to a range of modes to communicate mathematical thinking, including words, body movements, listening, writing, graphing, drawings, manipulatives, and music [18,21]. For example, activities where pupils use their bodies and gestures can enhance collective mathematical meaning-making [50] and open up new spaces for mathematical communication. Multimodality is particularly important in early years contexts, as pupils often develop their own ways of communicating through embodied activities, drawings, and manipulatives, rather than written mathematics [51]. Additionally, Sugimoto [52] employed a Language Demand Tool as a sense-making mechanism for prospective teachers focusing on a range of modes. The tool encompassed different sections for students to observe, including reading, writing, listening, speaking, and representing, alongside language support. The tool proved to be effective in redirecting prospective teachers' attention to the language demands in mathematics learning and also underscores the multimodal nature of mathematics classrooms.

In conclusion, teachers face particular challenges and opportunities in language diverse contexts, where they must make decisions considering the complexities, commonly referred to as "tensions" (e.g., [18] and "teaching dilemmas" [10,53]. Challenges include determining when to prioritise language over content, meaning making over discussion, and how to effectively support the use of multiple languages [10]. In this regard, the literature suggests that teachers might use strategies to only focus on language support and even minimise communication, which then limits how language diversity is utilised in mathematics learning [8]. This is related to teachers' beliefs and dispositions about language diversity in mathematics as some teachers might draw from 'deficit' approaches [54]. Therefore, it is crucial that prospective teachers are provided with opportunities to experience and reflect on such dilemmas or tensions to develop their repertoires for purposeful and socially just practices.

1.2. Context

1.2.1. Research Setting

This paper draws on Author 1′s involvement as a tutor on a module focused on subject knowledge in mathematics for year 1 students. The programme is a 3-year undergraduate degree in primary education studies without the Qualified Teaching Status (QTS), meaning that students can choose different pathways, instead of becoming teachers or follow an additional year of study to gain a QTS degree. If they choose to become teachers, they would be teachers of mathematics for pupils aged 3–12. Although there is a possibility that they might pursue different paths, we refer to them as 'prospective primary teachers' due to the common pursuit of becoming teachers.

The module under consideration aims to provide a foundation in key knowledge and understanding related to pedagogy and practice in early years mathematics teaching in Early Years Foundation Stage (EYFS) (ages 3–5) and Key Stage 1 (ages 5–7). One of the fundamental aims of the module is that students will be able to evaluate and identify good practices in teaching mathematics. The content of the module is mainly based on subject knowledge, but there are some insights about diverse needs in mathematics classrooms and how to address them. Nevertheless, although there are some references to communication and the importance of key vocabulary, language diversity is not explored within the module. Live Brief assessments are used as a formative assessment, which constitutes the main source of data (this will be explained in detail in the following sections).

1.2.2. National Context

In England, the importance of 'spoken language' is underscored, as teaching key mathematical vocabulary is a statutory requirement in the national curriculum of mathematics in England, which is the context of our research [55]. In regard to early years, Early Years Foundation Stage (EYFS) is a national statutory framework that sets the standards for the learning and development of children from birth to 5 years of age [56]. The EYFS framework has undergone a significant change in 2019 to be enacted in 2021, including changes in Early Learning Goals (ELGs) and a greater emphasis on communication and language. Similarly, the national curriculum for mathematics [57] mentions the importance of language in teaching and learning, including a list of key vocabulary for each area as either statutory or non-statutory requirements. (e.g., use the language of equal to, more than, less than (fewer), most, least). Pupils are also expected to reason mathematically using mathematical language; however, the insights from the curriculum seem to draw from a more cognitivist perspective, rather than socio-cultural lenses (e.g., communications to remedy misconceptions). More specifically:

> The national curriculum for mathematics reflects the importance of spoken language in pupils' development across the whole curriculum–cognitively, socially, and linguistically. The quality and variety of language that pupils hear and speak

> are key factors in developing their mathematical vocabulary and presenting a mathematical justification, argument or proof. They must be assisted in making their thinking clear to themselves as well as others and teachers should ensure that pupils build secure foundations by using discussion to probe and remedy their misconceptions [57], p. 4.

As for some statistics to set the broader context, 19.5% of pupils were recorded as having English as second language, which increases to 29.1% at the nursery level [58] in England. This suggests that the assumption of teaching in classes without language diversity should be challenged in teacher education programmes, where prospective teachers are expected to be more equipped to address language diversity through critical, purposeful, and socio-cultural lenses in teaching mathematics. With specific reference to the early years context, research suggests that although the use of language rich mathematics activities is correlated with pupils' broader mathematics skills, early years practitioners reported low levels of confidence in teaching mathematics in general and also a lack of opportunities to develop their pedagogy in mathematics [59].

2. Theoretical and Analytical Framework

We used Moschkovich's (2002) [3] three perspectives as a theoretical and analytical framework on the relation between language and learning mathematics in this study (Figure 1). These perspectives are lexicon, register and situated-sociocultural perspectives, which are powerful lenses to understand and improve teaching practices to enhance communicating mathematically, especially within diverse language contexts. Moschkovich proposed these perspectives to describe mathematics learning with particular attention paid to their relation to language. Each perspective emphasises particular aspects and practices of teaching and learning mathematics. She perceives these perspectives as relational, hence they are not mutually exclusive. More specifically, these three perspectives are nested and reflected into instructional practices with a dynamic relation to each other.

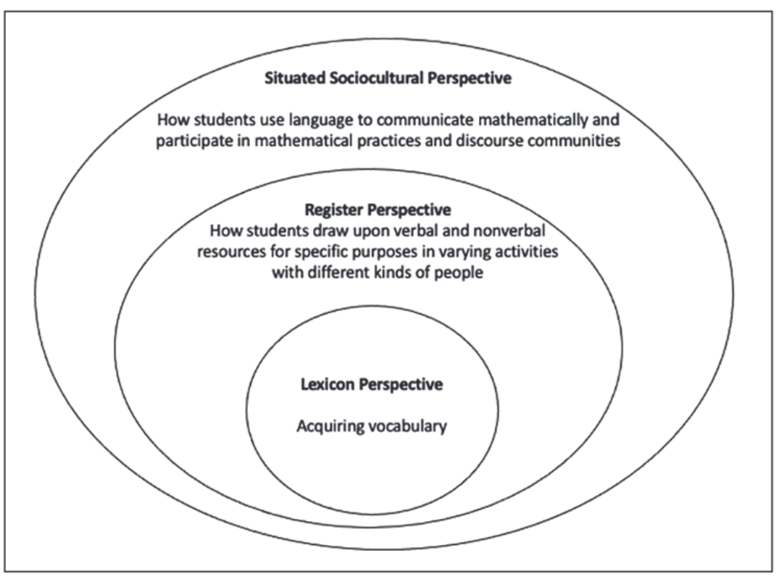

Figure 1. Moschkovich's perspectives on the relationship between language and mathematics [52] p. 179.

Lexicon perspective underscores the importance of explicit vocabulary teaching [60], which differs from a situational use of mathematical terms. An example of this could be teaching the word 'odd' as referring to numbers that are not divisible by 2. Moschkovich (2002) [3] argues that if teaching draws solely from a lexicon perspective, it will reflect a narrow view of language and subsequently limit how teachers may assess pupils' mathematical proficiency. This limitation arises because pupils may use various resources beyond verbal and written communication in the official language of instruction to illustrate their mathematical thinking. Additionally, merely having the knowledge of a set of key vocabulary may not be sufficient to participate in mathematical practices.

Register perspective refers to a language in which multiple meanings can be associated with certain terms [39]. For example, the word 'odd' may denote peculiarity, in addition to its meaning in mathematics. These multiple meanings can be confusing for some students in accessing mathematical knowledge and differentiating it from everyday language. However, Moschkovich (2002) [3] suggests that, in fact, the opposite might as well be true, that is students' using two different registers to communicate mathematically and assisting their mathematical sense-making. Nevertheless, how these boundaries are defined by teachers is open to debate. For example, Zahner and Sterling [42] proposed that Bunch's [61] pedagogical language knowledge framework with language access strategies, referring to pedagogical practices that aim to support accessing discipline-specific terminology, would be helpful. These strategies could be utilised to examine how teachers would draw such boundaries and teach everyday language with discipline-specific language.

Moschkovich's (2002, 2012) [3,4] third perspective, situated socio-cultural perspective, however, views mathematics learning as more than acquiring vocabulary. In Moschovich's [3] p. 197 words, 'A situated-sociocultural perspective can be used to describe the details and complexities of how students, rather than struggling with the differences between the everyday and mathematical registers or between two languages, use resources from both registers and languages to communicate mathematically'. For instance, if the word 'odd' was mentioned in two different social contexts, pupils had a better chance of deriving the meaning from those contexts and perhaps through using a range of resources including gestures, objects, diagrams and their home languages. This perspective suggests that students develop their own understanding mostly drawing from their sociocultural background and are engaged with multiple ways of understanding and mathematical conversation. It also acknowledges the complexities of teaching and learning, as well as its interdependency with the context in which students engage in mathematical communication. In other words, mathematical communication viewed as intercultural communication [29], using social, linguistic, and material resources to actively engage in mathematical practices [3]. As such, it goes beyond the mere substitution of words like 'tortilla' instead of 'bread' in mathematical exercises to engage Latina/o students [31]. These different perspectives imply that teachers would require a set of knowledge and skills, and perhaps employ complex, meaningful and nuanced perspectives to address complex challenges in accommodating the diverse language needs of their students.

3. Methods

3.1. Live Brief Task: Addressing the Language Gap in Early Years Mathematics

Live Brief is an authentic assessment method where university students work together to solve a problem or offer a development idea for an organisation. Live Brief assessments have been used in Higher Education to enhance collaborations between different organisations and to increase students' communication and teamwork skills through tackling real-life problems as a group [62]. Organisations, such as schools, communicate a problem or a development idea, which aligns with the specific learning outcomes of the module(s), to students to offer solutions and suggestions.

As part of the module under consideration, Live Brief assessment was utilised as a formative assessment opportunity for students to respond to a local school's development idea. Fundamentally, the module aims that students will develop subject knowledge of mathematical concepts, to identify and evaluate good practice, and to work effectively as part of a team to discuss theory and practice. In relation to these learning outcomes, Live Brief assessment offered a context for students to work as a group and identify good practices to address local school's development idea, which was about closing the language gap in mathematics. While the module did not cover this topic specifically, students had a chance to engage with various self-selected resources and readings. Due to structural limitations (e.g., time), students were asked to outline their suggested practices in a group PowerPoint presentation in English (the language of instruction).

More specifically, as part of the Live Brief task, the students were required to explain the role of language in teaching and learning mathematics, offer an action plan for the school and also suggest a whole-class, small group and a one–to-one activity. The school to which the students offered their ideas and solutions was in a socio-culturally deprived area where one-third of the students' home languages were different than the language of instruction, English. Table 1 presents information that was sent to students to address in their Live Brief presentation:

Table 1. Live Brief assessment information.

The context: The local school is a relatively small primary school with around 200 students and located in a socially and economically deprived area. One-third of the pupils speak an additional language, besides English. The largest multilingual groups are from Indian Heritage and Poland. According to the school's base line assessments, most pupils arrive at school with a vocabulary deficit and the school put vocabulary teaching at the heart of their EYFS curriculum. Vocabulary is taught in context and repeated in daily life. There are targeted speaking times devoted to developing pupils' vocabulary.
Tasks to complete: • A brief summary explaining the importance of language in mathematics • A roadmap/action plan aiming to close the language gap in mathematics • Three sample activities to go with this plan (one whole class, one small group and one one-to-one) **These activities should:** • Include suggestions to improve language rich environments • Suggest links to mathematics in those targeted speaking times • Consider students with different needs and backgrounds

3.2. Participants

The subject knowledge for teaching mathematics modules included 118 year 1 undergraduate students, and all of them participated in the Live Brief group presentation. In total, there were 19 presentations from three different groups/classes of students. Presentations were prepared in groups, so the data presented here illustrate collective proposed practices to address language diversity in early years mathematics. Author 1 acted as a tutor in the module and also took part in the formative assessment process for all presentations. Participants were not introduced to literature on the role of language in mathematics as one of their tasks was to research this area to be able to offer evidence-informed practices. The module was the students' first mathematics related module in their programme; they had previously attended modules that covered primary education, the development of children, primary pedagogy and subject knowledge in English and science and technology.

3.3. Data Generation and Analysis

Data were collected through the formative assignment submissions, students' group Live Brief presentations, for the module. In groups of 5–7, students were required to prepare a 10 min presentation, addressing the problem/development area that a local school presented for the Live Brief task. We have analysed all 19 presentations and present extracts from the ones we have permission to share.

We employed a reflexive thematic analysis [63] following the six stages below and illustrated our theme construction process in Table 2:

1. **Familiarization:** Author 1 had familiarity with the presentations through the students' live presentations. Nevertheless, we both read all the presentations independently and looked at one from each group together, thinking with Moschkovich's (2002) [3] three perspectives and related literature on the role of language in teaching and learning mathematics. This stage helped us to make meaning of data in light of the literature we have been engaging with. We made some preliminary notes for each presentation depending on our first interpretation of data and included them as 'group summaries' to inform our next stages of data analysis.
2. **Generating codes:** We chose 2 presentations from each group and started to draw links between Moschkovich's perspectives and related concepts independently with students' proposed practices. We also identified any hybrid perspectives engrained within some activities that could not be categorised into one perspective (i.e., occurrences of at least two different perspectives). We created a table (Table 3) where the occurrences of each perspective were recorded. We utilised some sensitizing concepts [64], in other words interpretive devices, that are derived from the literature (e.g., [38,42]) in generating codes. These concepts included lexicon perspective (e.g., explicit vocabulary teaching, pupils' writing their own definition); register perspective (e.g., multiple meanings in everyday life and mathematics, more focus on multiple registers instead of students' using such registers to communicate); situated-sociocultural perspective (e.g., participating, communicating, reasoning, making sense in mathematical practices using every day and mathematical discourses, language as resource). We had frequent meetings to discuss our initial codes and potential categories in light of the sensitizing concepts and our research question.
3. **Constructing themes:** Upon completion of coding all presentations and creating a summary of the occurrence of each perspective within the proposed practices, we started to think about possible ways of collating codes, categories, and practices to form themes (see Table 2). We dwelled on underlying factors that might have produced the proposed practices and some higher order concepts to capture the essence of data, addressing our research question.
4. **Reviewing themes:** We reviewed the themes, cross-checking the codes, categories, and the content of presentations to capture the dataset meaningfully and coherently.
5. **Defining and naming themes:** Three themes, explicit vocabulary teaching, scaffolding and the use of multi-sensory approaches were selected to account for the students' proposed practices in their Live Brief presentations. These themes are selected as they had the most explanatory power to capture the essence of the data and helpfully address our research question. Data extracts are chosen to illustrate each theme effectively.
6. **Writing up:** The final report has been produced collectively and reflexively, that included cross-checking the writing with the codes, themes and data set separately and engaging with discussions on the logical order of the themes and extracts from the dataset. In order to make data organisation and classification more manageable, we numbered the group presentations (e.g., G1), although we focussed on the tasks individually. We agreed that the most common practice, explicit vocabulary teaching should be the first one to portray the dataset.

3.4. Ethical Considerations

This research was granted ethical approval from the ethics committee at the university and complied with the British Educational Research Association's (2018) [65] guidelines. Students were provided with participant information sheets and asked to give their consent in an online form. Although the Live Brief presentation was formatively assessed, hence no summative grades were given, students' consent was sought at the end of the teaching period and after grades were released, to prevent any potential conflict of interest. All identifiable information (e.g., students' numbers, names) was removed from the presentations as Author 2 is external to the university.

Table 2. An illustration of theme construction.

Illustrative excerpt		
Singing number related songs: At the beginning of our lesson the whole class activity will be using songs this is a good way to close the language gap in mathematics in the EYFS this is because using music is good for active engagement with the class, it's also been shown that using songs and music within the classroom can have an impact on the way children's brains process information and enhancing their language skills and speech. In the class we will be singing the "five little ducks" alongside singing this song we will have puppets this is so we can give children a visual representation this will be especially useful for those children with additional language and needs. As we are singing the song as the whole class using puppets, we will have a video of the song to help children's cognitive recognition. Around the room we will have numbers on the walls to make it clear for the children if they need further help and guidance. This will then help to close the gap on mathematical language as it helps to explain the concept of 'one more' and 'one less'. *G9*		
Codes	**Categories**	**Themes**
• Using visual aids to scaffold • Using music for cognitive and emotional engagement • Repetition of counting • Using concrete materials (i.e., puppets) to scaffold	• Lexicon perspective (teaching counting using songs, visual aids, and puppets–providing opportunities to learn key vocabulary) • Register perspective (providing opportunities for sentence frames from daily life)	• Explicit vocabulary teaching (e.g., repetition of numbers, the numbers on the walls, the concepts of 'one more' and 'one less') • Scaffolding (e.g., the use of puppets) • Using multi-sensory approaches (e.g., the use of song and video)

Table 3. The number of occurrences of Moschkovich's perspectives in the proposed activities.

	Moschkovich's Perspectives		
Activities	**Lexicon Perspective**	**Register Perspective**	**Situated Sociocultural Perspective**
One-to-one activity	14	13	8
Small group activity	17	11	3
Whole class activity	19	11	3
Total occurrences [1]	50	35	14

[1] The reason why the total numbers do not add up to 57 (19 activities × 3 types of activities) is that some perspectives were evident as a hybrid form. We counted the occurrences of such cases for both perspectives.

4. Findings

In the following, we present an overview summary of the occurrences of the three perspectives in the activities proposed by the students. Subsequently, we illustrate a range of pedagogical approaches from the students' Live Brief presentations; namely, explicit vocabulary teaching, scaffolding, and the use of multi-sensory approaches.

4.1. Summary of the Occurrences of Moschkovich's Perspectives in the Proposed Practices

This brief section outlines a summary of the occurrences of Moschkovichs's perspectives in one-to-one, small group and whole class activities. While the table presents numerical data, we are not particularly interested in the frequencies solely. What we would like to achieve here is to illustrate a distribution of occurrences within each type of activity (i.e., one-to-one, small group, and whole class) to offer a potential starting point for future research and practice regarding addressing language diversity through pedagogy.

Table 3 demonstrates occurrences of Moschkovich's three perspectives within 57 proposed activities, that include one-to-one, small group and whole class activities (19 each). It appears that the majority of groups approached the role of language in mathematics mainly from a lexicon perspective and register perspective, although this was less prominent. Furthermore, a situated socio-cultural perspective was less significant in these practices. While this is not surprising on its own, findings suggest that although each type of activity manifested the three perspectives, a situated socio-cultural perspective was more evident in one-to-one activities compared to others. Considering the social and discursive characteristics of this perspective, we would expect it to be more prominent in small group and whole class activities. We will delve into potential reasons for this in the discussion, after examining the details of the proposed practices in the following sections.

4.2. Proposed Practices to Address Language Diversity

This section illustrates three themes to capture a range of pedagogical approaches that are proposed by prospective teachers. The first theme is about *explicit vocabulary teaching*, which captures a range of practices including offering child-friendly definitions, creating word walls for key terminology, and using repetition to expose pupils to the terminology. Secondly, we will capture practices that fall into the *scaffolding* theme, for example, teacher modelling and creating small groups to offer language support. Finally, we will present a range of *multi-sensory approaches*, such as the use of songs and concrete materials. In each theme, we will map Moschkovich's three perspectives and provide example activities from student presentations. These example activities aim to illustrate the theme under exploration and also how these activities diverge or converge with others.

Explicit Vocabulary Teaching

Teaching key mathematical terms explicitly emerged as a common practice in the students' presentations. They shared a common rationale for teaching vocabulary, aiming to address the 'language gap' in mathematics mostly because acquiring vocabulary was perceived as the central problem that pupils had, echoing national curriculum documents and policy rhetoric. G10, for example, underlined the significance of vocabulary development in fostering mathematical proficiency. Similarly, most groups put an emphasis on how the National Curriculum for Mathematics stated the importance of mathematical vocabulary for mathematical justification, argument, or proof [57]. While there was focus on the explicit vocabulary teaching as an activity mostly distinct from everyday life contexts, there were some instances where the groups alluded to key vocabulary as something to be utilised in real life. For example, G2 stated that 'By regularly being exposed to mathematical language, the students are more likely to understand and incorporate what they have learnt into their everyday dialogue.' As for how this 'exposing' is done, different approaches were evident, including offering child friendly definitions, creating word walls so that pupils can see key terms frequently and focusing on repetition for memorisation, as stated by G1:

> Teacher addresses the important key words for the lesson, whilst the children are gathered during 'carpet time'; ensuring the mathematical terms have been repeated and rehearsed collectively by the students multiple times to develop their memory and familiarisation.

The suggested activities typically involved teachers starting the lesson by teaching the definitions of words and subsequently referring to them, while questioning pupils' understanding of those words later. Mostly, the expectation from teachers was to lead vocabulary acquisition by creating opportunities for students. The following example illustrates this common practice (Figure 2).

The above one–to-one activity mainly draws from a lexicon perspective as it pays attention to creating opportunities to see, hear and say key terminology. However, it remains unclear how language diversity is supported. Arguably, incorporating visuals might have been considered as a support to learn key vocabulary. However, such approaches may lead to rote memorization rather than supporting students' active meaning-making processes. Consequently, connecting mathematical language to home languages and learning key vocabulary in context to communicate mathematical reasoning remained unaddressed.

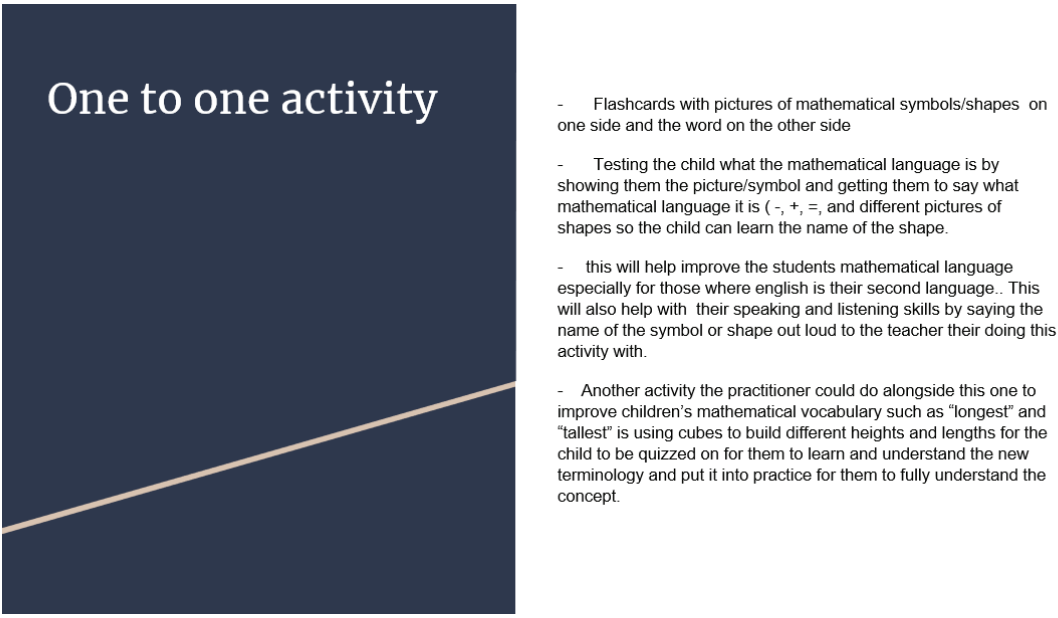

Figure 2. The use of flashcards to teach vocabulary.

We observed some variations in terms of how key vocabulary was proposed to be taught in the activities where a register perspective was manifested, including making connections to real life and referring to multiple meanings that some key words might have. For example, G2 commented on how confusing the word 'take away' might be, if it is used to mean 'subtract', as it can have other connotations (e.g., take away food). As presented by G19, the following whole-class activity offers a potential to teach vocabulary through storytelling, while making connections to everyday life concepts and facilitating students' own meaning making processes through mathematical communication (Figure 3). The activity below also creates opportunities and tools for pupils to illustrate their learning, mainly referencing a register perspective while alluding to a situated socio-cultural perspective, through providing opportunities for participation and social interaction by using a range of objects (e.g., 'porridge' bowls). This activity diverges from other whole class activities by at least attempting to include a situated-socio-cultural perspective. Nevertheless, it is not a strong example where a situated socio-cultural perspective is evident. One reason is the lack of involvement and consideration of pupils' socio-cultural backgrounds including their language diversity. For instance, it would be questionable how the particular book (i.e., Goldilocks and the Three Bears) and the choice of 'porridge' are relatable to some students. It is also important to note the lack of creativity here, as the practice refers to a common mainstream early childhood activity with minimal room for language diversity.

There were a few examples where groups suggested using translators to facilitate vocabulary acquisition of pupils with 'EAL' or, in fact, making connections to students' homes to foster vocabulary learning, as illustrated by G12:

> [Repetition of mathematical language at home] is a good way to help the children of Indian and Polish descents if English is not their first language as it gives their parents opportunities to communicate and translate with their children.

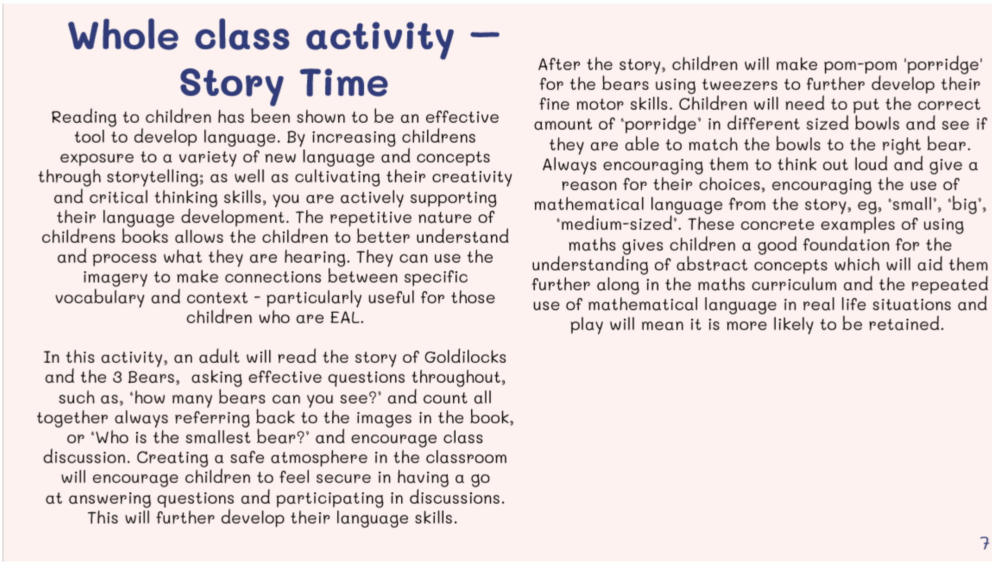

Figure 3. The use of storytelling to teach vocabulary.

The statement above illustrates good intentions to foster vocabulary acquisition through parental involvement; however, it does not seem to consider potential dynamics of home environment (e.g., parents' education level, languages, their work commitments, etc.) and seems to divert the teaching responsibilities to parents due to language diversity. This subsequently limits the ways in which teachers can be agentic in their practice, mediating language diversity within the classroom through meaningful and constructive relationships with parents. In fact, some groups highlighted the importance of incorporating pupils' home languages within teaching; however, it too stayed limited:

> Our Action plan to close the language gap in mathematics within the EYFS includes having labels in English, Polish and Indian as this will then incorporate the children's additional languages. By having labels, the children will then have a visual aid to help the students to understand mathematical concepts. (G9)

Although including visuals with words from pupils' home languages might be a constructive initial step, the ways they are utilised, whether following a lexicon perspective (i.e., simply teaching key words in both languages) or register perspective (i.e., focusing meaning making through communication) or situated socio-cultural perspective (i.e., involving mathematical discourse through social interactions) remains crucial to explore further.

5. Scaffolding

One of the common reasons why the groups suggested scaffolding activities was the importance of additional support from peers, teachers, and sometimes from additional resources, particularly in small group and one-to-one activities, for addressing language diversity. This support included teachers' modelling, differentiation based on students' perceived academic achievement, pairing students who speak the dominant language (English) better (as perceived by teachers) and offering a range of representations to scaffold students' learning. In contrast to the previous theme of explicit vocabulary teaching, we observed evidence of all three perspectives (lexicon, register, situated socio-cultural), often in combination and in varying degrees. This suggests that scaffolding, if designed meaningfully by prospective teachers, has the potential to address language diversity in mathematics classes by creating spaces where different communication resources might interact. There were only three small group tasks out of nineteen that referenced a situated-socio-cultural perspective, and one of them is presented in Figure 4. The activity reflects aspects of a situated-sociocultural perspective while also incorporating lexicon and register perspectives through its focus on vocabulary learning and encouraging pupils to use a mathematics register (Figure 4).

Figure 4. The use of scaffolding to teach about numbers and shapes.

Similarly, G14′s incorporation of the three perspectives is presented below (Figure 5). This is another small group activity that diverges from the rest of the small group activities where a few sentence frames are included (e.g., 'You need one more counter to make five.'), following a register perspective. The word of the day component aligns with a lexicon perspective, while providing a range of communication channels for pupils to demonstrate their mathematical reasoning can be related to a situated-sociocultural perspective. Nevertheless, home languages are still not considered within these practices.

Mainly, groups focused on the 'help' aspect of scaffolding, which was often directed to students who have diverse language backgrounds and needs. This help would sometimes come from teaching assistants as well as peers and teachers. In some cases, such help was perceived similarly to the help for students with special needs, indicating a potential deficit approach to language diversity in teaching and learning mathematics.

> To support children with English as an additional language/speech difficulty, a speech therapist will attend our activity and provide us feedback for those who may need extra support. (G4)

It is important to note that Bruner's (1975) [41] concept of scaffolding was envisaged to be about social interactions, reciprocal and active processes. For example, there was not any convincing evidence in the dataset in which pupils' diverse language backgrounds would be used as resources to scaffold mathematics learning within the classroom. In fact, the most common strategy for scaffolding was a whole-class question–answer strategy, which was led by teachers to create opportunities for engagement. We also noted that scaffolding in whole-school activities was mainly through additional resources that teachers would bring into the classroom, such as concrete materials, flashcards, labels including translated words, word walls to assist students' memory and retention. These resources offer a space and opportunity to communicate mathematically in different ways; however, how they are utilised remains to be seen.

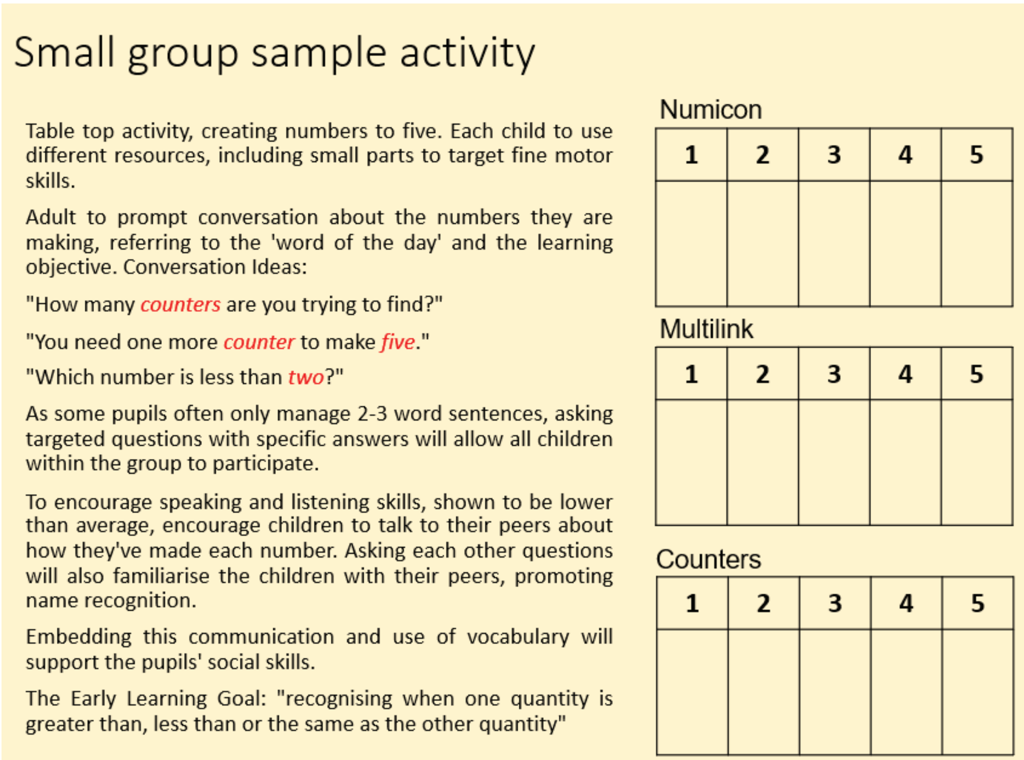

Figure 5. The use of three perspectives to teach about numbers.

6. The Use of Multisensory Approaches

Most students perceived the use of multi-sensory approaches, such as incorporating songs, the use of various manipulatives and representations including concrete materials (e.g., Numicons, Unifix cubes) and visuals, the use of play, especially in outdoor learning environments and cooking. Repeated rationales of those practices include making real-life connections, encouraging communication, and also addressing language diversity. However, there were cases where some groups attempted to use such approaches to minimize language-related demands to make the content 'accessible' for everyone. This is also a reflection of an approach that sees 'language as a barrier', and subsequently some groups proposed their solution as using language less. For example, G11 attempted to use visuals, potentially gestures, and body movements to create opportunities for addressing language diversity in their small group activity (Figure 6). This resonated with other small group activities in the cohort and resembled another mainstream activity that can easily be found online. Their suggested practice remained limited, and in fact, problematic, as the main aim appeared to be minimizing language exchanges. This was particularly evident in the following quotation: 'This helps with language barriers as the children are using their spatial skills rather than a language they don't understand'. Although embodied activities have the potential to address language diversity [50], it is imperative to explore prospective teachers' rationale and enactment of proposed practices in real life.

Small group activity

Lady-bug sheet- One side will be filled out by each student, they then pass it to the next student and must make it equal on the other side, this will help them with basic counting. As it is basic symmetry it requires less use of the English language, helping to support any students with a language barrier, it is visual learning, which also supports their fine motor skills. The children could also use peer support - (Vygotsky MKO) - as the more able students could help the lesser ability students in understanding and completing the task.

Giving out numbers to the children in groups of 5 (Numbers between 1 and 10), you then ask them to do a number of tasks to help develop their knowledge. The tasks should increase in difficulty, for example: Task 1 - Put themselves in order. Task 2 - Find the sum of their numbers. Task 3 - Separate the even from the odd etc. This helps with language barriers as the children are using their spatial skills rather than a language they don't understand.

Figure 6. The use of multisensory approaches to teach counting.

Nevertheless, there were other instances where the groups focused on the 'participation' element of using a range of multi-sensory approaches. For example, G15 suggested a play-based activity with large trays filled with coloured water and different sized, clear containers to teach about measurement. The participatory element included tasks such as interacting with those resources through independent play to a more structured play with guided questions, estimation (e.g., 'ask them to guess how many scoops do you think will fill this container?), and communicating their mathematical reasoning (e.g., how do you know which one is biggest?). While these practices were mainly small group tasks, there were whole class examples too. For example, G14 suggested that the class would sing the 'five little ducks' song while acting out like the ducks and answer questions such as 'how many ducks have gone away', 'if there is one less, how many will we have? Are there more ducks with mummy or less ducks?' This practice aimed to teach vocabulary, such as 'more than' and 'less than' through the song and role playing as a whole group. Those proposed activities, which draw mainly from a register perspective while tapping into a situated socio-cultural perspective (through offering situations where pupils can communicate mathematically through their bodies, gestures, for example), are essential to address language diversity in mathematics. This is mainly because the multimodal nature of these tasks enables different language repertoires to be activated, hence the increased engagement in mathematical communication. Nevertheless, home languages are still ignored to a great extent within these activities.

Following a multimodality trend, most of the activities were suggested to be outdoors so as to relate to different senses, such as touching, seeing, smelling, hearing, etc. There was an agreement among the groups that outdoor play would put key vocabulary in context, provide enhanced opportunities for communication and facilitate cognitive and emotional engagement. Another example of a multi-sensory approach was cooking and G17′s proposed practice below is a good example that captures the content of similar activities (Figure 7). In this small group activity, students were encouraged to communicate mathematically in order to prepare a fruit salad. Language diversity appeared to be addressed through pictorial cues alongside written instructions and also offering opportunities to illustrate mathematical thinking (e.g., sharing the blueberries). If home languages were incorporated within these activities (e.g., in the pictorial cues), there would be a greater chance to address language diversity.

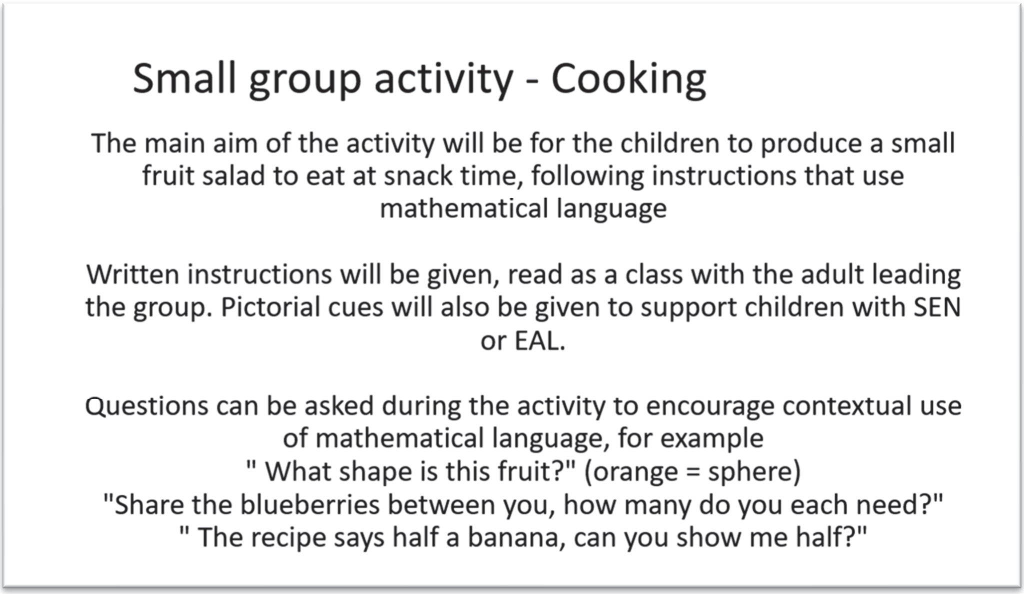

Figure 7. The use of cooking to teach about shapes and numbers.

7. Discussion and Implications

Drawing from our findings, we organise our discussion section focusing on two areas: how language diversity is seen and operationalised in the activities that are proposed by prospective teachers and how the nature of mathematics as a subject might be perceived by them. These two areas, we believe, are two strong starting points to zoom-out from our data and compare our findings with literature, to offer implications to teacher education, future research, and policies.

8. Language Diversity: A 'Problem', Resource, or a Neglected Aspect?

While some groups considered language diversity in the proposed activities, the majority either did not acknowledge it in their presentations or appeared to view such diversity as a problem to be resolved. This finding concurs with related literature that teacher education programmes are struggling to prepare prospective teachers with adequate knowledge and skills to address language diversity in teaching and learning mathematics [8,14–16]. After critically reflecting on our research context, it became evident that the module with which Live Brief assessments are associated includes limited content about addressing language diversity in teaching and learning mathematics. More specifically, the content of the module is mostly filtered through the dominant policy rhetoric (i.e., EAL), possibly leading students to see language diversity as a special need that is often classified in the same group with dyslexia and dyscalculia, for example. This might indicate that some Initial Teacher Education curricula may not be utilising research-informed practices effectively to address diversity [66] or may include very limited, if any, content to prepare prospective teachers in this regard [14,67]. Furthermore, while university tutors have no control over the specific content of the Live Brief assessment, it would be valuable to dedicate some time to explore the topic once it has been finalized by local schools. Reflecting on this, introducing current debates regarding language diversity in mathematics within the specific school context and also encouraging students to take risks, be creative and design their own practices (rather than mimicking mainstream practices) could provide a more effective starting point for students. Moreover, despite students being provided with contextual information about the school, they lacked information about the pupils themselves. If there were opportunities for our students to pose questions about the pupils, they could potentially design more meaningful activities to address language diversity. Without taking such steps and providing support to prospective teachers, relying solely on the implementation of Live Brief assessments as a means to address these issues by offering authentic real-life contexts may not fully achieve their intended purposes. Therefore, providing opportunities for students to practice, even on a small scale, appears to be essential.

The lack of professional support could also be exacerbated by limited guidance and direction through the nationally produced curriculum documents [8,17]. Findings suggested that the national curriculum for mathematics documents in England appeared to be students' first sense-making tool. Although there is a separate document outlining how to support pupils who have 'English as an additional language' [24] and there are some references to the role of language in the teaching and learning of mathematics, there are not any specific formal guidelines available for addressing language diversity in mathematics [57]. This opens questions around how the politics of 'diversity' discourses and practices are perceived by the government and how these perspectives might be translated into a range of curriculum making practices, including the production of guidelines. Prospective teachers should be encouraged to pose similar questions in their teacher education programmes to better understand how mathematics education is strongly connected to politics.

A range of interventions were evident including extra tuition by teachers, strengthening home–school connections, one–one activities to teach mathematical terminology in English, and having visual aids in students' home language, echoing the literature in this area [5,42]. Although some of the interventions have the potential to tap into more sociocultural aspects of teaching and learning mathematics (e.g., strengthening home–school connections, and incorporating students' languages within teaching), these perspectives often lacked criticality and remained superficial. This was particularly evident when students proposed diverting the responsibility of involving home-languages to parents, solely through parents' involvement in pupils' learning at home. Additionally, there were other groups attempting to make connections to pupils' home languages by using web-based translators during teaching or investing in flash cards where key vocabulary appears to be in different languages. This aligns with research indicating that early career teachers prioritize vocabulary in mathematics teaching practices, but are less likely to incorporate mathematical communication and discourse [38]. Although these steps are valuable and illustrate an awareness of language diversity, Gutierrez (2002) [68] argues that it is not sufficient to solely teach key vocabulary in home languages. Instead, teachers should honour pupils' diverse experiences, come to know their students through informal dialogues, avoid applying deficit approaches and provide rich opportunities for discussions in mathematics classrooms. For example, creating translanguaging spaces and challenging monolingual curriculum standards [22] through encouraging students to use home languages while solving mathematical tasks and discussing with peers, using their funds of knowledge, and avoiding deficit discourses are key in addressing language diversity. As such, we reiterate that Live Brief assessments might be limited without nuanced contextual information to outline such practices or a space for prospective teachers to imagine what translanguaging might look like in a unique context. It was striking, but not surprising (considering most of the current policy guidance), to see that the majority of presentations mentioned having English as a second language in a similar vein with students having special needs, such as dyslexia and dyscalculia. This implies that in such perspectives a deficit approach might be evident, as reported elsewhere [36,54]. It is, therefore, essential to explore prospective teachers' perspectives on and attitudes towards language diversity as these will be the building blocks of their future practices.

The proposed practices suggested that students tended to address language diversity more in the small group and one–one activities with an emphasis of scaffolding by using diverse teaching materials. The most common practice addressing the role of language was teaching vocabulary explicitly at the beginning of the lesson to a whole class. As aforementioned, this would have serious consequences in the way pupils are supported and assessed in terms of their mathematics competence (e.g., a limited view of language, not acknowledging and/or valuing other resources pupils might use). Moreover, while groups showed flexibility to adopt multisensory approaches for all types of activities, the integration of different senses lacked meaningful connections to mathematical communication and discourse. Although these activities (mostly from very typical early childhood practices that can be found online) were often presented as being relevant to real-life contexts, their actual relevance to students' lives remained uncertain. Therefore, it is suggested that prospective teachers should be provided with rich opportunities to develop their pedagogical repertoires and instances to interrupt their thinking about the role of language in teaching and learning mathematics. Nevertheless, some research argues that teachers might face dilemmas (e.g., [10] and tensions (e.g., [18]) to balance and address such a range of diversity in the classroom, hence, continuous professional development and willingness to improve practice for promoting social justice for everyone seem to be the key.

9. Mathematics: A Language Free Subject or a Communication Tool?

The nature and purpose of mathematics as a school subject has attracted much attention [69], especially in recent decades from a socio-cultural lens [26]. Although some teachers often perceive mathematics as a language-free subject [52], recent theoretical and empirical contributions in this area strongly disputed such long-standing beliefs [4,18,36,70]. Nevertheless, it appears from the groups' presentations that there are still some traces of such beliefs into the proposed practices. For example, there is an indication that teaching key vocabulary and symbols would be sufficient to address the 'language gap'. Furthermore, some tasks aimed to minimise verbal communication, most likely with good intentions that pupils with language diversity would be less challenged in terms of language and could divert their attention to mathematics content. However, the literature suggests the opposite, maintaining that mathematics teaching should be language rich, actively involve students' socio-cultural and language backgrounds meaningfully and focus more on students' mathematical reasoning, problem solving and thinking processes [4,35,70].

Furthermore, a situated socio-cultural perspective was evident more within one–one tasks, while we would expect to observe more occurrences in small group and whole group tasks considering the social and discursive nature of this perspective [4]. Prospective teachers might find addressing language diversity more manageable in their one–one interactions, potentially due to a lack of pedagogical repertoire of inclusive differentiation in whole-class activities and a lack of awareness of how to facilitate meaningful mathematical communication between peers. Additionally, students might approach language diversity from deficit perspectives, resulting in individualized interventions, rather than using language as a resource within small group and whole class discursive activities.

The aforementioned observations in the dataset can be a result of a lack of critical engagement with teaching mathematics, a lack of creativity and perhaps a lack of awareness of sociocultural aspects of teaching and learning mathematics. Additionally, the dominance of cognitive perspectives in teaching and learning, in general, might inhibit prospective teachers' noticing other aspects, including social, cultural, linguistic, historical, and economic [52], especially in a subject like mathematics, where most people think it is isolated or universal [71]. Consequently, this often leads to practices that minimise the use of a range of language repertoires students bring to the classrooms, with the belief that numbers and symbols can solely convey mathematical concepts without rich discursive opportunities. Hence, we were left with the question of whether the groups would propose different kinds of practices if the subject was different, for instance, English or Social Sciences, with the same purpose, closing the 'language gap'. Future research can investigate such differences or similarities to examine the influence of prospective teachers' perspectives and beliefs on the nature of the subject and the subsequent impact on the design of teaching practices to address language diversity.

10. Final Remarks

We would like to end this paper by offering future-oriented discussion points and questions, to be considered in research, practice, and policy. First and foremost, the students' proposed practices were limited in terms of addressing language diversity and lacked creativity and criticality. This issue should be further explored with a particular attention to students' long-standing beliefs about the nature and purpose of mathematics as a school subject and critical perspectives on addressing language diversity in the teaching and learning of mathematics. Additionally, this was a strong sign for us, especially the first author, who taught the module under exploration, to revise the curriculum with a particular attention to language diversity. There needs to be opportunities for students to interrupt their current thinking so that they can critically reflect on how they can design their practices in consideration to socio-cultural aspects and a range of resources pupils might use (including gestures, home languages, artifacts, diagrams, objects, etc.), apart from symbols and numbers. We are convinced that, as Moschkovich (2002) [3] p. 203 rightly stated, 'a situated socio-cultural perspective opens the way for seeing complexity and competence' in teaching and learning mathematics. Second, our research drew from the proposed practices, hence we did not have a chance to observe how these practices would be enacted in classrooms. Perhaps there would be a range of opportunities where our students would notice various dynamics, which subsequently would (or not) influence their actual practices. Therefore, it is essential to create opportunities for prospective teachers to move beyond the Live Brief assessment as a presentation, but in fact, obtain a chance to enact their proposed solutions in context so that pupils' voices and identities can also be taken into account. Finally, considering the influence of curriculum policies, including statutory requirements in the curriculum documents, it is crucial that the discussions around language diversity and how to address it through socially just perspectives should be at the heart of both policy discourse and practice.

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Informed Consent Statement: Informed consent was obtained from all subjects involved in the study. Citations, references and figures within the students' slides are deleted due to copyright issues.

Data Availability Statement: The data that support the findings of this study are not publicly available for confidentiality reasons, as they include information that may disclose participants' identity. The parts of the data that do not include personal information may be available from the corresponding author upon request.

Conflicts of Interest: The authors declare no conflict of interest.

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education sciences

Article

Exploring the Interplay between Conceptualizing and Realizing Inquiry—The Case of One Mathematics Teacher's Trajectory

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Abstract: Inquiry, an approach that departs from traditional mathematics teaching, empowers students through active participation and increased accountability in exploration, argumentation, evaluation, and communication of mathematical ideas. There is broad research consensus on the benefits of inquiry-based approaches to teaching and learning mathematics, including their potential to support equitable mathematics classrooms. While research has separately explored teachers' conceptions of inquiry and their efforts to enact the practice, little is known about the interplay between mathematics teachers' conceptions and enactment, and how it could be harnessed in professional development. In this study, we follow Alex, an experienced upper secondary mathematics teacher unfamiliar with inquiry, as he participates in a one-semester professional development course that draws on inquiry in multiple ways. His trajectory towards learning to teach through inquiry is revealed through patterns and shifts in his reflections and classroom actions. Our findings reveal significant developments in Alex's conception of inquiry and in how he realizes it in his classroom, identifying three paths that illuminate his inquiry trajectory: the teacher's role in inquiry interactions, a growing idea of inquiry, and orchestrating whole-class situations. In the interplay between enacting and reflecting, he moves from distributing authority separately between himself and 'the students' (as one unit) to fostering shared authority, a key aspect of empowerment, between himself and his students (as multiple voices) in both groupwork and whole-class episodes.

Keywords: inquiry-based mathematics teaching; conceptualization of inquiry; realization of inquiry; professional development; authority

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1. Introduction

Inquiry has received massive attention in educational research, mathematics curriculums, and professional development worldwide [1–3]. There is broad research consensus on the benefits of inquiry-based approaches to teaching and learning mathematics (e.g., [3,4]), including its potential to support equitable mathematics classrooms [2,5,6] Ernest [7] proposed the empowerment of students—with students experiencing a position of power through engagement in mathematics—as a goal for mathematics education, arguing that it is a step towards equity. Though it is a wide term, without a universal definition [8], inquiry is often referred to as an approach that departs from traditional mathematics teaching and empowers students through active participation and increased accountability in exploration, argumentation, evaluation, and communication of mathematical ideas.

In spite of the theoretical arguments and the policies supporting inquiry, this approach to teaching and learning is rare in day-to-day practice in science—where it originated—as well as in mathematics [1,9]. As researchers seek to understand the mechanisms at play in this phenomenon, several studies within science education suggest that teachers hold flawed ideas of what inquiry is (e.g., [9,10]), which seems to influence how inquiry is enacted (e.g., [11,12]). Summarized, "there seems to be confusion over what teaching science in inquiry really means and how that translates into classroom practice" ([13] p. 63). Research on mathematics teachers' ideas of what inquiry entails is scarce, and even less is known about how this might influence teachers' practices [14]. This article contributes to the field by exploring the interconnection between inquiry ideas and actions through a case study of an experienced upper secondary school mathematics teacher's path towards teaching through inquiry, drawing on his narratives and on observations of two lessons where he sets about bringing the approach to life.

What we do know from existing research is that it is in no way easy for teachers to shift towards inquiry-based approaches to mathematics teaching [14–16]. One way to support the teachers in this complex and challenging endeavor is through professional development (PD) courses. Based on a review of empirical research, not limited to the PD context, Stahnke et al. [17] suggested that teachers' knowledge and beliefs influence their in-the-moment decision-making and instructional practice. Recently, researchers have also looked at teacher development through the reflections teachers share after planning and teaching of lessons (e.g., [18,19]). Thus, PD courses should encourage teachers to experience and experiment with inquiry as well as to reflect on their views about mathematics teaching and inquiry [1,20,21]. Maaß et al. [14] stress that the ways in which teachers interpret inquiry-based approaches to mathematics after a PD course seem to be important in how they implement it in their own teaching, but that this possible connection is heavily understudied. Wee et al. [13] suggest that teachers participating in PD courses should also experiment with inquiry in their own teaching and, through this, develop their understanding of inquiry and how to teach through inquiry-based approaches.

Research thus emphasizes both reflecting on inquiry and enacting it as valued practices in PD. However, little is known about the interplay between these two practices, and about how this interplay contributes to teachers' professional development. We seek, therefore, to understand how inquiry is viewed and recontextualized by teachers over time, and further, how it is brought to life through their actions in the mathematics classroom. In doing so, we recognize that teachers' development occurs through iterative processes of experimentation and reflection, requiring a "careful study of the pathways teachers take as they grow as practitioners" ([22] p. 21). The findings of this study could provide insights into how teacher education institutions (in this case, PD) can support teachers' development. This is important, as inquiry "raises a quest for developmental work and professional development to support teachers in experimenting with and developing *their own* inquiry-based practice of mathematics teaching" ([21] p. 808, our emphasis).

A crucial argument for inquiry-based approaches in mathematics is that inquiry is a pedagogy supporting student empowerment [2,5,6]. Given the differences between the empowerment of students in traditional versus inquiry-based mathematics classrooms, power relationships in the classroom can significantly change when teachers transition towards inquiry-based approaches. In this study, we focus on the idea of shared authority [23]. More specifically, we are interested in the shift in authority relationships, i.e., who is in command, transitioning from the traditional classroom, where the teacher is the authority figure, to shared authority between teacher and students in inquiry-based approaches. We expect this lens to contribute to our understanding of teachers' pathways when learning to teach through inquiry.

The aim of this article is to provide a comprehensive picture of one teacher's trajectory, the pathways he takes as he engages in learning to teach through inquiry as he participates in a one-semester mathematics PD course, and how authority relationships are reflected in his trajectory. To our knowledge, few studies have taken this perspective. We will address this through the case of Alex, an in-service mathematics teacher with nearly 15 years of experience across subjects and grades, who still, in his own words, is unfamiliar with inquiry-based approaches. More specifically, we ask the following question:

> *What characterizes Alex's inquiry trajectory, interpreted through the interplay between his conceptualizations and realizations of inquiry?*

Conceptualization here refers to the teacher's formation of an idea of what inquiry entails, while realization addresses how it is brought to life in the mathematics classroom. Thereafter, based on our findings, we discuss the following question:

What are the connections between Alex's inquiry trajectory and the authority relationships in the mathematics classroom?

2. Theoretical Background

2.1. Inquiry in Mathematics Teaching

The following three facets form a natural basis for talking about and practicing inquiry as they are experienceable (can be observed and felt "in-action") and frequently highlighted across research: *the role of the students*, *the role of the teacher*, and *inquiry problems*. In Table 1, we have synthesized essential elements within the three facets based on a variety of frameworks and research on inquiry in mathematics (e.g., [2,16,21,24,25]).

Table 1. Theoretical framework for students' and teachers' roles in inquiry and inquiry problems.

Essential Elements of Inquiry in Teaching and Learning Mathematics.	
Students	Build on what they know to engage deeply with unfamiliar problems Collaboratively grapple with mathematical ideas Take on mathematical authority and responsibility
Teachers	Encourage and inquire into student reasoning Use student contributions to develop shared understandings and connections to formal mathematics Foster student empowerment through design, structure, and facilitation
Problems	Foster student engagement Are meaningful and relevant for students' daily lives Are related to mathematical ideas and concepts

Inquiry enables students to practice exploring the unknown through what is known, by connecting and building on their existing knowledge to develop what, for them, can be considered new insight and strategies [21,25,26]. As the social context can contribute to meaningful learning [16], e.g., enriching students' thinking [2], inquiry research often promotes collaboration. When grappling with unfamiliar problems together, students practice communicating, negotiating, and evaluating ideas [2,16,27] by actively engaging in each other's reasoning and working towards a shared understanding of the problem and its possible solutions [28]. Thus, we see students' active engagement in inquiry processes as exploration, argumentation, communication, and evaluation of mathematical ideas and relationships, where we consider all four elements as equally essential.

For teachers to promote students' inquiry-based learning, it is important to be curious about, encourage, and challenge students' explorations and argumentation through purposeful questioning and focusing actions that illuminate student thinking [2,16,29,30]. Such questions and actions "prompt students to explain their thinking and justify their solution strategies, with a focus on the reasoning the students utilized during the task as opposed to only the procedures used" ([25] p. 17). Argumentation is closely linked to the development of classroom norms [31], and an inquiry-oriented teacher is expected to promote collaboration and the sharing of ideas and argumentation between the students [2,25]. The structuring and orchestration of lessons is yet another highly important aspect of inquiry-based teaching [15,32]. Through the anticipation and monitoring of students' reasoning, teachers are better prepared to ask open and challenging questions with a learning purpose, not least to help students bridge their thinking to formal mathematics and the mathematical content they are grappling with. Connecting and sequencing students' ideas and arguments in a plenary discussion is one fruitful way to do so [2,15,25,32].

Teacher and student inquiry happens in interaction with a mathematical problem. In order to foster inquiry-based learning opportunities as described, the problems given to the students need to be related to mathematical ideas and concepts, as well as foster engagement and be perceived as meaningful and relevant by the students [16,21,24].

The three facets are artificially separated in Table 1; in reality, they are interwoven. For example, the way the teacher structures the inquiry activity—starting with selecting or designing problems that are cognitively challenging but still approachable through the students' existing knowledge [21,26]—lays a foundation for the students' active engagement in inquiry processes and for the teacher's further orchestration (cf. [15]) of the inquiry.

2.2. Teachers' Conceptualizations and Realizations of Inquiry

In this article, the term 'conceptualization' is used to capture a teacher's internal formation of an idea of what inquiry entails, revealed through the way they talk about inquiry, i.e., how one "describes and evaluates 'best practice' discursively" ([33] p. 315). We see conceptions as "personal constructs as they guide instructional decisions and impact the representation of the content (. . .) and that they are concept-centered and can be modified with additional information that adds to, challenges, or clarifies the conception" ([12] p. 1). Further, we view 'realizations' of inquiry as how inquiry is enacted in practice; more precisely, it refers to how "best practice" is brought to life in the mathematics classroom [33].

Even though teachers can *talk* about inquiry elements, this does not necessarily mean that these elements find their way into the classroom. Whitehead [34], first published in 1929, addressed the issue of ideas that are learnt but not used almost a hundred years ago, and many educational researchers have addressed inconsistencies between ideas and actions since (e.g., [33,35,36]). Much of the research on teachers' conceptualizations and realizations of inquiry, both in mathematics and science, has focused on comparisons and tensions between the two, and although there seems to be a strong connection between teachers' views about inquiry and how they enact inquiry in the classroom [9,14,17,32,37], researchers have also found misalignments between the two (e.g., [38,39]). Teachers might be able to conceptualize and plan inquiry practices but struggle to realize them [38], and they might be able to realize inquiry practices that they struggle to conceptualize [39]. There are also findings suggesting that even though teachers' conceptualizations and realizations are aligned, both are limited (e.g., [12,40]). Engeln et al. [16] found that even though most of the teachers in their survey study positioned themselves as having positive attitudes towards inquiry-based approaches, the vast majority reported using only certain inquiry elements in their practice, notably, mainly inquiry elements where the teachers remained in control. However, many of the above-mentioned studies rely on teachers' self-reported enactment of inquiry, which comes with a certain risk. For instance, Capps et al. [11] found that many teachers held flawed perceptions of what inquiry really was, which made them believe they were enacting inquiry practices in their classrooms when they probably were not.

Rather than focusing on alignments and misalignments between conceptualizations and realizations, this study focuses on how conceptualization and realization in conjunction, through cycles, can shape a teacher's trajectory for learning to teach mathematics through inquiry. There is a complex relationship between teachers' views, previous experiences, and practices [9]. The connection between views and classroom practices is non-linear and formed within a large system of connected factors [14,22]. In their study of four mathematics teachers' knowledge of inquiry and their inquiry practices as they engaged in a one-year PD program, through concept maps, interviews and observations, Chin et al. [41] found positive progress in both knowledge and practice among all four, but they were not causally related and there were no radical changes in practice over the course of the year. In contrast to most of the above-mentioned studies, they conclude that "our results taken as a whole indicate no obvious correlation between a person's knowledge of mathematics inquiry and her corresponding teaching practice" ([41] p. 859).

Many teachers conceptualize inquiry as an exploration process [9,13,42] more than processes of argumentation, evaluation, and communication of mathematical ideas. Kang et al. [42] studied 34 science teachers' conceptions of inquiry through a teaching scenario survey instrument. The teachers' conceptions were measured in terms of the characteristics they used to identify inquiry activities and compared to the five essential features of inquiry presented in *Inquiry and the National Science Education Standards* document [43]: (1) engaging in scientifically oriented questions, (2) giving priority to evidence, (3) formulating explanations based on evidence, (4) evaluating explanations in connection with scientific knowledge, and (5) communicating explanations. They found that the first three features were emphasized among the teachers, whereas the latter two were rarely used. The researchers claimed that the teachers' conceptualizations were thus limited to a traditional and narrow view of inquiry, due to the lack of connection between science content and inquiry teaching that lies in the evaluation and communication of explanations. Wee et al. [13] aimed to study how science teachers involved in a PD program focusing on inquiry-based activities changed their understanding of inquiry and inquiry teaching. They followed four teachers as they expressed their conceptualization of inquiry by drawing concept maps before and after they implemented inquiry-based teaching. They found that implementation did very little to improve the teachers' individual understanding of inquiry and their understanding of inquiry in the context of classroom instruction, particularly regarding the essential inquiry feature of communicating and justifying explanations, as also seen in the study of Kang et al. [42].

2.3. Inquiry and Shared Authority

Based on case studies, Ernest [7] hypothesized that empowerment is fostered by certain classroom experiences, such as mathematical risk-taking, experiencing success in genuine struggle, and collaborating. Inquiry radically shapes students' educational experience in this direction, as they formulate questions, mathematise, argue, prove, etc., and develop habits of mind with implications beyond school. It supports student empowerment not only because of the real-life relevance of the problems [6], but also because of the "vision of relationships between the different actors potentially involved within and outside the school system" ([21] p. 808). Given the differences between the empowerment of students in traditional versus inquiry-based mathematics teaching, we expect this lens to contribute to our understanding of Alex's trajectory. We limit our attention to one key aspect of student empowerment, a "shift in power relations so that the teacher listens to pupils in depth and allows them to make and express judgements and values their contributions" ([7] p. 13).

In the previous section, we argued that, despite the lack of a clear definition, inquiry-based approaches are recognizable by three interconnected facets: the problems, the role of the teacher, and the role of the students. While all three facets of inquiry are, in principle, interconnected, the teacher can more readily act on two of these: the task, and the role of the teacher; thus, these two can be perceived as more actionable by teachers, and therefore, more worthy of attention ([44] p. 12). Nevertheless, transitioning from traditional to inquiry-based teaching requires a shift in students' roles, too (e.g., getting to grips with different mathematical tasks, collaborating with their peers, taking ownership of mathematics). An obvious challenge for teachers is to identify and enact practices that enable students to meet the new expectations (e.g., working collaboratively [45]) and to cope with students' resistance to new practices (e.g., the use of challenging tasks or engaging with multiple solutions [46]). We are particularly interested here in the shift in authority relationships as Alex learns to teach through inquiry. Authority relationships differ significantly between traditional and inquiry-based approaches (see [6]), and the idea of *shared authority* is useful to capture this shift as we explore how a teacher learns to teach through inquiry. Following Amit and Fried [23], we understand an authority figure to be a person(s) whose statements and commands are accepted or obeyed without question (p. 147). Although inanimate objects such as textbooks and calculators can exert authority in mathematics [47], we limit our attention to people.

Empirical studies show that sharing authority is fraught with difficulties. For example, a case study of secondary mathematics teachers showed that sharing authority can be hindered by teachers' views of mathematics, with one teacher eliminating groupwork because of his conviction that his explanations were crucial to students' understanding [48]. More encouragingly, Ng et al. [49] showed that certain practices enacted by a teacher in whole-class discussions did help teacher in moving away from a pattern of positioning himself as the only authority. In contrast with these examples of the struggle to share authority, Arnesen and Rø [50] present a case where authority is shared between the teacher and the students, yet their analysis reveals a different set of challenges: considering the issue of shared authority in relation to the issue of supporting students' mathematical reasoning. Distinguishing between the potential of a teacher's moves to support shared authority and reasoning, respectively, they found that the teacher tended to prioritize shared authority at the expense of mathematical reasoning. Finally, tracking not one but two different agendas (connecting to children's mathematical thinking and connecting to children's funds of knowledge), Kinser-Traut and Turner [51] added a layer of complexity as they found that sharing authority is not a characteristic of a teacher's practice but is domain-specific. In our exploration of a teacher's conceptualization and realization of inquiry, the notion of authority enables us to capture the development of the teacher's scope of action on the student role.

3. Methods

3.1. Alex

Alex (pseudonym) was one of four teachers who volunteered to be observed and interviewed for a project looking at teachers' inquiry experiences and developments in relation to participating in a one-semester PD course offered by a Norwegian university to in-service lower secondary (grades 8–10) and upper secondary (grades 11–13) mathematics teachers. Elsewhere [52,53], we looked at all four teachers' conceptualizations of inquiry in terms of the teacher role, student role, and problems, both before and throughout their participation in the PD course. In this study, we focus only on Alex. Small-scale studies like this "are likely to remain a well-suited method for articulating the mechanisms of teacher learning" ([22] p. 24), which is in line with our aim.

Alex was chosen as our case because of his long teaching career combined with his self-reported newness to inquiry approaches to teaching mathematics. His teaching experience of nearly 15 years was mainly at the primary and lower secondary levels, but for the last year before entering the PD course, he had worked at an upper secondary school. Concurrently to participating in the PD course, Alex taught grade 12 mathematics for students who had chosen practical mathematics. (In Norway, upper secondary students choose between practical mathematics, social science mathematics, and natural science mathematics. Practical mathematics is considered as the least advanced option.) Many of the students knew each other and Alex from the previous year's grade 11 practical mathematics class, making their mathematics class a familiar environment for them. At the start of the PD course, Alex reported having a traditional, teacher-centered teaching approach; he described his typical lesson as an introduction by the teacher followed by individual work. However, he expressed being motivated to learn about inquiry and develop his practice accordingly, partly because of the centrality of the approach in the new Norwegian mathematics curriculum, and partly because he believed that inquiry could be fruitful for students at a range of mathematical attainment levels. Nevertheless, he voiced concerns about inquiry being an unfamiliar approach for his students not only in his lessons, but in their education in general. Accompanying this concern, Alex disclosed that inquiry was an unfamiliar approach for him as well, both from a student and teacher perspective.

While previous research points at in-service mathematics teachers' struggles with conceptualizing and realizing inquiry-based approaches (e.g., [38,39,41]), we want to contribute to a better understanding of the trajectory an experienced teacher—albeit a novice when it comes to inquiry—takes as he tries to develop inquiry-based approaches to teaching mathematics through repeated reflections and enactment.

3.2. The PD Course

Mathematics PD courses can support teachers in developing more sophisticated ideas of inquiry (e.g., [14] and positively influence their enactment of inquiry (e.g., [20]). In this context, facilitating authentic inquiry experiences has been emphasized [1,21]. The PD course in this study, taking place in the Autumn of 2022, engaged its participating teachers in two cycles of experiencing, reflecting, designing, and realizing authentic inquiry (Figure 1). These aspects are advocated for the design of PD promoting inquiry [1,20]. Through the two cycles, the teachers visited and revisited inquiry in mathematics from both a learner and teacher perspective in both PD and school settings. The PD course included three five-day seminars at the university (one in August, one in October, and one in December), focusing mainly on single- and multivariable calculus complemented with sessions and reflections on mathematical pedagogy. Between seminars, the teachers worked at their respective schools as well as following asynchronous and synchronous remote lessons and exercises related to the mathematical curriculum in the PD course. The last seminar was dedicated to repetition and exams; thus, we focus on the first two seminars.

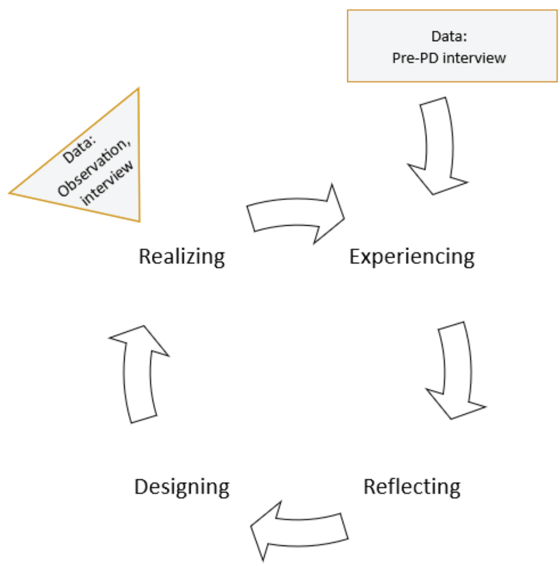


Figure 1. Model of the PD cycles and events of data collection in this study.

Experiencing. In each of the two seminars, the teachers worked in groups with an inquiry problem related to the mathematics curriculum in the course. The aim was for the teachers to participate in inquiry as learners of mathematics, to experience it in the ways that their students would (for example, by using their knowledge to collaboratively approach and grapple with an unfamiliar problem). The PD instructors modeled inquiry teaching, e.g., inquiring into teachers' reasoning, encouraging collaboration, and orchestrating whole-class summaries.

Reflecting. Subsequently, the instructors orchestrated sessions of reflecting on what inquiry in mathematics was. In the first seminar, the teachers discussed in their groups what they saw as essential for inquiry in mathematics, building on what they had just experienced and on their established perspectives of inquiry—a concept that had implicitly been circulating in the Norwegian educational curriculum for quite a while (see [14]) and in which the Norwegian term 'utforsking' is used in everyday language. This discussion was followed by a whole-class brainstorming session on inquiry. The collective identification of keywords could be seen as both soliciting and broadening teachers' perspectives. All contributions were encouraged and added to the blackboard without any validation by the instructor (Figure 2). (We include the picture of the board to give a sense of the number of keywords that were suggested. We are mindful that many readers will not be able to understand the Norwegian contents. However, it is impossible to offer a faithful translation by regarding the words (e.g., 'less structure', and 'understanding') isolated from the context, and an analysis of the class discussion is beyond the scope of this case study.) Subsequently, the first author synthesized the keywords into a mind map with three main categories: students, teachers, and problems (see Figure 3 for a translation to English). The mind map (in Norwegian) was given to the teachers, introducing the student role, teacher role, and inquiry problems as three facets of inquiry in mathematics that would guide future work in the course. By building on the teachers' perspectives and experiences, inquiry was portrayed in broad ways to allow for the teachers to develop their own variations [20].

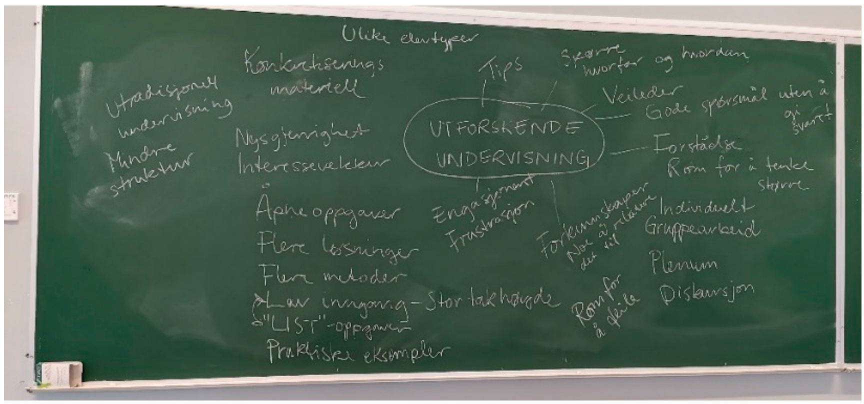


Figure 2. Picture of the blackboard at the end of brainstorming.

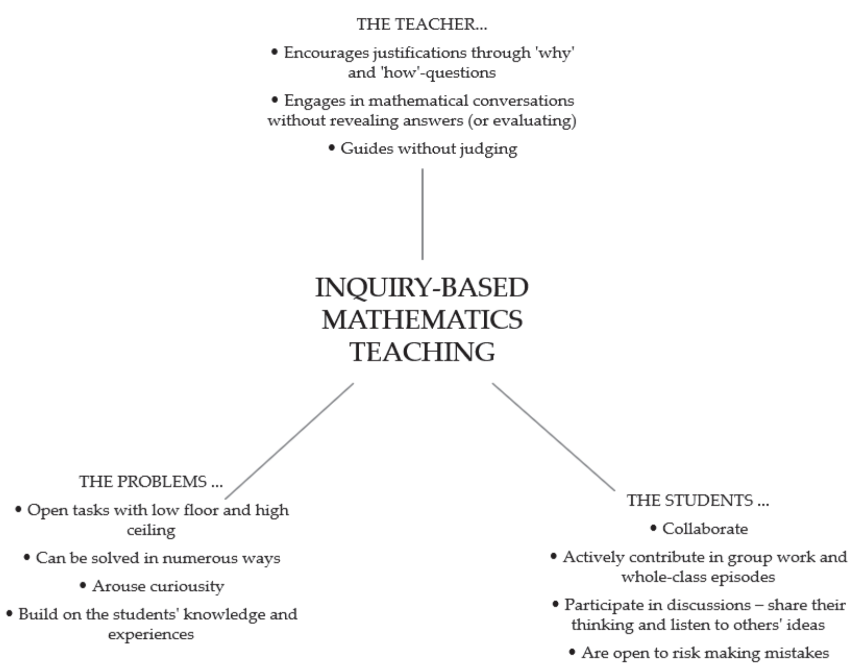


Figure 3. English translation of the synthesized mind map.

In the second seminar, the reflection session encouraged the teachers to reflect in their groups on the first cycle, and in particular, the design of an inquiry lesson and their realizations of inquiry by trying out this lesson in their own classrooms (see designing and realizing). This connected the PD course and the teachers' needs [1]. The teachers were encouraged to relate these reflections to the three facets (student, teacher, problem).

Designing. In both seminars, the teachers groupwise designed an inquiry lesson to try out in their own classrooms. In the first seminar, they were asked to design a problem for the lesson and agree on essential elements for realizing the lesson. The idea was to enable the teachers to design inquiry lessons that would be closely related to their own world, i.e., their practices [1]. Some support structures were offered, such as guiding questions (e.g., What makes a fruitful problem in your class? What specific aspects of the student and teacher roles do you want to focus on?), three versions of one example problem with varying levels of pre-decided strategies or procedures offered in the problem text [54,55], and the synthesized mind map in Figure 3. All these resources were given to the teachers in Norwegian. The problem Alex and his group designed was refined in collaboration with the first author, as the group only got around to making a first draft during the design session. The refined problem, subject to some modifications by Alex (e.g., adding some information) and translated to English by the authors, is shown in Figure 4.

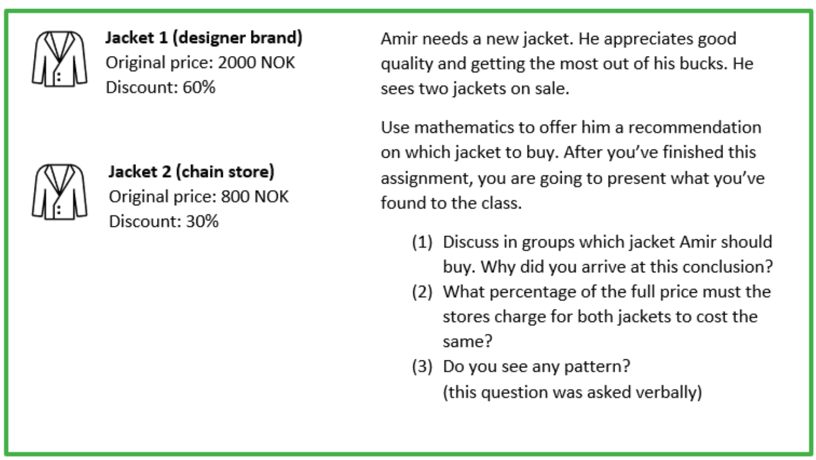
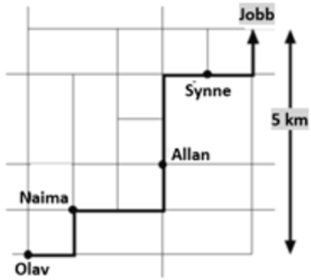


Figure 4. The problem for the first lesson.

The course instructors and the first author found that the first design session was too complex, with the teachers spending most of their time designing their problem and bringing little attention to what the students should do when interacting with the problem or on how they as teachers could facilitate the inquiry. Therefore, extra emphasis was put on the student and teacher roles in inquiry in the design session in the second seminar. A problem that could easily be adapted [20] and used in the teachers' classrooms was given to the teachers (see Figure 5 for a translation to English). In the same groups as before, the teachers tried to solve the problem in different ways, guided by the following questions: How might your student try to solve this problem? What ideas might they bring? And how would you as a teacher support them as they grapple with their ideas? Subsequently, all groups shared some of their solutions on the blackboard. The groups were then asked to design a lesson using this problem. Some guiding structures were the questions above and encouragements to discuss how they would use the students' ideas. Both the problem and the other resources were given to the teachers in Norwegian.



The figure above shows where each person lives, and the route Olav drives.

Olav has gotten his driver's license and drives to his summer job. Every day, he picks up Naima, Allan and Synne along the way. They also ride with him back to their homes.

The four of them decide to split the driving expenses, which are 1000 NOK.

Discuss in groups: How much should each of them pay? Remember to justify your suggestions.

Preferably come up with more than one suggestion. Provide arguments supporting what you think is the best way to split the expenses.

Figure 5. The problem for the second lesson, adapted from "Sharing Petrol Costs" from Bowland Maths, https://www.bowlandmaths.org.uk/materials/pd/online/pd_05/resources.html (accessed on 7 October 2022).

Because of this adjustment to the design phase, we did not obtain data on Alex's trajectory in terms of how his reflections on inquiry problems would be realized into a concrete problem for the second lesson. Consequently, our data regarding this facet was not as rich as for the other two facets (students and teachers). In this study, we therefore focus on the student and teacher facets.

Realizing. After each of the two seminars, the teachers tried out their designed inquiry lesson individually in their own mathematics classroom. The idea was for them to experiment and bring to life what they, inspired by their conceptualizations of inquiry and the experiences and reflections they had participated in in the PD course so far, saw as essential elements and actions for inquiry in mathematics. While they experienced inquiry as learners in the PD course, they now experienced it from a teacher's perspective. This type of personal experience with teaching through inquiry is key to teachers' professional development [1]. In the first cycle, they were also asked to hand in an individual reflection note on what they had focused on in their realization of the inquiry lesson, how it went, and what they had learnt from it.

3.3. Data Collection

Data were collected in all four phases of the cycles (experiencing, reflecting, designing, realizing) through recorded observations and interviews. Observations of teachers' pedagogical actions reveal the 'what' and 'how' of their teaching, but other perspectives are needed to encompass the 'why' [56]. To explore the qualitative aspects of, and ultimately understand, Alex's trajectory, we thus needed to combine and cross-validate observations from his realizations with reflections available through interviews [57]. We draw on data from three events: A pre-PD interview approximately two months before Alex started in the PD course, and observations of his realization of both the first and second lesson followed by interviews with Alex about his realizations.

The observations involved placing an audio recorder on Alex to record all interactions with the students and whole-class episodes. In addition, the first author was present at the back of the classroom, taking notes without interfering in the lesson. The aim of the pre-PD interview, conducted digitally and video-recorded, was to become familiar with Alex's current teaching practice and his conception of inquiry in mathematics before his participation in the PD course—a baseline. The two interviews after Alex's lessons were conducted in his office and video-recorded. The topics were (i) Alex's reflections on his realization of the lesson; (ii) his reflections on what inquiry in mathematics entails; and (iii) the connections between (i) and (ii). This gave us insight into his reflections on events that stood out for him from the lessons, in relation to his conceptualization of the student and teacher roles, and his ideas on new and refined inquiry elements to feed into his future realizations. All three interviews were semi-structured, allowing for Alex's conceptions and reflections to form the conversation. The interviews were conducted by the first author, whom we will sometimes refer to as the "researcher".

3.4. Analysis of Data

While previous research has often used scoreboards and schemes to score, or check the boxes of, inquiry elements present in a lesson or interview (e.g., [12,41,58]), our research looked at the presence (or absence) of inquiry elements together with the quality of the enacted inquiry elements. We achieved this through a thematic analysis [59] of the recorded data. Thematic analysis is especially fruitful when looking for patterns, and tracing developments, in and between observations and interviews [59]. This flexible analytical process provided us with space to explore the concurrent developments and patterns in Alex's conceptualizations and realizations of inquiry and how they together formed his trajectory.

The first author repeatedly listened to the recordings to identify critical events [60], which then were transcribed. As our focus is on Alex's trajectory in terms of how he conceptualized and realized inquiry, critical events included how he facilitated inquiry throughout the lessons and on interview extracts where he reflected on essential inquiry elements. Two of the authors coded the transcripts through iterative processes, moving between transcripts, coding, and recoding, supported by theoretically founded codes and supplementary codes emerging from the data. Table 2 shows the coding guide for the interviews. The details for the problem facet have been omitted, due to the changes in the PD design explained above.

Table 2. Coding guide for interviews. Codes that were inductively produced are marked with a *.

Inquiry Facet	Category (Codes in Parentheses)
Students	*Collaborative and communicative processes* (argue and challenge; build on ideas; discuss; evaluate; explain; share and listen to ideas; shared understanding) *Student thinking* (connect existing knowledge; explore; find strategies and solutions; see that there are multiple strategies and solutions *; use knowledge in new situations) *Authority and accountability* (actively engage; responsibility and ownership)
Teachers	*Interactions with student reasoning* (ask for justifications; challenge student thinking; direct *; encourage new solution or path; few prescriptions; foster collaboration; guide and support; inquire into student thinking; purposeful questioning *) *Brokering* (bridge student ideas and formal mathematics; connect students' thinking with each other) *Structure and planning* (anticipate student thinking; plan activity; select and sequence; summarize)

For the observations, we looked at how the teacher brought to life or tried to facilitate inquiry (e.g., the interview code 'argue' would have the observation code 'teacher encouraging student argumentation'). Any disagreements in coding were discussed and resolved. An utterance or observation could be assigned multiple codes if it addressed multiple inquiry elements. In addition to the assigned codes, the absence of some codes was equally interesting to us. From the coded utterances, we looked for patterns within an interview or observation, and based on the patterns, we wrote narratives of the respective events. These narratives guide the result section. From patterns across the narratives, we identified three main paths for Alex's trajectory. We let these paths emerge through the analysis before summarizing them in the discussion (Table 3 in Section 5.1) and discussing them to address the first research question.

We recognize the inherent asymmetry in the two facets of inquiry in Table 2; while both are equally accessible in teacher's conceptualizations, they may be seen as within the teacher's scope of action or not, depending on the actor they feature. This then has implications for the realization of inquiry. To capture such nuances that can explain what features of inquiry the teacher might choose to experiment with, we turn to the concept of shared authority. Inspired by Wagner and Herbel-Eisenmann [47], we characterize the authorities and their relationships in the narratives by interpreting who decides what happens during mathematics lessons and who decides what is true in mathematics.

4. Results

All excerpts in the result section are translated from Norwegian by the authors.

4.1. Pre-PD Interview

The main idea of inquiry, to Alex, was unstructured tasks where the students had to use and combine their existing knowledge. The tasks should have a low entrance point, so that "all students can accomplish something", but also include multiple solution paths. When elaborating on his conceptualization of inquiry, Alex continued to focus on the dynamics between the (individual) student and the tasks. He shared his ambitions for the students to discover what, for them, would be new ways to solve the problems (as opposed to just using formulas and strategies they already knew), to see how mathematics is connected across topics, and to become aware that "there are more ways to Rome" (sic). Alex did not reflect unprobed on what his role as a teacher in inquiry was, but was encouraged to share his views (Box 1):

Box 1. Excerpt 1.

Researcher: What do you think is the teacher's role in inquiry then?
Alex: Ehm, to have the toolbox. And then the students can get tools as they need them.
Researcher: Could you elaborate on that?
Alex: Yes, so (. . .) imagine you're at the woodwork room (. . .) [The students] come to the teacher and ask: "Do you have anything so that I can do such and such?". "Yes. What about this hand plane, it will make it smooth". (. . .) That you at least point them in the right direction of what they can look for to accomplish the task they are doing.

This conceptualization of the teacher role in mathematical inquiry portrayed the teacher role as steering the students in "the right direction". Similar views were also shared when Alex spoke about the students' role, where he talked about himself illustrating multiple ways to solve problems and the students choosing the one they liked the best.

4.2. Lesson 1

4.2.1. Observation Lesson 1

Throughout the lesson, Alex engaged in interactions with groups where he began making efforts to encourage students to explore (by finding multiple strategies and solutions to the problems) and communicate their mathematical ideas (mainly by telling Alex their solutions in groupwork and presenting their calculations and solutions in a plenary session). However, most of the student–teacher interactions remained on superficial levels. The excerpt in Box 2 illustrates the superficial level of his interactions and how he declined invitations from the students to be involved in their inquiry, which was another characteristic of Alex's interactions with the students throughout the lessons:

Box 2. Excerpt 2.

Alex: What have you found out?
Student 1: Well, I have found some arguments.
Alex: You have?
Student 1: Yes, do you wanna hear?
Alex: I want you to write them down, so that you can present them for the class later. Maybe even make something visual, maybe a PowerPoint or-
Student 2: Is this what we are gonna present?
Alex: Yes, and you need some arguments too.

In this excerpt, Alex emphasized the importance of supporting arguments but shut down the students' invitation for him to hear their arguments. Thus, he declined the opportunity to inquire into their reasoning. This happened both in group settings (where he declined two groups' efforts to share their arguments with him) and in whole-class episodes (where he quickly moved on from, or shut down, student suggestions to patterns). While the students inquired on the problem in their groups, there were also several episodes where students asked Alex questions or expressed confusion, only for Alex to walk away without reacting to their questions or confusion. At one point, a student reached out to Alex, telling him that his evasive responses did not help her, only to be shut down by Alex telling her that "the point is that I'm not giving you any concrete answers".

The few examples of Alex attempting to encourage students to support their ideas with arguments were almost exclusively general encouragements like the one in Box 2, and thus not related to specific aspects of their ideas. The superficial nature of Alex's interactions with the students was also reflected in how quickly he moved on from their inquiry. Most of his initial questions when approaching a group encouraged the students to explain their results, inviting them to communicate their solutions to him. However, once the students had responded to the question, Alex seemed to initiate a new process. At the start of the groupwork, Alex's go-to-response was "can you make a visual representation of this?" Later in the lesson, the go-to-response changed to "can you find another way to solve it?". Hence, the students' inquiry processes were not followed up or further elaborated on, resulting in few discussions between students or between students and Alex when he was present in a group.

Alex's efforts to support his students' inquiry processes when their progression paused seemed to include several instances of simplifying the problem and directing student processes. This happened both in groupwork and whole-class episodes. In Excerpt 3 (Box 3), we enter a whole-class episode in the lesson. All groups had, in random order, presented their answers to the first and second part of the problem, and Alex now wanted to make a table with the groups' answers to the second part (percentages paid for each jacket if they cost the same):

Box 3. Excerpt 3.

Alex: Now we're gonna make a table. (. . .) Did you arrive at any percentages where it [sic] was the same?

Student: We found that. . . Well, we didn't really find any percentages that were equal, but where the price was the same.

Alex: Yes, this was complicated. The jackets cost the same if. . .? If you have a 70% discount on the expensive jacket [jacket 1] and 25% discount on the cheap one [jacket 2]. Then what do you pay?

Student: What?

Alex: What do you pay? If you have 70% discount, then how much do you pay?

Students: 30%

Alex: Yes, right. So, if I pay 30% of the expensive jacket and 75% of the cheap jacket, then they cost. . .?

Students: 400.

As a student from one of the groups commented the imprecise question from Alex, he responded by reducing the problem to a standard task and directing the students to an answer. This type of interaction was typically seen where students' progression had paused and especially where the students expressed doubt.

Alex instructed the students to discuss in their groups as he launched the problem, but this focus was not maintained throughout the lesson. When Alex interacted with the students it was mainly in the form of short one-to-one interactions between himself and individual students, and he did not encourage collaborative actions.

4.2.2. Interview Proceeding Lesson 1

When reflecting on the lesson, Alex explained how he tried to distance himself from the inquiry processes (Box 4):

Box 4. Excerpt 4.

Alex: My thought was to give them minimum information. Minimum steering, like, I should steer them as little as possible. That they should try to find methods themselves. (. . .) Well, that was my thoughts on my- to hold back on the information, 'cause it's so easy to give too much information and direct them into one trajectory.

(. . .)

Researcher: How did you plan to hold back? Were there any questions you wanted to ask or-

Alex: To be encouraging and say "Yes! Now you're onto something, could you do more?". Like, that kind of opening questions. "Is there anything else you could do?".

Through such reflections, Alex seemed to conceptualize inquiry as an approach where the teacher provides little direction, and the students take on the responsibility of "finding methods". An essential part of the teacher's role for Alex seemed to be to encourage students to explore new strategies. When asked further about his interactions with the students, Alex expressed that he was unsure of what would really be purposeful questions to support the students' inquiry. He could not identify any questions he asked during the lesson that he felt were more fruitful than others but agreed, after some gentle reminding from the researcher, that "the combination of all those 'why' and 'what' and 'how did you find this', that's what makes the students progress". He concluded that he still had to figure out the "right" questions.

Alex took from the lesson that teacher preparation is essential for a fruitful inquiry process, stating that "if the problem is well thought through, you'll manage to guide them during the activity". To be better prepared for inquiry lessons, he had to "be aware of all possible methods". Thus, for Alex, anticipating different ways in which the students might approach a problem came forth as an important part of inquiry that he wanted to enact. Similarly, he addressed the need to improve his orchestration of whole-class episodes, as shown in Excerpt 5 (Box 5) about the various strategies and solutions chosen by the groups:

Box 5. Excerpt 5.

Researcher: Did you reflect on how to highlight this? That there were so many ways to find an answer?

Alex: I did, and that's an area where I can improve. I mean, I can be better at picking up what they do. And demonstrate that "alright, you guys used this method, and you guys used this, and you used this". (. . .) If I had been a little more focused, I could have picked up on all those different methods and weighted them against each other.

Researcher: So, what you're saying here is that helping the students to see connections between methods is a typical teacher task [in inquiry]?

Alex: Mhm.

Here, Alex shared a view of whole-class inquiry consisting of students presenting their methods and solutions, and the teacher making connections between them. His focus was on his own contribution to the whole-class episode.

As for collaborative aspects in his lesson, Alex disclosed that the class had not worked much with establishing what was expected of them when working together, but that he felt that active participation was implicit for collaboration. Alex emphasized that he had arranged the students in groups of three because he felt this was a good group size for everyone to be able to engage, and that he had purposefully assigned students to the different groups.

4.3. Lesson 2

4.3.1. Observation Lesson 2

The following excerpt (Box 6) illustrates how Alex's interactions with the groups included numerous efforts to inquire into their reasoning.

Box 6. Excerpt 6.

Alex: What have you found out?
Student 1: Ehm. . . Is it- each square is one kilometer?
RAlex: What-why do you think that?
Student 1: Because, like 1, 2, 3, 4, 5 [counts squares]. And it's longer if he's picking them all up.
Alex: Mhm. . .

Through opening questions such as "what are your thoughts?" and "what have you found out 'til now?" Alex showed interest in the students' inquiry as well as their results. We also noticed several episodes of both general encouragements to find arguments ("and then you should argue for your solution") and requests for argumentation that was connected to a concrete student idea (as illustrated in Box 6). Alex's inquiry into student thinking, together with some direct encouragement to engage in collaborative reflections (e.g., "what do you think, [name]?" and "does anyone have another idea?"), shows how collaboration and communication were promoted in this lesson. As before, he constantly encouraged his students to explore new methods and answers to the problem throughout the lesson. There were very few episodes where Alex directed the students' thinking; rather, he made several efforts to support student inquiry by trying to activate their previous knowledge. However, as illustrated in Excerpt 7 (Box 7), this was oftentimes met with confusion:

Box 7. Excerpt 7.

Alex: What have you found out 'til now?
Student 1: Ehm. . . We don't really know.
[Students talking about not wanting to charge their friends to ride with them]
Alex: Well, yes, but have you seen anything like this before? What is it that you are wondering about?
Student 1: It's. . . The whole thing doesn't make sense.
Alex: What doesn't make sense?
Student 1: Like, where have we seen this before?

Alex frequently asked the students if they had seen anything like the problem before or if they knew any information that they could use to solve their queries. Often, such questions were used as follow-up questions after inquiring into their ideas. Box 7 illustrates how Alex's inquiries into student reasoning at times were restricted to one question before abandoning the students' ideas to rather ask them if they had "seen anything like this before", and how this seldom resulted in any progress.

The orchestration of whole-class episodes took a different route than in the first lesson. Alex invited groups to share their ideas in plenary, in a sequenced order, and he built on ideas to progress the collective reasoning. In the following excerpt (Box 8), the class had agreed that each side on the squares in the problem figure equaled one kilometer:

Box 8. Excerpt 8.

Alex: And the question then becomes, how can I use that information? That's some of the information that we have. How can I use that to split the cost of driving to and from work? What are your thoughts on that? [Directs the question to a specific group]

Student: Well, we were a little unsure of that. So, our first solution was just to make it simple, though that's not fair to all four. But we just made that [suggestion]. We took 1000 NOK and divided it by 4.

Alex: Mhm. Why?

Student: Cause then everyone pays the same.

Alex: Yes, well, that's one method. Very good. Did you have any suggestions? [directs the question to another group]

[Different groups share their ideas on Alex's request]

As the excerpt shows, Alex sometimes challenged students' contributions by asking for arguments or mathematical calculations behind the suggestions. The students were also encouraged to build on others' ideas when Alex asked the groups to use the agreed information to find new solutions. However, only Alex questioned their suggestions; unless they were sharing an idea, the students were left as spectators.

4.3.2. Interview Proceeding Lesson 2

When reflecting on the lesson, Alex disclosed that he felt much more prepared this time (than for Lesson 1) in terms of purposeful questions, inspired by George Polya's work [61], which he had become familiar with when preparing for the second PD seminar. He also noticed how anticipating student reasoning to the problem together with his group in the PD seminar effected his ability to inquire into his students' reasoning, as he was familiar with the different ways that they might attack the problem or the challenges they could encounter. In Box 9, the importance of purposeful questioning was highlighted as Alex reflected on what he saw as essential for the inquiry-based teacher role:

Box 9. Excerpt 9.

Researcher: (. . .) So, what about your role in inquiry then? As a teacher. What's your job?

Alex: It is to ask the right questions. I get that now, more than- or, I've known it before too, but I haven't known what those questions were.

Researcher: So, you see a development there?

Alex: I see a development there. That now as I have sort of a script for it- Of course, I'll add something to it, I'll make my own formulations, but having something, a small script on what to ask the students. That was very nice.

It is not clear what Alex meant when he talked about "having a script", but it is natural to believe that he was referring to questions he discussed with his group in the PD seminars and questions from Polya's work (e.g., "have you seen anything like this before?"). His orchestration of the lesson, and his questions in particular, made him feel that his students were actively engaged: "I got more answers than I thought I would. And more students engaged when they got such open questions. Like, not the direct 'what did you find?', but [questions] about how far they had come, what was on the table at the moment, what was their thinking like. So, it was more including for the whole group". However, monitoring and summarizing the student inquiry after the questions had been posed was challenging to Alex, especially "walking around in the classroom and thinking quick enough to keep up with their thinking, (. . .) to not just stand there scratching my head and wonder like 'uhm, is this right?' but really (. . .) listen to what they're actually saying". In other words, moving from evaluating the correctness of their solutions to inquiring into their arguments and explanations was a big leap for Alex. He also disclosed that he was not entirely content with the processes his questions led to. He stated that "it was kind of fixed, like, it didn't proceed from those first questions". However, this was "partly because the students' progression didn't deepen". Therefore, Alex had "stopped the questioning because I didn't get anything more from it. We didn't make it to the next phases". He believed that encouraging the students to explore multiple methods and solutions and inviting them to share their suggestions with the class, on the other hand, had helped the students develop ownership of their ideas.

In the final excerpt (Box 10), Alex reflected on how his conception of inquiry in whole-class episodes had developed:

Box 10. Excerpt 10.

Alex: Well, I guess that it has become clearer to me that it's more of a conversation than me steering it. [The students] should contribute much more. (. . .)

Researcher: Ok. So, it's the students' ideas that are brought up in the plenary part, do I understand you correct then? [Alex confirms]. What do you do then, like, why are you there?

Alex: I'm a moderator, a summarizer, or. . . Yes.

Researcher: So, your job is kind of to decide who-

Alex: Well, yes, you control it a little bit, maybe, but the students should contribute more into that summary than normally maybe. Or, not maybe, more than normally.

5. Discussion

5.1. Alex's Inquiry Trajectory in Three Paths

We identify three main paths in Alex's inquiry trajectory throughout the semester: *the teacher role in inquiry interactions*, *a growing idea of inquiry*, and *orchestrating whole-class situations*. These three paths, summarized in Table 3, inform the rest of this section before we turn our attention to how shared authority is reflected in Alex's trajectory.

Table 3. Three main paths in Alex's inquiry trajectory.

	Pre-PD	Observation Lesson 1	Interview Lesson 1	Observation Lesson 2	Interview Lesson 2
The teacher's role	The "woodwork teacher" handing out tools	Removing himself from the inquiry (or directing it)	Acknowledging the importance of teacher questioning and preparation	Polya-inspired inquiry into students' work	Purposeful questioning as an essential element of inquiry in mathematics
Growing idea of inquiry	Inquiry in mathematics equals (individual) discovery	Finding multiple solutions and strategies Touching upon communication and argumentation	"How and why"—supplementing exploration with argumentation	Students exploring, explaining, and sharing ideas	Exploration and argumentation. Students as active communicators
Orchestrating whole-class situations	Whole-class situations not mentioned	Show-and-tell Teacher directing students towards right answers	The importance of good teacher summaries to connect student ideas	Selecting and sequencing students' contributions to display multiple solutions and strategies	Whole-class summaries as dialogues between students and teacher

5.1.1. The Teacher Role in Inquiry Interactions—From the "Woodwork Teacher" to the Curious Questioner

Our results illuminate how Alex's focus moves from directing the students (pre-PD), through distancing himself from their mathematical activity (Lesson 1), to exposing and inquiring into their ideas (Lesson 2). From an empowerment perspective, this is a major shift in how students are positioned [7] as contributors to the mathematical activity, their contributions being not only valuable for their own learning, but also non-trivial from the vantage point of the teacher.

Only when specifically asked to does Alex reflect on his own role as the inquiry-based teacher in the pre-PD interview, and when he does, it is in very traditional ways (Box 1). Alex's analogy of "the woodwork teacher" has few similarities with research characterizations of the teacher role in inquiry settings. Still, we see him feeding this perspective into his lesson when he seemingly believes the students' progress is at risk (Box 3). Data from his first lesson illuminate, however, how Alex mainly distances himself from the students' inquiry processes either by shutting down their efforts to share their inquiry with him (Box 2), not following up on his novice efforts to ask the students what they've found out (Box 2), or by simply leaving them with their questions unanswered. Providing the students with intellectual space to ponder on mathematical problems speaks to their mathematical empowerment [6] and is essential for inquiry [2,21]; however, students' intellectual space must be accompanied by elements of structure where the teacher supports the students in their struggles and promotes and elicits their inquiry [25]. We see that when Alex distances himself from the inquiry, his students become confused about what is expected of them, trying to infer the expectation (e.g., argue or present, as shown in Box 2) and even to communicate their confusion to Alex. His distancing is in great conflict with his view before entering the PD-course, and we speculate that he is facing a conflict between the transmission-based "woodwork teacher" and the guiding teacher he later emphasizes. If so, this is a conflict he resolves via two extremes: directing the students when they do not offer ideas and distancing himself from their ideas when they present some.

Alex relates his detached teacher role to giving students space to choose their own solution paths (Box 4), a signal that his reflections on the teacher role have changed drastically since he started the PD course. He also identifies anticipating student reasoning and preparing purposeful questions, identified as questions about 'what', 'how', and 'why', as essential elements for inquiry that concern him and that he feeds into his next lesson. His open questions in Lesson 2, inspired by the four steps of problem solving [61], are intended to illuminate students' inquiry processes and activate their previous knowledge, aspects highly emphasized in inquiry research (e.g., [2,25]). Alex's interactions with his students in this lesson suggest that he is now valuing and trying to inquire into their reasoning, positioning the students as key sources of the mathematical ideas in focus. However, our findings also reveal how he is only in the early stages of adapting to the inquiring teacher role, leaning on the 'what', 'how', and 'why' and Polya-inspired questions as a script (Box 9). The questioning techniques are not a natural part of his repertoire yet and he stumbles when trying to adapt them (Box 7), not knowing when to use the questions, how to continue his inquiry into the students' reasoning from them, or how to keep up with the students' ideas.

Changing from transmission-based teaching practices to student-active approaches like inquiry is in no way a simple task as it requires teachers to rethink their own role in the classroom [14]. However, looking at Alex's path, he gradually broadens his conceptualization of the teacher role and his realizations—though still wobbly—become more refined and focused on uncovering the students' inquiry. This becomes visible in the way he reflects on his actions, talks about changes he wants to make, and plans his next steps.

5.1.2. A Growing Idea of Inquiry—Inquiry Is More Than Exploration

Many teachers conceptualize inquiry mainly as exploration (e.g., [9,13,42], and this seems to also be the case for Alex in his pre-PD reflections. Inquiry, to Alex, is about discovering strategies, methods, and solutions. This perspective offers students some opportunities for epistemological empowerment [7] compared to traditional teaching (e.g., by strengthening their sense of autonomy), but not as much as in inquiry (e.g., by limiting their ownership of mathematics). In particular, no attention is paid to heavier analytical processes such as negotiation, argumentation, and evaluation, nor to collaboration, despite these aspects being highly valued in research efforts to conceptualize inquiry [2,21,25]. Alex maintains his explorative focus throughout both lessons, with frequent encouragement to find new representations, strategies, and solutions, which is also an expressed aim for him (Box 4). However, separating this from other research (e.g., [13,42]), we witness a growing awareness of the more argumentative and communicative aspects of inquiry. Alex takes steps towards sharing students' explorations in his lessons (in plenary in lesson L and both in plenary and via teacher–student interactions in Lesson 2) and gradually encourages the students to justify their suggestions (through general encouragements in Lesson 1 and more direct requests in Lesson 2). From our point of view, these aspects grow from Alex's experiences in the classroom and gradually receive more attention in the way he talks about inquiry, for example, in how the questions he asked in Lesson 2 made more students communicate their thinking, and his focus in Lesson 1 on how argumentation (the 'what', 'how', and 'why') is important for students' progress. Unlike his shutting down of students' efforts to share their arguments in Lesson 1 (Box 2), in Lesson 2, Alex specifically asks for them (Box 6, Box 8).

Argumentation is closely related to the development of classroom norms [31], which takes time and continued support; hence, Alex's early awareness and clumsy efforts to encourage argumentation can be seen as the first steps towards establishing expectations of argumentation in his inquiry classroom. Nevertheless, adding to a massive body of science research and in line with the few studies in mathematics, evaluative features [2,16,27] never become a part of Alex's path towards learning to teach through inquiry. Mathematical ideas are explored, shared, and sometimes supported with a statement addressing the *why*, but not assessed, as illustrated by the students' passive role in the whole-class episodes unless they are sharing ideas.

While previous research has focused on (science) teachers' conceptions and realization of explorative, communicative, evaluative, and argumentative processes of inquiry, often referring to the NCR [43] elements (e.g., [13,42]), our study also sheds light on Alex's reflections and actions towards the collaborative features of inquiry. Our findings suggest that while Alex talks about collaboration in positive terms, he struggles to feed this into his actions in the lessons, making it an idea that is talked about but not realized [33,34]. The students are placed in groups, but very little attention is paid to evaluations and negotiations—aspects of inquiry that are emphasized in research (e.g., [2,16,27]). We see the same regarding attempts to work towards shared understandings (as highlighted in, e.g., [28]). This is a question of limiting students' opportunities for epistemological empowerment through validation of ideas, and also a question of differential access. Especially in Lesson 1, we see Alex address individual students, engaging in one-to-one interactions, raising the question of equity—which students are genuine participants in the mathematical activity, and which are merely present? ([44] p. 33). We see some efforts in Lesson 2 to engage students by orienting them towards each other in the groups (e.g., by asking them if they agree with their group), as well as in the teacher–student interactions (e.g., asking if anyone on the group has other suggestions), which might suggest that Alex's ideas of the value of collaboration are developing and starting to take form in his realizations. Collaborative approaches such as inquiry strongly contrast with the traditional transmission teaching [14,44] that has guided Alex's practice up until his participation in the PD course, which might bring some context to the situation. We conclude that the individual focus in Alex's conceptualization of inquiry as individual exploration remains throughout the course, but is gradually complemented by an increased awareness of, and small experiments on, broadening this perspective and making small shifts in his actions.

5.1.3. Orchestrating Whole-Class Situations

Promoting and orchestrating episodes of whole-class discussions in inquiry can be challenging for teachers [15,32], and Alex is no exception. From an empowerment perspective, whole-class discussions are considered key practices for equity and the polar opposite of the individualized mathematical activity of traditional teaching ([44] p. 31). Before he starts in the PD course, whole-class discussion has no place in Alex's conceptualization of inquiry, but in his first lesson, he already invites the students to share their solutions with the class. Like so many before him, he realizes this as a sequence of unstructured, randomly selected show-and-tell with no connections made between student contributions. Structuring and orchestrating discussions is essential for inquiry-based teaching [15,30], to bridge students' ideas with each other and with formal mathematics [25,32]. Reflecting on this lesson, he recognizes the importance of preparing a summary to "pick up what they do" and compare the student ideas that have been presented (Box 5). Alex sees this as the teacher's job, a perception which allows him to remain in control of the situation (cf., [16]). Making summaries can be seen as a systematization of mathematical student contributions [32], but should then follow episodes of students comparing, contrasting, and challenging their own and each other's contributions [32]—Alex's reflections do not address student participation in whole-class episodes. His realization of the whole-class discussion in Lesson 2 illustrates that he now grapples with important elements for such events, such as selecting and sequencing [15] and trying to act as a broker [25] by using student contributions to develop a shared understanding of the problem [25,28] and encourage the groups to build on this in their further inquiry processes (as seen in Box 8). At this point, the brokering seems to be restricted to building shared understanding rather than connecting classroom mathematics to formal mathematics [25,32]. In his realizations, the students are left as a passive audience if they are not sharing an idea themselves; they are not invited to react to others' contributions (as we have reflected on in the previous section).

Alex, after the second lesson, shares with the researcher that he now sees whole-class episodes as a dialogue between the students and himself (Box 10). Looking at this path of Alex's trajectory, he gradually moves towards orchestrating whole-class discussions as they are portrayed in the research [2,15,25,32]. Menezes et al. [32] portray the teacher's role in whole-class discussions as "to coordinate the interaction among different students, orchestrating the discussion, promoting the mathematical quality of the presented explanations and argumentations" (p. 307). Though the interactions remain between individual students and Alex, his actions and reflections illustrate a growing awareness of dialogic structures and the importance of explanations and arguments being shared with the whole class. As we leave him, he welcomes students' ideas as important contributions, orchestrating ordered routes of eliciting student inquiry to build shared understanding and illuminate the various ways in which a mathematical problem can be approached.

5.2. *Alex's Inquiry Trajectory in Light of Shared Authority*

The analysis of two facets of inquiry (the teacher role and the student role) in our data enabled us to identify three paths of learning to teach through inquiry, paths we expect to be specific to Alex rather than shared by all PD participants. Our position is that Alex has the power to shape his paths; the enactment of inquiry practices is key to learning from professional development [1], but from a situated perspective, his enactment will be influenced by his responsibilities as a teacher and by the pedagogical reasoning underpinning his actions [56]. In particular, we assume that, during his two realizations of inquiry, Alex selected what aspects of inquiry he tried to learn about, valuing, as teachers tend to, actionable knowledge ([44] p. 13) and considering a smooth lesson to be a successful lesson (p. 15). In other words, we believe that Alex shaped his learning by concentrating on aspects of inquiry that allowed him to act (e.g., realizing the teacher role is more immediately actionable for him than realizing the student role), and decided on what aspects needed work based on key events that disrupted the smooth running of the lesson. We return with an interpretative stance to the narratives of Alex's paths, looking to understand what goals he might have set, and what implications these have for how Alex might attempt to realize inquiry. We do so through the lens of authority [23]. We argued that inquiry classrooms differ from traditional classrooms in terms of who is positioned as an authority figure [6], making this lens suitable to capture change for Alex, an experienced teacher in traditional approaches, but a novice to inquiry. We found that Alex chose specific aspects for his experimentation, aspects that reflected his concerns:

- Prior to PD, he believed that during inquiry, authority should be distributed between himself and the students (as a group) in separate agentic spaces. He was concerned with whether the students would play their roles, but he did not appear to see this problem as actionable and had no clear goals for his learning.
- In Lesson 1, he developed actionable ways of following up on his concern: reminding students of their roles and keeping quiet. After push-back from students, Alex revised his agenda to foster—through questioning—shared authority during groupwork and distributed authority during whole-class episodes.
- In Lesson 2, he experienced partial success in sharing authority during groupwork, through his more responsive questioning. Perhaps encouraged by this or overwhelmed by the burden on him in the distributed authority of whole-class discussion, he revised his agenda to sharing authority both in groupwork and in whole-class episodes.

We elaborate on these interpretations here, to characterize the three data collection points.

Pre-PD. Pre-PD, Alex envisaged authority in inquiry as distributed between the teacher and the students; they have authority while working on the task but relinquish it to him when they ask for help. The students were seen as a uniform mass, no distinctions were made between them beyond what is implicit in the aspiration that all may "accomplish something", and no reference was made to the interactions between students. This reflects his background of traditional teaching where authority is firmly in the hands of the teacher [6]. Alex reduced the teacher role at this time to manifesting intellectual authority when called upon (by identifying "the right direction" and offering the right help), a typical interaction pattern in classrooms where the teacher is the dominant authority [50]. However, he anticipated difficulties stemming from students not playing their part. He appeared to perceive this concern outside his agentic space, as he did not connect it to any action he might attempt.

Lesson 1. Two moves dominated Alex's actions during the first lesson, and both appeared connected to his concern prior to course start. First, he communicated explicitly and repeatedly that students have authority during groupwork and during the whole-class presentation. Secondly, he stopped himself from acting as an authority: he listened to the students who were stuck but provided no support, or he told students what to do (communicated solutions, produced visual representations, etc.) but failed to manifest intellectual authority (e.g., he listened to their solutions, and then, moved on without validating, challenging, extending, etc.). Alex's pre-PD concern was clear in his vocalization of the students' reimagined roles in inquiry [21], and perhaps also his search for an actionable way [44] to foster the students' roles. While keeping quiet does not appear on the surface as acting, in fact, it is a clear departure from the traditional teacher role [6] and it is not a trivial change [48].

These interaction patterns led to confrontation over what constitutes help in mathematics. The students openly expressed dissatisfaction, and, recognizing that he and the students shared the authority to define what was acceptable, Alex even explained his response ("the point is that I am not giving you any concrete answers"), hinting at constraints from his plan "to hold back the information", or from an external authority (the teaching approach itself, the researcher observing the lesson, or the mathematics teacher educators). Perhaps the confrontation, disrupting the smoothness of the lesson ([44] p. 15), prompted Alex to revise his agenda, seeking better ways of interacting with the students. This labeling of interactions as 'questioning' is consistent with his intention of not giving away the answers ("holding back" to respect student authority), but his agentic space had expanded to sharing authority between students and the teacher during groupwork, a notoriously difficult task [48]. Furthermore, he acknowledged that there are more than two authorities ('the students' and the teacher) in the classroom. Moves such as inviting students to produce their own solutions do contribute to positioning them as intellectual authorities individually, but not collectively [49]. During groupwork, Alex's moves led to the production of many different and potentially unexpected mathematical ideas. This challenged not only him (his questioning during groupwork), but also the students during the whole-class episodes. Following this revised interpretation, in the next iteration, he might attempt to distribute the authority during whole-class episodes between students, by giving them the authority to bring their own mathematical thinking, and the teacher, who has the authority to integrate these into a whole. Although his planned actions would fundamentally change the nature of the whole-class episodes, from show-and-tell to a phase where isolated ideas become integrated [2,15,25,32], it still positions the teacher as the main authority [50] and it is not clear that it would trigger development in Alex's trajectory for sharing authority between himself and the students. The awareness of students as a diverse group, however, is relevant for fostering shared authority among students.

Lesson 2. Following on from his concerns emerging from Lesson 1, Alex attempted to reconcile the competing demands of the teacher and students by sharing intellectual authority. He sought ways to say something helpful, without limiting the students' scope of action. At times, he was successful at relating, in some way, to the specific ideas brought up by the students, and at other times, he simply voiced a generic question without making a reasoned choice. A student's frustration ("it doesn't even make sense") with one of the new questions Alex was trying out ("have you seen anything like this before?") leads us to note that, although not conducive to inquiry, the episode created a space for the student to exert shared intellectual authority and disrupted the smoothness of the interaction, creating a need for improvement [44].

During Lesson 2, Alex refined his idea of students as authorities for knowledge production and validation. He not only invited students to take on specific roles (produce a solution during groupwork and present it during whole-class episodes), but by orienting students towards each other (to listen to alternative methods and build on them) through discursive moves that have shown promise in fostering shared authority among students [49]. The experience disrupted his assumptions on students' potential as authorities ("I got more answers than I thought I would"). Perhaps interpreting the smoothness of the experience ([44] p. 15) as a sign of partial success for his questioning, his agenda for the next inquiry-based lesson is to continue fostering shared authority, not only during groupwork but also during whole-class discussion. This goal is actionable [44] for him, through improved questioning. It is unclear what prompted Alex to reconsider authority relationships during whole-class discussion. There were no observable disruptions in the smoothness of the lessons that can be linked to this. We hypothesize that Alex's revised goal could be, in part, explained by the demands it places on Alex to remain the sole authority in connecting diverse student contributions. An alternative is that the situation is similar to that in the study of Kinser-Traut and Turner [51], where successes in sharing authority when connecting to students' mathematical thinking provided reinforcement of the teacher's conviction that it was the right thing to do.

6. Final Reflections

The study sheds light on how a teacher's conceptions of inquiry in mathematics throughout a PD course can influence how inquiry is brought to life in their classroom, a heavily understudied connection [14]. Moreover, it also highlights the opposite relationship, namely, how realizing inquiry lessons—and reflecting on them afterwards—can influence how inquiry is conceptualized. As researchers have sought to understand the mechanisms at play in mathematics teachers' efforts to conceptualize and realize inquiry, limitations in both conceptions and enactment have been addressed (e.g., [38,39]), as have alignments between the two (e.g., [32]). Separating from previous research, we focused on the trajectory shaped by the interplay between the conceptualization and realization of inquiry, identifying three paths that illuminate Alex's inquiry trajectory: the teacher's role in inquiry interactions, a growing idea of inquiry, and orchestrating whole-class situations. In this interplay, we glimpsed issues of student empowerment such as equitable participation and opportunities to validate and create mathematical knowledge, noticing particular tensions and synergies. Exploring their full complexity is beyond the scope of this paper. Our focus remained on the key issue of authority in mathematics. This lens allowed us to understand Alex's trajectory as driven by his own agenda for learning, with goals materializing through the interplay of his conceptualization and realization of inquiry, specifically by identifying actionable knowledge (the conceptualization of inquiry) and a lack of smoothness incidents (the realization of inquiry) [44]. He moved from distributing authority separately between himself and 'the students' (as one unit) to fostering shared authority between himself and his students (as multiple voices) in both groupwork and whole-class episodes.

In summary, our findings contrast those suggesting that realizing inquiry only has a minimal effect on teachers' conceptions of the approach, or that teachers' development in inquiry knowledge and practices after PD is small (e.g., [13,41]). As we have discussed, Alex lets his experiences, conceptions, and concerns feed into his lessons, and from the lessons, he reflects on and identifies new focus areas and concerns and refines his conceptualizations of inquiry in mathematics. This interplay forms his trajectory towards learning to teach mathematics through inquiry, which, for Alex, includes significant concurrent developments in both how he conceptualizes inquiry and how he realizes it. Ozel and Luft [12] argue that conceptions can be modified if challenged, clarified, or added to, and we suggest that trying to bring inquiry to life in the classroom can do just that—challenge, clarify, and add to the teacher's conception of inquiry. These modifications can again feed into enactment. However, our findings also suggest that this is a complex process; for example, Alex engaged in several unsophisticated and unreasoned efforts to implement what he saw as purposeful questions to inquire into the students' thinking but that, maybe due to his clumsy timing, ended in confusion. Further practice and refinements are needed to continue Alex's trajectory in learning to teach through inquiry. This argues for more research that does not simply count if or how many times an inquiry element is implemented into teaching but also analyzes its quality.

In the case of Alex, the analysis of shared authority provides a different lens for interpreting his trajectory—the goals he sets for himself and how he informs the revision of these goals based on his experimentation. This lens allowed us to explore potential mechanisms for the formulation of teacher goals, such as discerning between what is actionable and what is not, re-evaluating following disruptions in the smoothness of the lesson, reducing the load on the teacher, or persevering with what runs smoothly. However, Alex is a special case; he starts the course with a traditional teaching approach. Shared authority will most likely fail to capture changes in teachers who work collaboratively, and it does not account for the rich details of teaching through inquiry.

We acknowledge that Alex's trajectory was probably affected by the communities he was part of, for example, the group he and the other participants in the larger project formed, and the activities they participated in during the PD course. These influences on Alex's trajectory have not been thoroughly investigated in this work, and we welcome research that binds together PD experiences and the trajectories formed by the individual teacher. Though we cannot draw any conclusions on causality, we can pinpoint some connections between the PD course design and Alex's trajectory. PD should be relevant to teachers' daily practices [1]. Ensor [33] stresses that teachers should learn from activities rather than imitate them (and though he focuses on teacher education, we see this as also relevant for PD), and Hayward et al. [20] advocate for portraying inquiry in broad ways rather than through specific techniques. In the course in this study, the teachers were provided with creative and intellectual space to learn from their inquiry experiences both as students and teachers, and to, individually and collectively, refine their conceptions of the approach and choose what elements they wanted to emphasize in their efforts to bring inquiry into their classrooms. Though this was not our focus, we see that Alex, through the course, is provided several opportunities to, with the support from his peers and instructors, renavigate his inquiry compass. His paths are shaped by choices in the design of the PD course: the choice in mathematics education to build on teachers' pre-existing conceptualizations of inquiry (Figure 2), but structure these into three facets (Figure 3);

the choice to engage teachers in task design in the first seminar (Figure 4), and to work with a chosen task in the second (Figure 5); the choice to include compulsory classroom experimentation with inquiry between seminars, etc. Swan and Swain ([62] p. 175) stress that PD should support teachers' development by involving them in collaborative iterations where they "analyse, test and refine classroom activities that exemplify research based principles". As mathematical classroom activities consider how the students interact with tasks, each other, and the teacher [63], the iterations of experience, design, realization, and reflection in the PD course can be seen as supporting aspects that enable teachers to gradually refine their conceptualizations and realizations of inquiry in mathematics.

In this research, Alex volunteered to participate and was motivated to develop his teaching in an inquiry-based direction. He is in no way representative of any larger group of mathematics teachers; this trajectory is *his* trajectory. However, he provides empirical evidence not only that teachers' conceptions and enactments of inquiry can develop in a PD setting, but also *how* they can develop and on their possible paths and interplay. We encourage more research, to obtain a more nuanced understanding of the different trajectories teachers might take in similar settings, as teachers, despite having similar backgrounds, might have different paths of development in mathematics inquiry knowledge and teaching [41]. There is also evidence that teachers might only slowly, if at all, feed their PD learning and experiences into their teaching practice (e.g., [64]). This not only underpins the need for more and broader research, but also raises another question: Will the trajectory Alex has started on continue after his participation in the PD course? Artigue et al. [8] stress that this is an important question for inquiry-based mathematics education, as isolated activities and situations are interesting in themselves, but not enough for sustainable professional development.

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education sciences

Article

Overcoming Obstacles for the Inclusion of Visually Impaired Learners through Teacher–Researcher Collaborative Design and Implementation

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Abstract: Teacher preparation to address the needs of disabled learners in mainstream mathematics classrooms is quintessential for the implementation of the inclusive educational policies that governments are often committed to. To identify teacher preparation needs, we draw on data and analyses from the doctoral study of the first author, who endorsed sociocultural and embodied perspectives in an investigation—first exploratory, then interventional—of visually impaired (VI) learners' experiences and their teachers' inclusion discourses. Here, we focus on the intertwined contributions of physical and digital resources in the mathematical learning experiences of VI pupils, as these resources co-existed simultaneously in the observed mathematics lessons. We first summarise findings from the exploratory phase that highlighted inclusion issues related to resource use in the mathematics classroom. We then offer a critical account of the circumstantial and systemic obstacles that impeded the successful intertwinement of digital and physical resources and discuss teacher–researcher collaborative design and implementation of classroom tasks (auditory, tactile) in the intervention phase. We conclude by making the case that well-meaning individual teacher–researcher collaboration is a necessary condition for such interventions to succeed but not a sufficient condition for these interventions to be scaled up and have longevity.

Keywords: inclusion; disability; mathematics teacher education; mainstream classrooms; visually impaired learners; tactile and auditory mathematical tasks

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1. Introduction and Literature Review

There is recognition, in principle, of the importance of training teachers to address SEND learners' needs in international and national policy documents. Inclusive education needs to be implemented, not only because it promotes disabled people's rights to education [1], but also because it offers social and educational benefits to all learners [2]. In terms of social benefits, inclusive education makes disabled learners less stigmatised and more socially included while it also enriches non-disabled learners with tolerance, acceptance of difference, and respect for diversity [2]. In terms of educational benefits, inclusive education gives disabled learners access to a comprehensive curriculum, and it also leads to higher achievement than that found in segregated settings [2]. Simultaneously, inclusive education provides educational benefits to all learners through the changes that it brings in educational planning, implementation, and evaluation.

In England, Initial Teacher Education (ITE) policy concerning the inclusion of disabled learners includes statements such as: "trainee teachers must achieve professional standards before they can be awarded qualified teacher status. The standards ensure that teachers are able to help all pupils, including disabled pupils, to achieve their full potential" ([3] p. 73). More specifically, the policy states that "[t]eachers must learn to vary their teaching to meet the needs of all pupils, including those with SEN" ([3] p. 73) and that "[t]eachers must understand how pupils' learning can be affected by their physical, intellectual, linguistic, social, cultural and emotional development" ([3] p. 73). Such statements indicate that it is the responsibility of trainee teachers to learn how to teach disabled pupils so that the latter can achieve their full potential. Implicit in these statements is the role of the ITE programmes that provide teachers with the aforementioned training. Yet, these statements contain somewhat implicit references to this role, lack specificity and are aspirational rather than pragmatic. For example, there are scant references to the amounts of time needed for the aforementioned training.

This remains evident in recent policy documents in which the overall principles remain intact: ("the UK Government's vision for disabled children and young people is the same as for everyone else; to enable them to fulfil their potential in education, and go on to live happy and fulfilled lives.", [4] para. 12; "To ensure consistency across England our focus is improving the quality of Education, Health and Care plans (for those with complex needs) and of SEND support in schools and colleges.", [4] para. 13). However, there are very few statements that relate to Initial Teacher Training and Disability. Here are two examples of such statements.

The first statement prescribes the importance of training all teachers in adaptive teaching:

> *"Adaptive teaching is an important area in the CCF [Core Content Framework]. Alongside important content relating to the most effective approaches to adapting teaching in response to pupil needs, it sets out some specific content relating to knowledge and experience that all trainees must acquire relating specifically to pupils with Special Educational Needs and Disabilities (SEND). It is critical that all teachers begin their teaching career with adequate basic knowledge and expertise in this area, and all ITT curriculums, whatever the context, must set out specific content relating to SEND which trainees will learn and put into practice during training. As with all areas of the trainee curriculum, learning about SEND must be planned and specific, and there must be an assurance that all trainees have covered and learnt what has been planned."* ([5] p. 13)

The second statement stresses the importance of access to training for those choosing to specialize in SEND:

> *". . . alongside the universal SEND knowledge and expertise which all trainees should possess, there is scope for those preparing to specialise in SEND, either in specialist provision or in mainstream schools, to be able to access a specialist training curriculum that focuses in more depth on SEND-relevant knowledge and expertise. Such a training curriculum, which must be rigorously evidence based, should equally meet the expectations for detailed and specific planning, as should the expectations for school placement and mentoring, to ensure that the curriculum is delivered to trainees with the same standard of quality and consistency that we envisage elsewhere."* ([5] p. 13-14)

Our work aims to address the tension between intended and implemented policy and identify what teachers need in order to build inclusive mathematics classrooms. Research in this area indicates that there is limited teacher training (e.g., [6–9]). Implicit ableist narratives and a prioritising of the needs of a perceived "normal" student may underlie limitations in teacher training [10]. Ableism is "the network of beliefs, processes and practices that produce a particular kind of self and body (the corporeal standard) that is projected as the perfect, species-typical and therefore essential and fully human" ([11] p. 44) and holds a perspective on disability "as a diminished state of being human" [11]. The study that our paper draws upon [12] builds on this research. Our study aims to substantiate the benefits to all learners that are aspired to in international legislation as outcomes from the implementation of inclusive education. Such substantiation aims to highlight the significance of ITE around inclusion in England and beyond. Hence the focus of this paper is on investigating what teachers need in order to include VI pupils in their mathematics lessons and on how such inclusion can be of benefit to everyone in the class.

This study was divided into the following sections: the sociocultural and embodied theoretical underpinnings of our study [12] (2) are presented; the research design, context and participants of the study, methods of data collection and analysis are introduced (3); inclusion and resource issues in mathematics teacher education (MTE), circumstantial and systemic obstacles in implementing intertwined resources for inclusion and teacher–researcher collaborative design and implementation are presented (4); and the study concludes by discussing the implications for teacher education towards inclusive mathematics classrooms (5).

2. Theoretical Framework

The theoretical underpinnings of our study are sociocultural and include the social model of disability [13] and the Vygotskian sociocultural theory of learning, with a particular emphasis on the notion of mediation [14,15]. A pertinent role in the study's theoretical framework is also played by the theory of embodied cognition [16].

2.1. Influences from the Social Model of Disability

Our work endorses a social model of disability, according to which disability is socially constructed [13]. The social model considers disability as a problem imposed by society, which excludes people's full participation in social, educational, cultural, and other activities due to their impairments [13,17–19]. LoBianco and Sheppard-Jones argue that disability can be far less of an impediment when societies remove the barriers that disable individuals [20]. The social model of disability is associated with an inclusive approach to the education of disabled learners. According to this approach, it is the school that needs to transform its culture, policies, and practices in order to accommodate every individual's needs [2]. The social model of disability underpins the Convention on the Rights of Persons with Disabilities (CRPD) [1]. This convention considers that "disability is an evolving concept and [. . .] results from the interaction between persons with impairments and attitudinal and environmental barriers that hinders their full and effective participation in society on an equal basis with others" ([1] p. 1). Drawing upon Oliver's social model of disability [13], our study—which focuses on the inclusion of VI pupils—uses the constructs of "enabling" and "disabling". We use the terms "enabling"/"disabling" when a sighted member acts in a way that meets/does not meet the VI pupil's perceptual needs in a mathematics lesson. We define perceptual needs as the needs that relate to the pupil's accessibility to the mathematics lesson.

2.2. Influences from the Vygotskian Sociocultural Theory of Learning

The study's definition of "mathematical learning" is embedded in the Vygotskian sociocultural theory of learning [14]. In particular, we see mathematical learning as a social process which is characterised by the use of semiotic, material, and sensory tools that all comprise the culturally developed subject of Mathematics. While Vygotsky [14,15] explicitly considers semiotic and material tools as forming Mathematics, in his earlier formulations of the notion of mediation, which occurred while he was working with disabled learners, he implicitly considered parts of the body as sensory tools too, which—much as semiotic and material tools do—impact upon the individual's cognitive activity. This implicit consideration emanates from the tenet that body parts can be thought of as "instruments" used to sense the world: "the eye, like the ear, is an instrument that can be substituted by another" ([21] p. 83). Vygotsky's [15] attribution of the role of a psychological tool to elements of the body constitutes a strong allusion to the embodied nature of the human intellect. However, while the embodied nature of cognition is clear in Vygotsky's [15] works, Vygotsky was primarily interested in the sociocultural characteristics of said tools.

Vygotsky's ideas about knowledge mediation have their roots in his experimental work with disabled learners [15]. Vygotsky acknowledged that the language of a culture tends to be designed for the able-bodied. This implies that language may not be accessible to people who lack, or have limited access to, a sensory channel. He suggested that, instead of focusing on quantitative differences in achievements between disabled and non-disabled learners, a qualitative perspective can be enlightening. For Vygotsky, the inclusion of disabled learners in social and cultural activities can be fulfilled in the identification of ways to substitute the traditional mediational means with others, which are more suitable to the specific ways in which disabled learners interact with the rest of the world. For example, in the case of VI learners, Vygotsky posited that their inclusion could be achieved through substituting their eyes with another tool. As with the inclusion of any other tool in an activity, this substitution can be expected to cause a restructuring of the cognitive activity of the VI individual [15]:

> *"The positive particularity of a child with a disability is created not by the failure of one or other function observed in a normal child but by the new structures which result from this absence [. . .] The blind or deaf child can achieve the same level of development as the normal child, but through a different mode, a distinct path, by other means. And for the pedagogue, it is particularly important to know the uniqueness of the path along with the child should be led".* ([21] p. 17)

Therefore, Vygotsky considered disabled learners as different but not deficient. He focused on what these learners can do rather than on what they cannot do.

This understanding of inclusion and disability resonates with the understanding of inclusion as evident in today's international legislation (for example, [1,2,22]). In particular, Vygotsky [15] acknowledges the importance of designing education systems suitable for the able-bodied and disabled learners alike.

In the study that our paper draws on [12], we examine how the aforementioned sensory, semiotic, and material tools mediate the mathematical learning of VI pupils and consequently affect the inclusion and enabling of these pupils in the mathematics classroom [9,23,24]. Closely aligned with our investigation of the experiences of VI pupils is our focus on their teachers' preparedness to fulfil those learners' needs as well as on how the uses of said mediating tools are, or can be, beneficial to all learners in class.

2.3. Influences from Gallese and Lakoff's Theory of Embodied Cognition

While the theory of embodied cognition is not used by Gallese and Lakoff specifically for disabled learners, its tenets imply that disability is not a direct implication of an individual's physical impairment. This is extrapolated from Gallese's [25] and Gallese and Lakoff's [16] understanding of cognition: cognition is embodied, and understanding is multimodal.

The combination of these two tenets from the theory of embodied cognition [16] implies that a bodily impairment does not equate to disability: a sensory organ is one of the multiple modalities through which knowledge is constructed. Therefore, a limited function—or a non-function—of this organ does not by itself stop the individual from such construction, as there are other perceptual modalities to be utilised. In the theory of embodied cognition [16], what may make an impairment a disability would be a lack in the provision of multimodal activities to an impaired individual. Indeed, as understanding is multimodal, if a learning experience is reliant on the activation of a sensory channel with limited or no function, then the individual will be disabled. Therefore, within the theory of embodied cognition [16], disability can be seen as socially constructed. Implicit as well is the assumption that, as cognition is embodied, a necessary element of inclusion is the provision of opportunities that allow the impaired individual to construct knowledge. Therefore, the theory of embodied cognition seems to resonate with the understanding of inclusion in current international legislation (for example, [1,2,22]).

In the study that our paper draws on [12], we are particularly attentive to the multimodal elements in knowledge construction that the theory of embodied cognition draws attention to. We see cognition as both an intrapersonal and interpersonal process [25]: what we know and do is a result of our constant interactions with the world via our bodies and our brains. In tandem with our Vygotskian view of teaching as an intrapersonal and interpersonal process of engaging learners in discourses associated with the sociocultural activity known as mathematics, from an embodied viewpoint, the interpersonal elements of cognition are particularly important as they occur in the context of actions, emotions, and senses of, and with, others.

In the light of the theoretical influences from the sociocultural theory, embodied cognition and the social model of disability outlined so far, the research questions that our paper aims to explore are:

Research Question 1: What teacher preparation needs should teacher education programmes address, especially in relation to how teachers use resources in their lessons towards inclusive mathematics lessons?

Research Question 2: How does teacher–researcher collaborative design and implementation of inclusive classroom activities contribute to the benefit of everyone in class, disabled and able-bodied alike (in our case: VI and sighted learners)?

We see Research Question 1 as directly related to MTE programmes and Research Question 2 as addressing the need for on-the-ground, and ongoing, teacher engagement—and continuing professional development—with the fast-rising developments in the area of inclusive mathematics education research. We also note that our study sets out from the longstanding assumption within the participatory action research paradigm [26] that there exist vital benefits for research in teacher–researcher collaborations in shaping urgent research agendas as voiced by key stakeholders (learners and teachers) and in trialling applicable solutions to problems.

We now introduce the research design, context, and participants of the study that our paper draws on. We also outline the study's methods of data collection and analysis.

3. Methodology

3.1. The Research Design of the Study

The study [12] upon which this paper draws is the doctoral study of the first author and was supervised by the second author. The study investigated inclusion and disability in the discourses of teaching staff and pupils in English mainstream primary mathematics classrooms with VI pupils, first in an exploratory phase (Phase 1), and then in an intervention phase (Phase 2). By "discourses", we denote utterances—expressed through speech but also through gestures, facial expressions, and bodily expressions in general, which relate to inclusion and/or disability and which are expressed by the participants either during the lesson or outside the lesson. The discourses may signify the participants' attitudes towards—and/or experiences of—inclusion and disability [9].

In Phase 1, we investigated how class teachers, teaching assistants, and sighted pupils consider inclusion and disability in the context of the mathematics classroom, and with regard to VI pupils. We examined whether there are any variations amongst different participants in the same classroom in their consideration of inclusion and disability. We also identified how consistent the participants' discourses are with the discourses on inclusion and disability in the participating schools' policies; the SEND code of practice [27], which is the UK's educational code of practice for children and young people with Special Educational Needs and/or Disabilities; and international policies on inclusion and disability [1,2,22]. In Phase 1, we used classroom observations, focused group interviews, and individual interviews.

In Phase 2, we examined evidence from Phase 1 on inclusion and disability. With the aim of bringing the practice closer to endorsed principles in international legislation on inclusion and disability, we designed mathematics lessons with the participating teachers that the teachers then trialled in the classroom. These lessons aimed to be experienced as enabling and inclusive by the VI pupils and as beneficial to every pupil. The study explored, and aimed to provide specific evidence on, the social and educational benefits—which emanate from the implementation of inclusive education in the classroom—to all pupils. The lessons also serve as a platform for us to examine participants' potential discursive shifts regarding inclusion and disability. In Phase 2, we used written transcripts of the class teachers' contributions to the design of the three intervention lessons, classroom observations, focused group interviews, individual interviews, photographs of the pupils' work in the three intervention lessons, and pupils' evaluation forms of the intervention lesson in two classes.

Stylianidou's [12] study addressed the following research questions: How are inclusion and disability constructed in the discourses of teaching staff and pupils in the mathematics classroom? How do collaboratively designed mathematics lessons impact upon teaching staff and pupil discourses on inclusion and disability? The first research question was explored in both phases of the study while the second research question was explored in Phase 2.

For the purposes of this paper, we extracted data and analyses from [12] in order to answer the teacher preparation Research Questions formulated at the end of Section 2.

3.2. Context and Participants of the Study

Data collection was conducted in four primary mathematics classrooms (Y1, Y3 and two Y5 classes; pupils' ages varied from six to ten) in four mainstream schools in the county of Norfolk, UK. The VI pupils' presence and the willingness of teaching staff and pupils to participate in the study constituted the main criteria for the selection of participants. There is one VI pupil in three of the classes and two in the fourth. Most of the participating VI pupils had severe visual impairment and none of them was blind in both their eyes. Two pupils had congenital visual impairment while three had adventitious visual impairment ("Congenital" and "adventitious" have to do with the age of onset of visual impairment. Congenital VI are individuals who have been born with visual impairment while adventitious VI are the individuals whose visual impairment has appeared later in life). We collected data after securing ethical approval by the University of East Anglia's Research Ethics Committee and we ensured participants' consent as well as their anonymity, confidentiality and right to withdraw from the study.

Every class had at least one teaching assistant, but the teaching assistant's role differed from class to class. While two of the classes had a teaching assistant supporting the VI pupils almost exclusively; in the other two classes, the teaching assistants supported pupils who needed help in particular instances and their role did not focus on supporting the VI pupils specifically.

We coded the names of classrooms and of teaching staff and have used pseudonyms for the names of pupils.

3.3. Data Collection

We collected data through the observations of 29 mathematics lessons (33.5 h in total); individual interviews with five class teachers (six interviews, 2 h and 10 min in total); individual interviews with four teaching assistants (six interviews, 2 h and 15 min in total); focused group interviews with 35 pupils (16 interviews, 2 h in total); two ten-minute individual interviews with one pupil; written transcripts of the teaching staff's contributions towards the design of the three lessons that constituted the intervention phase of the study; photographs of the pupils' work in the three intervention lessons; and, pupils' evaluation forms of the intervention lesson in two classes. In one of the classes taught by two teachers—on different days—both teachers were interviewed. During the observations, written notes were kept for all lessons. Twenty-one lessons were audio-recorded, and 14 lessons were audio/video-recorded. All interviews were audio-recorded, except four, following interviewee requests. For these, written notes were kept instead.

3.4. Data Analysis

Our unit of analysis for the classroom observation data is the classroom episode. Our choice of episodes as analytical units resonates with the use of this method in [9,24] studies on the inclusion of VI pupils in the mathematics classroom. We define a classroom episode as a part of the mathematics lesson that has a starting and an ending point and thus can stand alone in the text with relative clarity, and that also has the capacity to convey a key point related to the focus of the study. Applying this definition to the classroom observation data, we broke each lesson down into episodes. In particular, in each lesson, we examined the classroom observation data, and we broke them into episodes.

The labels of the episodes illustrate the ways in which inclusion and/or enabling of VI pupils took place in the mathematics lesson, along with their impact on the VI pupils. The impact was elicited from the VI pupils' (re)actions during the mathematics lesson. We then collected all the labels, grouped similar labels together, and discerned the themes that the grouped labels fit to. Afterwards, within each theme and with the help of the labels, we identified the issues that concerned each theme.

In the data analysis sections of Phase 1, data from individual interviews and from focused group interviews were used to support, deepen, expand, or contradict issues that emerged from the analysis of classroom episodes. Data from the lesson design, classroom observations, individual interviews, focused group interviews, and evaluation forms informed the data analysis sections of Phase 2.

In what follows, we draw on the data and analyses in [12], first to identify inclusion issues on resource use in mathematics teacher education, and then to examine the circumstantial and—crucially for the MTE focus of this paper—systemic origins of these issues (Section 4.1). We then present evidence on how these issues fed into teacher–researcher collaborative design and implementation of inclusive classroom activities for the benefit of everyone in the class (Section 4.2).

4. Data Analysis and Findings

We first discuss the opportunities and challenges in using digital/physical resources in inclusive mathematics education, focusing on MTE issues (Section 4.1). We then present teacher–researcher collaborative design and implementation of inclusive mathematics lessons, offering a critical reflection (Section 4.2).

4.1. Opportunities and Challenges in Digital/Physical Resource Use for Inclusive Mathematics Education (and MTE Issues Thereof)

In Section 4.1, we summarise findings from the exploratory phase in [12] that provide evidence of inclusion issues on resource use in mathematics education. We use critical evidence from episodes that highlight where the effectiveness of teachers' practice can be further supported (in current practice as well as in MTE).

4.1.1. Teacher Positioning: When Teaching Becomes Inadvertently Inaccessible

While we have found that digital resources mediate VI pupils' visual access to the teacher's physical demonstration, we saw evidence of bodily discomfort in the VI pupil that was associated with the teacher's position in her physical demonstration. The teacher's position often appeared to impede the inclusion of the VI pupil (as evidenced by the Fred case in Figures 1–3). Her bodily position often resulted in aches in the VI pupil's back and arms as he held his iPad towards her. The bodily discomfort sometimes prompted the VI pupil to give the iPad up (see Figure 4). The teaching assistant often intervened by trying to include the VI pupil through the use of his iPad (see Figure 5).

The bodily position of the teacher constitutes a circumstantial obstacle. This obstacle is associated with Roos's finding regarding "the importance of the teacher when aiming for inclusion in inclusive mathematics classrooms" ([28] p. 240), especially, as Roos stresses, when inclusion is hindered because of reduced accessibility to the support offered by the teacher.

Figure 1. Fred sits up with his elbows in the air. He holds his iPad towards the teacher's physical demonstration.

Figure 2. Fred has his back stretched and is leaning towards the teaching assistant, with his right elbow on his table. He continues to follow the teacher's physical demonstration on his iPad.

Figure 3. Fred moves his body, as seen with his posture in Figure 1. He continues to hold his iPad towards the teacher's physical demonstration.

Figure 4. Before the teacher completes her question, Fred puts his iPad away and makes a facial expression of tiredness and disappointment.

Figure 5. The teaching assistant gives the iPad to Fred to follow the teacher's physical demonstration. Fred takes the iPad, making a facial expression of unhappiness.

4.1.2. A VI Pupil's Desire to Not Stand out and School Narratives about Disability as Deficit, Not Difference

Another issue that occurs in tandem with, and sometimes emanating from, the aforementioned issue of the teacher's positioning, is the VI pupil's preference for physical resources over digital ones for his inclusion in the teacher's physical demonstration. In our earlier example, despite the inclusive intentions of the teacher—she expects the VI pupil to use his iPad to follow her physical demonstration—and although these intentions are associated with the intertwinement of the VI pupil's digital resource and the teacher's physical resources, the iPad does not prove to be a satisfactory mediating tool for the VI pupil. When an opportunity to use the physical resources appears, the VI pupil grabs this opportunity (see Figure 6).

Inclusion, through the VI pupil's use of a digital resource, aims to be achieved also by the teaching assistant. The teaching assistant insists that the VI pupil should follow the teacher's demonstration from his iPad, despite seeing that the VI pupil is unwilling to use it and is interested in using physical resources. It is only when she sees the VI pupil continuously refuse to use his iPad (Figure 7), that she includes him through the use of physical tools.

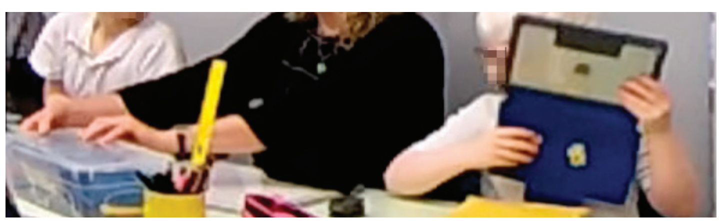

Figure 6. Fred moves the iPad case to cover its screen when the teaching assistant opens the box with the physical blocks.

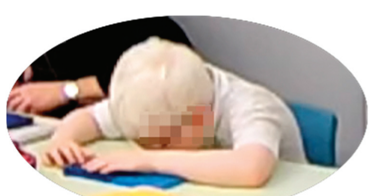

Figure 7. Fred works visually and tactilely with the physical blocks. The teaching assistant has given him 220 while the teacher has given 230 to the three sighted pupils—the teaching assistant missed the teacher's earlier giving of one block of Ten to the Tens pupil. Fred needs to lean that much to be able to see the blocks—otherwise he cannot see them.

While we found that digital resources mediate VI pupils' visual access to the teacher's hybrid physical–digital demonstration (see Figure 8), the teacher's emphasis on the VI pupil's use of digital resources is not as effective, due to the VI pupil's desire to not do something that makes him stand out amongst his sighted peers. Being the only pupil who is given a special mediating tool seems to upset the VI pupil. His looks towards the Interactive Whiteboard, from which he cannot access the teacher's hybrid physical–digital demonstration, seem to be a way to express his objection to being the only one who is asked to use a special tool for his inclusion. Unlike our previous examples, the iPad is not associated with any obvious, noticeable, external, physical (namely, bodily) discomfort for the VI pupil, it is associated with inner, emotional discomfort (namely, the feeling of being singled out from the rest of the class).

The VI pupil's reluctance to use his iPad constitutes a systemic obstacle because it is rooted in how the pupil sees his school's consideration of difference. The school may not always seem to cultivate positive connotations of disability as difference (not deficit) with regard to visual impairment and this narrative seems to be endorsed by the pupil who simply does not want to stand out in any way. Providing special digital resources to the VI pupils—iPad and computer—and being asked to use these in almost every part of the mathematics lesson is one way in which the VI pupil is made to feel different from the rest of the class (we acknowledge of course the well-intended underpinning of this provision, which is to help the VI pupil access what their sighted peers access). Another way in which the VI pupil is made to feel different from the rest of the class is the insistence on using resources that mainly try to mitigate limitations in sight. Again, the VI pupils' difference is not celebrated, as the VI pupils are asked to use their limited vision to construct mathematical meaning. They are rarely asked to use other sensory channels—such as touch, to which they have fuller access—in their mathematical learning.

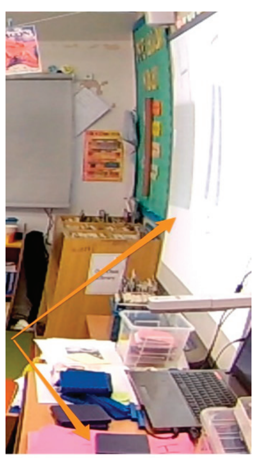

Figure 8. 127 demonstrated with physical blocks on the visualiser and projected on the IWB.

This systemic obstacle has repercussions on pupils' sense of belonging which, according to Rose and Shevlin [29], constitutes another key aspect of inclusion: the reluctance to use the iPad is an indication by the pupils that they do not belong to the classroom community, and their feeling of not belonging needs to also be considered when we aim towards a more inclusive mathematics teacher education [30]. This obstacle also resonates with Buhagiar and Tanti's point ([31] p. 72) that disabled pupils "were physically present in the class, but they did not seem to be part of it".

This systemic obstacle is also associated with Roos's [28] finding regarding the dislike of mathematics as a hindering issue for inclusion and how this dislike has sociopolitical underpinnings. As Roos [28] indicates, the label "special needs students" creates obstacles for inclusion, and the ideological way of using inclusion at school often generates exclusion. Implicit in Roos' statement is the problematic notion of the "normal" student. Schools cannot become inclusive when they are underpinned by the notion of the "normal" student: this notion legitimises exclusion, since it separates students who differ from the sociopolitical connotations of this kind of student as problematic and in need of remediation [10].

While we have found that hybrid physical–digital resources mediate VI pupils' visual and tactile access to physical resources, the teacher's limited awareness of the VI pupil's visual needs constitutes an inclusion issue on resource use in mathematics education. Despite the inclusive intentions of the teacher—she expects the VI pupil to use his visualiser to access the physical worksheet—and although these intentions are associated with intertwinement of the VI pupil's hybrid physical–digital resource and his physical resource, this intertwinement is sometimes not experienced by the VI pupil. Therefore, the inclusive intentions of the teacher become problematic for the VI pupil because he is invited to be included in a way that he is not comfortable with.

The appropriate adjustment of a physical resource to the needs of the VI pupil constitutes a systemic obstacle. This obstacle is systemic because it is rooted in the school's consideration of inclusion and, in particular, in the class teacher's role with respect to the inclusion of VI pupils. While pre-service and in-service training on the inclusion of VI pupils for class teachers is limited, training on the inclusion of VI pupils is provided to teaching assistants. These two facts indicate the systemic view that there should be a 'special' person—not the class teacher—responsible for the VI pupils. This may suggest an institutional narrative about inclusion as a transplantation of special education in mainstream settings [32]. Instead, as Noyes [33] and Ingram [34] also stress, knowing the pupils, and teaching in accordance with pupils' needs, constitute key factors of productive and sensitive inclusion.

While we have found that physical resources mediate VI pupils' visual and tactile access to the teacher's digital demonstration, the VI pupil's unfamiliarity with using physical resources constitutes an inclusion issue on resource use in mathematics education. This unfamiliarity stems from the teaching staff's emphasis on digital resources: the VI pupil is mostly asked to follow teacher demonstrations through a digital resource, and he is rarely asked to use physical resources to access these demonstrations. Even in the rare cases in which he uses physical resources to construct mathematical meaning, he is not encouraged to use touch as a sense to construct mathematical meaning: he instead employs his limited vision—and this is, again, related to tacit sociomathematical norms established in the classroom that pertain to the privileging of vision in mathematical learning. The VI pupil's reliance on his limited vision is often associated with a mistaken following of the teacher's digital demonstration (see Figure 9). We speculate that, had the VI pupil relied on touch, the mistakes could have been avoided.

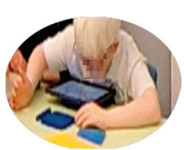

Figure 9. Fred pushes 4 blocks of Ones—instead of 3—with his right hand, without looking at his iPad.

4.1.3. Coordinating a Teacher's and a Teaching Assistant's Interventions in Assisting a VI Pupil

Despite the inclusive intentions of the teaching assistant—she assists the VI pupil in his mathematical work with physical resources (see Figure 10)—and while these intentions are associated with intertwinement of the teaching assistant's whiteboard and the teacher's digital demonstration (see Figure 11), the teacher shared with us after the lesson that the teaching assistant's intervention makes the VI pupil merely more dependent upon the teaching assistant's presence as a substitute for mere access to what he cannot see. Again, the support provided to the VI pupil neither enables nor celebrates his access to sensory channels other than sight (e.g., touch). Implicit narratives about the under-valued role of (e.g.,) touch in making mathematical meaning with VI pupils are examined in [24] who report that VI pupils are rarely provided with opportunities to use touch and in [6], from a student's point of view,

> *"The students had explained to us that it was rare for them to interact with representations of geometrical shapes, and an important aspect of designing the tasks was to produce tactile materials that would make this possible"* ([24] p. 134)

and from a teacher's point of view,

> *"According to the two teachers the lack of materials had a great impact on Nefeli's haptic apprehension and for this the researchers prepared and provided the teachers material following exactly the activities suggested in the school textbook".* ([6] p. 129)

The aforementioned obstacles indicate that "inclusion is a complex process of participation where both ideological and societal issues, as well as individual and subject-specific issues, must be considered in the educational endeavour" ([28] p. 244). To optimise the intertwinement of physical and digital resources, we need to overcome these, and other, circumstantial and systemic obstacles. We note that both—particularly the latter—are harder to overcome, as they are located in deeply rooted institutional narratives about what constitutes a legitimate mode of mathematical learning. In Phase 2, we and the class teachers attempted to overcome these two groups of obstacles. We report on our collaboration in what follows.

Figure 10. Fred follows the teacher's working out of the calculation from the teaching assistant and her whiteboard. The teaching assistant looks at the teacher's demonstration on the IWB.

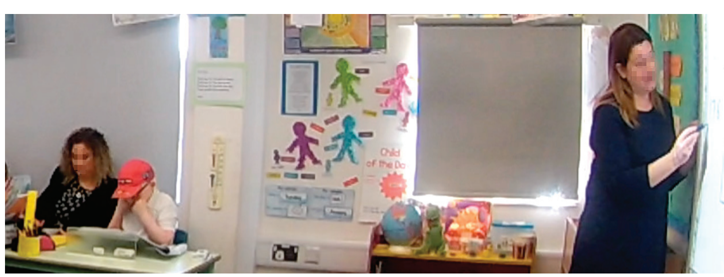

Figure 11. The teaching assistant writes the column addition on her whiteboard while the teacher is writing it on the IWB. Fred continues to rub his eyes.

4.2. Teacher–Researcher Collaborative Design and Implementation of Inclusive Mathematics Lessons (and Critical Reflection Thereof)

Here, we discuss how the class teachers and we (primarily the first author and doctoral researcher) tried to tackle the obstacles—that were identified in the exploratory phase—in our collaborative design of the intervention lessons. We present our collaborative efforts with the teacher and reflect critically on these efforts. We do so to propose that, beyond pre-service teacher education, ongoing collaboration/professional development is a necessary and potentially productive way forward for the sustainable efforts needed to create inclusive mathematics classrooms. We summarise findings on resource use that arose after the implementation of the intervention lessons: we focus on the 'what' and 'how' of the design with the teachers as well as on the implementation of the design (with a particular focus on resources used towards the support of VI pupils' mathematical learning).

4.2.1. Overview of Design Priorities and Issues

In the design of the intervention lessons, the class teachers and the first author decided not to involve the teaching assistants in the inclusion and enabling of the VI pupils. Instead, we decided to design the lessons in such a way that the class teachers were primarily responsible for the inclusion and enabling of these pupils. Cases where the class teachers are responsible for the inclusion and enabling of VI pupils are also seen in the literature (e.g., [7,35,36]). For example, Sticken and Kapperman [36] report on the complexities emerging out of limited coordination when inclusion is implemented by the class teacher and the support teacher.

The obstacles that arose in the exploratory phase with regard to the inclusion of VI pupils through digital resources prompted the class teachers and the first author to shift towards exploring alternative ways to include these pupils. These involved the design of tactile and auditory resources that do not require vision to be accessed, but that invite the use of other sensory modalities by the class. Our design of tactile and auditory resources for VI pupils was also informed by the literature (e.g., [6,37,38]).

The obstacles that arose in the exploratory phase with regard to the VI pupils having mistakenly followed the teachers' digital demonstrations with their physical resources prompted the class teachers and the first author to shift towards designing mathematical tasks that are experienced through touch—and not just by the VI pupils [39]. We aimed to increase the familiarity of the whole class with touch and to show the significance of touch in mathematical learning. In this respect, our focus on tactile mathematical tasks addressed to every pupil differed from that in the literature (e.g., [6,37,38]). The literature focuses on addressing these tasks only to the VI pupils, with the sighted pupils working on visually based tasks. In other words, the literature focuses on 'translating' sighted pupils' tasks to the needs of the VI pupils under the principles of adaptation/differentiation/accommodation. However, we focussed on designing tasks that address every pupil's needs under the principle of universal design for learning [1].

Our collaborative design of the intervention lessons was on the mathematical topics and learning objectives that the class teachers had planned to be working on the day of the lesson implementation. In addition, the lessons were co-designed in a way that teachers felt comfortable with. As the lessons were implemented by the teachers, and as they aimed to trigger long-lasting changes in the classroom, they needed to be substantiated with the teachers' contributions and agreed upon with the teachers.

In what follows, we present two auditory and two tactile mathematical tasks (and mention a third one briefly) that the class teachers implemented in the Phase 2 lessons. We first present two auditory tasks, both of which are based on number sequences: the first task was co-designed by a teacher at School 3 and the first author; and, the second task was co-designed by a teacher at School 4 and the first author (and was based on preliminary findings from the implementation of the first auditory task). We then present two tactile tasks. The first task is based on number sequences and was co-designed by the teacher at School 3 and the first author. The second task is based on shapes and was co-designed by the teacher at School 2 and the first author (and was based on preliminary findings from the implementation of the first tactile task).

4.2.2. First Auditory Task on Number Sequences

In the Y1 class at School 3, the teacher plays a single, low sound on a xylophone and tells the class that, when they hear this sound, it is a Ten. She then takes another xylophone, plays a single, high sound, and tells the class that, when they hear this sound, it is a Unit. Afterwards, she plays various sounds, sometimes by using one xylophone and sometimes by using both xylophones. Each time, she asks the class what number she plays. She then plays number sequences with a pause in between two successive numbers and asks pupils to move towards her and play the next numbers. Afterwards, she asks the class if the sequences are increasing or decreasing and by how much.

In this task, the teacher expected the class to discern number sequences through the use of hearing—by listening to number sequences represented via musical instruments. She expected to hear mathematical contributions commensurable with Y1 curricular requirements [40]:

- discern numbers;
- discern number sequences;
- say what the next number is in the sequences;
- explore place value;
- say if the sequence increases or decreases—and by how much.

The primary aims of this task were: to invite the class to experience number sequences through the sense of hearing—by the teacher incorporating music into Mathematics and to investigate the mathematics elicited through the auditory experience.

The use of the auditory construction of mathematical meaning contributed to the teacher's realisation that she should distinguish between focussing on mathematics and looking at the teacher/board. Before this task, this teacher—as well as other teachers—confused these two situations. More specifically, they posited that, if the VI pupil does not look at them or the board, he is not focussed. In this task, she realised that the fact that the VI pupil (Ned) does not look at her does not necessarily make him lose focus: this is because, in this task, mathematical learning is constructed in the auditory—not visual—modality. The teacher particularly reported:

> *"I did think music still allowed him to access that, so often Ned is not focussed—I mean he is not looking at the board, so he is missing key learning—but, because he wasn't looking necessarily up, I think maybe he can still listen to what was going on, so he could still kind of grasp what was going on."*

The active involvement—and the use of musical instruments—in the representation of number sequences by the teacher, as well as the invitation to the class to play the next number on the musical instruments, were particularly beneficial for both the sighted and the VI pupils.

Both mathematical and social benefits arose. With regard to mathematical benefits, we indicatively report the following: ease in discerning between Tens and Ones through music; translation of each sound into its place value representation; clarity of the patterns in number sequences via music. With regard to social benefits, we indicatively report the following: pupils realised that music and Mathematics are not necessarily disconnected from each other, but rather that music can be productively used in Mathematics; the auditory task made pupils relax—they associated music with Mathematics and considered relaxation as the effect of music upon them.

4.2.3. Second Auditory Task on Number Sequences

In the Y5 class at School 4, the teacher asks the class to work in pairs. Each pair needs to create a number sequence of five numbers, with the first number having two digits. The teacher tells the class that they will need two sounds for each number: one sound to represent a Ten; and, one sound to represent a Unit. Each sound will need to come from a different musical instrument. One pupil in each pair will represent the Tens and the other pupil will represent the Units. The teacher then asks the class to take musical instruments and to start creating, and then practising, their number sequences in their pairs.

Afterwards, each pair plays their number sequence to the rest of the class and the rest of the class tries to work out what the numbers are—and what the rule is—in that sequence. The teacher tells the class that it is up to them how they record the number sequences.

In this task, the teacher expected to hear mathematical contributions commensurable with Y5 curricular requirements [40]:

- discern number sequences
- express the rule in the sequences
- create number sequences

The primary aims of this task were to invite the class to experience number sequences through their sense of hearing—by the teacher incorporating music into Mathematics, to investigate the mathematics elicited through the auditory experience, and to invite the class to construct number sequences in pairs and then represent these sequences through musical instruments.

In this task, the teacher suggested asking pairs of pupils to construct, and then play, a number sequence using musical instruments. We now explain why these were good suggestions.

Her invitation towards pupils to work together reinforced pair work, which was missing from that class in Phase 1. We indicatively report the following benefits from pair work for VI and sighted pupils:

- The VI pupil (Ivor) was better included in the class, he was no longer a separate member from the sighted community of learners. Ivor particularly acknowledged that today's mathematics lesson was "[t]otally different" to the one he normally has. One of the differences that he pointed to was that "we were put in partners today". He pinpointed that he likes working with a peer because "[t]hey can help one another". He found it "[k]ind of easy and odd" that he did not work with the teaching assistant tin the lesson: "Easy because it was just like stuff and sequences, and I just knew what sequences were";
- There was mutual appreciation between the VI pupil and his sighted peer (Frank), both mathematically and socially. For example, Ivor helped Frank with the last number in their sequence: While Ivor correctly did not play any Tens for "6", Frank did not play any Ones—he possibly thought that it was Ivor's turn and he did not look at the number. Ivor made a facial expression to Frank showing that it was Frank who had to play that number. Frank played it. Therefore, while in Phase 1 Ivor was helped by others and appeared to be weak and distracted in Mathematics, in Phase 2 Ivor helped his sighted peer. This finding from Phase 2 illustrates Ivor's very good understanding of place value and also the very good collaborative skills between Ivor and his partner;
- Pupils liked the collaborative production of mathematical ideas.

We now present benefits from pair work as reported by the teacher:

- All children participated in and were actively engaged in the lesson;
- There seemed to be no pattern in the work of High Achieving Pupils (HAPs), Middle Achieving Pupils (MAPs) and Low Achieving Pupils (LAPs) in the auditory task. This task helped blur the boundaries across ability groups (and cast some doubt on the utility and purpose of such groupings). Specifically, some LAPs found it easy and some HAPs found it hard. This finding raises the need to discuss on which terms "ability groups" are decided and how accurate these decisions are.

Benefits from pair work were also reported by the teaching assistant: real engagement and excitement of the children, albeit somewhat noisy; good work in pairs; appreciation of our principle with mixed-ability pairs and our choice of pairs; the VI pupil was very much included and concentrated.

The teacher's invitation of pairs of pupils to play a number sequence using musical instruments reinforced:

- the pupils' active involvement in the construction of mathematics through music;

- the rest of the class's development of the auditory modality in the construction of mathematical meaning. For example, Ivor told the first author that "listening to the bits of music" and "listen[ing] [. . .] about the sequences" were what made him concentrate. In Phase 1, Ivor told her that he felt distracted and did not concentrate much in the lesson when he did not work with the teaching assistant. In the Phase 2 lesson, he told her that he concentrated even though he did not work with the teaching assistant. He also told her that, in the Phase 2 lesson, he did not find it hard to follow the lesson from the class teacher. In Phase 1, he told her that he found it hard to follow the maths lesson from his class teacher and that he instead found it helpful to work with the teaching assistant in mathematics. He specifically said: "I couldn't keep up with the teacher but now I can".

The mathematical and social benefits that arose in this second auditory task are similar to the ones reported in the first task.

4.2.4. First Tactile Task, Number Sequences

In the Y1 class of School 3, the teacher introduces to the class an A3 worksheet which includes four number sequences in a landscape format. Above the printed numbers, Wikki Stix is stuck so that the number sequences can be felt. The teacher moves around the classroom and invites pupils to close their eyes, feel numbers on the worksheet and tell her what the numbers are. She then asks some pupils how many Tens and how many Ones they need for the numbers that they have felt.

In this task, the teacher expected to hear mathematical contributions commensurable with Y1 curricular requirements [40]:

- 'read' numbers. We enclose "read" in quotation marks because this term is contextualised in this task differently—not through the visual sense. More specifically, "read" is contextualised as discerning numbers through the tactile sense;
- discern number sequences;
- explore place value.

The primary aims of this task were to invite the class to experience number sequences through their sense of touch by becoming familiar with Wikki Stix, which is often used by VI pupils, and to investigate the mathematics elicited through the tactile experience.

4.2.5. Second Tactile Task, Shapes

In the Y5 class of School 2, the teacher holds a bag with a range of 2D plastic shapes. He moves around the classroom and invites pupils to put their hands in the bag, pick one shape without taking it out of the bag and without looking at it, feel it and then describe it to the rest of the class. The teacher is the only one who has visual access to that shape, which remains in the bag.

In this task, the teacher expected the class to describe the given shapes through touch. He expected to hear descriptions of shapes commensurable with Y5 curricular requirements:

- name particular shapes (e.g., "rectangle", "hexagon");
- discern whether these particular shapes are 2D or 3D;
- name the properties of these particular shapes with regard to
  - sides: number of sides, if any sides are equal to each other, if any sides are parallel to each other, if there are straight and/or curved sides
  - vertices: number of vertices
  - angles: number of angles, types of angles.

The primary aims of this task were to invite the class to experience shapes through their sense of touch and to investigate the mathematics elicited through tactile experience. For these two reasons/aims, the teacher was open to the mathematical contributions of the class that stemmed from their tactile experiences, and he did not strictly request the class to respond with regard to all the mathematical properties listed above.

This task was suggested by the teacher, in resonance with our design principle asking the entire class to explore mathematics through touch. It complements the task we present in [39] where Wikki Stix is used to make and describe shapes.

The mathematical benefits that arose from this task reinforced the mathematical benefits of the tactile task that the first author had suggested. More specifically, pupils reported mathematical benefits pertinent to touch and to its characteristics. For example:

- Touch allowed pupils to “count the edges, sides, corners and vertices”;
- Touch allowed pupils to realise “the hidden facts on the shapes”;
- Touch allowed pupils to “have to feel around to get it”;
- Touch allowed pupils to “realis[e] [. . .] the differences”;
- Touch allowed pupils to “move the shapes”.

The pupils reported they “liked” the following in their engagement with this task:

- “touching the shapes and describing them”;
- “closing my eyes and feeling the shapes, trying to figure out what they were”;
- “picking into the bag and describing it”;
- “the different way to learn about shapes”;
- “feeling the shape and getting it correct”;
- “feeling” the shapes/the “feel” of the shapes;
- the “weird” feeling that touch generated for them.

This task also reinforced:

- gestures as a tool for construction—and expression—of mathematical meaning. Gestures were particularly used by sighted pupils, the VI pupil, and the teacher in the construction and expression of mathematical meaning;
- the verbal mathematical language that is elicited through tactile experiences (e.g., “it feels”).

As a result, the class and the teacher appreciated the opportunity to use touch in mathematical learning and experience, first-hand, the mathematics that touch can help to create.

To sum up, a range of findings arose during the implementation of these tasks. First, the tasks were experienced by both VI and sighted pupils with palpable excitement. No special resources were needed for the VI pupils, and everybody accessed the tasks using a sense to which they have full access. Compelling mathematical contributions by both VI and sighted pupils also arose from the use of these types of resources.

Apart from benefits to the pupils, the tactile and the auditory tasks were also of benefit to the teaching staff. The teaching staff also experienced visible excitement with the use of these resources. They acknowledged the mathematical and social benefits of these tasks, and evaluated touch and hearing as senses that have relevance and potency in mathematical learning.

In Phase 1, the teachers considered vision as the prevalent sense in mathematics, with rare use of touch and hearing implying that they are seen as being of secondary importance. In Phase 2, they were more open to, and appreciated, touch and hearing as valued sensory channels for mathematical learning. This openness made them diverge from the institutional norms that emphasise vision as a dominantly relevant sense in mathematical learning. They considered other senses as intellectually valid, and therefore moved away from considering certain senses as more intellectually valid than others. This openness may enable the design of more tactile and auditory tasks in the mathematics classroom, for the benefit of everybody.

5. Towards Inclusive Mathematics Classrooms: Implications for Teacher Education

Our co-designed intervention lessons with the teachers suggest that ongoing collaboration and professional development constitute necessary and potentially productive ways forward for the sustainable efforts needed to create inclusive mathematics classrooms. However, we argue that well-meaning individual teacher–researcher collaboration is not a sufficient condition for such interventions to scale up. It is necessary that inclusion issues should be addressed in teacher education (pre-service and in-service) so that the changes towards a more inclusive mathematics education are communicated to all pre-service and in-service teachers rather than at a small scale (a researcher and a few teachers, doing a bit of work together). We conclude by discussing implications for teacher education towards inclusive mathematics classrooms.

In this paper, we aimed to answer Research Question 1 (teacher preparation needs with a particular focus on the use of resources for inclusion in mathematics lessons) and Research Question 2 (the pedagogical potentialities of teacher–researcher collaborative design and implementation of inclusive classroom activities that benefit VI and sighted learners). We identified specific and particular limitations in the preparation of teachers reported more broadly in the literature (e.g., [24]) that are associated with inclusion issues on resource use under the following themes: teacher positioning; VI pupils' desire to not stand out versus school narratives about disability as deficit, not difference; and coordinating teachers' and teaching assistants' interventions in assisting VI pupils. We identified circumstantial, but mostly systemic, obstacles associated with institutional narratives about inclusion, disability, and mathematical learning. Teacher–researcher collaborative design and implementation of inclusive classroom activities [26] was found to be a successful way to address those issues and, ultimately, create inclusive mathematics lessons. Within this teacher–researcher collaborative paradigm, we substantiated the mathematical and social benefits to all learners that are aspired to in international legislation (e.g., [2]) as outcomes from the implementation of inclusive education. Amongst the benefits, we found clarifications of, and confidence in, mathematical topics, teamwork, and the active involvement of all pupils. While the scale of our efforts is modest, our findings nonetheless highlight potential ways forward as to how MTE programmes may better prepare teachers—and how they can do so at a larger and more substantial scale.

We envisage that the following implications will contribute towards MTE that pays more—as well as more specific and better tailored—attention to inclusion in mathematics lessons. The teachers' and the teaching assistants' support for VI pupils needs to be better aligned and orchestrated towards a common goal. Teacher training should facilitate teacher awareness of the VI pupils' perceptual needs and capabilities. It should also cultivate positive connotations of difference (e.g., with regard to visual impairment, with regard to pupils' mathematical contributions that differ from those seen as standard). Last but not least, teacher training should encourage the design and deployment of classroom resources in a way that VI pupils, and the rest of the class, can use a multiplicity of sensory channels, such as touch and hearing, for mathematical meaning making and communication. Further research needs to focus on continued collaborative work with teachers towards the design of more multimodal mathematical tasks for the benefit of everybody in class, disabled and able-bodied learners alike.

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education sciences

Article

Teacher Learning towards Equitable Mathematics Classrooms: Reframing Problems of Practice

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Abstract: This study responds to the debate on understanding and evaluating teacher learning in professional development programmes, with particular reference to the development of equitable mathematics classrooms. Conducted in the context of a year-long PD mathematics programme for primary teachers in Norway, designed to disrupt teachers' assumptions about mathematics pedagogy and how it relates to students' mathematical thinking, this study takes teachers' entry goals as its point of departure. Sixteen teachers participated in interviews at the end of the course. Recognising the situated nature of the development of pedagogic judgement in our analysis of teachers' reflections on their learning, we report on the shift in their "problems of practice" towards actionable concerns about student inclusion. We argue that this shift underpins a fundamental change in their assumptions about teaching and learning and a critical stance towards their own professional practice, suggesting an important indicator of what constitutes sustainable professional development for critical mathematics education.

Keywords: professional development; inclusive education; pedagogical judgement; equitable mathematics; teacher learning; problems of practice; framing

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1. Introduction

Success in school mathematics has a major impact on students' futures in terms of both academic opportunities and participation in society. However, while equity in mathematics instruction is a widely shared aim, there is little consensus on how to achieve it. As mathematics teacher educators, our main interest in the issue is how to develop sustainable change in teachers' practice, given the pressures of high-stakes accountability cultures that frequently undermine moves to equitable instruction [1]. In this article, we report on teachers' learning in a year-long professional development programme that aims to challenge and redirect their existing pedagogic practice, with particular respect to connecting with children's mathematical thinking. The programme is part of a national strategy in Norway, the Competency for Quality initiative (Kompetanse for Kvalitet, KfK), which addresses the fact that primary school teachers are generalists and that many lack specific training in mathematics. In operation from 2014 to 2025, the KfK strategy includes courses on a broad range of subjects and at multiple universities, with each institution having autonomy over the course design within some set parameters. In the particular version of the mathematics KfK course described here, the designers aim to disrupt teachers' preconceptions about their students' mathematical knowledge and thinking and develop a dialogic and inquiry-based approach to teaching for inclusion. We situate our study of these teachers' learning trajectories as a contribution to our understanding of how we can move away from the snap-shot evaluation of professional development in terms of the implementation of "best practice" to a focus on teacher learning as the development of context-sensitive and critical pedagogical judgement.

2. Literature Review

Educators' and researchers' understanding of what constitutes equitable mathematics teaching has developed over time, and we adhere here to the multidimensional perspective proposed by Gutiérrez, combining issues of access, achievement, identity and power ([2] p. 21). Raising the question of what can be done for the realisation of equitable classrooms, she proposes that general initiatives targeting teaching resources, teacher knowledge, policy and so on are not sufficient, while attention to the particular contexts in which teachers work is key. This concern for the incorporation of the situated nature of teachers' work into initiatives for equity raises serious questions for professional development programmes that focus on the top-down dissemination of principles for equitable teaching while ignoring the nature of teacher learning and the need to recontextualise the programme content in local school cultures [3]. Teachers are central in ensuring that values of equity permeate the reality of classrooms. For example, how they see students' capability for learning influences the cognitive demand of tasks they choose and how they present them. Thus, seeing students as "weak" or "slow" creates apparent mismatches that lead teachers to doubt their capability for rich tasks [4], leading to simplification in order to reduce what they perceive as harmful challenges to learners who need "protective nurturance" [5]. Task design and selection may also privilege some students over others in terms of their background knowledge [6], specifically in terms of the use of context, which presents multiple challenges in representing diversity of gender, class, culture and so on [7], or unfounded assumptions that certain everyday contexts suit certain groups best [8]. Thus, Gutiérrez ([2] p. 24) provides a vivid account in which disadvantaged (poor, working class, minority ethnic) students' academic success could be traced to a community of teachers who developed "a deep understanding of their students", avoiding assumptions that "pre-scripted contexts for Latina/o students such as tortillas instead of bread" were key to addressing excluded student identities (p. 27). Similarly, Kinser-Traut and Turner [9] report how a teacher committed to building on students' funds of knowledge struggled to incorporate this into her teaching, ending in appropriating the knowledge for herself as she attempted to share authority in the classroom, thus undermining the principle of building on students' home knowledge. These issues underline the context sensitivity of connecting with students' mathematical thinking and the need for awareness of the cultural and ideological dimensions of teacher noticing [10].

This emphasis on the importance of understanding teachers' equity-based judgements in context is underlined in Goldsmith et al.'s [11] systematic review of 106 studies on the professional learning of practising mathematics teachers from 1985 to 2008. Reflecting the state of the field at the time, professional learning was defined in the review as a change in knowledge and/or in practices, as well as a change in beliefs and dispositions that might influence knowledge or practice (p. 7). While the research on teachers' actual professional lives was already becoming more prominent, with more than twice as many studies considering changes in practice than changes in knowledge, changes in teachers' pedagogical reasoning were largely overlooked (just 8 out of 106 studies consider teachers taking on an inquiry stance as learning). Although rare, some studies (e.g., [12,13]) reported on the contextual embedding (school discourse, district policies, etc.) that hindered or supported changes in pedagogical practice (p. 15). The authors of the review concluded that it is problematic that studies tended to seek an answer to the question of *whether* a programme is effective, rather than *how* it works, particularly how it works in different settings ([11] p. 21). The few studies that reported on *how* found that the process was far from direct in the sense of moving "from a single professional experience to a change in practice to improvement of student outcomes" but rather was "often incremental, nonlinear and iterative, proceeding through repeated cycles of inquiry outside the classroom and implementation in the classroom" (p. 20), with small advances in one domain (e.g., knowledge, beliefs or practice) triggering a new process leading to another small change.

Questioning the meaningfulness of evaluating teachers' practices against theoretical "best practice" without examining the why behind the actions, Horn [14] proposed *pedagog-*

ical judgement as a concept that acknowledges the complexity of change in teachers' practice as far more than the acquisition of new skills or knowledge. Rather, teacher learning involves the development of a situated judgement, which includes the interpretation of local contexts and concerns; teacher learning is not just about conceptual change, but it "also involves developing ecological, holistic, integrated views of these aspects of teaching and continually exploring their interdependence in different situations" ([15] p. 46). This integration of multiple contextualised understandings that are in continual development is operationalised by Horn and Garner in terms of teachers' ongoing refinement of "problems of practice"—the stories that teachers tell from their professional practice about the problems they encounter. The framing of such problems of practice (what is the reason for the problem?) is consequential for how teachers envisage possible solutions [16]. Babichenko et al. [17] argue that the resulting "actionability"—is there something that can be done?—is a factor in defining teachers' agentic space. This research highlights the fact that framing in teacher talk is not fixed, as both the person raising the issue and others can contribute to its reframing and thus, implicitly, to the envisaged solutions. Thus, the framing and reframing of problems of practice is likely a productive site for teacher learning [18] when teachers engage with the issues and refine their pedagogical reasoning in a process that is closely related to shifts in teachers' *noticing* as they make sense of classroom situations, including what their attention is drawn to and, depending on the definition used, how they interpret what they see and how they decide to act on their interpretations [19,20]. While this potential for reframing is not necessarily realised, even in settings that are intended to promote professional development, there is evidence of examples of how such concrete problems from one teacher's practice can engage colleagues in processes of generative learning that transcend the specifics of the situation [21,22].

Thus, problems of practice define teachers' own agendas for professional development, and Horn and Garner describe how teachers move from a view of their own learning as filling "knowledge/skills gaps" to a more complex understanding of teaching whereby "no amount of additional information would sufficiently account for all sources of variability" ([15] p. 147). In the absence of a single or final solution to a problem of practice, teachers' pedagogical judgements are therefore based on iterative relationships between reasoning and actions, which are both connected to and contextualised by institutional obligations and ethical commitments, as depicted in Figure 1.

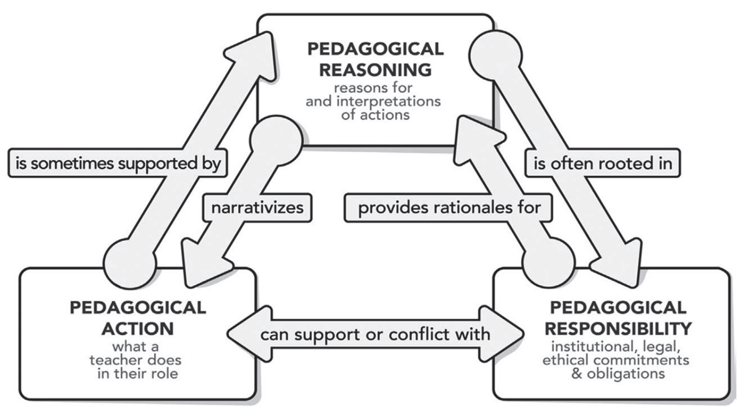

Figure 1. The components of pedagogical judgement and their relationships with one another. From Horn and Garner ([15] p. 56).

In this paper, we focus on shifts in teachers' framing of problems of practice in connection to the KfK PD course, understanding this reframing in terms of their evolution of pedagogical judgement. Thus, we pose the following Research Questions:

- How did teachers' problems of practice change during the course?

- How does their reframing of problems of practice relate to the development of pedagogical judgement?

3. Methodology

This study is part of the larger research project *Inclusive Mathematics Teaching: Understanding and developing school and classroom strategies for raising attainment (IMaT)*, financed by the Research Council of Norway Grant 287132. Ethical approval for the overall project was obtained from the Norwegian national body for ethics in research (NSD, now Sikt—Norwegian Agency for Shared Services in Education and Research). We report on data collected from teachers enrolled in a mathematics professional development course for grades 1–7 (ages 6–13) in the academic year 2020–2021 at a large teacher education provider in Norway. All participating teachers worked at the time as teachers in schools; most held formal teaching qualifications, although some did not, having been hired as teachers under a special exemption granted to schools because of staffing shortages. Some of the qualified teachers had completed a half-year unit in mathematics education as part of their training, but others needed to top up their mathematics teaching education to meet new government requirements introduced retroactively for primary school teachers in Norway. Consequently, some teachers participated in the course for continuing professional development, while others enrolled to fulfil their licensure requirements.

The course was taught by two experienced mathematics teacher educators (MTEs), Silje and Daniel, working together in a two-teacher system, where Silje was the course designer. Teachers, working in groups through the year, attended six "gatherings" evenly distributed over two semesters, where each gathering was organised as two full consecutive days of teaching focusing on teachers' lived experiences of being in the classroom and connecting this with research-based perspectives on mathematics teaching and learning. Furthermore, course assignments ("missions") required teacher groups to create sites of professional inquiry in their school contexts (rather than implementing ideas supplied by the course tutors), reflecting critically in their groups on their observations, particularly student thinking, in light of issues raised in the course. They submitted jointly written reports, with mathematics teacher educators nurturing teachers' criticality by giving feedback that raised new issues, pressed for justification or suggested alternative interpretations based on the empirical data included. Both the course design and the implementation were centred around the MTEs' own "theory of change", grounded in the stance that subject knowledge per se is not a main consideration for the facilitation of practical, inquiry-based and theoretical work that values and develops students' mathematical knowledge and mathematical thinking. We summarise the theory of change here, focusing in particular on how the course designer aims to use confrontation to disrupt teachers' preconceptions [23] and promote a radical change in their practice (for a detailed account, see Eriksen and Solomon [24]). Drawing on a Realistic Mathematics Education philosophy [25], Silje and Daniel aimed to enable participants to develop a reform-based approach by:

- Discussing authentic student work and noticing what students can do, considering what lies on their closest learning horizon, and how the teacher can challenge them to develop their thinking by:
    - Discussing the learning landscape/student learning trajectories;
    - Analysing and discussing the work of their own students ("missions");
    - Becoming familiar with, working with and accounting for informal and preformal methods that can eventually be used as teaching tools.
- Drawing on a range of theoretically driven approaches to support analysis and reflection to develop conversational features/rich discussion.
- Asking teachers to work in an investigative manner in and with mathematics throughout the course.
- Modelling the practice advocated by the course themselves.

As part of the larger project on inclusive mathematics teaching, data collection during the PD course was extensive. We draw here on two datasets drawn from the participating teachers. Dataset one, the entry data, was gathered from all course participants (31 out of 40) who responded to Silje and Daniel's e-mail request prior to the course start asking them to explain in writing their motivation for enrolling, their expectations of how the course might support their mathematics teaching, and their vision of what they would ideally like their mathematics teaching to be. The second and third authors asked the teachers for permission to use these data at the end of the first gathering, which they had joined as researchers engaged in the project. Dataset two, the exit data, consists of audio-recorded interviews with all 16 of the 40 course participants who responded positively to our request for interviews towards the end of the course. Thirteen of them had also provided entry data. All participants worked as teachers in schools, leading to various time constraints; to accommodate these, we interviewed 3 teachers individually, while the remaining 13 were organised into four group interviews (see Table 1). Interview topics were provided to each teacher in advance and covered their background as mathematics teachers; comparisons between their earlier experience in teaching the four mathematical operations and the approach used in the course; and the teaching of mathematics through inquiry.

Table 1. Overview of individual and group interviews.

Individual interviews	• Jenny • Ina • Mathilde
Group interviews	• Sofie, Rikke, Hedda, Vilde • Rita, Ulf • Gaute, Hedvig, Trude • Aurora, Solvor, Nora, Gina

All interviews were conducted on Zoom in Norwegian and were transcribed and translated into English by the authors, who include both native Norwegian and native English speakers, with the aim of capturing sense in plausible English rather than producing a literal translation. Appropriate translations were extensively discussed between the authors. Dataset one was also written in Norwegian and was similarly translated and discussed.

We began the analysis of both datasets by identifying problems of practice, subscribing to the assumption that these are productive sites for teacher learning [21]. Specifically, we identified issues in professional practice that teachers brought up as problematic in some sense, be they actionable (there is a non-routine solution that teachers can develop within their own sphere of action) or nonactionable (solutions are obstructed by immutable student characteristics or structural constraints). Unlike Horn and Little [21], we included problems-of-practice issues that may not appear to be directly related to classroom activities (e.g., teachers' efforts to communicate with colleagues). Our view is that, since teachers were asked about classrooms, by bringing up other school issues that may not physically be located in the classroom, they signal to us their perceived interrelatedness.

Having systematically looked for problems of practice in both datasets, we continued our analysis of dataset one by identifying teachers' assumptions about their cause and what would constitute "help" (in terms of learning), that is, their framing of their particular problems of practice. In the analysis of dataset two, we drew more interpretively on Anderson and Justice's definition of disruption as "an analytical construct that allows for the investigation of how individual learning and changes in local practice mutually influence the other within a purposefully designed learning context" ([23] p. 401). In doing so, we identified disruptions as described by the teachers when comparing differences between their previous experiences of, and beliefs about, teaching and the teaching approaches and activities advocated and used in the course. When teachers reframed their initial problems of practice, we endeavoured to pick out connections to the course design (trying new teaching strategies and tasks out in their own classrooms, group discussions and analysis of students' work) and references to pedagogical judgement, as described in Horn and Garner's [15] model (explanations/justifications of pedagogical actions in the classroom, resolution of ethical dilemmas).

Any qualitative study raises issues of trustworthiness, and this is heightened in the context of potential power imbalances, as in this study, where participants were aware that the researchers and their instructors were close colleagues. While this was not an issue for dataset one, which was produced for the instructors rather than the researchers, the interviews in dataset two could be seen to be overshadowed by this relationship. We are optimistic that this did not interfere with participants' responses on the basis of (1) the overall ethos of the course in encouraging participants to express their opinions openly in discussion and (2) our actual experience of participants expressing views that they might have been hesitant to say to an instructor ("I do this because I have to") or being critical of some aspects of the course. Another issue concerns our use of a mixture of individual and group interviews and whether these can be treated similarly in the analysis since group interviews include multiple voices. We found that problems of practice were at times co-constructed, with participants building on each other's contributions (e.g., one might describe a situation and another would raise a question on how that situation might be handled). Since this is an important site of learning, we highlighted such co-constructions in our analysis. However, we argue that even individual interviews are co-constructions of more than one voice [26], and in our analysis, we attempted to capture teachers' productive building of ideas in both individual and group interviews while focusing less on discussions deviating from our intended main topics. All three authors were involved in identifying themes from the data, with extensive discussion of extracts and their meaning and fit with the theoretical framework.

4. Findings

In this section, we first report on teachers' accounts of disruption as a result of attending the course, noting how their initial assumptions that it would provide them with missing skills and knowledge were replaced by surprising revelations about mathematics teaching, particularly the role and value of informal understanding. We note, too, teachers' enthusiastic embrace of the course principles as a feature of discussion with colleagues. We then move to an analysis of the ways in which they reframed their problems of practice, paying particular attention to the distinction between problems that were deemed actionable and those that were not.

4.1. Initial Problems of Practice and Underlying Assumptions

Responding to the written questions sent by the mathematics teacher educators ahead of the semester start, teachers expressed goals for learning that ranged from a simple wish to learn something "new" without a specific motivation ("*get tips about different teaching methods in mathematics*", Thea) to more complex motivations related to particular problems they saw either in themselves or in their interactions with students. Many initial problems of practice simply focused on becoming more confident and competent teachers, something to be achieved by enlarging their skillset with new techniques and example tasks:

> I imagine the course can give me a robust knowledge base, together with specific didactical approaches, methods and activities [. . .] I want a good balance between exploration and repetition/retrieval for each student. (Emma)

Jenny was very confident as a mathematics teacher and wanted the course to provide "*confirmation that what I do is right*" but also to equip her with "*new tools*" and up-to-date knowledge:

> I also want the course to give me a clearer understanding on how to teach maths following the new curriculum, instead of my trying to interpret it on my own, without knowing if I understand it right or not.

A recurring problem of practice was *helping everyone*, an issue that teachers frequently framed in terms of students' emotional experiences of success, enjoyment and interest:

> [I want my lessons to be] motivating, inspiring. That all students, irrespective of their starting point, [should have] the experience of being successful and the opportunity to develop their curiosity about this subject and across subjects. (Anna)

> I hope to be introduced to teaching methods and tools that can help me vary my classes, make students interested and help them experience a sense of being successful in maths. (Serena)

Linnea wanted help with a similar problem of student engagement, framed this time in terms of difficulty in connecting theory and practice:

> How to inspire a whole class in mathematics [. . .]. A practical lesson, going from theory to practice without creating chaos. The students being engaged but still listening to the teacher.

A similar framing saw student engagement as an issue of the apparent distance between school mathematics and real life; teachers wanted to help students "*to see connections between the maths and the real world*" (Jenny), the teachers' role being "*to make maths relevant for the students, so that they feel they need it later in life*". Nina also wanted to foster a positive relationship with mathematics through practical relevance:

> I want students to experience maths as something concrete and practical. In an ideal world my classes would include more discussions that included all students and more practical tasks.

Nina's mention of "more discussions" that "include all students" introduces another aspect of teachers' framing of student needs. Several teachers focused on the ideal of classrooms driven by sense-making, where all students take ownership of the mathematics and are challenged, for example:

> I want to have time and knowledge to meet each child where they are in their mathematical understanding and guide him to reach his full potential. I want a classroom where everyone has enough mental and emotional surplus to wonder together about puzzling mathematical observations through a well-developed mathematical language. (Iselin)

Often, however, this problem of practice was framed in accordance with two assumptions: students have a *fixed ability* in mathematics, and *different students need different methods*. Having raised her ideal of the inclusive discussion-based classroom, Nina admits that "*I found it difficult to [make this happen], being alone with 28 students in mathematics—naturally on very different levels*". Solving this problem is a main aim for her:

> I hope to expand my understanding of how to help students who struggle with maths, to get more practical approaches and strategies.

Mathilde and Klara adopt the language of levels to frame the problem and, hence, what they want from the course:

> Give me more methods and techniques to use in my teaching so that each student gets help on his own level. 'I don't understand why someone doesn't understand. (Mathilde)

> I want to adapt my teaching to the levels in the classroom and not let the lack of skills hold students back from [solving the tasks]. (Klara)

Trude is even more specific in describing the strugglers and what they need:

> The course should give an overview of good sources for alternative mathematics teaching, so that one can reach all students, also those who struggle with thinking logically.

Perceptions of students such as Trude's are common [4]. One of the main aims of the course is to disturb such assumptions about student ability and related preconceptions about teaching and the nature of mathematics. In the following sections, we show how the course appeared to achieve this: in their exit interviews, the teachers recounted numerous destabilising experiences and new insights that caused them to rethink their approach to mathematics teaching, and their ideas about what students bring to a classroom and the role of informal ideas in mathematics.

4.2. Challenging Assumptions and Initial Problem Frames

In the exit interviews, teachers gave frequent examples of how their professional reasoning had been disrupted (e.g., "it was a revelation", "before I used to . . . but now. . .") by certain experiences of the course (the missions in groups, the gatherings, the research-based readings). Three such shifts were most prominent in the data: how teachers saw mathematics teaching, the role of students' informal methods and the value of productive friction among colleagues. We discuss these in the following sections.

4.2.1. Seeing Mathematics Teaching Differently

Many of the teachers talked about the ways in which the course disrupted their thinking about mathematics teaching, causing them to reframe the issues around what is involved in doing mathematics and the role of how children think in learning. For many, this led to a new criticality of their former teacher-centred approaches and their own schooling. Typical were Gaute's comments about the impact of the course's teaching style, in which teacher participants were encouraged to engage with mathematical problems and explore their own informal strategies: "*already at the first gathering—it was a very aha experience in how to talk and think maths*". He describes how the experience of the course gatherings was very different from his own education in mathematics, where

> we drilled, and we memorised numbers, . . . I don't remember . . . having a lot of mathematical conversations, or different methods.

Jenny describes the inquiry approach of the course as "*mind blowing*", because it made her see mathematics differently. She also recognises how traditional schooling means that

> you have never thought about doing it this way [. . .] And all of a sudden, they [the course tutors] show that things can be done in a different way. I remember the first gathering, it was . . . mind blowing. Because they showed maths from a completely new angle.

This was a major revelation for her:

> I lacked a sort of an overarching understanding, right? I didn't see it [mathematics] from above, and [didn't] understand how things are tied together, right? I lacked a connection between all the issues, together.

Hedvig's "aha" experience focuses on her reframing of the potential range of student thinking and what this means for changing traditional teaching:

> . . .I have had a lot of aha experiences in a way . . . Talking about the world's biggest numbers and stuff like that . . . there's no limit to how children think, then. And that you kind of have to use what the kids think. Don't just follow the textbook.

Rikke has also noticed the emphasis of the course on understanding how children think and hinted that, combined with the disruption that the course "missions" brought to her habitual practice, this has improved her relationship with mathematics and thus her confidence as a teacher:

> Even though they are young children, I notice that I'm not always confident in those teaching situations, and I want . . . to stay in [my old ways]. [Then I'm] more confident about it. And [now we are] finding out a bit more about how children think, and ways of thinking and. . . yes. The whole process then, not just

> drawing two lines below the answer [to highlight the final solution to a problem]. And I think it's been really nice that [the course is] very focused on that.

She is struck by the way that "right answers" are not the point of the course gatherings:

> it's never scary to answer [here]. Because if you answer incorrectly here, you kind of don't get "caught" for it. [The MTEs] are more interested in thought processes—how you think—rather than [saying] "That's wrong". And I think that's a pretty nice thing to bring into the teaching as well.

Hedda echoes these ideas, highlighting a change in how she sees mathematics teaching in terms of no longer feeling that she ought to know everything: there are different ways of knowing, which the course has made clear:

> what I notice has happened is ... that now we know that students think in different ways ... And I think that—I'm not so afraid of not knowing everything now... That it's no big deal [that I haven't] heard of that particular way of thinking before. Or if I can't keep up with it.

4.2.2. Recognising the Value of Informal Understanding

These new perceptions of mathematics lead to a reframing of informal understanding as legitimate and valuable mathematics. Gaute reflects that although he used informal strategies when doing mathematics himself, "*I don't think it was a conscious part of my maths teaching*". The course has caused him to think consciously about the connection between informal methods and algorithms and its relevance to teaching:

> we've been given tasks ... and then we've had to drill into how we're going to do it. Especially those on the division algorithm, how to set it up. I've never thought about how it's based on equal sharing [...] even when you just draw up when you want to divide something by three [...] then you hand out, a thousand first, and then a hundred afterwards. I've never seen that before, and I haven't connected: that's what the algorithm actually does. So it was very much an eye-opener for me. That whole way of thinking about maths, especially when dealing with children.

Jenny also talked about how encountering two different additive models gave her tools to engage in her students' thinking more deeply:

> ... what was the best, was that I understood the two principles, right? Linear model and grouping. And I knew which students worked within which model [...] all of a sudden I had all the dots above the i-es ... I understood how things connect, and I can work within [the model] that I know suits the students.

This stood in stark contrast to how she saw it before, where her attempts to do things differently had been too random:

> When I worked with the students, I used to just try to show them what I thought would work for them. [I did this] without knowing what those things [linear model or grouping] are ... It was at a very intuitive level. I tried to guess how to do it.

Less mathematically trained and confident teachers such as Solvor talked about how they had realised that "helping" students by teaching them algorithms, as they had themselves been taught, was pointless:

> I've been thinking like that in terms of addition, you've thought that *that* algorithm—you want to teach them *that* quickly—because it's so convenient, right? You think of it as a help for them. But then you see now that it's counterproductive if you rush towards that algorithm.

Gina now realises that "It's not the goal to get there as quickly as possible":

> But I didn't think that before. Before, I just thought, to help them [with addition, you need to say]: '... Look here: Just put them one above the other. Then you

> add them up. Right? So easy and great and straightforward'. But it's not easy and great and straightforward if you don't realise what you're doing.

Instead, she realises that it is important to capture the informal stages that underlie the algorithms. Hedvig also notes that the power of algorithms has created a kind of trap, which the course has challenged:

> You just don't have to get in there [using algorithms] as soon as possible, and that's. . . the trap that I think maybe I have fallen into, not knowing that it was a trap. I've sort of said that; "Learn this method, and you'll be fine". But that's it—the understanding is not in using algorithms.

She argues that this different way is more inclusive:

> Allowing kids to think in different ways to come up with an answer [...] so that more kids get it. That's the point. And then they will [get it]—when they realise what they are doing—no matter which way they have got there. . .

4.2.3. Embracing Productive Friction

All teachers celebrated these new ways of seeing mathematics and their students and their new abilities to argue about pedagogical judgements. Solvor feels that the course has enabled her to articulate why "what's underneath" is so important for inclusion:

> . . . maths teaching has perhaps been adapted to those who get things easily, quickly, like arithmetic. Whereas now, we're learning to have a maths lesson that targets everyone. I think. There's a difference, and especially that there are different solution paths, because I had a discussion last year, with the neighbour saying, "No, but you just have to memorise multiplication tables. Some things you have to memorise." And then I was like, "No, you don't have to." But I couldn't argue why. But now, I can. Now I say, "No, you can use [relational thinking]".

Sofie is outspoken about the importance of teaching for understanding at her school:

> When they're talking about standard algorithms—I've become quite an opponent of it—and I don't think it's right either, but I'm questioning the maths teacher, like that; why do you do that, and. . .

She rebuffs responses that defend the top-down teaching of algorithms:

> They think it's important, so [. . .] I lost the discussion the other day, because they think that you have to [do it], because you have to find an effective strategy eventually, anyway. So if the student doesn't understand something, in order for them to move on, [. . .] you should [do it]. At least you've given them a tool to do the maths when they get to middle school. . . . But I stood my ground: "No, they have to know why they do what they do!"

Hedda also disagrees with other teachers' practices in her school now, describing a colleague as "old school" when he focused on setting up a standard algorithm rather than listening to the students' strategies. She is aware that this is exactly what she would have done in the past:

> And it was really such a good example of how I probably would have done it if I suddenly had to be a maths teacher before taking this course here. I wouldn't have been interested in hearing about different ways, because I thought I'd show them a standard algorithm. . .. And then I wasn't . . . didn't really know that it's totally [reasonable] to spend a lot of time hearing from different students how they think . . . And that it's really just as important [for the students] as continuing to calculate in your book one by one.

Vilde comments on the difficulty of talking to colleagues about what she sees as bad mathematics teaching, yet she is confident that, at some point, she will confront them and open the matter up for discussion: "*I think that I can't keep my mouth shut forever. I'm going to have to confront, or be curious and ask: 'Why are you doing that?'*"

4.3. Reframing Problems of Practice

As the previous section showed, the teachers developed strong positions on what kind of teaching they felt was right—fostering student understanding, promoting discussion and explanation, avoiding algorithmic teaching and recognising varied student contributions. Embracing these principles and their importance in equitable mathematics teaching led to new problems of practice: teachers were concerned with exploring the details of how they could put them into practice. These new problems of practice were not, however, always seen as actionable, and teachers struggled individually and jointly with these problems.

4.3.1. Refining Pedagogical Judgements—How Do We Make This Happen?

The teachers reflected on a number of problems in the classroom as they worked on how to incorporate the course principles into their daily practices, ranging from the need to elicit and understand students' thinking, to refining questioning and discussion and promoting new classroom cultures, to grappling with problems of mixed class teaching and student progress.

A number of the teachers commented on the need to prepare carefully for lessons, because, as Sofie says, "*now we want [to see] much more into the head of the student*", and this means being armed with strategies that are new to them, as Rikke points out:

> What do you want to know and why? What kind of follow-up questions should be asked of students when they are stuck? [. . .] It shouldn't be the right answer, but how did you think? How did you arrive at the answer?. . . Can you prove it?. . . In other words, the good questions.

Mathilde frames her particular problem of practice in terms of needing to think about what the students know and how to incorporate that, given that mathematics is more than right or wrong answers:

> . . . first of all, I have to—in preparation have a bit more focus on the students' strengths and skills and knowledge—in order to be able to facilitate. . . . And then I have to prepare myself for how to start a conversation like that with them. And I have to be aware of what might come up. I have to be a little bit more ahead of the curve than before.

Gaute comments that he has become more skilled at asking good questions in classroom discussions in order to elicit student thinking, but he struggles to throw off his embedded ways of dealing with mathematics and the resulting desire to present the "easiest" method:

> I'm definitely going to be better at asking for justification from students as to why that is. Maybe say: "Yes—can you show us on the blackboard? Has anyone else solved it any other way?" [But] I'd probably feel . . . a desire to give an easier method. If someone had drawn it, and sort of done it that way . . . there's a voice that says "actually, you can skip this, and just. . . do like that. Then you've halved your workload".

Gaute questions when it is appropriate to stop his students' exploration and say, "*This is a very, very good way to do it*", wondering how one can do this "*without somehow overriding [the students'] understanding and exploration*". He struggles with striking a balance in terms of being "*open to other ways of thinking, or the child's, the student's way of doing it, without in a way encouraging strategies that are really very convoluted*". For Gaute, the challenge is "daring" to let students sit longer, daring to give more open tasks and "*giving students room to explore, to try and fail . . . Now you are free*". He is particularly exercised by issues of summative assessment:

> It's a bit more demanding to . . . get a summative assessment—at the end. . . . when do you know that you have actually learned what you are supposed to? And when have you got . . . when have the students got the strategies they need to do it effectively then? You don't always have the time and opportunity to make a drawing to solve something you can actually do in a very simple and

> straightforward way. When do you say that understanding is kind of good enough, and you can give a bit more. . . tools based on recipes. Or algorithms?

Hedvig ruminates on similar issues but focuses on the pressure of time in forcing decisions:

> How long do you stay, and dwell on the same topic, and task. When does it stop giving something more? And we very rarely have the time to stay very long with fractions. No, we have to move on because we're going to have something else, ... When have the kids learned it? Or when have they learned enough. . .

Mathilde focuses on a related issue, which is getting students to change their expectations about how they should participate in class:

> I think it's difficult because the students are very used to the fact that there is either a right or wrong answer. . . . I noticed that the last time I was doing geometry that they're sort of not used to that [new] way, so they get very silent and quiet. Because they probably expect me to be looking for the right or wrong answer. So it's a challenge, and then it's challenging to be able to drive the conversation forward, when they stop or don't say anything. Getting them involved, it's been challenging.

4.3.2. Refining Pedagogical Judgements: Can We Make This Happen?

Some teachers framed problems of practice in terms of student characteristics or contextual constraints which caused them to perceive problems as inactionable. Some talked about their struggle to sustain and act on the idea that all students can benefit from their new strategies. Hedvig finds it hard to adjust her framing of the issues, since she sees the "nurture group" students that she teaches as unable to maintain engagement in a mathematical problem:

> [I have] children who have quite a short attention . . . span. Which may only be a short time with a mathematical problem. If it gets too difficult, we have to . . . [simplify the problem] or do something else. And I quickly resort to games.

She is keen to teach in mixed whole classes, where she can draw on the stronger students to help the others and reduce teacher authority at the same time:

> I kind of want to try it out, because there are some strong kids in one class, and there are quite a few who have to hang on and need some help. So you can use the strong mathematical brains to explain to the others. Because sometimes it's a bit like the teacher is standing and talking, and then they're going to do some tasks. So I kind of want to try to distribute some responsibility, around to the kids themselves.

Trude does not see this as a solution to the problem of how to include weaker students, which she sees as basically inactionable. She struggles to act on the course message in favour of mixed teaching and has decided to put students into homogeneous groups. She believes that all students benefit from investigative activities but argues that the stronger students (she struggles to find the vocabulary to describe them) will be held back otherwise:

> obviously, I could let the [. . .] brightest, or call it what you will, help those who were less so. But, my experience is that the ones who had come a long way had more fun. Because they got farther. They sort of got it, and that's a group of students who get short-changed in regular maths lessons. [Because] they are already there, and don't get enough challenges, and think maths is getting boring, and they're the ones who... there's hope they may move forward [in the future] with maths.

While Trude's hesitation about putting a principle into operation was clearly connected to one specific instance, Rita is more generally sceptical of teaching through inquiry. She tries to work out how the teacher and student roles have changed:

> Whereas before I would have presented a few alternative methods, more . . . unambiguous strategies, as an adult now you have to hide it more. So that they

> get to think about it. Justifying is nothing new, nor is understanding. The parts about formulating how they would solve it, and the part about figuring out things themselves are new.

This analysis leads her to conclude that the approach disadvantages "the weakest students" because the teacher has to be "secretive", and these students cannot then participate without help, which descends into mere funnelling. This problem of practice thus appears inactionable to her. Echoing many of the teachers' earlier problem framing, she holds on to the assumption that struggling students need a different type of mathematics teaching, and she questions the principles:

> Even if they say that maybe the weak benefit as well—you have some that just don't figure out any [. . .] way to solve it. You can keep hinting [. . .] and keep going one step down [. . .] When can you stop letting the child [laughs] figure it out? It's silly, it will just confuse the student who won't get anything out of it. Say—how do you use the numberline? Or find a correct answer. Because there will be a lot of guessing and trial and error and [they] still [won't] figure it out.

Rita and Ulf, who were by chance interviewed together, both work with immigrants who recently arrived in Norway, teaching "welcome classes" at the primary school level: Norwegian education policy stipulates that all children entering the country should be prepared for integration into mainstream classes at their own age level within two years, regardless of their circumstances and prior language/educational achievements. Rita and Ulf's job is to enable students to cope with Norwegian as a language for learning, as well as ensure that they are up to speed with the mathematics curriculum level appropriate for the school grade that they will eventually end up in. These twin aims create dilemmas for what to prioritise in their limited time with these students. Understanding Rita's situation, Ulf attempts to reframe her problem by contextualising it in terms of Norwegian education policy ("*It's the eternal problem of adapted education, isn't it? Cater for everyone*", Ulf) and, therefore, highlighting that it is difficult to work out in its generality. But she rejects the premise that all students can benefit from time for inquiry, acknowledging that her claim is controversial ("*I am throwing a lit torch*"):

> Still, . . . I am throwing a lit torch here, but for exactly these students, trying to let them figure things out even though they can't just . . . come up with mathematical ideas, "Oh, I can do it this way!" and give them so much time, too, and maybe still not come up with anything anyway. It's a pity, I think. That they can't figure it out. But in the end they have to just learn to add two-digit numbers, if they are sixth-graders. And so I'd end up having to just show them how to do it on the numberline.

Her position that inquiry is not suitable for everyone is challenged by Ulf. He sees the students as empowered by the approach rather than being left to struggle alone ("*It's not just about right and wrong [...] Students are taken more seriously nowadays, [...] Before it was the teacher who stood at the board*", Ulf) and envisages the teacher's role extending beyond individual interactions:

> I agree very strongly [about the problems] with Rita. But I also see that there are [. . .] students who find things difficult and then find them easier once they see the strategies of their classmates. And they are no longer afraid to go and share their thoughts because [they see it's not about] right or wrong.

The disagreement was not resolved during the interview, but Rita is open to discussion and admits that she falls short of arguments. She is able to list numerous arguments supporting the value of mathematical processes for children's education, yet questions their value for her students' learning:

> It's a good thing, and a useful thing. They learn to argue and to formulate [their thoughts]. And dare to stand up and say something. And try to convince others. And think outside the box. Creativity. There are many good things and I have

> no really strong argument for not using it. The only thing is that it takes time that needs to be taken from something else. [. . .] And it's a class for learning the language in at most two years. And even if I know it helps with their language development, I still have to take the time from something. And it doesn't always work.

Although the disagreement between Rita and Ulf is not productive, in the sense that it does not impact Rita's framing of the problem, it is one of several examples that speaks to the typical interactions in the course where different points of view are discussed. We return to the role of inactionable problems as sites of learning and productive friction in the discussion.

5. Discussion

This article contributes to ongoing efforts to understand teacher learning from a situational perspective [15,16]. Understanding learning as the development of pedagogical judgement [14,15], we have focused here on equity in the sense of catering to all students, the aspiration shared by participating teachers at the start of the course. However, our findings show that the course played a part in the significant reframing of teachers' problems of practice, from seeking to acquire "best practice" for children "at their own level" to seeking to understand and build on children's informal and productive mathematical ideas within the inevitable tensions of their own particular contexts.

Addressing Research Question 1—How did teachers' problems of practice change during the course?—we identified a shift in their expectations that the course would simply "plug the gap" in their skills and knowledge (see Section 4.1) to recognising the importance of drilling down into their classrooms and considering multiple avenues for addressing the problem (see Section 4.3). These expectations were challenged by their experience of the course, as teachers describe in Section 4.2. Looking back on the year, they attributed their learning to disruption, rather than just new knowledge and skills. They described "aha" moments that were prompted by their participation as learners in the gatherings, their participation in making sense of student ideas during gatherings and in interaction with real students, and their participation in argumentation and productive friction with other teachers on the course. They remarked on the new experience of mathematical discussion, the irrelevance of "right" and "wrong" and the revelation of the value of informal thinking, and their new compulsion to reject standard algorithms as frequently excluding some children and thus to press "old school" colleagues for justification of their practice. We thus found that at the end of the course, teachers had embraced the principles of the course as global aims for their teaching. More importantly, these global aims raised many new questions, leading them to reframe their problems of practice. Our teachers voiced the struggles that were associated with this contextualisation; since the course had required them to experiment in their own classrooms, they were able to raise new, actionable questions on how to cope with specific problems and why: for example, Gaute questioned how he could judge when students had a "sufficient" understanding for him to be able to move on in a lesson, and Trude hesitated when forming groups, as she felt pulled between her ethical responsibilities of organisational inclusion (mixed groups) and realising individual potential.

The shift in teachers' expectations of what might be useful ways of dealing with problems of practice is significant, as it recognises teacher learning as an ongoing recontextualisation in complex situations, rather than the addition of new knowledge ([15] p. 147). This interpretation opens up sustainable opportunities to learn in teachers' own professional communities, moving past the typical sharing of teaching ideas, provided of course that their colleagues have the same stance. This prerequisite is difficult to meet; teachers may distrust the value of probing colleagues' professional reasoning [3,15] or hesitate to voice alternatives to the main discourse of their community, especially when in a position with less power [27]. Our analysis of the changes reported by our teachers and the links that they make with particularities of the course led us to identify disruption as key. This is precisely Silje's intention in her course design, which aims to confront teachers' pre-existing assumptions [24] rather than relying mainly on the more common approach of fostering reflection and smoothly coordinating new ways of thinking and acting when intervening in professional practice [28].

Turning to Research Question 2—How does teachers' framing of problems of practice relate to the development of pedagogical judgement?—we found that teachers drew enthusiastically on the general principles of the course, but, despite their criticism of colleagues, they did not see these principles as "rules" for best practice. Rather, they articulated the difficulties that they experienced in putting them into practice, emphasising in the interviews the new questions that arose for them. As we have seen in Section 4.3, some problems of practice were seen as clearly actionable, but addressing them was not straightforward: the teachers assumed that all claims about classrooms are subject to multiple interpretations, and that actions need to be justified within their particular context. However, some problems were seen as inactionable; Rita struggles with how to fit in practices that she agrees are beneficial but that take time away from the major goal of preparing students for mainstream school in a time frame that she has no control over. Despite her adherence to the pedagogical reasoning behind the core course principle of valuing inquiry learning for all students, Rita's struggle with the mismatch problem identified by Horn [4], which makes her doubt the capacity of her students for inquiry mathematics, ultimately seems to lead to an assumption that her particular problem of practice is inactionable. However, her learning in the course appears to have led to a willingness and expectation to argue for her position, as we can see in her interaction with Ulf, as well as in the multiple voices she uses ("I don't have any good arguments", "I am throwing a lit torch. . ."). Rita positions herself as willing to engage in productive friction with colleagues, and this plays a part in the development of pedagogic judgement.

As we have argued in the literature review, teachers need to incorporate principles for equitable teaching into their own practice: they need to recontextualise the course content in their own settings [2,3]. In terms of Horn and Garner's [15] model (Figure 1), not only do teachers need to develop new pedagogical actions, but these need to be connected to pedagogical reasoning and tuned to local pedagogical responsibilities. As Goldsmith et al. [11] argue, the important issue for studies on teacher learning is not to show that learning happened, but to shed light on how. Building on our findings with respect to Research Question 1, we can see evidence in the teachers' narratives for four possible mechanisms that enriched their pedagogical reasoning and its bonds to pedagogic action and pedagogic responsibility:

- Disrupting previously held assumptions;
- Holding the expectation that principles for, and actions in, teaching are justified;
- Habitually unpacking classroom experimentation into actionable problems of practice that may well require the mitigation of conflicting priorities rather than the enactment of "best practices";
- Embracing productive friction with colleagues.

These observations return us to our argument that teacher learning is not necessarily evidenced by a change in teacher actions. We see the main learning achieved by our teachers as being the pedagogical reasoning that they clearly appreciate: they see it as confirming what they do and providing a basis for going forward. Further than this, though, the problems of practice that are presented by their local contexts provide opportunities for the exercise of reasoning, which we can understand as sustaining learning rather than limiting it. Learning is not about enacting a practice perfectly but about experimenting: principles sometimes come into conflict, and principles do not always trump all other considerations. This is perhaps best summed up by Trude's observation that orchestrating lessons where students are at the core is a principle that is easy to adhere to but difficult to realise. It requires focus and practice:

> [This is] essential for me, because it is about such an extreme presence and awareness as you stand there and [words] fall out of your mouth. What [words]

> do you say and how do students [. . .] then understand the mathematics? Being so aware, so present when twenty-five students [buzz] and I think about [teaching English next, and about them washing their hands for lunch] and you have such a short time to engage in these processes. So it's this extreme form of being present, when you must ask the questions that make students think mathematically and so on. I need to practice, and I don't know who to practice on. We barely have time to do that. So it's something I have to focus on.

6. Implications for Future Research

In this article, we have explored the nature of teacher learning from the point of view of the development of pedagogic judgement as the product of relationships between the pedagogic actions and reasons promoted by professional development and local institutional ethics and responsibilities. We have suggested that the four mechanisms identified above are central to the development of pedagogical judgement in a complex system such as mathematics education. We see the disruptive nature of the course described in this article as a key element in teachers' development and productive friction as a means of sustaining learning beyond the course.

Future research and PD development need to explore these mechanisms further. While this article focuses on the entry and exit data, further analysis will turn to data gathered during the course to develop insight into how this professional development promoted the mechanisms listed above. In particular, we believe that the structure of the assignments sending our teachers on "missions" in the classroom, requiring them to discuss and report in groups, even when they experimented in different school contexts, together with feedback on their reports from the MTEs, impacted teacher development at each stage. Such exploration can contribute to ongoing efforts to understand teachers' learning in context and the design of experiences that support such learning, such as Japanese Lesson Study adaptations around the world [29], video-formative feedback cycles [15] or the Australian Learning from Lessons project [30].

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Article

Teacher Development for Equitable Mathematics Classrooms: Reflecting on Experience in the Context of Performativity

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Abstract: In this article, we chart the development of one of us—Sue Hough—from a teacher who wanted students to understand to one who gained new critical understandings of student thinking, pedagogy, and the very nature of mathematics. We comment on the role of research interventions and learning communities in this development, with a particular focus on Sue's encounter with Realistic Mathematics Education and the connections it makes between informal and formal mathematics through the pedagogy of guided reinvention. Development towards teaching that enables all learners to make sense of mathematics requires fundamental changes in pedagogic practice and a reconceptualisation of progress. Bringing about such radical change relies on one further aspect of Sue's story—the freedom to experiment and learn as a teacher. We note the remoteness of this possibility in a climate of performativity and marketised education, and we discuss the implications of Sue's journey for our pedagogical responsibilities in professional development today.

Keywords: mathematics education; teacher learning; Realistic Mathematics Education; professional development; education policy

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1. Introduction and Background

A critical approach to mathematics teacher education involves challenging deeply embedded ideas and approaches on several fronts. In this article, we focus on the everyday challenge of including all students in making sense of mathematics, recognising that the outcome of traditional teaching is frequently the reproduction of privilege rather than understanding. Turning this state of affairs around is a question of thinking about how mathematics instruction is organised (and what mathematics is), how classroom cultures support learning, and how perceptions of students position them as 'able' or 'not able' in mathematics. These three aspects of mathematics education are situationally interrelated—they work together within the social/political system within which teachers and students work and learn. Hence, we are well aware of the difficulties of initiating teacher change in systems in which education values are dominated by performance measurement [1]. This situation is widespread: in the Anglo–American–Australian context, mechanisms of accountability are now normalised to the extent that teachers understand the quality of what they do as solely defined in terms of student test performance [2]. Pressure to perform is even exerted in less test-dominated education cultures such as Norway and Sweden due to international comparisons such as PISA [3], which raise political expectations and feed neoliberal discourses, compounding narrow definitions of good mathematics teaching despite a background of humanist educational ideologies [4]. In this context, the development of teacher action for equity is particularly challenging for us as researchers and teacher educators committed to fundamental change in what is valued in mathematics education. In this paper, we draw on Sue's account of her development from a newly qualified teacher to a teacher educator committed to developing equitable mathematics classrooms in which all students can participate. Building on her account of the particular events, ideas, and interventions that influenced her thinking, and the role of these reflections in the design of teacher education, we consider the implications of her story for the challenges we face in developing professional development towards inclusive mathematics classrooms in the marketised, accountability-driven context in which we now work.

2. Pathways to Equitable Mathematics Classrooms

As we note above, our interpretation of critical mathematics education is one that recognises the role of social systems in the reproduction of a mathematical 'elite' (who may in fact be simply good at following rules but nevertheless possess the examination 'ticket' to a future) and a large underclass of failures (around 30% of 16-year-olds in England fail its high-stakes national examination in mathematics every year [5]). We attribute the problem to the restricted nature of traditional mathematics education, which prioritises a purely formal conceptualisation of mathematics that has little connection to students' informal mathematical actions in the world. In this section, we explain our understanding of what needs to change, drawing on Horn and Garner's [6] work on teacher learning of ambitious and equitable mathematics instruction, and making links to Realistic Mathematics Education as instructional design for making sense of mathematics.

Horn and Garner emphasise the situated nature of the development of classroom spaces where learner conceptions of mathematics are heard, valued, and gradually developed. In particular, they note the need for three shifts in teachers' conceptualisation of teaching in relation to their particular work context, their 'beliefs, values, explanations, definitions, and ideologies' ([6] p. 26), and their resulting pedagogical judgements. The first shift concerns the organisation of instructional activities and a move away from the teacher-centred presentation of procedures towards an emphasis on problem-solving, argumentation, modelling, and proof ([6] p. 27). The second involves a corresponding change in classroom discourse towards whole-class discussion in which teachers remove themselves from the centre, questioning in order to seek information rather than eliciting and evaluating known information. Listening to students and seeking explanations of their thinking is paramount. These moves necessarily generate and maintain a classroom culture of listening and explanation in which students must also invest. The third shift concerns a change in teachers' perceptions of students and of the nature of mathematical competence itself (p. 36). Rejecting hierarchical views of ability and easy reproduction through a focus on calculation, Horn and Garner argue that:

> . . . expanding mathematical competence does not mean watering down mathematics. Indeed, the opposite is true: Expanding mathematical competence means rendering authentic mathematical smartnesses both visible and consequential in classrooms. Looking at mathematics as a field, we see that its great accomplishments have not come about from quick and accurate calculations, but from other kinds of insights, creativity, and intelligence: asking good questions, making astute connections, working systematically, seeing patterns, illustrating representations, and so on ([6] p. 36)

For us, Realistic Mathematics Education (RME, [7]) ticks the boxes of equitable mathematics education in its emphasis on a mathematics which is generated in a 'bottom-up' manner from students' informal actions, rather than being a 'top-down' given presented by a teacher in authority. This is, we argue, an equitable mathematics because it is rooted in realisable and meaningful contexts and most importantly remains so—RME never abandons the learner to meaningless forms. The instructional design underpinning RME thus emphasises the role of emergent mathematics [8]. Students move from models of their informal activity in carefully chosen contexts towards strategies that they can justify in ways that are accessible to themselves and others, thereby developing what Yackel and Cobb [9] describe as intellectual autonomy in mathematics. The shift away from teacher authority towards a mathematics that notices, appreciates, and builds on students' own understandings necessarily creates a different role for the teacher—they must develop the technique of guiding learners [10,11] as they move towards forms that can be articulated and 'taken-as-shared' in the community of the classroom ([9] p. 460). RME thus emphasises a process of mathematisation in which 'acceptable explanation and justifications [have] to involve described actions on mathematical objects rather than procedural instructions' ([9] p. 461).

This emphasis on informal models has implications for the dominant conceptualisation of progress as indicated by the successful application of procedures in standard mathematics topics. RME identifies two ways in which students engage with mathematics; on one level, they solve the contextual problem under consideration ('horizontal mathematisation'), while on the other, they work within the mathematical structure itself by reorganising, finding shortcuts, and recognising the wider applicability of their methods. This 'vertical mathematisation' [12] is indicated when students recognise similarities between their models of a number of related contexts and how these can be generalised to other problems. Thus, they may move from a 'model *of*' a particular situation (for example, a subway sandwich represented as a squared-off bar) to a generalised 'model *for*' solving various types of problems which may span diverse topics [13], such as a double number line or a ratio table. Progress is redefined as the progressive formalisation of models [7] but this does not prioritise the formal method; rather, it prioritises the explicit act of connection between 'model of' and 'model for' that underpins it. This reconceptualisation of mathematics and mathematics learning feeds into inclusion as envisaged by Horn and Garner: RME not only engages all students through its explicit connections to what is imaginable, but it also promotes an inclusive student-centred pedagogy. Most importantly, RME demands a different view of what mathematics is and who can do it, and, therefore, what progress is. Indeed, test results based on procedural knowledge do not reflect, or even demand, the deep understanding built by RME. As Sue's account in Section 5 shows, RME has played a significant part in her development as a teacher and critical teacher educator, and it remains a central plank of our work with teachers now in terms of providing opportunities not only for student access and engagement but also teachers' understanding of mathematics itself.

3. Teacher Learning in a Climate of Performativity

The explicit move away from procedural mathematics advocated by Horn and Garner and embedded in the principles of RME presents a number of challenges for teachers raised in post-performative times, whose practice is inextricably bound to prioritising test performance [2]. Even teachers who do not subscribe to a procedural approach find that demands for accountability compromise more exploratory aims [14,15]. The emphasis on outcomes measurement consequently reduces the impact of reform-based professional development [16,17]. This situation is exacerbated in England, where much professional development is now government-funded and centralised, and there is a perceived need for consistency across the sector in dictating what is to be learned as received 'best practice' [18,19].

On the contrary, teacher learning is an incremental and context-sensitive process which cannot be blue-printed [20,21]. In a performative climate, moving towards equitable mathematics instruction requires the development of critical noticing [22] which questions discursively fixed conceptions of competence/ability and how these position students, opening the way to different pedagogical actions (see also Horn [23]). Importantly, such actions need to be underpinned by pedagogical reasoning but also supported by what Horn and Garner [6] call pedagogical responsibility—a teacher's sense of what constitutes appropriate teaching in a given context. This comprises not just ethical principles (such as wanting all students to gain the mathematics they need to be able to participate in the world) but also responses to situational constraints. Horn and Garner point to the fact that teachers may feel that they must prepare students for high-stakes assessments, whether or not they think they are legitimate. This means that teachers must navigate their way through the potentially conflicting demands of teaching for deep understanding versus making sure that students can spot what is needed in an exam question. In addition to these genuine dilemmas, we see performative cultures and the marketisation of education as creating additional day-to-day obstacles for teachers. Education in England is increasingly 'delivered' by multi-academy trusts—not-for-profit companies that run groups of schools, often supported by sponsors, including businesses—which are focussed on public 'successes' and branding, fostering business rather than educational priorities [24]. The push towards marketisation is not limited to England; for instance, deregulation is now causing a focus on branding in Norway [25]. Loss of teacher autonomy in this context is a concern [26,27]. Together with the growth of nationalised best practice professional development programmes, this situation has presented us with complex new 'problems of practice' [6] in our work with teachers.

In this article, we are particularly mindful of how, as teacher educators, we need to acknowledge our own pedagogical responsibility in terms of the demands and constraints exerting daily pressures on teachers in the current climate of high-stakes testing and its impact on their students, invoking highly emotional responses for some [28]. Now, more than ever, we need to identify what is important about teacher development for equitable classrooms and how this can be incorporated into our training today.

4. Reflections on Teacher Learning: Telling the Story

The inspiration for these reflections lies in our work over several years researching student inclusion and the impact of RME in classrooms, and our awareness of the considerable pedagogical changes involved in using RME materials. The focus of this Special Issue on critical mathematics teacher education led us to consider the importance of telling a teacher's story in order to capture 'the view from the other side' and embed this reflectively in the context of our work as teacher educators. Thus began a process of digging into Sue's teaching past, collecting 'data' that frequently drew on her masters' degree work in 2003, but also included reflections—sometimes with accompanying student worked examples, RME materials, or professional development plans—as told to Yvette. Our process in constructing this narrative was for Sue to write about key events in her career, illustrated with her previous work and examples. Yvette read these accounts and asked for more details or explanation; our conversations were often long and completely novel—we had never discussed some of these issues before. A story emerged that seemed important to us in terms of Horn and Garner's [6] account of the shifts required to move towards equitable mathematics classrooms. It had RME at its core, showing its impact on Sue's conceptualisation of mathematics and her classroom practice.

At the same time, we were aware of the context in which we work and its impact on professional development. While the teachers we have worked with often bring stories back to training days of new student engagement and understanding as a result of the work we do with them, an increasing number have also talked about the need to justify their practice to school managers who worry about the impact on their school performance data of lessons in which rote learning and practice are nowhere to be seen. The increasingly performative context of education and professional development outlined in the previous section is important, and it has impacted how the story is told—Sue looks back on her experience of becoming critical with the hindsight of much greater RME knowledge and experience than she set out with, but she also sees this experience through the lens of the challenges she faces as a teacher educator now.

In the next section, Sue takes over the narrative to describe her development over a period of 15 years towards a new understanding of the meaning of 'making sense of maths', identifying critical moments and reflecting on how these raised issues for her regarding what mathematics is, what students bring to the classroom, and what this means for pedagogic practice. Central to her story is the need for a critical response to traditional accounts of mathematics and conceptions of progress, and what this means for professional development today. She draws on a collection of memories and illustrations from her participation in formal professional development, locally organised teacher communities, participation in funded projects, small-scale study of her own classroom, and finally her early experiences as an RME teacher educator. We have chosen to tell Sue's story uninterrupted and in her voice alone.

5. Moving towards Equitable Mathematics Classrooms—Identifying Key Events

5.1. Questioning Traditional Teaching

I began teaching in 1987, using a modern textbook series interspersed with investigations, group work tasks, and activities recommended by the Cockcroft Report [29]. But for the most part, like many other teachers, I settled into models of teaching that mirrored the way I had been taught: exposition of a formal method by the teacher followed by pages of student practice to 'acquire' the method. But planning for teaching lessons often meant that I needed to think more deeply than I had at school, and I learned for the first time what lay behind many of the 'rules'. I vividly recall the moment I realised that multiplying length × width to find the area of a rectangle was actually a way of counting the rows of squares that could be drawn inside it. I found such revelations shocking and felt embarrassed that I had come this far in learning mathematics (I had a bachelor's degree in mathematics) without making such a connection. Why hadn't my teachers told me about this? I became committed to explaining my new-found enlightenment to my classes, keen for them to share this understanding. Nevertheless, in marking my students' tests, it was apparent that the transfer I hoped for had not happened. I first blamed my students—'we had covered the work, they had good notes, they couldn't have bothered to revise the work'. Later, I read Jaworski's [30] critique of transmission teaching and the expectation that knowledge can be simply handed over through good exposition. In blaming my students, I had failed to recognise that successful teaching relies on learners making their own mathematical constructions, and that this is unlikely to happen through 'teacher telling'. This realisation gave me a different perspective: looking back, my own teachers had probably provided justification for the rules they taught me, but hearing their explanations was not the same as developing my own constructions and sense-making. Ironically, here I was, now the teacher, trying to transmit my newly constructed explanations to my own classes—by telling. Something needed to change.

5.2. Learning to Listen to Students

In 1997, I joined the Mathematics Enhancement Project (MEP) [31] as an intervention teacher, working with Year 10 students in their penultimate year of compulsory schooling, which would end with the national GCSE examinations at age 16. MEP arose from England's participation in the international Kassel Project, which highlighted the relatively high performance of several Eastern bloc countries compared to that of the UK. Lesson observation of one of the highest performers, Hungary, revealed that classes in Hungary worked together on problems guided by an 'on the go' teacher with an emphasis on whole-class interaction and student debate. In contrast, teaching in the UK focussed on correct answers rather than the details of the method, with teachers working with students individually rather than with the class as a whole. The subsequent MEP intervention was modelled on the Hungarian system, focussing on whole-class interaction, precise use of language in writing and speaking mathematics, numeracy, applications, and homework as an integral part of the learning journey rather than just an 'add on'.

It was inspiring to watch video of Hungarian students walking to the front of their classrooms, pen in hand, to write their solutions on the board and describe to the class what they had done. The class teacher invariably positioned themselves at the back or to the side of the classroom, asking questions of the student presenting their solution. This practice had a major impact on my own teaching. I began lessons outside my classroom, standing at the door, with students lined up, open book in hand, to show me their homework as they entered the room. As they filed in, I would select around three students to write their homework solutions on the board. Standing towards the back of my classroom, I would invite them to describe to the rest of the class what they had done. I would try to read faces and direct questions to other students to see whether they could follow their classmate's approach in a homework review phase that could take over half an hour.

By now, I was practising being neutral, thanking students for their solutions, commenting on them as 'interesting', but otherwise presenting a blank face. This teaching strategy became a common part of my practice when working with solutions that students brought to the classroom. I see this phase now as one in which I was trying to create a classroom where learners spoke more than me. I would ask students to describe what they had done and to ask questions of each other without needing to go through me. If students did not talk loudly enough, I would repeat their words, but I was conscious to do this verbatim so as to keep the focus on their thinking, not mine. At times, I would sit in a student's seat in a deliberate attempt to lower my presence and status in the room. When working at the board, some students were hesitant and needed support, so I developed less intimidating instruction. 'Explain what you have done' became 'Describe what you did', and 'Compare these two methods' became 'What's the same? What's different?'. All students needed encouragement to realise that their audience was the class, not the teacher. Gradually, student–student interactions increased, so that a student at the front of the class might notice a blank face and ask: 'Did you get that?'. I noticed a shift in mathematical authority in the classroom as students began to scrutinise and question each other's mathematical statements or arguments, no longer looking to me to tell them whether it was correct or not. My teacher role was now one of facilitator, enabling them to think, describe, draw, listen, speak, and question mathematically.

My choice of solutions to showcase was based on mathematical criteria for selecting a solution. I picked out (i) common misconceptions, (ii) standard solutions, and (iii) unusual approaches that might contain a diagram or more long-winded strategy. I was not focussed on whether such answers were correct or not but rather whether they were accessible to other learners. I learned a good deal about the way learners thought mathematically and found novel solutions that bore no relation to the method I had taught, although they made sense to the student when described. There were times when a student might describe what they had done and other students would follow, but I, the teacher, would not. This was a reminder that my way of seeing mathematics might be far away from theirs.

5.3. Recognising the Need for Informal Mathematics

Following my MEP experience in 1998–2002, I joined with a number of teachers in my local area who were interested in developing innovative pedagogies, particularly whole-class interactive teaching, and were working with tutors at Manchester Metropolitan University. We videoed each other and jointly watched and commented on the lessons. This activity helped to develop our awareness of teacher actions such as where we stood, how we presented ourselves as neutral, what sorts of prompts supported student contributions, and what actions helped students reveal their inner mathematical thinking. One member of the group was particularly focussed on developing student-centred approaches and was strongly influenced by Gattegno's claim that "everyone can be a producer rather than a consumer of mathematical knowledge. Mathematics can be owned as a means of mathematizing the universe" ([32] p. 2). Watching one of his lessons, described below, had a profound effect on both the way I began to see learners and how I viewed the school mathematics curriculum:

> Aidan is teaching a bottom set Year 8 group of 10 students. The first question in his quick quiz starter is presented as the multiplication 16×12 written above a rectangle accurately drawn 12 down and 16 across. The side lengths are not labelled, and the outline is drawn on a faintly squared blackboard. The students set to work drawing the rectangle on squared paper, and one enquires if it needs to be 'dead accurate'. Aidan circulates to gain a sense of student approaches. After six minutes, he stops the group and talks to them about how to behave when observing each other working at the board. He reminds them to use manners, be respectful, and remember that they are on a journey to becoming mathematicians.

He talks about 'us' and 'we', and that we are looking for shortcuts, spotting patterns, making connections, and recognising that we are all at different stages of that journey.

Aidan invites a student to come to the board to show how he found the number of squares. Beginning with the bottom left-hand square, he touch-counts each square, writing '1' in the bottom square through to '12' at the top of the column (see Figure 1). The rest of the group watch. The counting is slow. To me, it feels slightly tedious but apparently not to the class and certainly not to Aidan who comments on the precision with which the student is counting and the hard work that it takes to work in this way. Another student comments that 'you could have just written in 12 because it says it's 12 down the side'. The first student, seemingly not ready to make this connection, continues counting down the second column of squares, writing in numbers as he goes, "13. 14. 15. 16. 17. 18. 19.". Aidan instructs him to stop there and asks 'Where is it going to end? What is the last number he will write in this column?'. He reminds the class not to shout out in order to allow thinking time. After a 30 s pause, Aidan accepts a student's suggestion of 24 with the response 'How did you know it was going to be 24?'. The student refers to 1 and 1 making 2, 2 and 2 making 4. Aidan responds with 'Ok' and moves on to hear from another student. We return to the student at the board and repeat the sequence of counting, stopping part way and hearing other students' rationale for what goes in the last square of the third column. Aidan provides a meta-commentary on their strategies, noting that they are pattern spotting. He carefully selects who comes to the board, based on the strategy they have used. The next student numbers only the bottom square of each row to reveal a total of 192 squares. A third partitions the original rectangle into 4 smaller sized rectangles, but her partitioning does not match the 100/60/20/12 totals she writes in each mini rectangle. Their methods are all based on counting the squares inside although some have developed short-cut ways to do this. None of the class are attempting to multiply 16 $\times$ 12 using a standard algorithm. Eighteen minutes of the lesson is given to sharing strategies, with Aidan commenting on them and directing some students to try others' strategies when it comes to the next lesson.

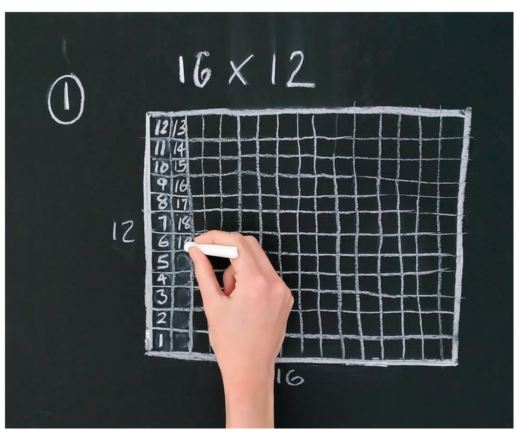

Figure 1. Touch-counting strategy used to find the total number of squares of a 16 $\times$ 12 rectangle. Reproduction of classroom board work.

Observing and analysing this lesson was revelatory for me. It raised more questions than answers and threw up a wealth of contradictions, forcing me to look again at my own practice and indeed my beliefs about what and how my own students were learning. To begin with, the pace of the lesson felt slow, but what did that actually mean? Yes, it was slow in terms of traditional pace metrics such as the number of practice questions a student completes or the amount of different techniques shown to the class. But what if pace is understood as pace of thought or of making connections across representations? Secondly, what about the level of challenge? I had just seen Year 8 students touch-counting columns of squares aloud, slowly, and deliberately. Surely they knew you could just multiply the length by the width of a rectangle to find the total number of squares? Surely they had quicker methods for performing the calculation 16 × 12? What became apparent to me in this lesson was that no, these students did not realise that counting these squares in columns amounted to the same total as multiplying the length by the width. And no, performing the calculation 16 × 12 was not easy or straightforward for these learners.

I was forced to question many of my long-held assumptions. These students would first have encountered strategies for finding area using multiplication four years ago in Year 4, and again in Years 5, 6, and 7. Nevertheless, despite all of this curriculum time and experience, they did not connect the multiplication of lengths strategy for finding area with the counting squares version. This was more than simply not remembering—it felt like there was a major gap in conceptual understanding between their concrete, long-winded but accessible strategies and the formal algorithm. If this was true in this case, what other chasms were lurking in my own classroom? Not only that, but what could I do about this?

5.4. Developing Teaching That Values Informal Understanding

In response to the revelations in Aidan's lesson, I investigated the topic of area further, initially conducting a study in my own classroom that was designed to improve the ability of lower attaining students to find the areas of various shapes. Informed by my own understanding of area and my awareness of research showing standard student misconceptions, I set out to intervene in students' approaches by teaching them to visualise squares when thinking about area. As I will show below, my later encounter with RME made me radically rethink this approach and the role of informal understanding in mathematical concepts.

5.4.1. Intervening in Student Thinking—The Area Project

In 2003, as part of my master's degree, I conducted a small-scale study with two lower attaining Year 9 groups. I carried out a short pre-test requiring students to find the areas of a rectangle, triangle, parallelogram, and compound shapes, all presented on plain paper, with side lengths labelled. The test also included four non-standard figures presented on squared paper. The average score was 3.5 marks out of 12. A close inspection revealed that students were most successful when finding the areas of shapes presented on squared backgrounds, using strategies of piecing together half squares to make full squares before totalling or by counting and accumulating full and half squares, as shown in Figure 2a,b, respectively.

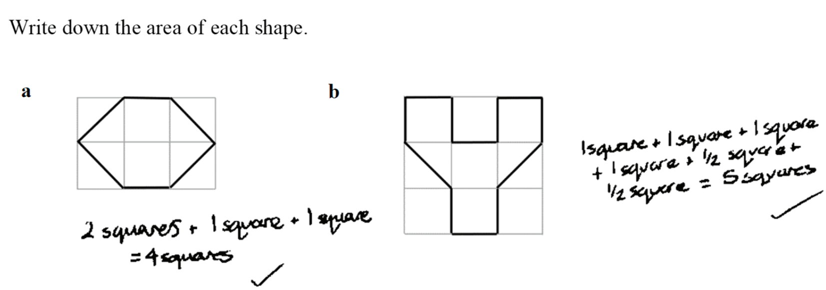

Figure 2. (**a**,**b**) Finding area using a squared background: two different Year 9 student strategies.

Difficulties mainly arose when finding the areas of shapes drawn on a plain background. Some students, as illustrated in Figure 3, consistently found the perimeter rather than the area, while others added the two dimensions given in each question, whatever the shape. Others consistently multiplied the given dimensions, and one repeatedly multiplied the given dimensions and then halved the result. There was no indication in these solutions that students were thinking about the space that could be fitted inside the plain figures. Finding the areas of compound shapes (such as an 'L' shape) proved to be particularly challenging. Several students found the perimeter, while others appeared to be 'doing something with the numbers', be it adding, multiplying, or a combination of both (even when this resulted in an unrealistically large number), again apparently unable to connect their calculation with the space-covering aspect of area.

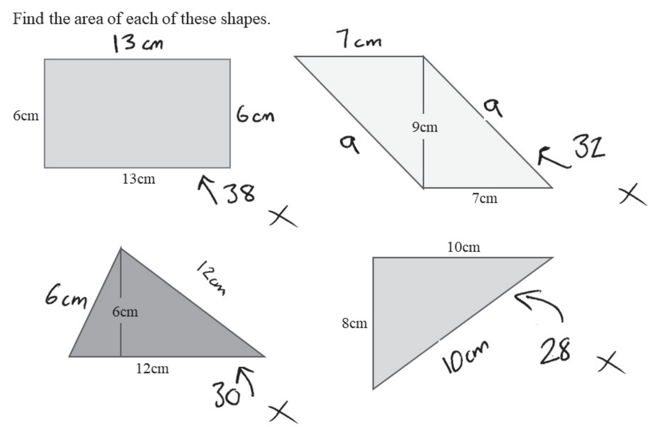

Figure 3. Calculating perimeter rather than area: a common student strategy from the Year 9 pre-test.

It was reassuring to realise that my own observations concurred with the findings of several researchers, particularly their findings on the effect of squared paper. The Assessment of Performance Unit study [33] noted that around one-fifth of the students tested gave perimeter when asked to find area, but that this was less likely to happen if a grid or unit square key was provided. The Chelsea Diagnostic Mathematics Tests study [34] reported similar findings—while only 33–50% of students could correctly evaluate the area of a right-angled triangle presented on a plain background, 78–91% were successful when the triangle was drawn on squared paper. I began to reflect on why my students were experiencing such difficulties. Counting squares is a transparent process closely linked to the informal understanding of area. The concrete nature of this strategy makes sense as a means of comparing one space with another, as a way of quantifying that space, but the traditional trajectory of school mathematics moves students quickly from informal square-counting strategies to multiplicative methods. The disappearance of squares leaves learners with some algebra shorthand and shapes outlined by lines marked with lengths, which give no hint of the small unit squares contained within. I decided to bridge this gap by devising a method that built on counting squares as a way of encouraging students to visualise the squares inside a shape. I rejected the idea of using squared paper or a transparent grid overlay, as these methods had been used before, were not available in an examination, and provided too much of a 'quick fix' in that the student did not have to fully engage in the process or make connections with the side lengths of a shape as corresponding to a row of squares sitting on that length. Instead, I decided to develop their ability to visualise and draw the squares freehand, using the dimensions labelled on the sides of the shapes as a cue for how many unit squares they needed to fit and draw into a row. Having created a row of squares to match the length dimension, students could use the height measurement to partition the vertical dimension and reproduce the required number of rows of squares. Having filled the shape with unit squares, students could then count them to find the area.

I tested the students again on the same questions at the end of my intervention. The images in Figures 4–6 illustrate some of their approaches. For triangles, the preferred method was to draw in the squares to make full rows even though these extended beyond the triangle boundaries, as shown in Figure 4. Students were then able to argue that the triangle used up half of the squares.

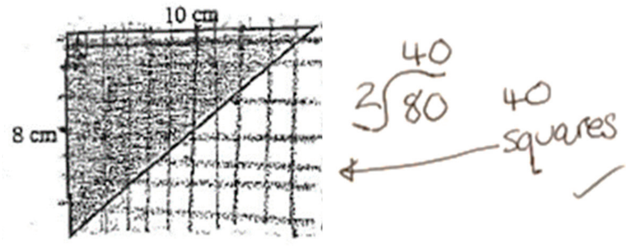

Figure 4. Adapting the drawing in squares method for a right-angled triangle: a student strategy from the Year 9 post-test.

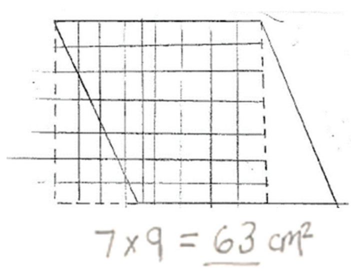

Figure 5. Adapting the drawing in squares method for a parallelogram: a student strategy from the Year 9 post-test.

Similarly, for a parallelogram, students drew a full row of squares along the top length, as illustrated in Figure 5, and repeated the rows directly underneath to form the rectangle of equivalent area.

However, there were difficulties in acquiring this method. Partitioning a side length into the required amount of squares proved to be challenging for many students, even more so if the side length was an odd number. Some recognised when halving was appropriate and made efficient use of this strategy, but others guessed and checked, adjusting the sizes of their squares to fit as they approached the finish line. Some struggled with the relationship between the number of squares and number of dashes. When faced with dividing a line segment into, say, 12 parts, some students were naturally inclined to put in 12 dashes between the endpoints, creating 13 gaps. The guess and check method of estimating the size of one gap could take time with much rubbing out, while test items not drawn to scale could lead to inconsistencies in the sizes of squares. As illustrated in Figure 6, the student began by marking eight spaces on the top line. As stated in the question, the diagram is not accurately drawn and the eight equal spaces in the top line do not correspond to the 3 spaces on the bottom line, hence the inconsistency.

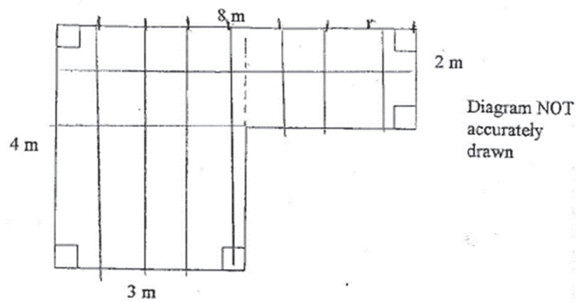

Figure 6. Struggling to fit same-sized squares when the image is not drawn to scale: a student strategy from the Year 9 post-test.

My intervention appeared to have had an impact, however, with the students introduced to the 'drawing in squares' strategy making post-test gains of 5.6 marks out of 12 compared to +1.3 marks for the traditionally taught 'textbook' group. I concluded that there was definitely something to be gained by teaching lower attaining students to draw in the squares. The method was time-consuming, but it seemed to prompt engagement with the meaning of the dimensions marked on the boundaries of the shape and, in turn, it built up a picture of the squares inside and how these filled the space. Once realised and visible, the students were able to evaluate the amount of squares. Many elected to touch-count each square, a lengthy, primitive strategy, but at least this was a sense-making strategy owned by the student. Some moved on to multiply particular dimensions as a way of determining the amount of squares contained within a particular block, thereby using the formal operation present in all area formulae—that of multiplication. Crucially, this was not because they had been told to do so; rather, it was because it made sense to them as a way of quickening the square-counting process. My encounter with RME later that year underlined the importance of learners constructing for themselves the connections between side lengths realised as squares, through counting squares, and finally to multiplication, and the implications for the role of models in a truly student-centred pedagogy.

5.4.2. Encountering RME: Redefining Progress

My first encounter with RME was later in 2003, when I was recruited as a teacher in Manchester Metropolitan University's pilot study of the Mathematics in Context (MIC) materials, an American version of RME produced under the guidance of Thomas Romberg as a collaboration of the University of Wisconsin in America and The Freudenthal Institute in The Netherlands [35]. I readily abandoned my school's intended scheme of work and began teaching my Year 7 high-attaining group using the *Reallotment* booklet [36], a unit designed to provide students with guided opportunities to work on their ideas of area. The unit was intended to take around four weeks of study. This was September. In mid-December, I reluctantly decided that I had better move on to the next RME module, even though there were still several pages of *Reallotment* left. The experiences of working with that class on an RME-designed curriculum were both exhilarating and shocking, and I was really beginning to question what I believed at the time constituted effective teaching and learning.

In my classroom, unusual events occurred: students willingly volunteered to demonstrate their strategies to the rest of the class; they challenged one another's thinking with newfound confidence and genuine interest; they became active participants in reading text, starting problems without the need to be shown what to do; they instinctively requested materials such as tracing paper and string because they could see how these tools might help them to solve a problem. Ultimately, they saw questions as genuine problems that they could comment on, contribute to, and so have authority over. One student later said, 'It doesn't feel like you are doing maths'. Likewise, for me, the mathematics certainly felt different. My awareness of what was important mathematically changed. I came to recognise the informal strategies of fitting in/reallotting parts of a shape/seeing one shape as a fraction of another as valid in their own right, and I began to recognise that the contexts and associated informal strategies underpinned a learner's developing understanding of area as 'space covering', as in Hughes et al.'s [37] and Foxman et al.'s [38] work. I also realised that tasks such as finding the area of triangles drawn on a 9-pin geoboard [38] may have generated similar strategies, but in the past I would not have valued students' informal thinking, focussing instead on the area of a triangle formula endgame.

I recognised now that there was a world of mathematics underneath the area formulae. I learned new ways of thinking about problems through studying a picture, drawing a picture, and applying intuition (as opposed to a formula). I saw imagery that helped bring the space-covering aspects of area into focus. I appreciated the roles of movement (of tracing paper, of parts of shapes) and of matching same-sized areas in developing an understanding of conservation of area. I began approaching mathematics problems at any level with my eyes open wide to imagery, to drawing, and to thinking about alternative representations, using my new-found powers of mathematical common sense. I already had the pedagogy developed through working with Whole Class Interactive teaching communities to support these ways of working, but RME gave me an even better awareness of student thinking and a greater expectation that students could access these problems across the whole attainment range. Through RME, I now had the questions to prompt multiple strategies, provide open access to starting problems, provoke different opinions and ways of seeing, and develop a classroom culture in which these approaches could flourish and take root. However, working with RME brought disturbances too: I had several questions and concerns about progress. How would my students perform in tests? What was behind the design of *Reallotment*? I knew my students were thinking, learning, and becoming empowered, but how did their informal, naturally developed solutions to the contextually-based area problems link with formulaic approaches? Indeed, what constitutes progress in an RME classroom?

5.4.3. Connecting Informal and Formal

In a standard textbook, formulae appear early and trigger exercises designed to practise their use. Success in these indicates progress. In RME, progress towards formal representations is seen as a longer-term aim stretching over weeks, months, and even years. It is not until lesson 7 of *Reallotment* that students are introduced to formulae, and the focus is different: formulae are constructed by the students themselves from carefully chosen images that enable them to reason about how the areas of a bare generic rectangle, triangle, and parallelogram connect to each other. Far from practising how to apply these formulae, the next question offers a slanted triangle and states that sometimes you have to use another strategy to find the area of a triangle. The problems in *Reallotment* appear to be in a random order if we look through the traditional lens of what constitutes progress. We expect to see a build-up of questions, containing harder numbers, or the application of more than one procedure, or the use of symbols as an extension of the use of numbers, in a process known as 'progressive complexity'. In RME, students make progress through progressive *formalisation*, where the level of question difficulty remains the same, but students have the opportunity to move from using informal, intuitive approaches through a variety of pre-formal strategies to using more contracted, efficient and abstract strategies [12].

In Figure 7, a question taken from the *Reallotment* Teachers' Guide asks students to find the price of pieces of wood given that the rectangular piece costs \$18. Students are not required to know any area formulae—this is not an application problem. This context and the presentation provoke a range of naturally occurring but conceptually different approaches. Strategy 1 is to draw in the background squares, figure out the price of one square of the wood, and recognise that the triangle covers three of those squares. Srategy 2 is to draw a line splitting the original triangle into two smaller right-angled triangles, then see each separate triangle as a half of the corresponding rectangle. Strategy

3 provides a logical reason for seeing the black triangle as half of the original rectangle. In my class, students offered this range of strategies and it was possible to see some students shifting towards more contracted/global strategies, e.g., the area of *any* triangle is half of that of the surrounding rectangle/parallelogram. Through exposure to several context-based problems and by sharing their informal approaches under the guidance of the teacher, learners gradually become 'a little bit more advanced in mathematizing the problems' ([7] p. 15).

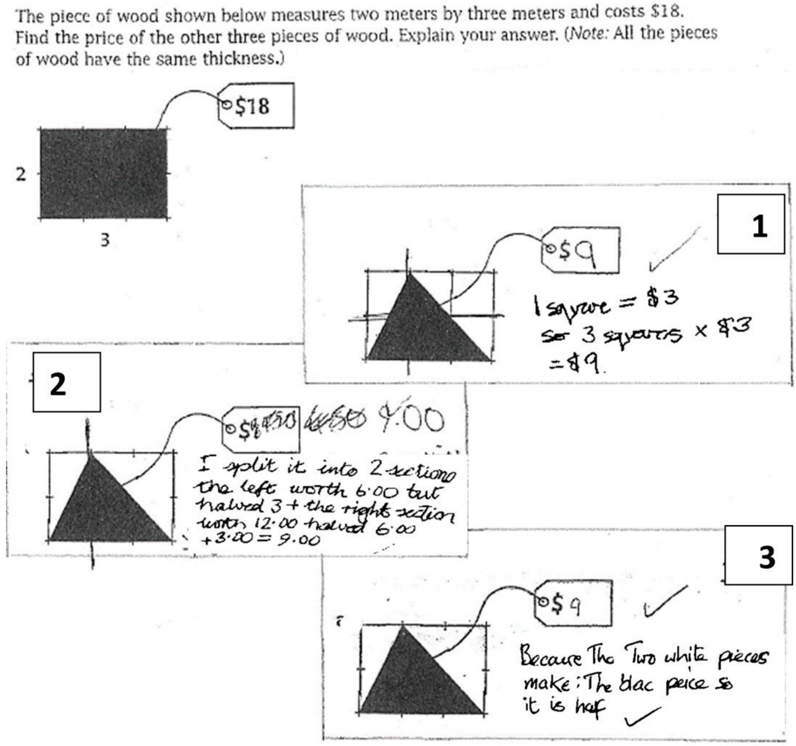

Figure 7. Progressively formal strategies (1 to 3) for answering the wood-pricing problem.

In RME, the formal world of mathematics (in this case, the area formulae) is most definitely available to all learners but the route to it looks very different. It is this shift towards more formal methods and representations that are rooted in context and sense-making that constitutes progress in RME. This understanding enabled me to reflect back on my intervention in students' learning of area. The method of instructing students to draw in the squares could be viewed as a 'top–down' strategy imposed by the teacher rather than generated by the students. But it could also be argued that the original shape (e.g., a rectangle presented on a plain background) is the context and that students are creating their own 'model of' the context by drawing in the squares. While students stayed with drawing in the squares and touch-counting each one they were using informal strategies closely linked to the context and, as such, they could be said to be mathematising horizontally. However, during the course of the intervention and with guidance, it was possible to see some students *vertically* mathematising. These students shortened their approach to drawing in the squares, drawing only the top layer of squares or partitioning down the side of the enclosing rectangle and moving to multiplication rather than touch-counting, as illustrated in Figures 8 and 9.

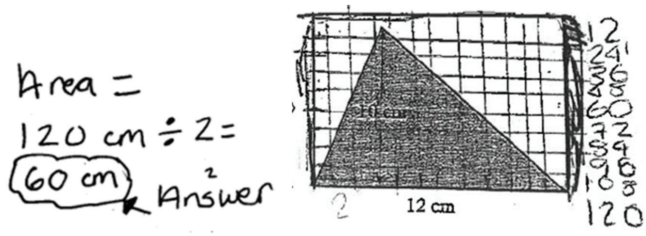

Figure 8. Drawing in squares and counting up in multiples of 12: student solution strategy from the Year 9 post-test.

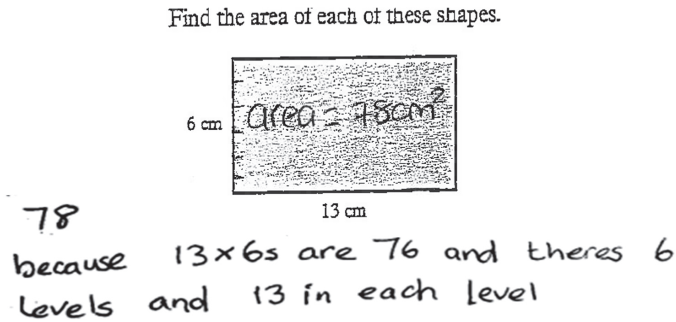

Figure 9. Partitioning the height dimension only and reasoning why multiplication will find the total number of squares: student solution strategy from the Year 9 post-test.

When I devised the drawing in squares approach, I knew that students needed to connect the dimensions of an 'empty' rectangle with the unit squares required to cover the space inside. However, I did not have knowledge of the contexts that could generate strategies which would naturally enable students to engage with the space-covering aspect of area. The contexts in *Reallotment* develop awareness of unit square coverage in subtle ways, such as using the price of a standard tile as a mediating quantity, asking for a comparison of one land area with another, or presenting shapes as 'coloured in' on a squared background visible only around the edges so that the learner engages with ways of seeing and counting the squares inside the shape even though it appears as 'empty'. RME gave me knowledge of informal strategies and helped me to recognise their potential for learner development, supplying the tried and tested contexts that enabled these strategies to emerge rather than being 'delivered' to students.

RME also helped me to become aware of a vast number of informal strategies of which, in 15 years of teaching and as a learner of mathematics myself up to the age of 21, I was not consciously aware. I now saw that these informal models were a legitimate part of mathematics in their own right. Previously, if I had seen learners using long-winded strategies (such as drawing 120 dots and circling them in groups of 8 in order to work out 120 divided by 8), I would have shown them that there was a much quicker way to do it. Now, I recognised the importance of their approach and the need to consider what problems to use in order to shift their thinking slightly, rather than destroy their confidence by offering a formal method that made no sense to them. I could make a long list of informal strategies, including 'matching', 'reallotting', 'comparing', 'creating', and 'using bar models and ratio tables'. These strategies and my awareness of the need to work with what learners bring to the classroom were to become fundamental parts of my work with trainee teachers and professional development programmes over the next 20 years.

5.5. Using My Learning in Teacher Education

My work as a teacher trialling RME in 2003–2004 led to my recruitment as a trainer for other teachers joining the project in the following three years. I was strongly influenced by the key events from my own personal journey outlined above in both designing and leading professional development. I knew from the area project the value of looking closely at student work, even their crossings-out. I knew from using RME in my own classroom how important it was for the teacher to experience the materials as a learner and feel the uncertainty of not knowing what answer or method is required. I was aware from working with whole-class interactive teaching strategies the importance of students sharing their solutions at the board, with the teacher responding in neutral ways, acting as a facilitator, and gradually shifting the authority over the mathematics to their students. Working alongside other teacher trainers, I drew on my experience to design RME-focussed professional development tasks, along with a belief that as trainers we too needed to take on the role of facilitator. We provided teachers with tasks designed to help them notice the difference between their students' responses to traditional teacher-led mathematics tasks and how creative their students could be when tackling an RME problem for themselves freehand. We modelled how RME materials could be used in the classroom, asking teachers what they noticed and inviting them to try. In teacher reflection sessions, we listened, remained neutral without judgement, invited participants to respond to other participant concerns, and encouraged them to negotiate solutions as a group. We tried to provide professional development that replicated RME classrooms, whereby instead of telling our learners what they should see or think or do, we set the task and made space for sharing ideas and reflection.

Table 1 lists some of the core aims of our professional development, which emerged over many years of working with teachers on RME-based projects, and provides examples of activities supporting our aims.

Table 1. Our professional development aims and corresponding activities.

Aim	Activity
To develop teacher awareness and knowledge of students' natural informal approaches	Teachers study student solutions to mathematics problems, analysing approaches, looking for connections across methods, linking student answers to how they are taught, ranking solutions from informal to formal
To enable teachers to experience RME both as a teacher and as a learner of mathematics	Trainers model RME lessons in which teachers are positioned as students and trainers provide meta-commentary on their pedagogic decisions. Key strategies: bring learners to the board to showcase a range of their solutions; remain neutral; focus learners on solutions with directions such as 'Say what you see', 'Can you draw something?', 'What's the same, what's different?' [about these solutions].
To develop teachers' appreciation of how RME uses context and models to build a different view of progress	Focus on how a particular model, such as the bar, emerges from many contexts to become a model that learners can apply elsewhere across many topic areas, even to non-contextual, bare number questions
To provide models of teaching that shift the role of teacher from transmission orientation to that of a facilitator	Video observation emphasising noticing and accurate, non-evaluative description. Focus on, e.g., what neutral teacher responses look like and what teachers see as mathematical 'progression'

We knew from my own experience the importance of teachers having time to work with RME in their own classrooms and the freedom to fail. In our early professional development programmes, we set expectations that teachers would spend at least a year using RME materials with one of their classes for 90% of the curriculum time. This was not seen as an issue by the teachers or their schools at the time. In the early 2000s, practice-based classroom research and experimentation was welcomed and admired, bringing kudos to a mathematics department. The fact that this might mean a class had not covered exactly the same curriculum as other classes was considered inconsequential. Experienced teachers were well aware that 'covering' a topic in a traditional way did not translate to many students being able to successfully answer test questions on that topic, and there was plenty of curriculum time for students to encounter topics again and for teachers to identify and work on any gaps in students' knowledge prior to their final examinations. School senior leaders recognised the importance of innovative approaches to mathematics teaching and prioritised it. They sought to employ mathematics curriculum leaders who could deliver innovation and trusted the judgement of these subject experts. However, over the last 15 years, the shift of focus towards accountability and systems of monitoring, measuring, and judging schools and students has undoubtedly stifled the era of experimentation. We discuss this further in the final sections.

6. Discussion: Developing and Delivering Professional Development for Equity in the Current Education Climate

Sue's journey began in the 'progressive' context of the late 1980's when teachers could experiment with new pedagogic approaches, sanctioned by the influential Cockcroft Report [29], which advocated classroom discussion, investigation, and problem-solving in order to develop the full potential of every student. Her participation in research projects, her membership of a community of teachers, and her introduction to RME fed and fostered successive moves in her 'problems of practice' [6] over a number of years in the non-linear, situated, and incremental process of teacher learning described by Clarke and Hollingsworth (2002). As her narrative shows, she developed expertise in ways of teaching that built on students' informal understandings and led to a reconceptualisation of mathematics itself as doing rather than knowing. In terms of Horn and Garner's [6] three shifts, she developed an understanding of instructional design, a repertoire of strategies enabling her to decentre her role in the classroom, and a recognition of the value of every student's contribution. Together, these contributed to the development of pedagogical judgement which went far beyond the acquisition of 'best practice'.

For us as teacher educators, the challenge is now how to 'reinvent' Sue's journey for today. We see a major problem in doing so, which is the increasingly rigid approach to coverage in a curriculum that prioritises formal mathematics. Sue's story highlights the central role of RME in challenging this approach; it not only engages students but also enables them to access mathematics in its fullest sense. Despite Sue's development of a student-centred pedagogy and her awareness that informal approaches were somehow important, it was only her engagement with RME that joined these elements together in terms of a new mathematical understanding that escapes reliance on mere procedures. Using RME materials, and understanding their design, was crucial to making this shift. It is important to note that this learning takes time. In a recent randomised controlled trial of RME in England (2017–2021), we were able to ask teachers to spend at least a quarter of their lesson time working with RME materials. Although this was much less classroom time than in earlier projects, we saw teachers changing in similar ways to Sue's learning—they began to reconceptualise progress and their classroom cultures became more inclusive. But, as we note above, some teachers reported pressure from senior leadership about the coverage of their schemes of learning. This kind of pressure is increasingly evident in our current professional development activities, and it has led us to consider our pedagogical responsibilities towards the teachers we work with. We now know that, in many schools, asking teachers to commit to using RME materials over any length of time is unrealistic and arguably unethical because of the pressure it puts on them.

However, our commitment to challenging what we see as deeply inequitable mathematics education has made it difficult to move away from our previous professional development model of supporting teachers to engage with RME materials and pedagogy over a period of one or two years. Although we still model RME lessons in training sessions, we know that teachers may only be allowed to try out one RME lesson rather than a whole module. School managers are insistent that all classes in a year group are taught topics at the same time, with the perception that missing a lesson from the scheme of learning amounts to dropping behind. Going back to where Sue started, we now focus on providing teachers with ways of working in their classrooms that make small but important inroads into classroom culture, without necessarily having to change the materials they are already using. We have developed 'pedagogies to keep in your back pocket', such as remain neutral, think about where you stand in the room, ask questions about images such as 'Say what you see' and 'What's the same, what's different?', and sharing student solution strategies. Teachers quickly see the potential of such strategies for encouraging student contributions and gaining insight into their mathematical thinking, but without RME materials, the range of methods tends to be less varied or novel. To compensate, we have chosen elements of our materials and attached particular teaching strategies. For example, a photograph of the finish line of a running race with the caption 'Say what you see'. In professional development sessions, we emphasise that the teacher's role is not to explain or say what *they* can see but only to scribe verbatim at the board what students say. This device is helpful because it forces teachers into the role of listening carefully to their learners and of capturing exactly what they say. Teachers see how it can be applied to their own school curriculum, so that traditional questions containing a graph, table, or shape can be stripped down to the image alone, along with the invitation to 'Say what you see'. This alleviates teacher tension about curriculum coverage. They appreciate that investing in 'Say what you see' for photographs of contexts and the resulting shifts in student access and classroom culture have the potential to impact test success.

Like Gore et al., we have worked on making space for teachers to reflect on their classroom experience, opening up 'spaces of freedom ... to experience accountability more productively, still with a focus on student outcomes' ([16] p. 454). 'Gap tasks' in between professional development sessions ask teachers to focus on a single strategy that they can support with an RME lesson if they so choose. Sue's story highlights in particular the importance of looking closely at student work, and joint reflection on these tasks with other teachers is a major way of creating space for noticing more about their students' thinking and recognising for themselves the value of the informal mathematics that underlies the formulae. This is not pure RME, but how to address this within our pedagogic responsibilities is our current problem of practice.

7. Implications for Future Research on Professional Development

In this article, we have made space for Sue's narrative in order to highlight her learning in terms of a continual reframing of problems of practice, but it has also been important to situate the narrative in its political context over time. What is unique about Sue's story perhaps is its depth of pedagogical reasoning in challenging accepted versions of progress and the nature of mathematical thinking and learning. The role of these challenges in underpinning pedagogical actions for equity has become even more apparent to us in writing this article as we have thought through the implications for professional development in our marketised world. We suggest that future research should extend Gore et al.'s [16] work by exploring how teachers might develop equitable mathematics classrooms within the constraints of accountability, and the implications for sustainable professional development. We recognise the compromises we are currently forced to make. We're working on it.

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Article

Maintaining Tensions: Braiding as an Analogy for Mathematics Teacher Educators' Political Work

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Abstract: Although the field of mathematics education has made gains in centering the need for justice-oriented approaches and antiracist teaching practices in teacher education, much of this work remains in its infancy. Moreover, research focused on this area highlights teacher candidates' knowledge and dispositions and often ignores the role of the mathematics teacher educators facilitating the process. We contend that mathematics teacher educators must pay more attention to how intersectional identities, contexts, Mirror Tests, and principles of Rehumanizing Mathematics manifest in teacher education to better understand how teacher candidates develop political knowledge in teaching mathematics. To this end, we introduce a framework of considerations, which we call a compass, that identifies four dimensions (or strands) and offers guiding questions for mathematics teacher educators to consider. We offer examples from a multi-site research study to illuminate each dimension and build the case for the necessity of braiding the four strands together as we engage in this line of work. Implications for practice and future research are discussed.

Keywords: teacher education; teacher educators; mathematics education; intersectional identities

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1. Introduction

Teaching is political. Mathematics teaching is uniquely political in that mathematics is over-represented in high-stakes testing, serves as a gatekeeper for more rigorous academic tracks, and is often used as a proxy for intelligence. However, the general public views mathematics as absolute and culture-free [1–3]. As such, understanding and navigating the politics of teaching is a critical aspect of the profession. For instance, without understanding the history of the eugenics movement and standardized testing [4], teachers may not understand or be able to articulate to parents and others why such exams create inequities by race, gender, and socioeconomic background. However, teacher candidates (TCs) are not likely to develop political knowledge by simply being "on the job"; they must be actively supported to do so [5–7]. Unfortunately, most teacher preparation programs are not designed to prepare TCs to recognize or navigate the politics of teaching and learning mathematics or to dismantle oppressive systems within mathematics education, even though this work has been clearly articulated as central to mathematics teacher learning [8]. As such, TCs are underprepared to take on this work once they are in full-time teaching positions, and many find it difficult to resist the pressures of their schools to "teach to the test" and "cover standards" [5,9].

Our research asserts that there is parallel work involved for mathematics teacher educators (MTEs) to learn how to dissect and navigate the politics of teaching to successfully support their TCs to do the same. That is, even when individual MTEs are committed to supporting TCs' development of political knowledge, it is not clear that they have the full support of their TCs [10], their program [11], or sufficient expertise to carry out this work in ways they intend [12]. For example, featuring the kinds of dilemmas that arise for nineteen MTEs in contexts across the nation as well as fifty-seven commentaries from additional MTEs, White, Crespo, and Civil highlight the ways that race and identities arise for MTEs and how they grapple with equity- and justice-oriented dilemmas [13]. While these dilemmas are raised in the White et al. book, and their associated microaggressions (for MTEs and TCs) are evident, the authors do not include overarching guiding questions, frameworks, or resources for MTEs in doing this work. This is one example that motivates the need for explicit tools to help MTEs both prepare for these dilemmas and make sense of dilemmas that are unique to themselves and their settings [14]. In fact, a significant gap persists in understanding how MTEs' intersectional identities and contexts are intertwined as they plan, implement, and reflect upon experiences designed to cultivate TCs' political knowledge.

We take the position that our community cultural wealth as three women of Color scholars [15], our experiences navigating teaching and teacher education, and our commitments to justice-oriented work in the highly politicized context in the United States combine to offer a unique lens for our current research project that focuses on developing political knowledge with K-12 TCs. Because the United States does not have national standards nor curricula for P-20 education (preschool through post-secondary education), aside from Advanced Placement and International Baccalaureate courses, each of us draws on the 2017 Standards for Preparing Teachers of Mathematics, as articulated by the Association of Mathematics Teacher Educators (AMTE), considered to be the leading United States professional organization for MTEs. Given this, we identify several themes and questions that MTEs might attend to as they endeavor to support TCs in promoting equitable teaching (AMTE Standard C.2.1), understanding the history of power and privilege in mathematics education (AMTE Standard C.4.4), and advocating for themselves and their students (AMTE Standard C.4.5), all of which we see as elements of cultivating TCs' political knowledge for teaching mathematics in the United States.

This article offers an identity and context compass we call A Compass for Preparing Teacher Candidates with Political Conocimiento, which we later discuss and present in the article, based on themes that emerged through the analysis of our research planning meetings, analytic memos, and reflection sessions. We offer examples from our data to illuminate how we attended to identity and context as we planned, implemented, and reflected on our research experiences with TCs. We conclude this article by articulating implications for the field to consider as we continue to understand how intersectional identities and contexts impact our planning, implementation, and reflection on political knowledge development for TCs.

2. Theoretical Framework

Political Conocimiento in Teaching Mathematics (PCTM) is a theory that helps explain the kinds of interlocking knowledge bases necessary for teachers to be effective, especially with students who have been historically marginalized by schooling [16,17]. Building upon Pedagogical Content Knowledge (PCK), which combines knowledge of content (mathematics), knowledge of pedagogy, and knowledge of students [18,19], PCTM does not merely add on "politics" as a fourth knowledge base; it reframes the way we think about knowledge itself [16]. That is, rather than viewing knowledge for teaching as objective/universal—something that one accumulates and then applies to one's context, thereby separating it from affective domains, like beliefs, dispositions, emotions, and ideologies—PCTM positions knowledge as relational, embodied, based upon experience, and co-created with others. This framing captures the fact that knowledge is always subjective/situational and negotiated with others. As such, knowledge of mathematics is considered knowledge "in connection to" the sanctioned version of mathematics; knowledge of students is knowledge "with" students and their home communities; and knowledge of pedagogy is experiential knowledge of specific pedagogies with others.

Moreover, PCTM assumes that having knowledge of mathematics, pedagogy, and one's students is not sufficient without knowledge of the politics that can arise from making particular choices in one's practice; the four knowledge bases interlock and operate within the histories of society and the comrades with whom we engage in El Mundo Zurdo (the left-handed world of activists) [20].

Within the framework of PCTM, "politics" means power dynamics that arise in interactions (e.g., with colleagues, administrators, and families) and in response to structures, where teachers feel pressure to follow authority (e.g., abide by policies). For example, a teacher who advocates for de-tracking might be met with resistance from those who benefit most from inequitable tracking practices (e.g., wealthy families of dominant backgrounds). Similarly, MTEs who advocate for racial healing within mathematics might be met with resistance from TCs who are mathematics majors and believe the system of school mathematics should maintain a focus on rigor, content, and logic, as those tenets have produced "good, hard-working people" like themselves. Second, teachers are not prepared to "question existing educational systems that produce inequitable learning experiences and outcomes for students", such as "personalized" computer-based learning, often framed as "equity-oriented", but may exacerbate inequities. Third, policies, such as teacher evaluation tied to students' standardized test scores, create pressures for TCs and teachers to teach to the test, rather than using rich tasks that promote long-term understanding and are sometimes not aligned with such standardized tests. In other work, we offer a heuristic for TCs to use as they work to deconstruct the narratives and politics in play in mathematics teaching and learning [21]. Here, we assert that MTEs face similar pressure(s) to teach to assessments (e.g., Educational Teacher Performance Assessment, abbreviated as edTPA) that are required by state and national licensing bodies [17] and need specific tools to support them as they engage in political work.

To prepare TCs to navigate such politics, MTEs must also be supported to develop this knowledge themselves, in partnership with their TCs. Rather than acquiring a predetermined set of skills, PCTM focuses on a teacher's way of being, holding oneself and those with whom one interacts accountable to a set of principles that promote justice-oriented mathematics learning [16] and support human flourishing [22]. We use the "Mirror Test" [23] to refer to the ongoing vigilance with respect to holding oneself and others accountable to these principles. That is, rather than looking to external entities, such as the Danielson Framework [24], edTPA [25], or student test scores to decide whether one is an excellent teacher, MTEs and teachers need to develop the ability to look internally and hold themselves to a higher ethical standard. For further elaboration about the Mirror Test, see [26]. In turn, we ask ourselves, as three critical women of Color MTEs, if our actions reflect our mirror tests.

3. Methods

In writing this article, we engaged in a deep examination of our experiences as three women of Color scholar–activists working with primarily white TCs to foster PCTM. We used narrative inquiry [27,28] as a methodology and analytical tool in our research, as it allowed us to capture the multiplicity of our experiences engaging in justice-oriented work with TCs in ways that foregrounded our stories, lived experiences, and collaboration across spaces. The uniqueness of this method is that it allows us to think about the process that women MTEs of Color engage in with each other and with our TCs to understand better how our intersectional identities manifested as we planned for, facilitated, and reflected upon four PCTM tools.

4. Narrative Inquiry

We chose narrative inquiry for two primary reasons. First, narrative inquiry values stories and storytelling as a way to gain insight into how people live and make sense of their experiences [29]. Our research team met monthly during the fall 2020 semester and then weekly for three to four hours during the spring 2021 semester to discuss the planning, implementation, data collection, and reflection on our PCTM tools. Data for this article were derived primarily from our planning meetings held in the spring of 2021. Much of this discussion and planning entailed telling stories about our contexts, our interactions with TCs, personal experiences regarding the COVID-19 pandemic, and the changing political landscape. Narrative inquiry as a framework allowed us the opportunity to interpret ourselves, our stories, our scholarship, our current research project, and the mathematics methods courses (required mathematics pedagogical courses for state licensure) through our past and present experiences of preparing ourselves to best prepare our TCs' development of political knowledge.

Next, narrative inquiry moves beyond a simplistic analysis of the story as if the story or narrative is an object to be analyzed. Instead, narrative inquiry situates the researcher as thinking with stories to allow the researcher to consider how they act upon the story and how the story acts upon them [30]. As researchers, we constantly considered how we were making sense of the tools alongside our TCs. That is, the tools are not "finished", just as we and our TCs are not "finished" in our learning and development. We are constantly in motion and drawing upon each other to remake ourselves through story. As such, what it means for us to be MTEs of Color is being co-constructed with our TCs, who are developing meaning around what it means to be a teacher of mathematics, given their intersectional identities. This element of narrative inquiry is particularly salient for us as it presents a direct connection to the "C" in PCTM, conocimiento, which situates knowledge as existing with students and communities instead of knowledge of students and communities [17]. In this way, we see community and solidarity as core tenets of narrative inquiry and conocimiento. Therefore, we, as three women of Color scholars, *live* our stories. We consider our positionality and how our contexts shape us and our stories. We also highlight the importance of how our stories exist with each other in the community of our research team and the broader field of scholars of Color engaging in this work and how those stories reflect the ways we continue to develop our ability to prepare our TCs with PCTM.

5. Researcher Identity Statements

5.1. Author 1: Marrielle Myers Is Also Known as "A Southern Belle Turned Southern Bull"

I identify as a Black, cisgender woman who does not live with any disabilities. I grew up in a middle-class home in the South and was raised in a Christian family. I was taught to "respect" authority, and part of that meant not asking questions even when I did not understand. This led to me internalizing many of the narratives teachers had about me at a young age (e.g., too talkative or bossy), even though I disagreed with them. So, I did what I was told and experienced "success" in school. I was so indoctrinated in the system that it was not until I was teaching in a Title I high school that I realized I was actually reproducing the same structures that created my suffering. I remember choosing to teach at that school because I thought "those students" needed me. I was so wrong. I needed them. My students facilitated my growth and liberation. My high school teaching experience, coupled with experiences in my graduate program, helped me understand that both marginalization and privilege existed within me. I find myself constantly trying to understand my intersectional identities better and unpack my journey while simultaneously creating experiences for my TCs to engage in this work. In many ways, I see myself as vastly different from my TCs, yet when I think back to my upbringing, I am faced with the fact that we may be more similar than I am ready to admit. Recognizing our similarities gives me hope that change can occur. Yet, I am aware that my TCs may miss the message because I am the messenger. So, I am left wondering how to express my commitment to disrupt and dismantle the system that causes harm to so many with enough "Southern sweetness" to make it palpable to my audience.

5.2. Author 2: Kari Kokka Dancing with and around the Insidious Nature of Whiteness

I identify as a fourth-generation Japanese American cisgender nonqueer woman who does not live with a dis/ability. My family's unconstitutional incarceration experience during WWII and witnessing inequities in my public high school (e.g., fueled by standardized testing, tracking, lack of supports for all students, etc.) in San Jose, California, with approximately 90% students of Color, fuel my commitment to justice in education. I was a teacher activist, mathematics teacher, and instructional coach for ten years in a Title I New York City public high school with over 90% students of Color. I often feel the need to assert my identity as a woman of Color, given how white supremacy positions Asian Americans in proximity to whiteness. For instance, in my first institution as an assistant professor, I found that colleagues did not understand that I identify as a person of Color and that I grew up with and worked amongst people of Color, and I was often viscerally uncomfortable in the predominantly white space of the university. When I reflect on my work in the predominantly white institution of my previous position, where I was employed when writing this article, I realize that I may have centered whiteness by preoccupying myself with white people's feelings and discomfort when exposing my TCs (and colleagues) to interlocking systems of oppression, e.g., white supremacy, cisheteropatriarchy, imperialism, capitalism, ableism, etc. My inclination to center whiteness may have been influenced by my role as a pre-tenure assistant professor and because I understand that the institution is an actualization of whiteness and that my (white) colleagues would be determining my tenure case and reading my (white) students' evaluations. I tended to overly preoccupy myself with my teaching evaluations because my identity is wrapped up in being a teacher. Being a high school math teacher was my most fulfilling, rewarding professional experience, and I am still in touch with my high school students. I often dream about engaging in future research with high school student co-researchers where we can build meaningful, lifelong relationships.

5.3. Author 3: Rochelle Gutiérrez Is a Nepantlera Working to Embrace Her Contradictions

I identify as a Xicana, cisgender, nonqueer, white-passing bilingual woman with Rarámuri roots. I am also not living with a dis/ability. Growing up in an activist family on Muwekma Ohlone lands, my father, Rubén (an electrician), taught me the power of union organizing, and my mother, Josefa (a previous farmworker, turned stay-at-home mother, turned community college counselor), modeled creative insubordination and how to advocate for others' rights. As a secondary mathematics teacher in a Title I school, I sought to challenge and embrace my primarily Latine and Black students. I wanted them to bring their full selves to the mathematics classroom, something I did not experience while in the mainly white "gifted program" of my K-12 years. But, I struggled with that kind of teaching, as I had lived most of my life code-switching (home/school, English/Spanish, cultural significance/academic rigor). I knew how to do one or the other, but not both simultaneously. Only later in life could I embrace my contradictions (privilege/oppression) to weave together the multiple worlds I belong to and become a nepantlera. In working with TCs, I do not always admit to sharing their privileged stances (e.g., HS math team member, knowing songs about SOH-CAH-TOA), mainly because these are painful parts of my schooling. Instead, I often expect them to drink through a firehose in learning abolitionist teaching and Rehumanizing Mathematics [31] approaches because of my sense of urgency about addressing the injustices and trauma that many historically oppressed students experience in school. Yet, it took me decades to get beyond my code-switching ways toward more translanguaging. In embracing my contradictions, I aim to empathize with my TCs and recognize they have their lives ahead of them to grow into the kind of teacher they want to be, as political clarity does not come in a semester or year.

6. The Tools

We used four scenario-based tools developed and piloted by the Political Conocimiento Development Tools (PCDT) team at the University of Illinois to support TCs in cultivating

PCTM. Funding for the tool development was provided by the University of Illinois in the form of a Campus Research Board grant (Political Conocimiento Development Tools) awarded to Rochelle Gutiérrez. Graduate student members of the development team included: Alexandria Cervantes, Theresa Dobbs, and Shafagh Hadinezhad. The suite of tools captures various elements of political knowledge and has multiple entry points. The tools (named in parentheses) help TCs: (1) identify aspects of equity in their workspaces (How Would You Classify It?) based upon four dimensions [32,33]; (2) consider how certain scenarios might feel riskier to deal with than others (Difficulty Sort); (3) reflect on how some scenarios feel similar and might lead to general approaches for student advocacy (Similarity Sort); and (4) prepare for and rehearse the kinds of politics they might face in teaching so that they can be strategic in their actions (Mapping and Rehearsal), which might require creative insubordination [34,35]. Using the tools places TCs in somewhat uncomfortable spaces, as they need to reflect on their identities, the kinds of microaggressions that have repeatedly harmed students (and perhaps themselves), and the knowings (e.g., mathematics, pedagogy, students, politics) they carry or with which they are familiar. Engaging with these tools raises issues for them to consider whether they actually act upon what they say they stand for in terms of their teaching and their commitments to Black students, Indigenous students, and students of Color, as well as other historically marginalized students and teachers. Elsewhere, we provide empirical results of TCs' PCTM cultivation [36]. Here, we focus on how we, as women of Color scholars and MTEs, are placed in the position of needing to navigate our identities, contexts, and positions with our TCs when using the tools and co-constructing knowledge with our TCs.

7. Context, Data Sources, Collection, and Analysis

This multi-site study occurred at three different teacher preparation programs in various locations (East Coast, Midwest, and Southeast). The three sites varied in size/enrollment, degree programs (e.g., B.A., B.S., M.A.T.), grade-band focus (elementary, middle, and secondary), and commitments to antiracist principles throughout the program. Our research team engaged in monthly meetings during the fall 2020 semester and weekly three-hour meetings during the spring 2021 semester.

Data Reduction and Analysis

We began our data analysis process by listening to over 24 hours of recorded Zoom meetings, reading transcripts of the recordings produced by Otter.ai, and reviewing Google document files that we used to take shared meeting notes. Next, we identified transcript segments that focused on planning or reflecting upon using the four tools. We coded these segments using open coding to identify emergent themes in our conversations [37]. This process led us to identify the following themes: MTE identities, TC identities, the Mirror Test, Rehumanizing Mathematics [31], programmatic connections, creative insubordination, current climate, and MTE pedagogy.

Next, we created data tables where we organized the transcript segments by tool, listed which of our voices were present in the segment, and recorded the results of our initial round of open coding. We then sorted the segments into two groups: teaching and tool implementation versus research/scholarship considerations. An example of a quote that incorporated discussions of our identities and context but focused on research and implementation was when Kari noted that we should modify the interview protocol to talk about ourselves before asking our TCs to talk about themselves, stating, "Oh, just that I always felt awkward. I just feel a little extractive when I'm like, 'Oh, tell me all these things about yourself. And I'm not going to share anything about me'". While Kari noted that doing this was important to humanizing the research process, this particular quote was not relevant to how MTEs consider their intersectional identities and contexts in planning classroom activities; rather, it was a methodological consideration. This sorting process left us with eleven transcript segments, which we then coded again.

The second round of coding allowed us to collapse and categorize themes under four larger headers (intersectional identities, Rehumanizing Mathematics teacher education, MTE Mirror Test, and contextual considerations). For example, we combined MTE identities and TC identities into a broader code of intersectional identities. We also collapsed programmatic considerations and the current climate under contextual considerations, as that code was more expansive and honors the range of contextual concerns (e.g., program, grade band, regional location, political climate, virtual learning) that MTEs should consider when engaging in this work. This process resulted in creating a compass that highlights the complexity of our work as women of Color scholar–activists seeking to develop PCTM with TCs. In the next section, we frame, present, and offer robust examples of our stories to illuminate the compass.

8. The Compass: Braiding Our Multidimensional Work

Because there is a gap in understanding how MTEs' intersectional identities and contexts are intertwined with their political work, we sought to develop a framework of considerations for MTEs to foreground the influence of context and intersectional identities. Our framework for considerations, which we call A Compass for Preparing Teacher Candidates with Political Conocimiento (see Table 1), characterizes how we engage in our work and includes four strands (or dimensions): intersectional identities, Rehumanizing Mathematics teacher education, MTE Mirror Test, and contextual considerations. Within each dimension (strand), we offer empirically derived questions for MTEs to consider as they engage in justice-oriented work with TCs. It is important to note that although these strands are presented in a linear format, we do not assign any rank order of importance based on the order in which the dimensions are presented.

Table 1. A Compass for Preparing Teacher Candidates with Political Conocimiento.

	MTEs	TCs
Intersectional Identities	a. What are my intersectional identities? b. How might my intersectional identities influence how I engage with the tool (e.g., understanding scenarios related to being an Asian American woman, living with autism, etc.)? c. How might my intersectional identities influence the ways I facilitate this experience for TCs? Specifically, how will my identities and TCs' identities influence how they perceive me, the course and its content, PCTM, the tool, etc.? How might I modify the tool delivery (e.g., racial affinity grouping) for my specific TCs?	a. What are my TCs' intersectional identities? How do I know this, or how do I respectfully gather this information? b. How might my TCs' intersectional identities influence how they make sense of the tools? c. How might my TCs' intersectional identities influence how they make sense of the tool in relation to other people (e.g., being open-minded, hopeful, resistant, skeptical, worried, etc.)?
Rehumanizing Mathematics Teacher Education	a. How will I care for myself in this experience with my TCs (e.g., recognize and attend to my emotions and needs) without disrespecting TCs' emotions? b. How will I use a healing-informed approach for TCs and myself (when using the tool, particularly considering how the scenarios may be triggering for me)? c. What resources, colleagues, comrades, friends, networks, and support systems do I have?	a. How will I care for my TCs in this experience (e.g., who may feel (re)traumatized or who may be of dominant backgrounds and use emotions, such as tears, to center themselves)? b. How might I involve my TCs' perspectives? Whose perspectives might be centered versus marginalized? c. How can I create spaces to learn about and with my TCs in authentic ways? What challenges might my TCs be facing, and how can I support them?

Table 1. *Cont.*

	MTEs	TCs
MTE Mirror Test	a. How do I make sure I am consistent with the principles I value and the communities with whom I hold myself accountable? b. What do I do when promising or problematic perspectives and/or actions of TCs arise? c. How will I recognize and manage tensions in this work? d. In what other ways might I work toward my justice goals?	
Contextual Considerations	a. How might my context support and/or constrain me, my TCs, and my work with them (e.g., my institution's policies and requirements for licensure, student teaching school placements, and cooperating teacher relationships)? b. How might the political climate in my country, state, city, or region (e.g., Black Lives Matter movement, Capitol insurrection, elections, police brutality, Critical Race Theory bans, teacher surveillance) influence this experience?	

As we share examples in the next section, we highlight how we are constantly holding multiple strands (or dimensions) of the compass simultaneously. We liken this holding and constant tension to braiding hair [38]. When you braid or plait hair, you begin by gathering strands of hair (see Figure 1). For us, these strands are the four dimensions of the compass (intersectional identities, Rehumanizing Mathematics teacher education, MTE Mirror Test, and contextual considerations). Once you start the process of braiding, it is imperative that you hold each strand until you complete the braid. There is never a time while braiding that your sole focus is on one strand such that you disregard the other pieces. The level of focus or priority may shift slightly as you grab one strand and thread it in with the others, but if you do not maintain tension with all of the strands (dimensions), you will lose the braid. Likewise, MTEs should consider what it means to consider intersectional identities, Rehumanizing Mathematics teacher education, their Mirror Test, and contextual considerations such that all strands remain engaged, despite being moved from the foreground to the background depending on the particular tool or experience. The strength and beauty of the braiding process—in this case, in cultivating TCs' PCTM—depends on the MTE's commitment to acknowledging and maintaining tension with these four strands.

As three women of Color scholars who are acutely aware of our positionalities as we engage in political work, we offer this compass as a framework of considerations that rejects knowledge of how to prepare TCs' political knowledge (e.g., read, plan, do) and centers knowledge with our contexts, mirrors, and TCs. As three women of Color scholar activists, we must constantly explore and interrogate our stories and intersectional identities while always seeking to maintain tensions, as articulated through the braiding analogy. In our identity statements, each of us briefly explained our personal herstories (we use the word "herstories" to center our experiences as women), connections to, and tensions in this work. Additionally, we described our commitments to justice-oriented teaching that foregrounds antiracism. These commitments are our mirrors. And, in doing this work collaboratively, we are each other's mirrors. The examples that follow highlight the complexity of this type of work and how we, in our work, embrace and maintain tensions in the different strands.

Figure 1. Braiding. We honor the creativity, brilliance, and power that Monique "Mo Thunder" Bedard shares via their artwork. We presented an abstract of this paper to Mo and requested permission to use this image in our piece, as this image not only captures the concept of braiding but also Indigenous wisdom and stars, which are critical to the identity of our research team. Mo graciously accepted our request. We are honored to be in community with Mo and to share their artwork [39].

9. Examples from Our Work

Our tools, which focus on broader narratives around historically marginalized learners and teachers taking risks in an increasingly toxic social and political climate, made us highly conscious of our racialized and gendered identities. Because this work is inherently complex, and we construct knowledge with our TCs, you will notice several strands of the compass that emerged throughout our discussions. As we unpack our quotes, we support the reader in drawing connections by bracketing the relevant strands of the compass.

9.1. Gaining TC Buy-In

Each of us considered several factors when planning to use the tools. Kari often considered the whiteness of her institution (where she worked at the time of writing this article) in relation to her positionality as an Asian American woman. When she began her professorship there, she was the only faculty member of Color in teacher education, with all-white secondary math TCs and a few TCs of Color in her elementary mathematics methods class. Because white supremacy positions Asian Americans in proximity to whiteness, Kari felt that her TCs may perceive her as "less threatening" than if she were Black, Brown, or of another marginalized group of Color. Kari danced with and around whiteness, centering the essential question, "What actions can I take as an antiracist/abolitionist teacher?" for her courses. Other times, Kari feared (white) backlash and therefore softened her pedagogical goals for the comfort of her (white) students. Rather than using terms like "backlash" or "risk-taking" when discussing advocating for students, Kari used "pushback" and "navigation of dilemmas". In this dance, she privileged the strands of TCs' identities and rehumanizing teacher education for them in negotiation with her Mirror Test.

Rochelle also grappled with when to foreground language about abolitionist teaching, Rehumanizing Mathematics, or creative insubordination. Rochelle's students had already had her as the instructor of their social justice foundations course where politics were front and center, and they came from a range of backgrounds, not all intending to stay in education. But now, these students were mathematics majors in a methods course, obtaining licensure and a minor in education. In one of our meetings, Rochelle explained how she framed the work to her students:

> "I will expose you to perspectives you haven't had access to". Our purpose as MTEs is not to present a list of teaching practices. There is no destination or prescribed way to get there. It's more about, "Do you pass your Mirror Test? Are you doing the right thing for students? If not, what are you going to do about it? In our foundations classes, we learn about the difference between education and schooling. Are you educating, or are you schooling? What does that mean for the students you teach?" Our job as MTEs is like being a good parent is to put healthy food in front of our TCs, but we can't force them to eat it.

As a Brown woman who has taught these students a course 1–2 years earlier where her activist politics were front and center [MTEs' intersectional identities], Rochelle considered how her identity needed to be braided with her TCs' identities, recognizing that they have already had exposure to some social justice ideas, but they may not have made explicit connections as mathematics majors [contextual considerations]. Rochelle drew upon language that softened the idea that she might be trying to "indoctrinate them". By framing it as a Mirror Test and no destination, Rochelle was able to ask, "Are you educating, or are you schooling?", which perhaps might feel less threatening. The strand brought forward is the identity of the TCs, though Rochelle's intersectional identities and Mirror Test were also at play.

Marrielle taught a K-2 math methods course that focused on teaching mathematics for social justice, multilingual learners, and addition and subtraction problem types. Because of previous TC feedback and course evaluations indicating that some course activities "were not mathy enough" and that the professor was too focused on being a "social justice warrior", Marrielle struggled with how to position herself and her commitment to developing PCTM. These concerns, coupled with preparing elementary TCs who sometimes think that K-5 classrooms are "sunshine and rainbows", create unique challenges for conducting justice-oriented work in elementary grade bands. Marrielle described how she communicated her approach to TCs:

> My approach seems to be more providing my TCs with food for thought, looking with different lenses. An offering. And while I recognize that my TCs will do what they want to do when they go into schools, I hope they will put these lenses (e.g., anti-racist teaching and advocacy) on and carry them with them. My former colleagues and grad school professors offered things for us that we didn't know we needed and we realized we could have been doing things that were harmful. Even today, we (MTEs) continue to have dilemmas.

Marrielle is holding several strands of the compass in this excerpt. As a Black woman who has been previously accused by TCs of having an "agenda" [MTE intersectional identities], Marrielle, too, softened the language against indoctrination by noting that she will offer several lenses for TCs to consider throughout the semester. Second, Marrielle connected her experiences to the TCs' experiences by indicating that we do not always know what we need as novices, and that she continues to face political dilemmas as a professor that require the use of specialized knowledge and creative insubordination [Rehumanizing Mathematics teacher education and contextual considerations].

These examples highlight how the authors softened their desire to want TCs to "drink through a firehose" related to developing PCTM. While the sense of urgency is still there, these quotes highlight a balance between wanting to relate to students (e.g., we all face dilemmas) and wanting to honor TCs' agency (e.g., we are providing choices but not forcing anyone), while also advancing elements of our teaching philosophies (e.g., Mirror Test and different lenses).

9.2. Considering Our Identities and Contexts in This Moment

The next excerpt is from a conversation where we were reflecting on engaging our TCs with How Would You Classify It? In planning for this tool, we each reviewed a set of scenarios that Rochelle used in previous iterations of this work. We used ten identical scenarios across our three sites, and we each created two additional scenarios unique to our grade levels and local contexts. The scenarios included Indigenous students, Black students, and students of Color viewed in deficit or harmful ways, such as teachers or administrators discounting their perspectives, their abilities in mathematics, or their funds of knowledge. In the conversation that follows, Rochelle initiated a conversation with our research team about the potential impact of our current climate on our work, saying:

> Here's my question. . .does doing these kinds of scenarios, and with everything else that's going on with Black Lives Matter for example, is it making some of our students who wouldn't normally be conscious of, the kind of role of this extra violence that's happening when people have to relive these things? Or when people are being reminded of these things? Is there something about this moment, that's also influencing how TCs are making sense of the scenarios, separate from just, how do pre-service teachers make sense of these scenarios? We have individual students (TCs) who've had those traumas.

Here, we see that Rochelle highlighted the world context [contextual considerations] and especially needing to care for her TCs of Color [rehumanizing for TCs] who might be triggered by the violence when reading the scenarios. Marrielle built on the notion that the current context may be causing more people to reflect on the moment and added:

> Our students are also having to navigate their own positions at this moment. Some of them are trying to do that and think about "what is this going to mean, for me as a teacher, right, how do I handle this for myself? And how do I handle this as a teacher?" Whereas before, I don't know that some students have even had to think about where they stand on some of these issues, right? Because. . .depending on who you're friends with on social media. . . it's pretty feasible that you really hadn't heard of these social issues, but now. . .if you're paying any level of attention to any news source. . .you don't really have much of a choice. And, so I think, there is this level of just heightened awareness. I had a student say, "Can I ask something?" And he said, "I'm concerned that I'm a white male, and I know that I have privilege and haven't experienced other things people have experienced", and he said, "I'm scared, like, what if I get this wrong?. . . At first, I. . . didn't necessarily see this as my role, right? Now I'm thinking, Okay, this is something I should be doing. But what if I messed it up?"

Marrielle continued to consider the impact of the context of "heightened awareness" around racial justice (e.g., Black Lives Matter movement), maintaining tension with the strand of [contextual considerations]. She then connected context with how TCs are navigating their positions as individuals and teachers and more broadly in society, thereby bringing forward the strand of [TCs' intersectional identities]. Rochelle responded, saying:

> What does it mean TCs to be in this moment, trying to figure things out, maybe for some of them feeling like they don't want to mess up or get it wrong, in front of three women of Color who are running these activities, right?

In this excerpt, Rochelle reminded us of our original question of what it means to engage in political work in this moment [contextual considerations] and for TCs to be grappling with the notion of "doing this work the right way" [TCs' intersectional identities], thus highlighting the tension between how TCs see themselves in this work at this time. Rochelle highlighted that the tensions TCs were feeling could have been exacerbated because they were navigating their identities in front of women professors of Color [MTEs' intersectional identities], thus bringing the strand of our intersectional identities to the fore. This was important to note, given that most of the TCs we worked with across all three sites were white. Kari built on this notion of the whiteness of her teacher education program and what this means for her work with TCs and stated:

> All of the TCs in my class are white. And my institution has this tendency to place them in all white schools. All of my TCs grew up in only white communities. And they have been very thoughtful around what is my role as a white educator? And I think the trickier thing is, what is their role as a white educator who works with white students? And so, I've included some readings around the Rethinking Ethnic Studies book so they've been thinking a lot about, how do I do social justice pedagogy or culturally relevant pedagogy with white students?... In terms of "messing up"...they don't want to culturally appropriate [...] So we've been having similar conversations.

Because Kari's TCs did not have foundations classes in their program, she supplemented her methods course with additional readings [contextual considerations]. While Marrielle and Rochelle were considering their social and political contexts, Kari's use of contextual considerations highlights her attention to the whiteness of the space along with the structure of the teacher education program, which represents how her localized context became a dominant strand here. Like the white male TC in Marrielle's class who noted he was scared to be "wrong", Kari's students were also worried about "messing up", but they were more specific and used the language of cultural appropriation [TCs' intersectional identities]. This distinction is important, as Kari's TCs were able to name that phenomenon because of her commitment to preparing TCs to understand culture and privilege even in the absence of a foundations course [MTE Mirror Test]. Rochelle echoed this statement by noting that her TCs recognized the need to be careful when engaging in justice-oriented work, given that they did not come from the same cultures as their 6–12 students [TCs' intersectional identities].

10. Implications

In the examples we presented above, we highlighted multiple dimensions of the Compass for Preparing Teacher Candidates with Political Conocimiento (See Table 1) and how various strands were foregrounded or backgrounded as we engaged in our work. Although the questions in each dimension are offered as a list in the compass (See Table 1), our examples dispel the notion of linearity and underscore the tensions that are raised and maintained as we braid these various strands together. For example, we encourage MTEs to consider the questions from the intersectional identities strand: "What are my intersectional identities?" and "What are my TCs' intersectional identities? How do I know this, or how do I respectfully gather this information?" We also encourage MTEs to simultaneously consider questions from the Mirror Test strand, such as, "How do I make sure I am consistent with the principles I value and the communities with whom I hold myself accountable?" These questions from the compass, in addition to those in the other dimensions, aim to support MTEs' work to consider multiple questions simultaneously, maintaining the tensions that hold the braided complexity of this work together.

We argue that attending to the questions in the compass is essential if MTEs are going to be prepared to develop mathematics TCs to engage in justice-oriented pedagogy by promoting equitable teaching, cultivating positive mathematical identities, drawing on students' mathematical strengths, understanding power and privilege in the history of mathematics education, and enacting ethical practice for advocacy [8]. Gutiérrez asserts that conocimiento, or political knowledge, is not objective or concrete [16]. Conocimiento is subjective, and each person has their own relationship, lens, or ways of knowing a construct. Conocimiento represents knowledge *with* students, schools, and communities. As such, we did not treat the tools, scenarios, or implementation as fixed. Instead, we treated each of the scenarios in their current form as entry points. We read, evaluated, and revised scenarios before using them to account for our intersectional identities, TCs' intersectional identities, how the scenarios allowed TCs to make assumptions or "read" them through their lenses, as well as individual contexts. We also considered how our intersectional identities and contexts influenced how we presented the tools, how we were perceived by our TCs, and what TCs felt "safe" to share during the activities.

If we do not think about and attend to all of the ways we are entangled within each other's lives as teacher educators, teacher candidates, and teachers, we miss the opportunity to co-construct new forms of knowledge that are appropriate for our contexts. Furthermore, without being intentional and maintaining tensions with the strands in the compass, MTEs (including ourselves) run the risk of implementing critical work in ways that uphold white supremacy and capitalism (e.g., preparing TCs to challenge explanations of gaps that occurred during remote learning but still citing "learning loss" as the problem) instead of preparing teachers to dismantle oppressive systems and advocate for students.

We urge MTEs to consider how we (as MTEs) change as we engage in our work. MTEs should expect to be challenged, grow, and change as much as they seek to foster growth and change with TCs. Here, we return to the braiding analogy to highlight that one way MTEs can engage in going deeper into themselves is to consider the four dimensions of the compass. We invite MTEs to consider if their justice-oriented work with TCs includes braiding. Moreover, we argue that it is critical for MTEs to consider which strands of the compass they may privilege, move to the background, or ignore as they are braiding and how this impacts their recreation of self and their Mirror Test as they engage with TCs. This process of change and transparency can be challenging, and we encourage the field to consider the emotional labor required to do this work [40], how we care for our healing and emotions as we engage [41,42], and that this process is integral to rehumanizing the research experience for MTEs and TCs.

11. Closing Considerations

We acknowledge and center tensions that emerge in our work. While we offer this compass for MTEs' work, we also encourage MTEs to consider the strengths, limitations, and implementation challenges of the compass in relation to their own intersectional identities and contexts. For example, we created and used this compass as a team of three women of Color scholar activists with different intersectional identities and varying contexts. In doing so, we were able to recognize how our unique contexts (e.g., grade level bands, foundations courses, structure of student teaching) influenced what we had conceived of as "normative" and offered up options for framing things differently from the view of other contexts, such as considering teaching in another country where there is less emphasis on teacher surveillance. But what happens if an individual MTE uses the compass? Are there other sources of community that could support dialogue and learning from others' perspectives? For example, might Facebook, Instagram, TikTok, or #MTBoS Twitter/X posts help individual MTEs recognize the perceived constraints of their locations and intersectional identities and understand how teachers in other countries might offer creative ways to navigate those constraints? Furthermore, how might MTEs with additional intersectional identities (e.g., gender expression, sexual orientation, language, ethnicity or race, social class, caste, religion, immigration status, or living with a dis/ability, etc.) find value or drawbacks in the compass? Additionally, are there certain strands of the compass that MTEs should consider first (e.g., a scaffolded approach) as they engage in this work? The compass should be used and studied to understand other relevant dimensions of intersectional identities and global contexts more fully as MTEs and TCs work to collectively develop PCTM. By utilizing the compass and braiding (while maintaining tensions in the strands), we position the field to move forward by co-creating new knowledge around how MTEs' intersectional identities and contexts manifest in developing TCs' PCTM.

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education sciences

Article

Pedagogical Imagination in Mathematics Teacher Education

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Abstract: After providing a brief summary of what has already been said about pedagogical imagination, data are presented showing how prospective mathematics teachers can become engaged in such imaginations. With reference to this data, the notion of pedagogical imagination is explored further by relating it to dialogue, social justice, mathematics, hope, and sociological imagination. To illustrate these relationships, different episodes from the data are highlighted. Finally, the central role that pedagogical imagination can play in mathematics teacher education is discussed.

Keywords: pedagogical imagination; dialogue; social justice; mathematics; hope; sociological imagination; critical mathematics education

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The notion of imagination does not contain a well-defined semantic nucleus. One can talk about a child having a good imagination when making a drawing, about a criminal demonstrating imagination when inventing an alibi, and about a person showing imagination when cooking a delicious dinner.

To be used in the context of mathematics teacher education, the notion needs careful elaboration. It may help us to articulate what it might mean to do research from a critical perspective. It is important to study what is taking place, say, in a classroom setting, and to ground this research with careful observations. However, when doing critical research, it is equally important to identify alternatives to what is taking place. It is important to research possibilities. When doing so, pedagogical imagination plays a crucial role.

Pedagogical imagination is not only important for educational research; it is also important for developing educational practices. It is important to invite prospective teachers to look beyond the given curriculum and beyond what is presented in textbooks and to involve them in conceptualising teaching-learning processes that are not part of the school mathematics tradition. This way, prospective teachers might become prepared for assuming a role in changing educational routines.

In the following, we *first* provide a summary of what has already been formulated about pedagogical imagination. *Second*, we present data where prospective mathematics teachers were invited to engage in pedagogical imagination. *Third*, with reference to these data, we outline different features of the notion of pedagogical imagination. *Fourth*, we summarise the central role that pedagogical imagination could play in mathematics teacher education.

1. About Pedagogical Imagination

The importance of pedagogical imagination in mathematics education research was recognised in a supervision session that took place in 1996 in Durban. It was two years after Nelson Mandela had been elected president. The Apartheid period had come to an end, and South Africa had turned into a democracy.

Black and Indian PhD students participated in the supervision session; they belonged to the first generation of PhD students in mathematics education in post-apartheid South

Africa. The supervision of this group was organised through a South African-Danish project of cooperation in mathematics education, and Ole Skovsmose acted as the supervisor.

One of the principal issues for the supervision session was the planning of the data production for the PhD studies. There were many things that could be captured by the data. The Apartheid regime had a devastating impact on the educational system, imposing a strict separation between black, Indian, and white students. Although the apartheid period had come to an end, what can be referred to as the "topology of apartheid" continued almost the same. While laws can be changed overnight, material structures cannot. By the topology of apartheid, we refer to, for instance, the location of neighbourhoods, schools, workplaces, hospitals, and roads. As part of the apartheid policy, the centre of Durban (like in most cities in South Africa) was designated as a white area; the neighbourhoods for black people were positioned as isolated enclaves far away from the city centre, and the Indian neighbourhoods were located in between as a kind of buffer zone. The schools were distributed accordingly: the extremely poorly resourced schools were in the black neighbourhoods, the richly resourced schools in the white neighbourhoods, and the badly resourced schools in the Indian neighbourhoods. This topology remained even after the apartheid laws were abandoned.

It is crucial for research in mathematics education in South Africa to document how the topology of apartheid has had, and continues to have, a destructive impact on black and Indian students' possibilities for learning mathematics. However, the PhD students felt strongly that it was *also* crucial to identify alternatives to what had taken place—and what, to a great extent, continued to take place. They found it important to research not only what *did exist* but also what *did not exist* not yet.

But how can we research something that does not exist? What alternatives could one imagine? Would it be possible to research such alternatives before they are actually implemented? Such questions were addressed during the supervision session in 1996, and it was recognised that a *pedagogical imagination* might constitute a crucial component of research that does not only address what actually takes place in the classroom but also tries to conceptualise what could come to take place. One particular issue that the PhD students felt was urgent to explore was the multicultural classroom. But due to the topology of apartheid, no such multicultural classrooms were in sight. So, how does one research learning opportunities that might be established in a multicultural classroom?

Pedagogical imagination is not only relevant to the post-apartheid situation in South Africa; it is relevant to any situation where one wants to investigate alternatives to what is taking place. The general features of research possibilities have been presented in [1,2]. These features can be expressed by means of a triangle (see Figure 1).

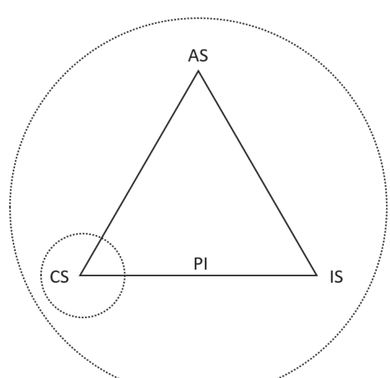

Figure 1. Features of research possibilities.

One corner of the triangle, CS, refers to the current situation. It refers to what is actually taking place in the classroom or whatever situation one is researching. It refers to what can actually be observed by a researcher who does not want to interfere with what he or she is studying. The imagined situation, IS, refers to what one could imagine taking place. This situation can be thought of as an idealised alternative to what is taking place. The arranged situation, AS, refers to what is possible to organise, considering the idealised alternative, IS, and the constraints set by the actual practical and organisational structures. Pedagogical imagination, PI, is a relationship between the current situation, CS, and the imagined one, IS. Pedagogical imagination is the process that brings us to think about possibilities and alternatives to what is currently taking place.

While naturalistic inquiry—such as that described by Lincoln and Guba [3]—and positivist research only focus on what is currently taking place, i.e., the current situation, CS, critical research explores the CS plus idealised alternatives, IS, and what can be managed in practice, AS. Pedagogical imagination is symbolised in Figure 1 by the side of the triangle that connects the CS with the IS. If one does not imagine anything different from the current situation, say, for methodological reasons, then the triangle collapses into one point, the CS. In this case, one is doing naturalistic inquiry or positivistic research. Doing critical research means researching the whole triangle. In Figure 1, the two different foci of research are indicated by two circles: the smaller circle with CS in the centre represents the focus of naturalistic inquiry and positivistic research, while the larger circle encompassing the whole triangle represents the focus of critical research. (In Chapter 16 in [4], pp. 211–221), one can find a discussion of doing critical research and its relevance for critical mathematics education.)

Vithal [5] applies a critical research approach to investigating the possibilities for learning mathematics in a multicultural classroom in post-apartheid South Africa. During the apartheid regime, the school system was strictly segregated, and this segregation continued in practice even after the apartheid system was abolished. To investigate multicultural classrooms meant researching something that did not actually exist but could be imagined as a future possibility. Vithal conducted her research applying the critical approach condensed in Figure 1. She studied what was currently taking place in educational practice, the CS; she engaged in formulating visions about the learning potentials of a multicultural classroom, the IS; and she managed to establish some temporary and heuristic forms of multiculturality where a few black students attended historically Indian schools.

Biotto Filho [6] applied the same critical research approach, although in a different context. His research focused on the learning of mathematics among a group of children from an orphanage in Brazil. It is important to recognise that the pattern for doing critical research can be applied in all kinds of contexts, but what can be conceptualised as idealized alternatives and what can be managed in practice is highly context-dependent. Helliwell and Ng [7] considered the role of pedagogical imagination in innovating in mathematics teacher education to address issues of sustainability. Carrijo [8] addressed the conditions for learning mathematics among immigrant students in Brazil, and in her presentation of possible landscapes of investigation that can facilitate interaction and dialogues between immigrant and non-immigrant students, she applies pedagogical imagination. Chapman [9] worked on mathematics teacher education, did not use the expression "pedagogical imagination", but only "imagination". For Chapman, many of the actions taken by future teachers had the aim of reflecting on concepts and initiatives already completed. In other words, reflect on the past without paying attention to the future. Imagination is a way of including the future in mathematics teacher education, allowing them to think of alternatives. The imagination process proposed by Chapman with 15 prospective mathematics teachers in Canada was based on three steps: imaging, imagining what, and imagining how. Allowing prospective mathematics teachers to reflect more on the future "can help them to be hopeful about the possibility of doing and making things better" (p. 86).

2. A Mathematics Teacher Education Practice

Pedagogical imagination does not only concern research processes but also daily-life classroom practices. For teachers, it is important to consider alternatives to what is actually taking place in the classroom. It is important to consider possible alternatives to the school mathematics tradition, which assigns special importance to the teachers' exposition of new topics and to the students' work with pre-formulated exercises.

In the following, we draw on data from [10], where prospective teachers worked in groups; they were asked to imagine some mathematics lessons for primary or secondary school students. Lima's study was guided by concern for inclusive education, and one of the conditions set for the prospective teachers' planning was that at least one student with a disability was present in the classes they imagined working with.

Here we concentrate on one of the groups of prospective teachers: Danilo, Denise, Isabel, and Kátia (all pseudonyms). The school they imagined working in was a state public school, which mostly took in students in situations of social vulnerability. The mathematics classes were imagined to use materials that are usually available in school. The classroom was conceptualised with ample physical space. There were computers that could be brought to the classrooms, but there were fewer computers than students. The classrooms were well lit and ventilated, but they were too small for the number of students. The desks were arranged in rows. There was a projector in each room. The school had an internet network for teachers. Each class lasted 50 min. The prospective teachers imagined that the school adopted traditional types of assessments: tests prepared by teachers as well as assessments prepared by the government. This imagination was built from the personal experiences of each of the prospective teachers.

The prospective teachers specified the class that they imagined working with as being a Grade 6 class from elementary school with 30 students aged between 11 and 13 years old. The students were lively, talkative, and curious. They asked questions about everything, even about what colour pen they should use when copying something from the blackboard. In the classroom, there were some small groups that did not separate. The prospective teachers imagined girls sitting in the first few desks and being very participative; they also imagined a group sitting at the back of the classroom not interested in participating.

Some students were specifically imagined. Bruno using aggressive language but still enjoying mathematics and participating in the activities. Thiago had a lot of difficulty with mathematics; he was nervous and afraid of making mistakes. Giovana had cerebral palsy with motor and speech difficulties; she wrote slowly and walked with crutches; she attended classes regularly and was very studious. Heitor was autistic and very good at drawing, which had not been encouraged at school; he did not have any friends; sometimes he had the urge to get up and walk. And Josué was blind, with his senses of touch and hearing being very developed; he read in Braille. Like Giovana, he sat at the front of the classroom and he liked to participate in the classroom activities.

For this group of students, the prospective teachers decided to create a sequence of three classes dealing with statistics. They should be introduced to data collection, sampling, mean, median, and the construction of tables and graphs.

We are going to present some episodes. (The original Portuguese version of the dialogue that we present is found in [10]. English translations of some of the dialogues are presented in [11].) The transcriptions we present are based on our translations from the original Portuguese versions. Our analysis of the dialogue is different from those presented in the previous publications, which concentrate on addressing problems related to inclusive education.) from the conversations between the prospective teachers Danilo, Denise, Isabel, and Kátia and the researcher Priscila. The episodes concern the articulation of their pedagogical imaginations. We are going to use these articulations in Section 3, "Features of Pedagogical Imagination", when we elaborate further on the very notion of pedagogical imagination.

2.1. Episode 1

To work on concepts related to statistics, the prospective teachers decided to consider an activity in which students would be working in groups. They planned for each group to receive a box in which each of the students should deposit a piece of paper with their shoe size written on it. Coloured balls of crumpled paper would be made available so that students could build a visual and tactile graph: for each occurrence of a shoe size, a ball would be glued in the column in the cardboard diagram referring to this number. The dialogue below shows concerns about including the blind student.

***Denise**: We have to think about the blind student.*

***Kátia**: Maybe we could make the axes with glitter paper, because it is very rough. We could also, on the axis that has the shoe sizes, cut out the digit in EVA (Ethylene-vinyl acetate) with glitter that he could feel it.*

***Danilo**: Yes, we could think of using an EVA with glitter and one without glitter to differentiate.*

***Kátia**: And so, we can do the same for all groups, because it is more beautiful. Furthermore, we do not differentiate only one. More attractive. More colourful.*

***Denise**: The good thing about EVA is that it is a cheap material. If it is not available at the school, one can buy it.*

***Priscila**: Yes. But one thing I kept thinking...You talked about cutting out the digits in EVA for the blind student to read.*

***Denise**: Wow, guys! But is this how he reads?*

***Kátia**: I think so. I have seen ready-made EVA digits for sale. I bought some the other day; they are cheap.*

***Denise**: But Kátia, it is not Braille.*

***Danilo**: That is what I just thought. It is not Braille…*

(Quoted after [10], p. 141–142. Our translation.)

After Priscila interfered, the prospective teachers realised that the blind student did not read the numbers by touching something textured, but rather by using the system of Braille. The prospective teachers searched on the internet about the instrument used for writing in Braille. They found information about the reglete, an instrument similar to a ruler that makes marks on paper. They decided that the texts should be written in Braille with the aid of the reglete.

2.2. Episode 2

The prospective teachers imagined each student would pick a piece of paper from the box, read it aloud, and paste a ball in the corresponding position on the graph. Thus, they would construct a bar graph for the distribution of shoe sizes. After the graphs were made, the imagined teacher would ask which shoe size had the highest occurrence and explain that this number is called the mode. Then the prospective teachers began to reflect on the teaching of the median, as they recognised the need for imagining strategies to include everyone. They considered, for instance, the possibility of asking the students to draw the shoes lined up in ascending order on a sheet of paper.

***Denise**: I think drawing can also be done. It could involve the students more... Or maybe not, because only one will draw... That's a difficult decision.*

***Kátia**: I think asking them to draw could lead to the issue of only one student drawing because we will only give out one sheet per group. Indeed, it would be difficult to think about involving everyone.*

***Denise**: The concern with drawing is related to the blind, right?*

***Kátia**: That thinking like this as a whole, it's better not to have the drawing.*

***Priscila**: Not to have it? But what about the boy who likes to draw?*

***Kátia**: But the blind one?*

***Isabel**: Right!*

Danilo*: How difficult it is!*

Danilo*: Difficult because we cannot focus on just one and exclude the rest as well.*

(Quoted after [10], p. 145. Our translation.)

The concern for including everyone made Denise suggest another idea, which everyone approved:

Denise*: Guys... I'm thinking of changing everything here. I think I'm the "mess aunt". I think it might be a lot of work, or maybe someone might feel uncomfortable, but at the time of the activity, we would have to think better. I'm thinking here of doing it with the student's own shoe.*

Kátia*: Can you explain?*

Denise*: One person says the sizes of the shoes and then the other students place their own shoes with the sizes organised in ascending order in a line on the floor as the peer says the numbers. The blind student knows his shoe size. One student can help the other… One can read the shoe size; another can pick up the colleague's shoe and put it on the line… That way, no one is excluded, not even the student with mobility difficulties nor the autistic student … Everyone may help and be helped.*

Isabel*: I think they would like that.*

Danilo*: I really like that idea, Denise.*

Denise*: Sometimes I think I have some really crazy ideas.*

Kátia*: It's not crazy—they will love it!*

Danilo*: Yes, they will get up—move around.*

Kátia*: They will get excited! I would never have thought of that!*

(Quoted after [10], p. 145. Our translation.)

The concern for including all students meant that the initial idea of making a drawing was replaced by another that prioritised groupwork, touch, and movement.

2.3. Episode 3

The prospective teachers imagined the class working with the concept of median, based on Denise's idea of lining up the students' own shoes. She sets out her idea in the following way:

Denise*: I think now we ask them to see the shoe sizes. Then they will see that this amount is equal to the number of students. So, we ask, whose shoe is in the middle? If the number of students is odd, visually it will be easy to see, right? Then a student goes there and looks at the size of that shoe in the middle. If it's an even amount, then we'll have to think with them that they'll have to add the two in the middle and divide by two. The blind student can feel the shoes with his hands and pick up what's in the middle...*

Then Kátia shared a preoccupation about a concept that was a prerequisite for this activity, namely the notion of average (arithmetic mean).

Kátia*: But look here, thinking about it now... If we haven't introduced the concept of arithmetic mean yet... When it's even* [the shoe size], *they'll have to add it up and divide it by two, right? So, they are going to have to do the arithmetic mean.*

Danilo*: Yes, they can do it together.*

Kátia*: But don't you think it would be easier if we first worked the arithmetic mean with them? Because then when they get to that point, they will see that if the number is even, the median is the arithmetic mean of the middle ones.*

Danilo*: I think it's better to do it at this time in the activity. Like when they have a problem and you're going to ask them to fix that problem.* […]

Kátia: *I understand. Do you mean that you think it awakens more interest in the student if we have already given them something to apply the method to?*

Danilo: *Yes!*

Denise: *Like throwing the problem to them, right? Like asking: and now, in the case of an even number, how we do it? What do you think you should do? I think we must guide students in this discovery, just like Danilo said. Because they will have something... they will have a problem to solve.*

Isabel: *I prefer it that way too.*

Kátia: *You are right. I also learned arithmetic mean by a mechanical way. Type: take the number of elements, add and divide by the number of elements. But I really didn't understand why I needed to do this. I just reproduced it. As time passed, I understood.*

(Quoted after [10], p. 134. Our translation.)

Based on these reflections, the prospective teachers decided to introduce the concept of mean when dealing with the median.

2.4. Episode 4

In all interactions, the prospective teachers were referring to Josué as 'the blind student', a fact observed by Danilo, who expressed his discomfort with doing so:

Danilo: *Hey guys, I'm sorry... we're saying: blind student... blind student... we're talking about the student's characteristics, but he has a name!*

Kátia: *You're right!*

Danilo: *Yes... It looks like we are ignoring that they're people.*

Kátia: *So, what's his name?*

Danilo: *Josué.*

Denise: *Let's be careful to call him Josué!*

Danilo: *I'm sorry, guys, but this was bothering me!*

Denise: *You're right!*

(Quoted after [10], p. 203. Our translation.)

After the reflection proposed by Danilo, the prospective teachers started to try to refer to Josué by name. This is important for ensuring that they see the student as a person before paying attention to his or her physical condition.

2.5. Episode 5

After discussing the median, the prospective teachers imagined that they would work with data from the entire classroom so that the students could compare the results of the groups and reflect on the sample. The work would begin by pasting the posters with the graphs prepared by each group on the blackboard:

Danilo: *I think it would be cool if we glued or hung the charts on the board. Then we could ask them to compare the differences on the results of each group and also to think about the graph of the whole classroom.*

Kátia: *Sure! Because on the board, it will be in a position that everyone will see! But... what about the blind student?* [Apparently, Kátia forgot to refer to Josué by name.]

Danilo: *Oh, yes, true!*

Kátia: *I'm thinking that way he won't participate.*

Priscila: *Let's think about it; what could be cool to do? Do you think you can paste the poster with the graphs on the board?*

Danilo: *Yes! It would be just letting him get up and touch; it would be his way of seeing.*

***Denise**: We have to be careful to place it at a height that the student can touch and see the information of the other groups.*

***Kátia**: But then he's going to get up and touch one by one?* [Said in a reproachful tone.]

***Denise**: Yes! And as each one will have already been made with subtitles in Braille, he will be able to touch each one. And the autistic student too... And if other people wanted to touch the graphs too... I don't see any problems.*

(Quoted after [10], p. 143. Our translation.)

The suggestion of the students getting up to touch and analyse the graphs would allow everyone to read them, but the disturbance that would cause in the room bothered Kátia. After some conversation, the prospective teachers realised that it was no problem for everyone to get up, and so it was decided.

2.6. Episode 6

To continue the analysis of statistical data in the whole classroom, each group should receive a printed table to record the collected information. The group with Josué would have its table presented in a textured way. The prospective teachers were concerned about making sure that Josué could come to do things like the others:

***Danilo**: You can put his hand on the chart so he can see the number of balls too.*

***Isabel**: He can count one of the columns and say the result aloud to the others.*

***Denise**: Yes, he can do that.*

***Priscila**: Since you're thinking about Josué, will he be able to do the shoes part?*

***Kátia**: He will! Maybe he will need his peers' help, but he will.*

***Danilo**: I think he will, but Giovana could have problems.*

***Denise**: She has motor difficulties, but I think if she's patient, she'll be able to put the shoe there.*

***Danilo**: And she can get help from other peers as well. This is important!*

(Quoted after [10], p. 151. Our translation.)

Imagining inclusive classes led the prospective teachers to talk about the importance of students helping each other, and they identified an important idea of inclusive education, namely that everyone can help and be helped.

2.7. Episode 7

The prospective teachers imagined that the discussion of the comparison of data from the groups would close the activities related to statistics. However, Kátia felt they were missing the teacher making a formalization, a kind of closure of the content:

***Kátia**: Can we come up with something to formalise on the blackboard?*

[...]

***Danilo**: Guys, this thing about formalising there... I'm thinking... when we do an activity like that, full of things to build, full of graphs, full of steps... I at least think it would be interesting for us as teachers to ask students to write what they thought, what they learned... like a self-assessment. They formalise the content in their own way so that we could know how they thought, if they understood the content... For us to evaluate them to know if they learned... Before we formalise something for them.*

***Kátia**: I think this is cool, but at this stage of schooling, 6th grade, if you ask them to write, some will write what the mode, the arithmetic mean, and the median are... But others will talk about the paper balls. So, if we did a simple questionnaire asking: what did they understand? What is mode? What is arithmetic mean? What is median?*

> ***Denise***: *I think that this questionnaire . . . at the beginning, they will be nervous. And then, if they don't succeed, their self-esteem goes downhill.*
>
> ***Kátia***: *Yeah... could be.*
>
> ***Denise***: *And I thought Danilo's idea was wonderful. They will have to tie the ideas together. I think we can help... direct. Like if they say: Teacher, I don't know what to write! We will say: Ah, write down what you have learned! What did you do? This is nice so they don't get lost and also know what to answer. And we do not interfere in the result.*
>
> (Quoted after [10], p. 154. Our translation.)

This dialogue led the prospective teachers to think about the need for formalization for a final word from the teacher.

2.8. Episode 8

When in the end the prospective teachers were invited by Priscila to evaluate the experience of participating in the pedagogical imagination process, Denise and Kátia made the following observations:

> ***Denise***: *In the part where we thought about activities, it was a matter of breaking down barriers in the mind, really. There was something I did not think about, but Danilo mentioned it, or Kátia, or Isabel. Then I thought: Wow! How could I not think of that? And the importance of sharing things too. When you put several heads together, you know?*
>
> ***Kátia***: *Yes, talking with colleagues here, at various moments I deconstructed many things that were in my mind.*
>
> (Quoted after [10], p. 209. Our translation.)

These comments indicate that the prospective teachers realised the importance of dialogical interaction in the process of formulating a pedagogical imagination.

3. Features of Pedagogical Imagination

We acknowledge that the notion of pedagogical imagination is not well defined, and we let it preserve its open nature. Still, we want to elaborate on the complexity of the notion by relating it to other equally open notions, namely *dialogue*, *social justice*, *mathematics*, *hope*, *and sociological imagination*. In order to provide this elaboration, we are going to refer to the different episodes presented in Section 2. We are going to use the episodes to illuminate aspects of pedagogical imagination rather than analyse the episodes as such.

3.1. Pedagogical Imagination and Dialogue

The importance of dialogue for educational processes has been pointed out by Freire [12]. Alrø and Skovsmose [13] try to bring some specification to the notion by identifying eight dialogic acts, namely *getting in contact*, *locating*, *identifying*, *advocating*, *thinking aloud*, *reformulating*, *challenging*, and *evaluating*. A dialogue can also be characterised "negatively" in terms of non-dialogical communicative acts. Faustino and Skovsmose [14] have identified eight such acts: ignoring, disqualifying, and lecturing being some of them. A dialogue can then be characterised as a communicative process where plenty of dialogic acts occur and few non-dialogic acts.

Dialogue is important for constructing and articulating pedagogical imaginations. Dialogic interactions might help to distance pedagogical imaginations from personal and private fantasies and turn them into collective ambitions.

The conversations we have presented in the eight episodes are rich in dialogic acts, while non-dialogic acts seldom appear. The conversation in Episode 2 concerned the possibility of students making drawings. Denise thought aloud: *I think drawing can also be done.* However, Kátia perceived difficulties and made a challenge: *I think asking them to draw could lead to the issue of only one student drawing because we will only give out one sheet per* *group. Indeed, it would be difficult to think about involving everyone.* Denise located a different challenge: *The concern with drawing is related to the blind, right?* She identified a different possibility: *One person says the shoe sizes, and then the other students place their own shoes with the sizes organised in ascending order in a line on the floor as the peer says the numbers.* She advocated for this by adding: *The blind student knows his shoe size. One student can help the other* [. . .] *That way, no one is excluded, not even the student with mobility difficulties nor the autistic student . . . Everyone may help and be helped.* Isabel followed up with an evaluation: *I think they would like that.* Danilo made his evaluation by stating: *I really like that idea, Denise.*

In Episode 3, Kátia asked: But look here, thinking about it now... If we haven't introduced the concept of arithmetic mean yet... When it's even [the shoe size], they'll have to add it up and divide it by two, right? So, they are going to have to do the arithmetic mean. She believed that for students to learn a new concept (median), it was necessary for them to master a concept that she considered a prerequisite, namely the average. She did so even if, as in the present case, it only involved a simple calculation. Through dialogue recalling previous school experiences, the prospective teachers realised that letting students try to learn a concept based on the need to solve a problem might be more meaningful for them.

In Episode 4, the prospective teachers reflected on the importance of taking care of the way in which they referred to a blind student: he should be referred to by name and not by his disability. They recognised that calling a person by name was an attitude of respect. In Episodes 5 and 6, the dialogue allowed the prospective teachers to think of strategies so that everyone could participate in the activities. From the dialogue in Episode 7, they came to consider the need and feasibility of formalizations on the part of the teacher. In Episode 8, Denise precisely defines the role of dialogue in articulating a pedagogical imagination: *There was something I did not think about, but Danilo mentioned it, or Kátia, or Isabel. Then I thought: Wow! How could I not think of that? And also the importance of sharing things too. When you put several heads together, you know?*

3.2. Pedagogical Imagination and Social Justice

We do not assume that a mathematics education for social justice needs to start out with a specific definition of social justice. We do not see mathematics education for social justice as being an education that "informs" students, prospective teachers, or anybody else about what social justice means. Instead, we see education for social justice as education that engages the participants in expressing what they find to be just and unjust. It is education that brings to the forefront the participants' concerns and ideas. One need not expect the existence of broad agreement about what justice is; the crucial point is that possible divergences are expressed and provide points of departure for conversations and dialogues. Conceptions of social justice are social constructions; they can take place in mathematics classrooms, both in schools and at universities. Dialogues are crucial for such constructions. (For a discussion of social justice as a social construction that can take place in the classroom, see [4].

Visions about social justice are formed through pedagogical imagination. Logically speaking, pedagogical imagination might be rooted in any set of educational and political visions. One could, for instance, claim that the management approach in education is guided by a pedagogical imagination directed towards a neo-liberal horizon. However, we do not intend to operate with an all-embracing and free-flowing concept of pedagogical imagination. We pay particular attention to conceptions of social justice and to ways of pointing out social injustices. Such injustices could concern economic inequalities, structural poverty, exploitation of workers, racism, sexism, any type of homophobia, and social exclusion. The notion of pedagogical imagination that we want to operate with tries to conceptualise educational practices that contest any such cases of social injustice.

Pedagogical imagination can encompass grand visions about social justice. However, it can also be concrete and concern particular cases. It was part of the whole contextualization of the prospective teachers' pedagogical imagination that students with different disabilities would be present in the classroom. Josué was a blind student. How do we meet his needs in the mathematics classroom? We are dealing with a particular issue of social justice. In Episode 1, Kátia made a proposal: *Maybe we could make the axes with glitter paper because it is very rough. We could also, on the axis that has the shoe sizes, cut out the digit in EVA with glitter that he could feel it.* Danilo followed up on this idea, and Kátia concluded: *And so, we do the same for all groups because it is more beautiful. Furthermore, we do not differentiate only one. More attractive. More colourful.* For the prospective teachers, it was important that Josué, like the others, could identify the axes of the coordinate systems in which graphs were depicted. Understanding that a blind person reads with their hands was essential for thinking about activities that would include Josué.

When Danilo, in Episode 4, exclaimed that the student had a name and they should use it, it concerned a specific feature of social justice. Not calling Josué by name but referring to him as "the blind person", is a way of dehumanizing him. It means seeing the disability before the person. The quest to call him by name is a humanizing act. Freire [12] highlights that humanization is a vocation of human nature. However, for Freire dehumanization is present both in individuals who have their humanity stolen and in those who steal it from others. The search for humanization is possible because dehumanization is not a given destiny, but the result of an unjust social order. In this sense, the search for the humanization of Josué was also a search for social justice.

The process of pedagogical imagination led the prospective teachers to conceive a dialogical classroom in which students interacted, respected, and helped each other. Denise's comments in Episode 2 explain this: *That way, no one is excluded, not even the student with mobility difficulties nor the autistic student. Everyone may help and be helped. Everyone may help and be helped.* A statement in the same direction was made by Danilo when, in Episode 6, the group was concerned about the autonomy of Giovana, a student with motor difficulties, in carrying out an activity. The prospective teachers found that stimulating collaboration among students was something positive, as Danilo pointed out: *And she can get help from other peers as well. This is important!*

3.3. Pedagogical Imagination and Mathematics

In a school context, mathematics might be considered to be fixed in the form of a pre-defined curriculum. A curriculum might be carefully specified in terms of topics to be covered and in terms of the chosen textbook. What can be referred to as the school mathematics tradition sets a definite agenda for what is taking place in the classroom. An important task for the teacher is to make an exposition of a new topic as presented in the textbook and to clarify students' doubts and uncertainties. The students' task is to solve exercises as they are formulated in the textbook. Finally, it is the teacher's task to check if the students have solved the exercises correctly. According to the school mathematics tradition, a principal criterion for students having understood mathematics is for them to be able to solve the relevant exercises correctly.

Pedagogical imagination signifies a readiness to conceptualise alternative educational possibilities. It is important that the mathematics teacher is ready to consider other ideas and themes than those defined by the curriculum. Working with landscapes of investigation is one such possibility (see, for instance, [15]). It is important as well that the teacher recognise ways of organising the teaching-learning processes different from those engraved in the school mathematics tradition. We see a readiness to construct educational alternatives as an important component of the mathematics teacher's professionalism; as a consequence, pedagogical imagination becomes an important ingredient in any mathematics teacher education programme.

The imagined classroom activities deviated from the traditional model of mathematics teaching; the prospective teachers looked for possibilities to engage their students in an investigative process. In Episode 3, the prospective teachers discussed the importance of teaching the concept of mean so that students could also find the median of a distribution with an even number of elements. Kátia's initial discomfort was dealt with dialogically, where, for instance, Danilo highlighted that it was better for the students to learn when they were involved in solving a problem—a fact that was corroborated by Denise. The discussion led Kátia to realise that, in fact, students' investigations could contribute to non-systematic and more meaningful learning processes.

In Episode 7, the need for a formalization made by the teacher was put on the agenda. Until then, the imagined classroom activities concerned the students' activities, a fact that worried Kátia. She asked: *Can we come up with something to formalise on the blackboard?* The prospective teachers considered the need for formalization, which, however, Danilo found might go in the opposite direction of what they had proposed up to then. He proposed that a formalization could be completed by the students: *I at least think it would be interesting for us as teachers to ask students to write what they thought, what they learned... like a self-assessment . . . Before we formalise something for them.*

The collective pedagogical imagination allowed Kátia to move away from a comfort zone. Unexpected things might happen when students get up to touch the posters (Episode 5), when the explanation of the concept of average comes before the students are faced with the calculation of the median (Episode 3), and when the students are recommended to sketch definitions for concepts learned before the teacher presents formal clarification of the applied notions (Episode 7). These imagined situations seemed to make Kátia ready to enter a risk zone.

According to Penteado [16], the practice of many teachers is located in a comfort zone where, most of the time, they can predict and control what is going to take place. The word "comfort" indicates, for instance, that the teachers do not risk facing mathematics questions that they cannot answer. Moving into a risk zone means that the teacher might lose control and predictability with respect to the students' activities and questions. However, moving into a risk zone also generates educational possibilities. Pedagogical imagination led Kátia to consider entering a risk zone and revealed that openness to risks generates possibilities.

3.4. Pedagogical Imagination and Hope

Freire [17] pays particular attention to the notion of hope. To him, hope is an integral part of a struggle for a better society. In Freire's words: "I do not understand human existence and the struggle needed to improve it, apart from hope and dream. Hope is an ontological need" (p. 8). Freire does not assume a Marxist-like determinism, according to which a classless society will emerge due to some pre-identified economic laws. Freire points out that the struggle for improving the world cannot be reduced to some "calculated acts alone", or to a "purely scientific approach" (p. 9). He highlights that, without a minimum of hope, "we cannot so much as start the struggle" (p. 9).

In the monumental work *The Principle of Hope* [18], first published in German in three volumes in 1954, 1955, and 1959, Bloch provides a careful discussion of the concept of hope. Bloch was a Marxist, but not in any orthodox way. In line with Freire, he claims that socio-political changes do not take place according to some economic laws that can be identified in advance. Political actions need to be fueled by hope. Bloch points out that hope concerns "dreams of a better life" ([18], Volume I, p. 11), and it needs to be added that he is talking about dreams of a better life *on earth*. In the *Principles of Hope*, Bloch uses the expression "concrete-utopian horizon" in order to unite utopian visions with real-life political actions ([18], Volume I, p. 146). To operate within a concrete-utopian horizon means to articulate visions about what could be desirable. However, such ideas need not be wild speculations; they can be directed towards the horizon of real-life possibilities. They can be formulated within a "concrete-utopian horizon". This is an important feature of pedagogical imagination.

When the prospective teachers started their project work, they were presented with a range of conditions that needed to be considered. It was made explicit that the class they were going to consider would include students with different disabilities. It was stipulated that it would be a public school, which in a Brazilian context indicates that resources might be limited. It was also stipulated that the curriculum must be observed. By adding such conditions, the prospective pedagogical imagination became formed within a "concrete-utopian horizon", exemplified when Kátia in Episode 1 proposed the use of a material and promptly recognised that it was cheap and therefore possible to use.

One could have invited the prospective teachers to apply their pedagogical imagination freely. They could be invited to consider any possible mathematics content and any form of educational environment. A pedagogical imagination can be elaborated as "free speculations", and this might invite a more fundamental critique of what is actually taking place in the mathematics classroom. Specific conditioning imposes limitations on the pedagogical imagination, but such conditioning might also help to intensify an imagination and ensure that it takes place within a "concrete-utopian horizon". A pedagogical imagination brings us beyond what is currently taking place, directing us towards realistic alternatives.

It was specified that the prospective teachers had to consider a classroom that included students with different disabilities. This made prospective teachers consider mathematics classes from an inclusive perspective. At no time was the non-participation of Josué, Giovana, or Heitor considered. Classes were imagined to be for everyone.

In Episode 1, the initial planning was changed in order to include everyone, and the students were lining up their shoes. However, it is important to be aware of the fact that whenever a mathematics activity operates with numbers referring to the students—like height, weight, body mass, or shoe size—new problems might occur. By putting things in numbers, some students might be pointed out as being, say, the tallest, the smallest, and the heaviest. Being pointed out as having the smallest or biggest pair of shoes might be a problem to some: "Making differences among students public in terms of numbers might provoke bullying and ruin the self-esteem of some students" ([4], p. 141). (See [19]) for similar remarks concerning the use of the Body Mass Index in the mathematics classroom.) In Episode 2, the danger of bullying was somehow indicated by Denise's remark: *maybe someone might feel uncomfortable*. This remark was not further unfolded in the dialogue. However, a pedagogical imagination is temporary and preliminary; it is always in need of being further elaborated upon; it is rooted in a dialogue that is always in need of being continued.

As part of the pedagogical imagination, the prospective teachers considered that all students were important actors and responsible for their mathematics learning. There was no discrimination. The pedagogical imagination made it possible for them to approach a pedagogy of hope, as formulated by Freire [17]. The prospective teachers shared the hope associated with an inclusive education, in which the classroom turns into a favourable place for encounters between differences. The process of pedagogical imagination invited the prospective teachers to express their hopes with respect to the mathematics classroom but also with respect to the organisation of the school and society in general.

3.5. Pedagogical Imagination and Sociological Imagination

In 1959, Charles Wright Mills published the book *Sociological Imagination* [20]. It was during a time when the positivism paradigm dominated sociology. According to this paradigm, the overall aim of sociology is to provide extensive and reliable descriptions of social facts. The descriptions should be objective in the sense that they should not reflect the perspectives of the researchers, and they should be neutral in the sense that they do not encompass political or ethical priorities. According to logical positivism, all scientific disciplines should contribute to the unity of science by providing such descriptions. The natural sciences—in particular, physics—were considered paradigmatic role models for sociology too.

Wright Mills did not agree. According to him, sociology should not only try to describe social realities; it should also try to present possible alternatives to such realities. This led Wright Mills to present sociological imagination as an integral part of sociological studies.

The notion of pedagogical imagination is closely related to the notion of sociological imagination. The purpose of articulating pedagogical imagination is to move beyond the descriptive paradigm in educational research. The aim is not only to describe educational realities as they might appear in schools and classrooms but also to formulate visions about alternative educational possibilities. The aim is not only to research what is, but also what could become. Such ideas bring the notion of pedagogical imagination to the forefront of what can be referred to as critical research. As indicated in Figure 1, positivistic research (and naturalistic research in general) focuses on the current situation, CS, while critical research also addresses what could be imagined, IS, and what could be managed, AS.

Pedagogical imagination is not only part of critical research; it also concerns daily-life classroom practices. Pedagogical imagination can be formulated by teachers when they consider possible alternatives to what is usually taking place in their classroom practices. Teachers can consider if there are new topics that they could address in their teaching. They can consider the students' possible reactions and also the parents' possible reactions to controversial issues. Teachers can consider different possibilities for covering the curriculum and alternative ways for preparing students for tests and exams. To address such issues means to engage in pedagogical imagination.

We find it important to engage prospective teachers in formulating pedagogical imaginations. Pedagogical imagination is part of teachers' professional expertise, and it is important to prepare prospective teachers for this.

4. Pedagogical Imagination and Mathematics Teacher Education

Pedagogical imagination emerges from a set of collective constructions. One can assume that pedagogical imagination can take the form of free-floating personal speculations, but this is not the notion we are operating with. We see pedagogic imagination as developing through processes of *dialogue*. We relate pedagogical imagination to visions about *social justice*. Pedagogical imagination also concerns *mathematics*. The mathematics curriculum might appear to be a given, and so might many of the accompanying classroom routines. But the mathematics curriculum and the classroom routines can be recognised as being not necessities but contingencies. Pedagogical imagination is an expression of *hope*. Hope can concern social and political development, but simultaneously, it may concern a different life in the classroom. The notion of pedagogical imagination is related to the notion of *sociological imagination*. However, while sociological imagination first of all concerns ways of doing research, pedagogical imagination concerns ways of doing education research *as well as* ways of making educational innovation.

It is important that mathematics teacher education does not coagulate as preparation for adapting to the given social and educational order. It is important to prepare mathematics teachers for moving beyond what is normally taken as given. As a consequence, *we see pedagogical imagination as a crucial component of mathematics teacher education*.

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education sciences

Article

The Role of Mathematics Teacher Education in Overcoming Narrow Neocolonial Views of Mathematics

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Abstract: Over the past 30 years, teacher education has changed to incorporate a larger emphasis on understanding students' sociocultural backgrounds, knowing that these influence their learning. However, in terms of mathematics and mathematics education in teacher education, less has been done to recognise the sociocultural mathematics backgrounds of students. An example is provided to show how entrenched colonial attitudes to mathematics have developed into neocolonial policies that influence mathematics education. This example is based on a large historic research project in Papua New Guinea (PNG) that aimed to document and analyse the nature of mathematics education from tens of thousands of years ago to the present. Data sources varied from records of first contact and later records, archaeology, oral histories, language analyses, lived experiences, memoirs, government documents, field studies, and previous research especially doctoral studies. The impacts of colonisation, post-colonial aid and globalisation on mathematics education have been analysed, establishing an understanding of the current status of mathematics education as neocolonial. Neocolonial education policies diminish cultural ways of thinking. Thus, teacher education has an important role in sensitizing preservice and inservice teachers to the impact of neocolonial approaches as well as in developing with students some ways of reducing this impact and encouraging more holistic, culturally relevant mathematics education.

Keywords: neocolonialism; ethnomathematics; language and mathematics; postcolonial education; Papua New Guinea; Asia-Pacific

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1. Introduction

Mathematics education in colonised countries tends to be imported from dominant, overseas countries, especially those that colonised them or subsequently provided significant aid. However, to understand this impact more fully, it is important to understand the mathematics that existed prior to colonisation which is still practised today in these countries, and this mathematics' historical collision with colonialism and neocolonialism. A case study of Papua New Guinea is provided to explore this impact on mathematics education.

Papua New Guinean societies existed from at least 40,000 years ago with several migrations from the north or west. They adapted to various changes such as the minor Ice Age and volcanic eruptions. There are several Papuan or Non-Austronesian language Families with hundreds of languages and a few Isolates (see Table 1). Around 5000 years ago, a major new wave of migration occurred and the Austronesian Oceanic languages developed starting in East New Britain and spreading around the coast and to Island Melanesia as far as Fiji [1]. Groups were relatively autonomous, managing to meet their needs through trade arrangements and intermarrying relationships. There was no central government. The 850 PNG cultures and languages were not influenced by Europe or the Middle East until the 1800s. All these groups developed different forms of technology and economies that required mathematics. This is called ethnomathematics, which varies with each cultural group and language. The environment and ecology influenced its development so there may be some similarities between some groups. Furthermore, they also exchanged knowledge [2].

Table 1. A summary of collected data on counting systems in Papua New Guinea showing the different types of systems described in terms of the cycles of frame words from which higher numbers are formed.

Austronesian Oceanic	Types	West Papuan	East Papuan	Torricelli	Sepik-Ramu	Trans New Guinea	Minor Phyla	Total
2	**(2)**	0	0	0	3	39	0	42
18	**(2, 5)**	0	1	16	5	86	1	109
	(2, 3, 5)	0	1	3	5	17	1	27
12	**(2, 4, 5)**	0	0	5	3	31	1	40
34	**(5, 20)**	0	1	2	17	52	7	79
4	**(4), (4,8)**	0	0	0	1	6	2	9
	(6)	0	0	0	0	5	0	5
	Body-Parts	0	0	0	8	58	4?	70
45	**(5, 10)**	2	12	0	3	4	0	22
19	**(5, 10, 20)**	5	0	0	0	4	3	13
73	**(10)**	1	8	0	1	2	0	13
3	**(10, 20)**	2	0	0	0	1	0	3

After colonisation by Germany in the north and England in the south in the 1880s, in 1902, Papua in the south and, after World War I, New Guinea in the north became Territories of Australia and hence an Australian colony. The funding for the colonies was very limited and so there was little money for education. Australia itself was a colony and for whom, like other colonies, this study has some relevance. Papua New Guinea (PNG) became independent in 1975 so the period of colonisation was relatively short compared to that of many other countries, allowing people to maintain and adapt their cultures. In a sense, its colonisation was condensed in time but had features similar to other places as well as unique features.

2. Research Aims and Methodology

The purpose of this research was to document and analyse the development of aspects of mathematics and mathematics education in Papua New Guinea from the past to the present. There are a couple of available bibliographies of education covering colonial times until the mid-1970s [3,4] but these do not focus on mathematics or mathematics education during this period nor from the time before European contact. Despite ongoing research within the country, there has been little on mathematics education per se after the mid-1980s when the Mathematics Education Centre declined [5]. Two exceptions in the 1990s were the doctoral studies of Kaleva [6] and Kari [7] which led to research through the Glen Lean Ethnomathematics Centre from 2000 to 2016 [8,9].

2.1. Data Sources

This historical research involved the extensive use of first contact and later documents and memoirs; archaeological and linguistic research from diverse areas and language groups; oral histories; lived experiences; field visits to villages; large research studies on number systems [10], measurement practices [11,12], and mathematical words in different cultures across the country; research studies on mathematics education [5] and teacher education [13,14]; government documents, especially major reports [15–17] and plans recommending changes [18–22] to education; syllabuses; and studies on the language of instruction [23–25].

Many documents, such as first contact documents and linguistic data, did not focus on mathematics per se but it was possible to connect many of these accounts to lived experiences over the past 50 years and students' reports on the ethnomathematics of their cultural groups to develop significant themes. The mathematicians and mathematics educators who carried out ethnomathematics research shared and facilitated this research consisted mainly of Papua New Guineans from different tribal groups. While colonised views of mathematics were a starting point, there was a general consensus among these researchers that the mathematical practices of their communities, that is their ethnomathematics, were describable, decomposable and able to be re-assembled as mathematics. Technical and ethnomathematical methodologies were important to these communities, embedded in cultural practices and relationships, and passed on from generation to generation. For these researchers and for the UoG teachers who went to their Elders for their research projects, their cultural identity was significant to their professional identity.

2.2. Themes and Key Findings from the Data

A grounded-theory approach was taken for establishing themes in this research. The points that kept emerging in the sources provided the main themes, such as the lack of funding or the impact of the use of English as a language of instruction. Other themes were the critical points made by Papua New Guinean education leaders in their reports, such as the importance of maintaining culture while the mathematics education researchers valued their cultural mathematics and noted that the mathematics taught in schools violated [26] their cultural understandings. The themes that emerged from these sources included the following:

1. The languages of mathematics in villages and in schools;
2. The use of visuospatial reasoning in mathematical thinking;
3. The valuing of both traditional mathematics for one's everyday life (once identified) and school mathematics for the dream of a job;
4. The dissonance of mathematics at home and at school.

However, as a historical study of mathematics education [27], there was an argument emerging regarding the impact of colonialism which resulted in the hegemony of educational practices for Papua New Guineans who had received an education from teachers, usually Australian, whose first language was English and well-educated, articulate, high-achieving Papua New Guineans who often received their education from English-speaking teachers. In addition, overseas aid advisers continued to recommend global trends in education from national outcomes-based education to standards-based assessments. The whole education system was affected by these trends. In mathematics education, there was an emphasis on problem solving and a standard, linear approach to mathematical topics which emanated from western curricula. However, when the teachers were school students, often due to a lack of books and equipment, they practised the rote learning of western mathematics and, due to a fear of punishment, failure and letting down their family, they learnt not to speak in their home languages and to follow the (usually male) dominating voice.

Hence, the key findings were as follows:

5. The depth and diversity of foundational/traditional mathematics learning;
6. The growth and sources of neocolonialism;
7. The limitations of neocolonialism;
8. Examples for overcoming neocolonialism.

3. Results

3.1. Languages of Mathematics in Villages and in Schools

When Lean began to collect counting words in 1968 from tertiary students and teachers who at the time were fluent speakers of their home languages (there was already evidence of these languages changing rapidly), he realised he also needed to carry out village fieldwork and to search worldwide written resources, such as European Enlightenment and Royal Anthropological Institute documents, British New Guinea and Australian Papua annual reports, German reports, and documents by missionaries, linguists and translators. He took 22 years to complete this huge task. Besides the common lingua franca Tok Pisin, he learnt Tolai (his adopted family's language) and had some familiarity with other languages. It became clear that there were counting words or ways of identifying numbers in the languages of PNG that were different from Indo-European counting systems. He was also able to identify how different systems may have developed and how some languages had influenced others. To do this, he undertook the intricate task of organising these counting systems and comparing neighbouring systems (see Table 1). As a result, along with archaeological linguistic research, which identified proto-languages, he was able to indicate that the Papuan Non-Austronesian mathematical systems had developed and existed for tens of thousands of years and the Oceanic systems for five thousands of years, as well as how they spread (not from the Middle East) and changed [10,28,29].

In my studies with Kaleva on measurement, it was found that there are many words for and grammatical ways to express length, area and volume as well as ways to express forces, comparisons and units of measurement [11,12,30]. However, school policies discouraged the use of home languages so often that there was limited understanding of concepts that belonged to village experiences. The rote learning of western mathematics using English words prevailed, sometimes with little meaning for the learner.

More recently, Bino, Muke, Sondo, Kravia, Sakopa, Edmonds-Wathen and I encouraged teachers to express mathematical concepts in their own language and found that this requires some discussion [31–34]. Nevertheless, most school concepts can be discussed or indicated in cultural ways. However, one of the major concerns has been the failure of teachers, students and communities to recognise the intrinsic mathematical ways of thinking culturally and to consider any mathematics in community activities as quite separate to what they see as mathematics, that is what they learn in school. Our research indicated that in fact mathematical thinking is constantly used in everyday activities [35–37].

3.2. The Use of Visuospatial Reasoning in Mathematical Thinking

When Alan Bishop visited the PNG University of Technology where Lean and I worked, he found that the tertiary students who had virtually no picture books (and no TVs or photographs) had difficulty interpreting images of objects. However, with minimal training on how to read these images, the students proved to be very competent. At that point, he decided that there were two distinct capabilities: visualisation and interpreting visual representations [38,39]. Lean and Clements [40] continued this work on spatial abilities with 3D objects and so, along with a number of other studies carried out in this fascinating area, I had a strong foundation for my research [41], culminating in a book [42]. Constantly I experienced villagers making decisions and students and people telling me they were doing it "by eye" or 'in their heads'. Sometimes they used objects such as ropes or steps to explain what they were visualizing. This was affecting all areas of mathematics from their understanding of number size to measurement practices, shapes, geometry, trigonometry and other ratios. For example, in making a house smaller, the Elders were able to visually decide on the horizontal and hypotenuse lengths for a house to keep the same angle. The equidistance of points from other points and points in a straight line were also managed by eye.

Some knowledge and ways of thinking were embodied. For example, parallel lines of a trapezium were understood when walking equidistant from each other between two non-perpendicular lines. The angle of equilateral triangles and the tessellation of these triangles were embedded in visuospatial imagery when planting trees at the vertices using two equal-length sticks. Diagonals were checked for equality when rectangular walls were marked out. A man dragging his foot with a taut rope tied to his ankle and the other end to a stick at the centre point would mark out a circle. A rope with two knots (separating 3, 4 and 5 units or 1, 1.4 (understood as just less than a half) and 1 units) was used by some villagers for obtaining right angles but usually some men were skilled in accurately determining these by looking and coming to a consensus within the group. Knowing the lengths of areas was sufficient to compare areas of roughly the same shape [12]. Ratios were used for comparing areas of grass needed for different roof areas or for comparing volumes of pigs or pig fat via their girth, length of body, height from the ground and foot area [11].

3.3. Valuing of Home and School Mathematics

Many discussions with mathematicians, mathematics educators and teachers suggested that home mathematics was not initially recognised without learning about the field of ethnomathematics. Nevertheless, when architecture students were making their first designed sculpture out of paper without glue or tape, they called upon cultural imagery, patterns and practices to create their small but beautiful sculptures [43]. Students at the University of Goroka remarked on the mathematical capabilities of their Elders when they were describing the activities that the students were asking about for their reports on cultural mathematics. They were honestly proud of their ancestors' mathematics, "even if they did not call it mathematics" [44].

Students also strived to do well in school mathematics. They knew it was a subject they had to do well in to go further in school. There was fierce competition for positions in Grades 7, 9 and 11 as well as in tertiary institutions. However, much of the mathematics was memorised and learnt by rote. Hence, it was clear that for different reasons, they valued both school and home mathematics. It seemed, however, that when they valued home mathematics, they had a greater sense of pride in knowing about their cultural mathematics and succeeded in school mathematics [44,45]. Their cultural identity influenced their views on mathematics and on themselves as mathematicians.

3.4. Dissonance between Home and School Mathematics

Without encouraging students to see their cultural mathematics, the students would simply take the view that there was school mathematics and there was different mathematics at home. Many student teachers regarded the mathematics one had to pass at university or school as not existing within culture or being irrelevant. It was only when students were encouraged to make links deliberately that this dissonance began to break down [44,46].

Professional development sessions with teachers [46] and projects of students at the University of Goroka studying the elective *Mathematics, Language and Culture* have shown the strength of recognising cultural mathematics [44]. The following example is from the teacher Mulock Mulung [47], whose lecturer was Wilfred Kaleva. The images are by Mulock, who became a mathematics teacher educator in 2016. It is based on a traditional way of trapping birds that is used in the village of Hotec as well as two other large villages and smaller surrounding villages in the hinterland behind Salamaua, Morobe Province. This is a diverse geographical area. For protein, they hunt and set traps. There are a variety of traps on the land, in the river, in the sea, and in trees. "There are pig traps, bandicoot traps, wallaby traps, snake traps, bird and cuscus traps". He described the making and using of the bird trap, *lek* in his language "*matec*" (his language words are used unless specified as Tok Pisin, the main lingua franca of PNG). It is a special bilum (Tok Pisin) (a loose, continuous string net with each stitch individually interlocking in a figure of eight). "It requires great skill to make it and is carried out by specialists. . . . Catching the birds is a very dangerous activity in terms of men's lives being at risk because they climb and stay in tall trees more than 50–80 m high just to set the trap" and keep a close watch on the birds coming to feed.

"The *habiyom* birds (black with red eyes) come in large flocks from May to September to eat the fruit of the trees used to make canoes". Mulung describes and illustrates the process of making the net. "Ropes are extracted from the bark of *akek* which is similar to the *tulip* (Tok Pisin) tree. . . . Once the bark is removed, it is dried in the sun" and then the fibres are carefully removed. "These are twisted into strong rope". Two sticks are set into the ground so they are 3 m high. The Y at the top of each stick holds a bamboo pole from which a metre-long loop is made and strengthened from which a net with a slight bag is made. The men latch wood to the tree for steps and select a sturdy, wide branch to rest on while waiting for the birds. (See Figure 1 for each of these steps). "Hundreds of birds are caught over several days and the women come and bring food and collect the birds".

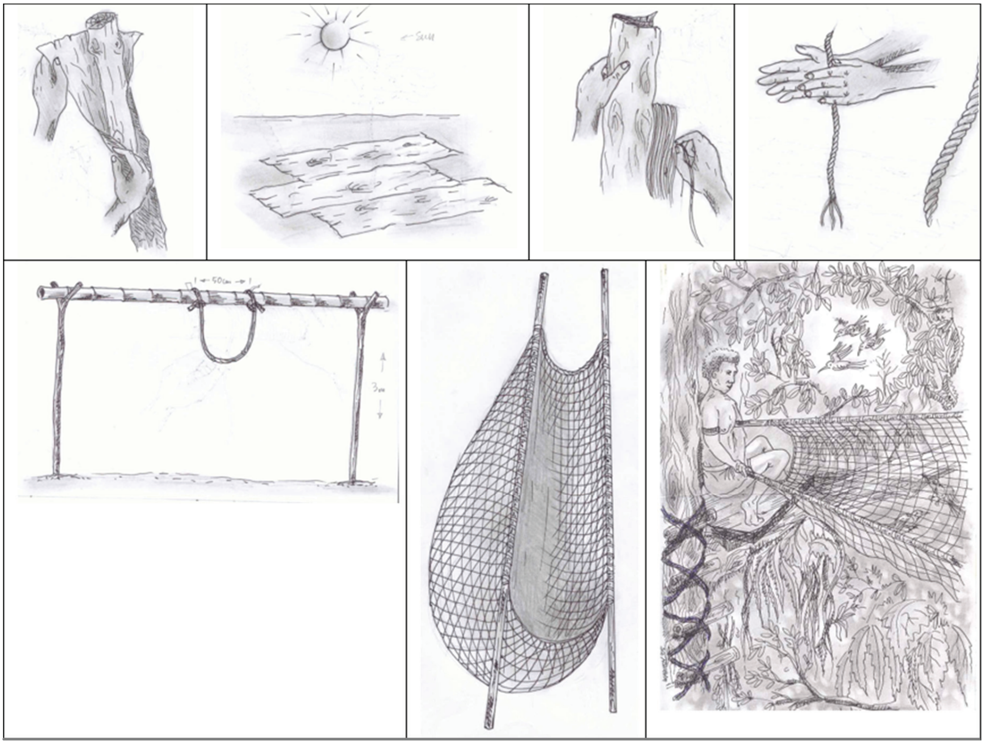

Figure 1. Making the bird trap net and capturing birds in a tree. (Drawing and information from Mulock Mutong (2005)).

Using this foundational knowledge, Mulung then prepared a series of examples for teaching the secondary course. He provided background school mathematics in an understandable way before applying traditional mathematics to school mathematics through examples, explanations and exercises related to the topics of area, ratio and rates, and trigonometry.

Steps. The removal of the bark, drying it in the sun, making it into fibres and twisting it into rope. The preparation to make the net and the finished net with the poles. A person who has climbed the wooden steps to the tree branch and waiting for the birds.

"Example. Calculate the trapezoid bed for the net trap (Lek) that has the height of 1.5 m and has lengths 4 m and 2 m respectively. . . .

Example. Those seasonal birds that fly to and fro following their routes fly 80 km in two hours. What is their rate of flight and how far will they fly in 5 h? . . .

Example. Net trap 1 (Lek 1) had caught a total 250 birds in three days and net trap 2 (Lek 2) had caught a total of 750 birds in three days. What is the ratio of birds in three days caught by net (Lek 1) and net (Lek 2).

Solution:

Lek 1: Lek 2 = 250:750

= 25:75 (simplest form)

= 1:3

Ratios are often used to express the composition of a mixture. Ratio of this type can also be used to determine the amount of each component in a quality of a mixture.

Example. A particular pot can hold 24 cups of birds (1 cup = 1 bird), 9 cups of pure water and 3 cups of Gravox chicken curry powder.

(a) What is the ratio of birds: pure water: Gravox (curry powder)?

(b) What quantity of each would require to make 5.0 m^3 of birds' soup. . . .

Example. The tree that people climb to set their net trap (Lek) is 80 m tall. The spot where women come to get the trapped birds in exchange for the food - from their men is creating 38 m with the top of the tree. Calculate the distance (Y) from the exchange spot to the base of the tree."

Through this project, this student has shown pride in his relatives' and neighbours' capabilities, strength, perseverance and courage. He also recognises that these are mathematical activities requiring mental mathematical capabilities that can be linked to school mathematics. In some respects, mathematics is associated with the physical and social environment familiar to the students in the same way that a mathematics trail or project is prepared for students to encourage their interest in mathematics by making it relevant to their everyday lives. Further examples are available in other references [24,36,37,42,48,49]. Developing or using such examples is an important aspect of mathematics teacher education if teachers are to provide examples and exercises relevant to students [50].

4. Discussion

While these themes were evident in studying the documents and other data, there were more profound considerations emanating from the data sources. All pointed to the fact that there has been and, in most cases, there still is evidence of scientific thinking, technological thinking and the necessary associated mathematical thinking in the various cultures of PNG. However, history has shown that these have not been encouraged in the school curriculum even when there is the desire for school education that reflects the values of PNG societies. This historical study provides some of the details of this occurrence.

4.1. The Depth and Diversity of Foundational/Traditional Mathematics Learning

Papua New Guinean societies used mathematics in technology, trade, social relationships, and understanding natural sciences tens of thousands of years ago. Much of this knowledge is still passed on between generations today using Indigenous ways of learning and teaching [27]. Most foundational mathematics is learnt from older men or women who gather under relational connections to share their knowledge in groups during everyday activities or special traditional activities [36,37].

A few remarks might indicate the extent and depth of this knowledge. (See also [10,36,37,51,52]). Seafarers had fishing and navigation skills, travelling over the horizon to distant places [53,54]. There were trading routes and reciprocity to negotiate with items often passed on to far distant places, crossing many language groups [55]. Kinship patterns were extensive, and again, reciprocity was significant [56,57]. There were tools and processes for carrying, collecting, fishing, agriculture, food and materials preparation, creating, building, playing and celebrating [36]. There were designs and patterns of cultural significance and replication of objects such as canoes [37,42], pots [58,59], drums, baskets, string figures [60], shields, bows and arrows, axes, or house walls and roofs [36,42]. All the details for attaining designs, curves, thicknesses, lengths and strengths of objects were mathematical. There was extensive knowledge related to medicines [61] involving spatial knowledge in recognising plants and where to gather them, and knowledge of how to treat illnesses with different medicines and processes.

The classification and sets of designs were sophisticated and related to culture [51,62]. These are evident for the shapes of objects and on the various parts of canoe boards, house boards [52], shields [36], other carvings, leadership symbols [2], food containers and pots [58,59,63]. Actions, their order and links between them have been studied in string figures [64,65] but also in making other items, such as string bags (bilum) [35,66,67]. They are remembered but also reorganised to create new designs. Patterns occur in gambling practices [68], weaving and making string bags. Numeral systems are varied, with some being unique and some shared with neighbours. Some are linked to collecting, measuring, trading or classifying [28]. Importantly, counting often has sophisticated systems and cultural importance, as indicated by Owens and Lean [29] and with Paraide and Muke [10] who also provided theses on their own languages [24,69] respectively.

Mathematics teacher education needs to ensure that teachers do not restrict these mathematical concepts to Euclidian geometry or base 10 counting systems. There is a wealth of cultural examples that indeed extend current school curriculum ideas on classification, pattern, design and numeral systems. Furthermore, these mathematical approaches link students spiritually to mathematics.

4.2. The Growth and Sources of Neocolonialism

Neocolonialism in PNG is a result of colonial education policies and practices but also the continuing expectations and practices of aid advisers and nationals.

4.2.1. Historical Developments

In the late 1800s, a few anthropologists visited PNG (e.g., Mikloucho-Maclay [70]), European sailors navigated its waters [71] and a few German business people began plantations or recruiting for other plantations in the Pacific region [72]. Missionaries soon followed, sharing the gospel of Jesus in the vernacular languages, often in a religious format but also assisting villagers, especially with health issues and education [72–74].

Governments felt the need to set up administration and controls. The German government in the northern mainland and islands soon set up administrative centres, laying claim to it as a colony in 1884. This prompted the British to lay claim to the southern side close to Australia, leaving the colony of Queensland and later Australia to administer Papua. One issue of the early administrators was the exploitation of 'the natives' as they were called. This encouraged them to provide a basic education. Mostly it was through supporting the mission schools but then they began requesting that schooling be in English so that the administrators could converse with the natives. Money was attached. After World War I, the League of Nations passed the northern section to Australia as a Trust Territory. Gold mining was exploited as was already occurring in plantations. This provoked many foot patrols into the virtually unknown, unpacified highland areas which were then opened up since aircraft were able to fly there. Importantly, this impacted the local economy as quantities of kina shell money were imported to pay the workers from the areas and encouraged people to travel to other people's land for work. Using pounds (made of paper like a 'leaf') and shillings was suitable to the digit tally (5, 20) cycle counting systems of many. These terms and translations for this 'money' continue to today (field visit to Malalamai, 2006) [75].

In the Australian Territories, English was stipulated as the language of instruction for government funding [4]. Not only the locals but also the Germans were required to have schooling in English. However, overall, little money was available to support the two Territories' colonial administrations [76,77]. Already the dominance of English as the valued language had begun. Teachers now need to make an informed and concerted effort to use a local language and not slip into a lingua franca such as Tok Pisin. However, too little has been done to explain basic mathematical concepts, such as arithmetic operations, in terms of local languages. These were rote learnt but widely used in employment.

Interestingly, in early British New Guinea and Papua administrative reports, basic word lists of the local languages were recorded as new centres were set up. However, most of the language work was carried out by churches. In Port Moresby, Lawes [78] and colleagues had written down the Motuan language by 1885 and used it in large schools for the local people [4]. Other village languages were also used, especially Dobu in the Papuan islands, Tolai in East New Britain, Bel in Madang area, and Kôte and Yambim in Morobe and beyond for churches and schools [73]. Students completing the two levels of the basic curriculum or later Grade 6 would be recruited as teacher assistants in schools. South Sea islanders also came as pastors and teachers [4].

During and after World War II, there was one administration [3] but no national curriculum so teachers taught what they or their senior teacher knew as mathematics from their home countries. In the 1960s, the Australian Prime Minister started to talk about autonomy [79] for the Territories. Around the world, more and more colonies were becoming independent. However, education in PNG was very limited and insufficient for autonomy, let alone for an independent country. Hurriedly, high schools were set up. By then, many Australians, often quite young, were recruited as kiaps (administrators in charge of areas) and teachers to remote areas as well as towns and coastal centres. Teachers' colleges trained both Papua New Guineans and Australians [80,81]. Some students were selected for studies in Australian secondary schools and universities (as per personal communications with such students between 1970 and 2016). By 1966, the University of PNG was set up in Port Moresby, and the beginning of the PNG University of Technology was not long after [82]. There were graduates by self-government in 1973 which preceded Independence in 1975. Research into education, particularly mathematics education, was strong and began influencing worldwide research [5,83]. Teacher educators and senior high school teachers were mainly from overseas but, for a decade, PNG teacher educators were trained as a group for two years in Australia [13]. Following that, there were Australian and New Zealand Awards for Masters degrees, in-country Masters, and more recently a couple of intensive courses in PNG.

Further details can be found in the bibliographies previously mentioned [3,4], Paraide et al.'s book [27], an earlier summary by Owens et al. [84], and another long-term education researcher, Weeks [85].

4.2.2. Colonial Impact on Education and Languages

The administration of the colonising countries focussed on law and order, taxes, and keeping records of businesses and other groups such as churches [85]. Initially in the early and mid-1900s, funds went to government schools and to missions if English was the language of instruction and were proportional to students' achievement in English and mathematics examinations set by Queensland (an Australian State). Missions or churches dominated education training and still do today; all but one of the primary teachers' colleges are run by churches ([13], see Appendix of 27) with the Institute of Education mostly concerned with early childhood education. The University of Goroka also provides Certificates and Degrees in Early Childhood Education and degrees (including Masters) in education for all sectors [5].

Before self-government was set up, Australia instigated a 6-month training program in Rabaul, mainly for Australians. In following years, ASOPA in Sydney provided some understanding of cultural diversity and respect for students undertaking school education, certificates and degrees [86] for teachers going to PNG. Before and after Independence, there were committees to advise on curricula for primary and secondary education and teacher education. The college staff members were able to be in touch and share their ideas and strengths. Some overseas mission staff members were in the country for many years while others came for short terms [5,13]. Recognising the needs of village children in primary teacher education was evident, and it was partially reflected in the official use of the term Community Schools after Independence. Nevertheless, the local PNG teachers tended to think schooling was the way they were taught by the Australians, and for mathematics, this involved considerable use of rote learning, although Dienes influenced a number of schools [81,87,88] with the idea of mathematics as logic and the use of games with apparatus for teaching. Like many good ideas, most of the materials sat idle in school storerooms up to the 1980s as there was not sufficient professional development for teachers. This coincided with an increasing number of Grade 10 students completing two years of teacher education, and by this time, expatriates were not expected to hold primary school teaching positions (International Schools fell under the government's International School Agency which continues today. Students and teachers are Papua New Guinean).

There have been schools using local languages for teaching, e.g., Tolai in East New Britain, Tok Ples or local church languages in remote Morobe, Enga, Milne Bay and Bougainville [89,90]. However, school students from 1960 to 1985 reported they were punished for speaking languages other than English in both government and mission schools [23]. Since the education system meant that students often left their village for a small centre, they were already beginning to use a non-home language. The children then went to high school, senior high school, teachers or other college or university where English and Tok Pisin were the main languages between students. After years of education away from their village, teachers might or might not go back to their village area to teach. Many students struggled to keep their culture and vernacular language and to acquire their family's foundational knowledge. Despite this, they still had strong connections and pride in their family and their family's foundational technological and mathematical knowledge [43,44]. Was the loss irrevocable?

4.2.3. An Indigenous Voice

Before Independence, a committee of educated Papua New Guineans chaired by Alkan Tololo prepared a report for the Department of Education [15,91]. They recognised the importance of students valuing their culture, knowing how to live in their villages, and connecting village knowledge and school knowledge. The second version was more nationalistic in presenting a philosophy recognizing cultures and languages and aiming to preserve the numerous societies of PNG through education [92]. However, there was still an Australian responsible for the Territories, and he could not see how this report could be implemented so he went to the expatriate Dean of Education at the University of Papua New Guinea who hurriedly prepared another education plan [84,92]. There was perhaps some concern that the capable and elite Papua New Guineans should have the opportunity for a western education without the expenses of international schooling [93]. The Australian curriculum schools were replaced by International Agency Schools. These schools enrolled expatriate children, mixed-race children, and the children of professional and business Papua New Guineans.

By this stage, it was recognised that different cultures counted by different cycles and were not all base 10 systems. Since the late 1970s, teachers' colleges encouraged their student teachers to learn basic words of the language of their students if practicing in a village school. This included the counting system, arithmetic operational words and ways of measuring. The Mathematics Education Centre at the PNG University of Technology, the Education Research Unit at the University of Papua New Guinea and the Department of Education drew together many mathematics research studies, resulting in a special issue of the *Journal of PNG Education*, Indigenous Mathematics Project in 1979 [5]. The textbook for secondary schools was called *Mathematics Our Way*. The team leader was a New Zealander, the team members were highly committed PNG curriculum writers. Then the expatriate Oxford Press came with textbooks and teacher guides. Such glossy materials did not last in schools, and curriculum changes were made, perhaps trying to raise standards.

The idea of a preschool education in vernacular languages, while successful in many places, was not supported in 1975 or over the next decade, although Provincial leaders advocated for it and other ideas about implementation. This was perhaps the only value in internal assessment comparisons which, as Weeks [84] pointed out, were almost an obsession with several detrimental effects when tied to a lack of funding, such as no increases in secondary enrolments and the rise of de facto secondary schools (distance education, as well as vocational and technical colleges). There were no new senior high schools as planned and the monies taken from the universities did not reach the schooling sector to increase enrolments at any level. This did not help students who were unable to find employment or the urban drift as some might have thought [84].

The opportunity to hear and develop the Indigenous voice was lost at this stage and indeed for 10 years until 1986 when another Indigenous committee, this time chaired by Paulius Matane, wrote a report for which plans were made [16]. The World Bank continued their financial support and educational reports. They supported the idea of achieving universal primary education through village schooling. Thus, cultures and languages were recognised in schools. There was an opportunity to incorporate village mathematics.

4.2.4. Attempts to Educate following the Indigenous Voice

The government accepted and began implementing the Matane report and the Reform period began [94], albeit rather slowly and inconsistently. The whole structure of education was to change as well as the curriculum. To have village schools, the community needed to provide the school and the teachers' houses. Teachers who knew the language and had achieved a Grade 10 could be recruited. PNG was attempting their own education and not just following externally dominated ideas. The structure of education was changed to three years in elementary schools (Pre-elementary, Elementary 1 and 2), six years in primary schools (Grades 3 to 8), and four years in secondary schools (Grades 9 to 12). The desire for universal education meant elementary schools in villages would use the home language of the children [95,96]. However, setting up this system especially in remote areas was problematic although changes were gradually made [97,98].

There were more PNG educators with higher degrees, and curriculum advisory committees had strong national representation from practicing fields, universities and schools. They set high standards for mathematics and teacher education within the constraints of time. However, they were not necessarily meeting regularly as they were before 1990 to share ideas [13]. In essence, it took about 20 years to implement this change but still there was insufficient teacher education.

4.2.5. Funding Affecting Teacher Education, Research and Materials

Funding was an issue. It was taken away from higher education. The maintenance of higher education institutions and research could not really continue as before, and even getting government funds for salaries was problematic (T. Chan, personal communication, 1997). However, the money did not reach the school sector in terms of implementing the Reform curriculum. There was no funding to assist with the necessary input from Elders into the languages of the schools.

To survive, elementary teachers worked for half a day, so they could tend to their gardens in the afternoon. Teachers first trained under the head teacher and were accredited upon inspection if they knew the local language, had a Grade 10 education and had undertaken training. Then they would be paid a full salary. However, for years, training was often not available and inspectors found it difficult to visit. Many teachers received no or inadequate salaries.

An Australian advisory team, whom it was said had too much say, was involved in developing the system of teacher education for elementary schools. The teacher education courses were set up as Self-Instruction Units with a short introductory workshop, often given as lectures to a large number of teachers in a village area. At first, teacher education was delivered by travelling Institute staff members and then by Provincial Education Officers with varying skills, training and experience. The motivation of teachers varied considerably (personal communication, T. Hamadi, lecturer from PNG Institute of Education, 1997). There was not a full unit on teaching bilingually and transitioning from the vernacular language to the English language and there was not a mathematics unit developed by the Institute using cultural mathematics. Teacher educators and teachers were not sure of how to establish cultural mathematics.

A lack of funding led to loans requiring repayments by the government and more overseas aid with more overseas advisers with their own (neo)colonial views.

4.2.6. Curriculum Changes

There were now two Australian-funded projects for curriculum changes. Both had highly committed expatriates, mostly Australians, and each had a PNG counterpart. One project was the Primary and Secondary Teacher Education Project. This developed new curricula for colleges and encouraged interactive learning, the use of computers for knowledge (e.g., mathematics textbooks and encyclopaedia) and some communication between lecturers (at least principals), and the implementation of gender equality (note the idea of equity was not even considered). There were certainly improvements in teacher education, and some senior staff members undertook Master's degrees. Nevertheless, evaluations suggested that, on the main issues of bilingual education and gender equality, there was still a long way to go and that there was still no compulsory or apparently taught cultural mathematics subject in primary teachers' colleges. Bilingual education was not even considered by mathematics curriculum lecturers [23,99]. Ten years later, both the language and the mathematics subjects were regarded as too difficult for the lecturers replacing those who were able to take advantage of the PASTEP project (personal communication, Jones, UK Volunteer Services Overseas (VSO) team leader, 2015).

It took 10 years after Tololo's committee first emphasised the importance of universal education suitable for students in their ecological environment before the committee chaired by Matane reiterated the same values. Syllabuses were beginning to appear in the early 1990s, but it was not until another aid project began, 25 years after Tololo's report, 23 years after Independence, that a concerted effort began to bring language and culture into the curriculum, especially in elementary schools. However, many issues were not adequately addressed by education authorities or aid organisations [84,100].

The group of Australian Aid advisers took over a year to actually involve PNG Curriculum and Assessment officers in their work. The advisory team from Australian Aid (Curriculum Reform Implementation Project) introduced Outcomes-Based Education (OBE), then common around the world, but the directive required short syllabuses. These proved to be inadequate, and after the elementary school level, there was no initial or strong Indigenous voice in the mathematics curriculum documents. The Teachers' Guides and expensive textbooks that were essential for implementation soon disappeared, just as the earlier books had disappeared. OBE began to be seen as the problem for education by the elite and others. The country was in a dilemma with its lack of funding and new neocolonial curriculum. Like many of the reforms in mathematics education, even going back to the introduction of Dienes blocks, it was inadequately supported by teacher education or inservicing [6]. Teachers and educators wanted their culture involved but could not see its implementation in the curriculum.

4.2.7. The End of Learning Cultural Mathematics in Home Language

In 2012, O'Neill was elected as the Prime Minister with the promise that English would be the language of instruction from the start. This was promised even though so much research supports learning mathematical and other concepts in one's home language and bridging them later into English as the best educational approach, although students were not doing well on Pacific standardised tests [27]. The elementary schools disappeared and were replaced by early childhood education centres for two years (having a play-based first year and picking up the pre-elementary syllabuses from the elementary schools), and then the students had to go to primary school for Grades 1 to 6. Mathematics was no longer called Cultural Mathematics. There were restructures, yet again, of the education school system [101]. In fact, instead of Australian colonialism, Japanese approaches to mathematics began. The English version of a Japanese textbook was now available for teachers to buy if they did not receive it from the Department of Education. Standards based assessments, following world trends again, were introduced.

4.3. Overcoming the Limitations of Neocolonialism

The Matane report was an attempt to overcome PNG's colonial legacy, and this Reform era had high potential. The issue of disappearing funds and the need for more overseas aid to implement changes made the process problematic. Nevertheless, significant changes were started.

The language issue for PNG was significant. Language and culture are closely interconnected. At the time of Independence, with 850 languages, most of which were still the children's home language, English was rarely heard in villages and homes. Since Independence, the lingua franca Tok Pisin began replacing home languages rather than English. The loss of language and of understanding what was being discussed in the classroom if English was the language of instruction was exacerbated by the amount of time students were studying away from home, even from the beginning of school. Due to contact with people from other language groups through schooling and migration to the towns, English and Tok Pisin were taking their toll on local languages and cultural practices, especially those that required young men to carry out the tasks. With the Reform, elementary schools were available in the village or nearby village and they required teachers to have the home language of the majority of students. There was mathematical terminology, especially counting, to be heard in the classrooms. There were few or no resources to assist the teacher and minimal teacher education. However, some of the earlier generation of school children had not heard these words and were excited and proud to hear them in the classroom (personal communications, 1999 to 2003). However, there was no clear implementation process for the policy of transitioning (bridging) from vernacular to English which was to occur towards the end of Grade 2 (the third year of school) and the first year of primary school (Grade 3) and continue throughout later schooling. Nevertheless, there were some good ideas such as the use of 'shell' books that told in pictures a probable village story and on which the local language could be written. These were used to read with the class. The schools that had the support of SIL volunteers (Summer Institute of Linguistics volunteers who were primarily recording the local languages and assisting with the translation of the Bible) were doing well in terms of using phonics for bilingual transitions, teaching materials, and mathematical language but there were still many languages without an agreed orthography. At least 400 languages were being used in schools.

Money was needed for all the language tasks but also for developing each culture's mathematical ways of thinking and discussing mathematical concepts. These tasks involve Elders who were already busy surviving in their rural environment, and it would take time, support and money to discuss and establish cultural mathematics and language. There were no government or aid projects implemented for this. Some teachers were able to bring local language into mathematics besides counting but this was a high expectation without considerable support. The lack of teacher education for elementary schools and the lack of resources meant that students were not adequately learning to read in Tok Ples or English and their mathematics was just as poor.

Educated Papua New Guineans, such as O'Neal, who had opportunities in Australia, still felt that the only good education was one that met overseas levels of education, including in terms of the curriculum and language of instruction. At first, when the standards were introduced, some thought that this meant English had to be used as the standard across the nation with the same lessons taught to every child. An effort was made to educate the advisers who educated senior teachers who were then to inform teachers about the standards. The Assessment and Evaluation Division of the Department of Education was informed of our discussions, and they began to realise that the standards were ways of measuring the achievements of outcomes. Early childhood teachers were already orally monitoring their students during classes. In reaching for the highest standards for their country and avoiding the stigma that Australia now had because of its connection with OBE, the country "looked north" (a former Chief Minister's slogan) to Japan.

The draft of the elementary curriculum was too hard for PNG elementary teachers to follow, and a teachers' guide was prepared based on earlier SIL materials with so-called scripted lessons. There was little connection between the two documents but gradually the syllabus was reduced, and a Japanese-based English textbook was made available to teachers. There were still no links to cultural mathematics. For example, there are many examples of line symmetry and rotational symmetry in PNG cultural artefacts and practices, but the textbook example was on tiles that are probably only found in exclusive hotels and are unknown to village children. Even though OBE and the use of local languages were being blamed for the students' poor level of reading, which was seen as an extension of colonialism, there appears, ironically, to have been even less emphasis and training on cultural mathematics models of concepts in the PNG curriculum than in that carried out within Australian Aid projects. Ethnomathematics was not considered and local language learning was discouraged, although teachers would use it in an ad hoc way if it assisted students to make sense of a concept [25]. Furthermore, in primary and secondary schools, external examinations still mattered for the selection of students for the next phase of education. Assessment tasks and criteria were, however, suggested in the syllabus and textbook for improving students' learning while teaching. A global mathematics system could be identified.

4.4. Examples of Overcoming the Limitations of Neocolonialism

The Reform attempt to introduce a PNG education with goals set by PNGians for PNGians was dismissed by the government due to a lack of vision for the need to support language work at the grassroots level, a lack of knowledge in overseas aid projects on local languages and mathematics, and growing neocolonial attitudes within the country. Nevertheless, a number of projects show what might be possible to overcome neocolonialism.

4.4.1. Teacher Education Units on Ethnomathematics

From 1990 until 2016, several lecturers supported the popular elective subject for teachers, *Mathematics, Language and Culture*, at the University of Goroka. These included Wilfred Kaleva, Rex Matang [102,103] and Charly Muke, who all had relevant Master and/or doctoral degrees. In 1996, they were supported by the American Richard Zepp and in the 2000s by the Australian Kay Owens. The students prepared research reports on the mathematics of their and/or another's PNG culture. They then made links to the PNG curriculum, usually the high school curriculum.

Over 230 of these reports were analysed for reference to mathematical measurement ideas, but in doing this, it was also evident that traditionally, people made use of visuospatial reasoning (see Section 4.4.3). Students were able to identify many areas of the mathematics syllabus (usually those of secondary schools) which related to their village activities and/or artefacts. These especially included designs and orders of steps with links to algebra; measurements, especially length, volume and angle; trigonometry; and geometry. Importantly, students were proud of their ancestors' mathematical thinking and capabilities even if they did not call it mathematics. From this sense of identity, students were appreciating how mathematics could relate to their community life, encouraging their mathematical identity [42,44,104]. This approach to professional identity through cultural identity was also evident in the aforementioned project by architectural students (see Section 3.3). The students proudly called on their cultural backgrounds to develop design, joints, balance and problem-solving skills [29].

4.4.2. Early Childhood and Early School Self-Instruction Unit

A research team (2014–2016) funded under the Australian Research Development Awards developed a Self-Instruction Unit on mathematics teaching and learning that was given to the Institute of Education [105]. The materials included a comprehensive model of teaching that incorporated culture and language, mathematics and early childhood mathematics education. It was very practical. It was accompanied by small books based on activities to be found in villages in PNG on concepts such as composite numbers, measurement of area and number patterns, which lead to the concept of multiplication. The pages could be translated into local languages. There were also videos of cultural mathematical activities, classroom games, and how to use the early mathematics assessment tasks based on Matang's work [102,103] and practiced in workshops. Teachers who joined in the remote workshops valued [106] what they learnt, but the workshops were only 3 to 5 days. This was too little, too late. SIL was beginning to make good inroads into teaching teachers how to teach bilingually and to recognise cultural mathematics (at least their counting systems) when the Reform was stopped.

The materials [105] were also valued by teachers in the Solomon Islands and Tonga, but again funding would be needed to implement the ideas more widely. There was evidence, however, from the Madang Province that the information that was given on computers in the later workshops, rather than that in hard copies, was not being utilised fully. However, the main issue was the difficulty of bringing about change when political changes counter the purpose of using local languages by making English the language of instruction and by changing the curricula.

After years of English (or Tok Pisin) education in the country, it is now difficult to arrest the loss of languages or the devaluing of home languages in education. The loss of cultural mathematics and its use in understanding school mathematics is also evident. Teachers did not have access to national or international research on the strengths of learning and understanding mathematics in home languages or learning in multilingual situations [23,31,33,107–110]. Expertise was not readily available for ongoing professional development. Teachers needed more support for teaching in their home language and transitioning to English. Nevertheless, Australian First Nations are reviving their languages which had been often considered lost due to Australian protectionist and assimilation policies. Perhaps it is not too late for PNG.

4.4.3. Recognising Visuospatial Reasoning as a Key of PNG Mathematical Thinking

Voices such those of Charly Muke and Patricia Paraide on learning in one's home language, teaching bilingually and transitioning to English were being drowned out. However, in 2022, Charly Muke, a plenary speaker at the International Conference on Ethnomathematics 7 (ICEm-7) (hosted online by PNG as well as other countries), said that teachers and administrators now need to do something differently because they were stuck with English. If English is decreed the language of instruction from early childhood onwards, then there need to be alternative ways forward. Muke noted how he sat in primary school not understanding a word but for mathematics, with concrete materials, he figured out what was going on in his own language in his head. Perhaps, said Muke, we need to consider how Papua New Guineans think mathematically when they are doing cultural activities that often involve science, technology, engineering and mathematics.

Firstly, we know they think visuospatially in these contexts. They often call it 'in my head' or 'by eye'. How do they do this? Already the work of the secondary school teachers mentioned above could be extended in discussing this way of reasoning mathematically. Owens noticed the use of ratios as mentioned in many examples in Section 3.2 above. The regular use of a bit of rope as a measuring unit to mark equal distances can be adapted to lessons on measurement. It is also used for circumferences, such when one is collecting and flattening out bamboo for floors, or making a decision on the sizes of pigs. Rope tied to a post and the leg of a man dragging his foot as he walks is used to mark out the circumference of a circular house. Its length and the ultimate volume of a house are visuospatially linked. Sticks are also used for measuring lengths as they can assist in equalizing or halving spaces between posts or spacing *morata* (made from sago leaves sewn over narrow planks of limbom palm) for covering a roof. For measuring shell money, a fathom from one's outstretched arms is used. Steps, hand spans, the fist to the elbow (especially for one's girth) and finger parts are commonly used. The height to one's arm pit, shoulder or head is commonly used for heights or parts of houses. Other readily available tools such as spades are used for deciding the depth of trenches and slopes. Estimates can be made of slightly longer and shorter measures by sight.

The physical embodiment of measures also aids visuospatial reasoning, such as using walked lengths and directions, feeling the swells when sailing, knowing the strength needed for bows, and marking the passing of time when doing activities such as sailing, walking, fishing or sleeping.

4.4.4. Recognising Ethnomathematics

Ethnomathematics is about mathematical processes. In Section 4.4.3, we discussed visuospatial reasoning which incorporates displays and representations. These draw meaning from another important aspect of ethnomathematics, that of learning mathematics through group work and discussion. In our field work, Owens noticed a discussion of the ratio of two sticks forming the sides of a right-angled triangle so that the slope remained the same when one of the sticks needed to be shorter. When an Elder was showing us how to make a difficult diamond pattern while weaving, another Elder pointed out where an earlier line was wrong, creating the mistake. The Elders were discussing the mathematics involved in weaving. One student teacher noted the large numbers of people who would gather for making some decisions such as bride-price or land distribution [44]. Most decisions involving mathematics at a cultural and community level are discussed, so the encouragement of thoughtful discussions mathematically would help mathematical learning in schools. Our workshops asked teachers to notice who was doing the talking in their classrooms [33]. Following the workshops, we found teachers implementing activities to encourage discussions and inquiries. Group work was not just for practice but to enable a discussion or establish more than one answer to a question.

Using representations for numbers is commonplace. Muke noted that his father marked parts of his body for different decades. Bodytally systems can be found in several western provinces [10] (see Table 1). Pig tusks, shells, bamboo pieces and knots on ropes often mark numbers while leaves may be torn from a palm frond to indicate the passing of days.

Muke also recommended the use of traditional games in teaching. He illustrated how their betting game with stones involved number operations and probability while cat's cradles illustrate the ideas of polynomials, sequences and shapes. He also recommends studying their counting systems and representations (see Section 3.1) [19].

Teacher education for multilingual classes and cultural mathematics needs to be compulsory. Our work [10,36,37,42] provides sufficient examples to be used along with students continuing to provide their own examples to encourage this practice as a matter of course. Muke's [25,111] study is illustrative of how good teachers are likely to use local languages for explanation but much more is needed for strengthening the mathematical register in these languages. Using a transliteration for the word 'multiplication', for example, does not really provide the meaning of the word. However, there are numerous expressions for 'equal groups' that would strengthen its meaning. Likewise, establishing the meaning for 'division' can easily be discussed, for example, in sharing long lengths of shell money with relatives [37,96]. This example also could provide the notion of unequal lengths, since more connected people might receive more shell money, and the notion of ratio. Paraide, who initially learnt mathematics in her vernacular Tolai and was exceptionally good when she started school in English, experienced another issue regarding colonialism in the absence of an emphasis on cultural mathematics. Because her parents did not go to school, her expatriate teachers made her feel uncomfortable at school, and this was exacerbated by not having other students to speak her language to discuss problems in mathematics [27].

During the education Reform sparked by the Matane report to use local languages and cultures, notably, there was little work done on local languages for mathematics outside of the counting words. Lean's work, though available in teachers' colleges, universities and Education Departments, was not being utilised to strengthen an understanding of counting systems, cycles or bases. There was no systematic education on how to record mathematical terminology in a local language. Implementing this would be costly and require many skilled people. Could the Teo Māori experience be repeated even in a small way [112]? A list of mathematical terms for primary school were translated into local languages in workshops by teachers and Elders in discussions but often only a few terms were explored in the short time available [31].

Our research over decades on ethnomathematics in PNG is mostly summarised in our books [8,19,28] and papers, for example, [30,51,113].

5. Conclusions

There is a way forward to address Muke's suggestion that schools need to implement mathematical ways of thinking that have a cultural basis. In other words, knowing and valuing ethnomathematics and associated mathematical ways of thinking and learning could be modelled in terms of school mathematics and teaching it [114–116]. Furthermore, teacher education can support these changes and, in the process, change teachers' understanding of mathematics, cultural mathematics, language, the ways of teaching mathematics, and the politics of education in a neocolonial country.

5.1. Ethnomathematics and School Mathematics

Modelling requires the recognition of classes and systems often associated with patterns. The various classes or parts have relationships through the systems.

5.1.1. Classifications

There are sophisticated classification systems for counting, design, art (on cultural artifacts), gambling and kinship (see above and cf. [117,118]). Classification in school geometry, for example, is simplified and its relevance reduced by not having a spatial and cultural component. Every language has classifications. Canoe decorations provide one example [51]. In PNG, many counting systems, especially among the Austronesian Oceanic languages and some neighbours, are based on classifications [119].

5.1.2. Space and Geometry

People's knowledge of places and a mental map of large areas are held in their heads as they traverse forests or seas [35,120]. This knowledge involves position but also their visuospatial knowledge of trees, soils, water movement, winds, reefs, fish, sharks, dugongs, shell fish and other creatures that inhabit different areas. The interconnectivity of the mathematical aspects involved, such as position, shape and vectors, has a purpose. Having a purpose is a main driver for learning, remembering, and making connections between mathematical ideas [42].

People's knowledge of complex trade, intercultural relationships and reciprocal agreements [118] involves complex accounting systems covering many goods and money (PNG kinas or traditional money, e.g., shell *tabu*). Pairs, matching, equality and inequality, and increase and decrease are central to these systems. All these are mathematics concepts. Some mathematical knowledge is recorded, often on the body in some way or by objects and displays [10]. Representations include tattoos, body parts, displays, *bilas* (body decorations), the demarcation of land, house sizing and design [24,28,121,122]. All cultures have mathematical thinking for activities—counting, measuring, designing, locating, playing, explaining [83], understanding, interpreting, inventing and reasoning [123]. These are techniques for and models of cultural ways of thinking mathematically.

5.2. Implementing Ethnomathematics in Schools

Listening and working with Elders is essential [124–126]. Money is needed for this. First, a range of mathematical activities needs to be discussed and the mathematics needs to be teased out and represented, as in mathematical modelling. The mathematics might not easily fit into the school curriculum but could be used for patterns and relations. For example, string figures show algorithms and inventions, while canoe boards show classifications and patterns. Designs, e.g., *kapa* (round leadership symbols made of hard shell and tortoise shell) have diverse symmetries, patterns and angles. Ways of counting have systems, and many can easily be coded (personal communication, Kari, 2003), while others indicate intricacies related to cultural practices. Each basic counting system can be classified using frame words (the basic words from which others are made), cycles (which indicate the systems for making high numbers). In most cases in PNG, this is a more appropriate approach than using the term base. Many are digit-tally systems with (2, 5, 20) cycles [127] (see Table 1). Appropriate teacher education is essential to assist teachers to analyse the counting systems of their students and others in PNG in order to make links between systems including base 10 [14].

In addition to the work on foundational/traditional mathematics given in the two chapters of Paraide et al. [36,37], Owens [32] discussed cultural implications for discussing large numbers, groupings, time and work patterns, transactions, classifications, art and design, and Bino [67] indicated mathematical thinking for model canoe building and sailing. More importantly, she showed the significance of this ethnomathematics for social justice in providing a means for equality and money in a rural situation. In cultural practices, people discuss problems and situations that need resolving. They share their conceptual understandings which are generally associated with visuospatial reasoning which is a holistic way of presenting the problem. Concepts, comparisons, memories of the past related to a problem or object, patterns, parts, size and shape are all considered visuospatially and ecoculturally. An environment supports and constrains patterns of activities and the diversity of responses. These sophisticated ways of thinking need to be expounded more by teachers, villagers, researchers and curriculum writers. This idea of mental mathematical thinking which generally includes visuospatial reasoning [28,35] needs to be captured in mathematics and these thinking skills brought to the fore in school mathematics in PNG if neocolonial losses are to be overcome.

5.3. The Importance of Teacher Education

The study by Quartermaine [13,88] noted the significance of involving teacher educators in decision making and having regular contact between them for generating quality teacher education. In this way, new approaches to mathematics education could be introduced. A decade later, Tapo [11] who was looking at effective teacher education to implement the recommendations of the Tololo and Matane reports, also recognised the importance of curricula changes and the professional development of teacher educators. There is no doubt that, around the world, quality teacher education is at the heart of quality teaching in schools. With the constant turnover of staff and the poor state of teachers' colleges in terms of their facilities, low salaries and gender equity, as well as opportunities for reading research and carrying out research, there is scope for improving teacher education.

However, it is essential to highlight two areas of the curriculum. First is the need for recognising ethnomathematics. In Australia, all teacher educators undertake an awareness of Indigenous education and must achieve competency in this area. There is a limited voice for ethnomathematics but there is a strong voice for Indigenous education. These include the Aboriginal and Torres Strait Islander Mathematics Network in which mathematics, business and education are connected; Yunkaporta's [128] eight ways adopted by the NSW Department of Education; the international group Indigenizing University Mathematics; the Aboriginal Education Consultative Groups who are keen to promote Indigenous education in general; and the Stronger Smarter Centre which focuses on teachers and teacher education encouraging this in students rather than a deficit approach. There are some parts of the National Curriculum for Mathematics in which First Nations are recognised.

A former Queensland Center (called Yumi Deadly) and Chris Matthews used the Goompi model in which reality is abstracted to mathematics through creativity, symbols and cultural bias, and then this mathematics is reflected upon to create a new reality through the same processes. In Australia, as in PNG, often good work in ethnomathematics has been done through projects that run only when a grant is available. It is important that ethnomathematics is part of the curriculum and teacher education for it to remain influential in mathematics and teaching.

The second aspect of ethnomathematics is mathematics teaching and learning. Morris [129] noted that the Goompi model is also applicable to teaching. In particular, a teacher needs to respond to the cultural background of the students. For example, students may share their stories to present their reality, and their teacher needs many teaching strategies to be creative in responding to the students and for them to abstract the mathematics. Their reflection, as prominent in Indigenous cultures, also encourages further application to the real world and to mathematics. She notes the following:

> "The Goompi Model provides an excellent framework for teachers to enact this and follow a cycle of learning that takes students from their everyday reality to the world of mathematics and back again by connecting maths with culture. Throughout the whole process, students' cultural backgrounds are supported and reinforced while also seeing themselves as mathematicians for tomorrow" [129], p. 192.

Ethnomathematics has strong research groups in other countries with displaced and disadvantaged communities, such as in Brazil, Peru, the USA, and African countries, especially Mozambique and the Republic of South Africa. Interestingly, both Nepal and Indonesia have ethnomathematical research studies underway. Ethnomathematics has made a considerable difference in Hawaii where it is recognised at the University providing higher degrees in this area.

Ethnomathematics needs to be a compulsory subject of teacher education in all countries where there are First Nations, colonisation and/or multiculturalism resulting from both the way that the country was formed and from immigration. When this became an elective subject in PNG teachers colleges, there were often various time and organizational constraints. Furthermore, ethnomathematics goes a long way towards meeting the PNG goal of universal education which would provide an education for rural communities without access to cities and paid employment.

Ethnomathematics paves the way for social justice for those not employed in salaried positions [48]. Ethnomathematics provides links to a cultural identity which, in turn, will improve people's mathematical identity which is needed in all places—rural, remote, city, suburban, and small town.

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education sciences

Article

The Fascists Are Coming! Teacher Education for When Right-Wing Activism Micro-Governs Classroom Practice

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Abstract: U.S. educational reform is often the harbinger of global demands on mathematics education practices globally. It behooves teacher education to 'catch up' on current trends, hopefully, to stave off the worst of the fascist tendencies of contemporary politics of education. Past foci on research-based 'best practices' and 'mathematics for all', grounded in liberal multiculturalism (confirming expectations from critical mathematics education scholarship), have become the targets of activists and politicians, turning once-exemplary teachers and their students into casualties. The four phases of *currere* are employed to study this phenomenon and to identify strategies and tactics for teacher education programs. The *currere* methodology indicates that the content of such programs must reduce time devoted to evidence and research-based practice in order to accommodate techniques and knowledge bases for the recognition of right-wing tactics, clowning, slogan parody, and political organizing. Teacher education must further place mathematics teachers' embrace of expertise, authority, and neutrality within broader perspectives on the politics of education, organizational infrastructure strategies and tactics, resource curation, and personal safety planning. Teacher educators themselves must prepare responses to threats on their careers, lives, and families, and proactive 'game plans' for the development of new program curricula.

Keywords: mathematics teacher education; critical mathematics education; socio-political issues; social justice

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1. Introduction

Mathematics education seems to have reached a consensus around the commitment to creating the best experiences with mathematics that are possible for the greatest number of people. Although a range of social, political, cultural, ideological, geographic, and other perspectives are represented in approaches to mathematics teacher education globally, professional associations share a general vision of successfully reaching as many learners as possible [1–5], with related recommendations for teacher education to support their goals [6–8], as well as for those who supervise mathematics teachers [9]. This is often framed within a rhetoric of "mathematics for all" [3,10,11]. A commitment to success and increasingly sustained engagement beyond some level of "basic knowledge" is typically understood as supporting mathematics education for democracies through an educated—that is, informed and enlightened—citizenry [11–13], in which all learners, especially those who are members of socially marginalized communities, deserve the best opportunities. Alternatively, such attention to 'all' is understood as providing knowledge and skills essential to economic progress and technical or industrial innovation [14–16]. Research on effective teachers of underrepresented groups in mathematics has recognized important instructional methods based on challenging the status quo and accepting the political nature of one's professional work [17,18]. Contemporary mathematics education theory has begun the spadework of explicitly connecting instruction and curriculum development to political philosophy [19,20]. Prospective teachers are often helped to identify how to garner the support of potential gatekeepers as they pursue reflective practice [20,21].

However, political movements worldwide have created the opening for direct attacks on forms of mathematics instruction considered to be in conflict with their beliefs by bringing conservative right-wing ideologies out of previously social margins into mainstream governance [22,23]. Most of the sources mentioned in this paper date from before this huge worldwide social movement, during the period in which teacher education practice was grounded in evidence from research amassed over decades. Across the wide range of existing motivations and goals for mathematics education, a general professional agreement coalesced internationally around assessment and instruction that facilitate meaningful mathematics learning, usually described as helping learners to understand mathematical concepts and to build skills upon a conceptual framework, providing opportunities to practice problem solving with mathematics in situations that are intended to model the real world or everyday life, and taking advantage of the funds of knowledge that students bring to the school experience from their home, family, and community life. To truly believe in "mathematics for all" is to commit personally and professionally to a shared project among mathematics educators. Esteemed researchers, award-winning teachers, and highly praised curriculum materials pursued a loosely agreed-upon commitment to this version of mathematics education.

Within the fascist-aligned activism that is emerging, however, those previous authorities have been negatively branded as "woke", "teaching non-math during math", and more. "Woke" originated in African American vernacular to communicate alertness to prejudice and discrimination. In its original sense, a "woke" mathematics teacher would be aware of, and actively attentive to, important societal facts and issues, especially issues of race and social justice. Right-wing politicians and activists subsequently seized this term as a focus for fomenting fear and as a way to rally public sentiment around its derisive ridicule. Mathematics standards and principles grounded in cognitive psychology, sociological understandings with decades of documentation, and data on increasing demonstration of mathematical competencies are now interpreted as "woke" and a serious threat to the public good. They emerged to support a view of mathematics for all that recognized cultural differences as relevant to school learning, the value of connecting mathematics concepts to students' everyday lives, the usefulness of problem-solving contexts, and critical thinking. They tended to be embedded in a generally neutral sense of multiculturalism and democratic citizenship. Such strategies and methods might provide a scaffold for the more political action required to survive the direct assault of fascist activists.

However, a nuanced distinction is that mathematics in school can no longer be merely used to educate active citizens who understand the rhetorical uses of mathematics in social and political life. While even informed civics is threatening to those who mean to manipulate, the recognition that the mathematics itself is embedded with the vestiges of colonialism means that our efforts are more than simple social justice. Politicians and the media that feed off them have fueled parent groups, ultra-right-wing militia groups, and some disgruntled teachers themselves, so that mainstream mathematics education recommendations are perceived as intermingled with threats to white supremacy and other conservative values. Because of this, the culture wars of mathematics need to be a central component of teacher education. What used to be the recruitment of gatekeepers to understand how meaningful mathematics practices can facilitate the gatekeepers' goals [21] has turned into the need to protect oneself and one's family from physical harm, rather than to solicit approval or direct forms of support. While a school administrator once would mediate family complaints about problem-based learning, for example, is now more often a short office visit to terminate employment and escort the teacher from the school. During a less overtly political time in recent history, a teacher would justify research-based practices with multicultural explanations and performance assessment data. Now, teachers have to decide whether they can live with the internal conflict between knowing what the best practices demand and the long-term harmful effects on reduced learning and participation in mathematics that are the known outcomes of the curriculum they are required to implement. Intersecting with such issues is the need for teacher educators to practice self-care and professional development, no longer relying on the assumption that their professional positions convey status, authority, immunity from danger, or sympathy for their personal fears.

All of this might seem like a predictable outcome of neoliberal policies and practices. Critical mathematics education scholars have been describing for decades how the constellation of corporate interests, liberal politics, neoliberal global economic interests and transnational competitions for status and its privileges create supposed "crises in mathematics achievement" buttressed by industries that "measure" achievement, industries that train and "teacher-proof" practices, and industries of scholarship that position academic researchers and practitioners, along with students and families as pawns in a perpetual coalescence of profit-driven needs construction [24–27]. A key point of my analysis to follow is that we need to move past critiques of the neoliberal–transnational nexus of power and act, now and without delay within all positions of power and knowledge, to a fundamental dismantling of mathematics education in all forms.

This article outlines an approach to teacher education that prepares teachers, and those who train and support them, for the political realities of our time. I focus heavily on the situation in the United States. This is a reasonable choice since many innovations and resources originate in the United States. Trends and fads often start in the United States and spread later around the world. The main idea is that U.S. educational reform is often the harbinger of global demands on mathematics education practices globally. It behooves teacher educators to "catch up" on current trends, hopefully to stave off the worst of the fascist tendencies of contemporary U.S. politics of education as it spreads worldwide.

2. *Currere* Methodology Applied to the Circumstances

The methodology guiding this research report is an analysis of news reports and other artifacts of scholarship guided by the four-phase *currere* method of inquiry [28]: (1) regressive: begin with an autobiographical inquiry; (2) progressive: turn toward the imagined future implications of current experience; (3) analytical: interpret the autoethnographic past, present and future; and (4) synthetic: use fragments of experience and artifacts within the other phases to understand the larger cultural and political context, and to make action decisions consistent with one's values, commitments, hopes and dreams. *Currere* work does not necessarily follow a linear sequence from one phase to the next. Some of the work from each phase seeps into the others as the analysis takes on a life of its own, often leading to questions, rather than starting with them, generating hypotheses within the work rather than as starting points for data collection. Scholars experienced in *currere* enter the experience through autobiography, not intending to write memoirs or life histories, but rather to tap into the artifacts of everyday life shared by culture and society. The artifacts do not need to be carefully selected, since any artifact is an opening into the themes, patterns, ideologies and relationships of power that become evident through the *currere* processes [24,25]. The four phases ideally lead to the sort of richer understanding of self-transformations necessary for work toward social justice [26]. As discussed below, teacher educators' reliance on the authority of science (evidence-based research and classroom practices) has mostly amplified the very problems inherent in the politicization of mathematics teaching and learning. If we view teacher education and the self-study of teacher education as disciplines in their own right, then alternative methodologies and perspectives such as phases of *currere* are ways to "stretch [them] from the inside to provide richer, more meaningful studies" [27], p. 290. Teacher education and teacher educator self-study are like mutually concentric circles of perspective sharing porous boundaries of relation among knowledge, institutions, expertise, authority, justice, and community. Both also share a particular relationship with time and space, in the sense that teaching and teacher education are always at once about the present, the past, and the future, with what happens in the moment always related to past experiences and imagined futures. The scholarship in this article is grounded in *currere* methods since they have the potential to focus attention on such "borders between what is and what is becoming, and between notions of the particular and the whole . . . often configured in terms of interpenetrating circles, or spheres of temporary coherence" [29], p. 93.

2.1. Regressive Phase of Research

In the mid-1990s, a year or so before my university was to make its decision about my tenure, my provost received a thick dossier documenting the ways that my scholarship was dangerous to all humankind. I was a strong advocate for new reforms in mathematics education grounded in problem solving, reasoning, communication, multiple representations, writing and literature in mathematics, and interdisciplinary problem-posing approaches. Conservative mathematicians and members of the general public sympathetic to their cause compiled this dossier because they believed that traditional "drill and practice" on basic skills should be the sole experience for all mathematics learners. I was part of a program of "citizen resistance" to the changes supported by mainstream mathematics education research and professional associations, which was slowly influencing policy and practice. In my own case, I was lucky at my fairly liberal, Northeastern U.S. State University. My provost forwarded the dossier to me, with a post-it note saying, "Congratulations on being at the center of controversy in your field." It is possible that this dossier ironically helped me to obtain tenure.

That frightening experience has many precedents in the history of (mathematics) education and has turned out to be a foreshadowing of far more terrifying and dangerous actions that have been percolating—and now taking visceral, real-life forms—in the current socio-political climate. To take one current example, I was horrified in March of 2023 to read of the eminent and esteemed Stanford University mathematics education scholar Jo Boaler, who posted an update on her blog, recounting the personal threats to herself and her family that have been mounting in specificity over the past few years [30]. I experienced an intense bout of PTSD. In her blog post, Professor Boaler supports serious and aggressive debate within a democracy: "Honest academic debate lies at the core of good scholarship." Yet, she asks, "What happens when, under the guise of academic freedom, a small cluster of aligned people distort the truth in order to discredit someone's evidence and boost their own allied position?" She had listed details about personal and academic attacks on her blog since 2012. Yet, these attacks had now escalated. Noting connections between the "disagreements" over research-based practices supporting equity and mathematics for all and her opponents' histories of racist comments, she felt compelled to share the escalation of their attacks, not only upon her, but now upon her family and the members and families of the other four authors of a newly proposed mathematics curriculum framework for the state of California that included realistic threats of physical violence. Approximately one-third of California students demonstrate proficiency in mathematics [31]. Data on high levels of mathematics show indefensible social and racial inequities [32]. Boaler's research has persistently demonstrated the efficacy of active engagement in mathematics learning in comparison with simply practicing procedures. This is consistent with an enormous body of supportive literature from other scholars [33]. Echoing my earlier experience with these folks in the 1990s, yet at a much more sophisticated and heightened level, the most recent and intense time of doxing and online harassment of Boaler included the sharing of her personal emails and very personal details of her daily life on Twitter, and rallying supporters on the Fox News Channel's Tucker Carlson program, leading to the reported direct threats from the many Twitter followers of the opponents and Carlson viewers. Repeated additions of slurs and attacks on her Wikipedia page led Wikipedia to delete them and lock the page. Stanford University police have determined it necessary to include her home in their daily patrols to ensure the safety of her and her family.

2.2. Progressive Phase of Research

Boaler's story is just one small piece of a larger social movement discrediting mainstream professional standards in mathematics education as connected to other politically charged yet research-based reforms in education. The state of Florida recently banned several highly regarded textbook series produced by well-established publishers [34]. Local school districts nationwide are following suit with similar restrictions on what teachers can and cannot do in their classrooms. Teachers in my graduate program courses at my own university tell tales of secretly reading a poem at the start of a lesson to motivate students, fearful that an administrator casually walking by their classroom might overhear and punish them for "not following the script." Others are required to attend so-called "professional development" workshops that "train them" on appropriate algorithms to demonstrate while recognizing "woke" mathematics to avoid in the teacher guides that accompany their text materials. *The New Yorker* magazine, usually devoted to literary essays, arts, and cultural commentary, found this movement so noteworthy that they published a lengthy, historically grounded feature article [35], tracing long-standing controversies to the ways that mathematics is often imprisoned by its associations with order and discipline, through the Sputnik era that sparked a democratization of mathematics beyond mere skills and the 1970s "Back-to-Basics" backlash, the 1980s NCTM unleash of "problem solving", the 1990s backlash labeling problem solving "fuzzy math", and finally the current versions of "social justice" that struggle to link school mathematics with life opportunities beyond low-level skills.

Mathematics educators have always understood the ways that their curriculum can support or challenge contemporary political movements [36]. Textbook writers as diverse as David Eugene Smith [37], World War II German text writers [38], and contemporary writers of mathematics textbooks for social justice [39–41] are excellent examples of what is possible. The unfortunate aspects of the current politicization of mathematics take their most frightening form in the personal attacks on teachers and curriculum developers. Nevertheless, the more pressing concerns have to do with the ways that these attacks are physical reductions of a misunderstanding of research-based practices. As Brian Lindaman, Chair of the California Framework Committee, is quoted in *The New Yorker*, it is indeed important to ask teachers to take on the challenge of addressing equity by finding ways to help all "students find the joy and beauty of math early" and "Many of them do, it's just that somewhere along the line that gets taken out and they stop seeing the beauty in mathematics. And we think that has something to do with the way it gets taught." Yet, the opponents of research-based practices have reduced all of the proven effective pedagogical techniques into a simplistic misrepresentation of research's support for "the importance of children seeing real-world applications of what they were doing, in a way that made sense to them" [34] with a naïve misconstrued misconception of "wokeness". Since wokeness is linked to every malady from mass shootings to lower military recruitment, inflation, youth unemployment, and more, the public micro-management of classroom practices from outside the classroom demonstrates a social fear of science-based policy. Boaler directs a research center at Stanford that works jointly with neuroscientists on brain development and brain-based classroom practices. The movements' rigid adherence to mathematics as purely utilitarian skills rather than a meaningful collection of concepts through which life can be enriched also limits the potential for creative lessons. In the Pennsylvania town of Perkasie, the school district felt it necessary to apologize to parents and students after one of its high school teachers assigned mathematics homework that included what the district is calling "adult content without a proper context" [42]. The assignment focused on the highly regarded autobiography, *I Know Why the Caged Bird Sings*, by the acclaimed and celebrated poet Maya Angelou. Despite Angelou's stature, one parent declared, "I still think it is a problem . . . It shouldn't be on a math test" [42].

2.3. Analytical Phase of Research and Tentative Conclusions

One type of response taken by mathematics educators to this public disparagement of their work is to dig further into the research literature that supports their recommended practices. Boaler itemizes the attacks on her work to be specifically about Research on Timed Testing and Math Anxiety, Research on Mindset, and Neuroscientists' Studies of Brain Responses to Mistakes [43]. This strategy is highly ineffective as a reaction to the manufactured controversies, since the threatening opponents dismiss any academic, research-based evidence as further examples of what they oppose. The anti-intellectualism of this dismissal is historically linked to fascism and terrorism [44]. A country that is not officially fascist in its doctrines can experience fascist politics. Those politicians base their efforts on dividing society and demonizing specific groups. Anti-intellectualism is one of the "pillars of fascist politics", along with myths of certain subgroups as superior to others. The micro-management of mathematics education practices is one example of the overarching social tensions plaguing the United States and other countries around the world today. Stanley [44] describes the intimidation of scholars into silence as one key feature of fascist politics. Gender studies scholars, scholars of Islam and the Middle East, and those, for example, in African American Studies and Indigenous Studies whose work threatens a mythic, rosy picture of the past they wish our education systems to present as objects of veneration, are equally attacked alongside mathematics education researchers and policy innovators. By harshly attacking those who seek to show the truth in its full complexity, writes Stanley, the activists threatening them undermine the search for truth and valid, academically informed professional practices. These overarching social tensions come full circle in establishing right-wing activist methods, platforms, and discourse that are readily applied to school mathematics in addition to other disciplines and general school policies. At the same time, popular views of mathematics as utilitarian skills make it possible for engaging in meaningful mathematics practices to be experienced as threatening to the false security of rules and memorized, "correct" ways to behave. Options and nuances in mathematics, when they accompany flexible inclusive and culturally welcoming school policies, seem like the ground is shifting under one's feet.

Much as we mathematics educators want to focus on known best practices for classroom teaching and learning, the implications of the current political trends are that we must simultaneously gain the skills and practices to fight the fascist takeover of our political institutions as well as the media coverage of our classroom practices. Right-wing extremists have two main strategies. Prospective teachers need practice in recognizing and responding to these strategies.

One is to set up situations where they can play the victim and increase sympathetic interest in their cause, or at least polarize and confuse the issues. Their other favorite tactic is to threaten and use violence to increase the fear level of their opponents. Symbols are less costly than actually injuring and killing, and so they like to use symbols like clubs, tiki torches, burning crosses, or dressing in sheets or military-style uniforms. By getting there first, they set the tone, but they do not win just by doing that. Their victory comes when their opponents respond in a like manner and try to out-intimidate the intimidators [45]. Successful anti-fascist strategies involve a commitment to re-framing the issues and confronting the right-wing escapades with a strikingly different tone that confounds them [45–47]. Boaler's attempts to calmly list the substantial evidence for her work set a different tone. She maintains a stance of expertise and cultivated neutrality, clothed in the historically rhetorical use of science and the scientific method grounded in data as defining truth. This fails to counter the fascists because they begin with a different rhetoric; the fascists begin with faith in certain beliefs, and use their faith to determine their own truth, distinct from data that they have no confidence in. Rochelle Gutiérrez [48], in contrast, leaps directly to the motivations behind, and social consequences of, traditional skill-based mathematics education practices: "The majority of the messages I received (A sample from Gutiérrez' article: 'Whites will always rule and always achieve . . . I suggest you go and shoot yourself . . . Someone as evil, stupid and racist as you should be barred from teaching . . . I'm going to spearhead a media campaign to have you fired.' [48], p. 72) were racist, misogynistic, and vulgar. Responders did not seek dialogue, analyze the argument I made, or even maintain a context of mathematics education. Instead, the messages were an attack on the person, a smear campaign" [48], p. 70. A case study of Boaler's experience should be a central feature of teacher education in mathematics so that future teachers can understand why explanations grounded in research are not well received and serve only to fuel the flames of right-wing activists.

Other professionally informed mathematics educators use the strategy of clowning—resisting authorities not by direct resistance or arguments, but by participating in ways that make it clear how those in power are ridiculous and silly. For example, they might show up in enormous numbers at every public event, such as school board meetings and government policy debates, in absurd costumes that mock the idiocy of the right-wing positions. Clowning puts the fascists in the awkward position of not being able to claim victimhood or a need for aggressive behaviors. Stealing the empty fascist slogans and replacing them with highly similar but silly alternatives repeated loudly often defuses the originals. Or, mathematics educators might satirize expectations or policies in events with exaggerated parody. Imitations of what right-wing politicians and school administrators demand can be performed in public places in ways that make the outrageously harmful or demotivating forms of classroom practice visible. Memorial events for the victims of right-wing actions (horrible test scores, loss of job prospects, lower incomes, increased health risks, and so on) create public displays counter to the fascist messages. Establishing safe houses to repair the damage wrought by fascist extremism (community-run mathematics circles using state-of-the-art practices) can protect the defenseless victims of fascist policies while telegraphing to the wider public how similar the expected practices are with more overt forms of abuse that require "safe houses". Prospective teachers can practice such actions. They can gather with activist teacher groups at public venues to clown, create posters and protest signs for school board and local government meetings about school policies and programs, and invent their own satirical events that highlight the dangers of bad practices that are finding increasing prominence. Teacher educators and their protegees might study the Orange Alternative activists in 1980s Poland [49,50] for inspiration and examples of humorous tactics that defused personal risks of violence at the same time as communicating a deeper and more substantial critique than would have been possible with direct evidential confrontation. The Orange activists dressed as silly clowns and gave the police who were intending to perform crowd control flowers and kisses; they placed small gnomes all over the city without explanation overnight, to gently use non-threatening decorations as a sign that dissenters were everywhere, ready to sprout and be seen at a moment's notice. Experiential curriculum using Theater of the Oppressed techniques for working through alternative options for classroom practice has been highly successful in teacher education [51,52]; teacher educator self-study groups, coursework in teacher preparation, and professional development workshops can use the same techniques to role play, problem solve, and imagine action options in proactively anticipating or responding to right-wing hostilities and policy development.

Mathematics educators have 'grown up' in an ideology that believes in the efficacy of critical thinking. Our own successes with mathematics have often led to admiration for what we think of as our intelligence. We often consider a good mathematics student to be "better than just a good student". We are comfortable with a position of power and privilege tied to the assumption that mathematics is an essential school subject. We take pride in that presumed necessity. Since few people are as knowledgeable about mathematics as we are, we assume that we should be able to define our work and judge how well we are at it. Most of us believe that mathematics is neutral, that it is people applying mathematics that create its political uses, rather than imagine that there are ideological or political bases to our mathematics. Ethnomathematics, critical mathematics education, and social justice mathematics each challenge that naïve standpoint. Yet, it is not simple even for us to interrogate the underlying ideology of our discipline. The complex history of school mathematics, with its European origins, overlaid upon a global and transnational variety of national school systems, with their own histories of colonialism, Indigenous communities, social movements, and cultural and political conflicts, is challenging and often overwhelming. In the current climate, scholarship and social movements describing and challenging the oppression that many have suffered (post-colonial nationalism, queer rights, women's rights, Indigenous rights, etc.) become direct threats in the zero-sum culture war wielded by aggressive, right-wing activists. School mathematics is the ultimate example of how the legacies of colonialism are entrenched ideology, since it is usually believed to be universal and neutral, despite its very specific history. Teaching traditional mathematics curricula perpetuates the epistemicide and erasure of other mathematical traditions and cultural practices worldwide [53]. Ethnomathematical attempts to integrate local mathematics practices with traditional school mathematics often perpetuate coloniality by only understanding local practices in terms of the traditional curriculum, rather than using local practices to reconceptualize what mathematics is in the first place [54,55]. To act on these realities really would change the nature of school mathematics! Informed mathematics educators would seem to be ethically obligated to act on this understanding, that is, to accept the threatening nature of what they are doing when they make it possible for school mathematics to be meaningfully learned by all students. In other words, what we believe in is, we need to admit, threatening to fascists. The fascists know this better than we do [36].

It is time to double down on the important collective work of clowning and safe houses, and zero in on the re-framing that we otherwise would not think we should need to do. We cannot merely claim our expertise and expect folks to accept our pronouncements about best practices. We must mock the opponents, and protect each other from their personal violent attacks. They are coming. Believe me. I know. What skills must we teach teachers, as they come under attack? Gutiérrez [46] suggests the following: that political knowledge is equally important to cognitive psychology and subject matter content; how to find curricular materials that challenge dominant paradigms of mathematics as a neutral collection of concepts and skills; organizational infrastructure strategies and tactics (tools and techniques of documenting attacks, countering them, and protecting colleagues personally and professionally); and resource curation (networks, public media contacts, sympathetic policy wonks, etc.). This is not a huge leap from some of what we already do. Yet, it demands that we think more about how to shift from trying to convince teachers with little to no experience as learners in model mathematics classrooms to use "best practices"—introducing multiple representations, building understanding through classroom conversation and students' listening to each other, and integrateing mathematics with the arts, reading, and writing. This shift requires apprenticeship beyond an embrace of new instructional methods in the political action and defensive tactics required to survive while following those recommendations for best practices. Ladson-Billings [17] could gently talk about teachers seeing themselves as political beings. Now that is not enough. One must actually act politically just to be able to do the most highly recommended work.

3. Synthetic Phase and Results

The first three phases of *currere* took us on a journey from personal autobiography into a process of extrapolation and alarming trajectories, demonstrating the personal dangers that mathematics teachers face in the midst of right-wing conservatism. These personal dangers reflect dismal demands that the teachers implement less than optimal instructional strategies, combined with potentially harmful forms of assessment and evaluation. The consequences of such micro-governance of classroom curriculum and instruction include knowledge gaps, or ignorance, regarding self-care and evidence-based best practices and a naïve adherence to an ideology of expertise, which promotes ineffectual resistance. These consequences point to the need for techniques of self-care and political action training for both teacher educators and their students.

3.1. Knowledge Gaps

Through questions about the historical context that guided a genealogy of the circumstances surrounding contemporary teacher education, it became evident that liberal commitments to diversity and "mathematics for all" are inadequate as principles or standards for mathematics education. Teacher education needs to reconsider its positionality in terms of social justice, supporting high achievement for all students, and training teachers in evidence-based practice. It is not that evidence-based practice is bad. What is problematic is how the attention to practicing professionally recommended instructional methods creates knowledge gaps. Both teacher educators and the new teachers (indeed, most mathematics teachers) presume an authority originating in their subject-related and pedagogical expertise. This presumption does not prepare them for criticism, nor for threats on their lives and the lives of their family members. Critical teacher education has tended to emphasize the use of evidence-based and research-based justifications for instructional innovations [56]. This emphasis leaves teacher educators and teachers ignorant of the strategies they need to protect themselves and to provide students with a socially just mathematics education, as well as the ways that seemingly good commitments merely sustain and perpetuate forms of inequity tied to the legacies of colonialism [53].

3.2. Critical Mathematics Teacher Education Leaves 'Believers' Unprepared

Programs that encourage such an orientation to mathematics teaching hope to produce mathematics teachers committed to "mathematics for all" and other socially responsible goals tied to social justice and environmental stewardship [57,58]. Such a perspective on mathematics teacher education buttresses the ideology of expertise, ironically feeding into the perception by right-wing activists that mathematics teachers are misguided crusaders for "social justice" and, thus, in need of strong motivations to change their practices. The perspective further leaves teachers unprepared for hostile takeovers of the curriculum and school policies, because it does not think of political readiness as an important component of teacher training. For example, Skovsmose [59] clarifies critical mathematics education pedagogy requires two criteria, subjective and objective. Teacher educators use these criteria to provide prospective teachers with techniques for selecting appropriate mathematics problems [60]. In other words, critical mathematics education and other versions of mathematics for social justice help teachers to further believe they are highly trained and knowledgeable, and that the knowledge that matters most is the knowledge that helps them to teach specific content through well-chosen content-based activities. The subjective criterion requires that the problem appear relevant to students within their conceptual understanding. The objective criterion requires the use of data and detail to view an existing social issue in order to facilitate deeper understanding. The integration of mathematics and social justice is claimed to potentially spark meaningful conversations about issues impacting local communities and beyond, preparing individuals for citizenship. Neither criterion helps when conceptual understanding, the integration of social justice values, and active citizenship are both ridiculed and feared by members of the community.

Critical mathematics education as an orientation promoting critical citizenship and democracy [60–62] might have circumvented the current situation. The present moment requires something other than a critique, a style of coalition building across the critical democracy–liberal divide that recognizes a potential destruction of school mathematics altogether, with teachers and students as the primary causalities, and researchers as bystanders. My argument is to begin with the political action training I sketch in the next section, while saving the more nuanced political critique.

3.3. Political Action Training Can and Should Be Major Components of Teacher Education Programs

Several specific forms of training in strategies and tactics have been highlighted for responding specifically to fascist activism. These include, first of all, techniques for recognizing when curricular changes or education policies as ultra-right-wing attempts to micro-govern mathematics teaching and learning, such as setting up situations where conservative members of the community can play the victim to polarize and confuse the issues while increasing sympathy for their cause; threatening and using violence to increase fear; spreading communities with symbols that evoke the threats while not necessarily carrying them out; and intentionally using rhetoric that does not include evidence-based research. Teacher educators and their protégées are further aided by the understanding that rhetoric of expertise and shock at the attacks on their authority grounded in knowledge are useless as tools of resistance to the threats. Successful organizing that can be practiced within teacher education includes clowning, parody, the establishment of "safe houses" and the rehearsal of responses informed by the traditions of the Theater of the Oppressed [63].

Malcolm X [64] described the problem of liberal do-gooders in 1965: they are no different from right-wing reactionaries except that they are more hypocritical. They pretend to be friends with the oppressed in order to use them as tools in an ongoing game of power with the reactionaries. Smart community organizers know that this level of self-awareness is best nurtured through political action with a common agenda [65,66]: "we need to be planning . . .—organizing ourselves, strengthening our networks, building resources and models, setting precedents, and creating infrastructure and policies . . . and professional organizations" [66], p. 20.

4. Discussion

Mathematics education theory development has struggled to find a place for political interpretations of practice. Certainly, there has been valuable scholarship contributing to our understanding of both the content and the orchestration of curriculum as political [67–76]. Yet, this steady accumulation of scholarship does not prepare even highly esteemed scholars of mathematics education for a political onslaught against "mathematics for all". It once seemed like our project was to, first, create models of mathematics education that recognize the diversity of the populations that they serve [77]. Tony Brown [77] suggests that "discourses of mathematics education research often aspire to cultural and historical continuity whilst simultaneously operating on the notion of a consensual ideal dependent on the future achievement of social models with adequate levels of resources". Such discourses, he argues, "rest on oversimplified models of social change that inflate the operative role of individual teachers and mathematics education researchers in affecting broader teaching and learning cultures" [77], p. 1. Ethnomathematics-informed teacher educators recognize deeply entrenched legacies of colonialism that place Tony's well-formulated argument in an even more challenging context [59]. The *currere* experience reported in this article further suggests that the nuances of ethnomathematics also simplify teacher education discussion by not attending simultaneously to the impending backlash that will accompany the successful integration of social justice with mathematics education. Political action training is "the next step".

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