 DOI: https://doi.org/10.1016/j.geomphys.2021.104288

COUNTING MAXIMAL LAGRANGIAN SUBBUNDLES OVER AN ALGEBRAIC CURVE

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Abstract. Let C be a smooth projective curve and W a symplectic bundle over C. Let $LQ_e(W)$ be the Lagrangian Quot scheme parametrizing Lagrangian subsheaves $E \subset W$ of degree e. We give a closed formula for intersection numbers on $LQ_e(W)$. As a special case, for $g \geq 2$, we compute the number of Lagrangian subbundles of maximal degree of a general stable symplectic bundle, when this is finite. This is a symplectic analogue of Holla's enumeration of maximal subbundles in [14].

1. Introduction

Let C be a smooth projective curve of genus $g \geq 2$, and V a vector bundle of rank r and degree d over C. For $1 \leq k \leq r-1$, a rank k subbundle E of V is called a *maximal subbundle* if $\deg(E)$ is maximal among all subbundles of rank k. Consider the following enumerative problem.

What is the number of rank k maximal subbundles of V, when it is finite?

Classically, Segre [27] and Nagata [22] proved that if $d \not\equiv g \mod 2$, then a general stable bundle of rank two has 2^g maximal line subbundles. Later, Holla [14] gave an explicit formula enumerating maximal subbundles in general (see also [19], [23] and [28]).

The goal of this article is to give an analogous result for symplectic bundles. To pose the problem, let us recall some basic notions. Let L be a line bundle of degree ℓ. An *L-valued symplectic bundle* is a vector bundle W on C equipped with a nondegenerate skewsymmetric bilinear form $\omega\colon W \otimes W \to L$. Such a W has rank $2n$ for some $n \geq 1$. From the induced isomorphism $W \cong W^\vee \otimes L$, we have $\deg(W) = n\ell$. In fact, it can be shown that $\det(W) \cong L^n$ (see [3, § 2]).

A subsheaf $E \subset W$ is called *isotropic* if $\omega(E \otimes E) = 0$. By linear algebra, $\mathrm{rk}\,(E) \leq n$. If $\mathrm{rk}\,(E) = n$ then E is said to be *Lagrangian*. A *maximal Lagrangian subbundle* of W is one whose degree is maximal among all Lagrangian subsheaves of W.

Let W be an L-valued symplectic bundle over C. For each integer e, let $LQ_e(W)$ be the Lagrangian Quot scheme parameterizing Lagrangian subsheaves $[E \to W]$ with $\deg(E) = e$; equivalently, quotients $[q\colon W \to F]$ with F coherent of rank n and degree $n\ell - e$, and $\mathrm{Ker}(q)$ isotropic. The scheme $LQ_e(W)$ is projective, and contains the quasiprojective subscheme $LQ_e^\circ(W)$ consisting of Lagrangian subbundles. By [8, Proposition 2.4], the expected dimension of $LQ_e(W)$ is

$$D(n,e,\ell) \ := \ -(n+1)e - \frac{n(n+1)}{2}(g-1-\ell).$$

Based on results in [9], we will see the following.

Proposition 3.2. *Let L be a line bundle of degree ℓ and W an L-valued symplectic bundle of rank $2n$ which is general in moduli. Write*

$$e_0 \ := \ -\left\lceil \frac{1}{2} n(g-1-\ell) \right\rceil .$$

1. *A maximal Lagrangian subbundle of W has degree e_0, and $LQ_{e_0}(W) = LQ_{e_0}^\circ(W)$.*
2. *If $n(g-1-\ell)$ is even, then $LQ_{e_0}(W)$ is a smooth scheme of dimension zero.*

This indicates how Lagrangian Quot schemes enter the picture. Our problem reduces to evaluating the integral

$$\int_{LQ_{e_0}(W)} \Theta_0 \ .$$

where Θ_0 denotes the fundamental cycle of the zero dimensional scheme $LQ_{e_0}(W)$.

To compute this integral, more generally we find a closed formula for integrals $\int_{LQ_e(W)} \Theta$, where W is an arbitrary symplectic bundle, e an integer and Θ a certain 0-cycle on $LQ_e(W)$. (We work with cycles rather than cohomology classes; see the paragraph subsequent to Definition 1.1). To obtain the desired formula, we follow essentially the method of Holla [14] for the case of vector bundles. An important ingredient in the argument of [14] is the fact, proven in [24, § 6], that for small enough values of e, the scheme $\mathrm{Quot}^{r-k,d-e}(V)$ parameterizing subsheaves of rank k and degree e in W is of the expected dimension, and that a general point of any component corresponds to a vector subbundle. For the present work, an analogous statement on Lagrangian Quot schemes is required. This follows from [8]. (We mention that in both [24] and [8] the respective Quot schemes are even shown to be irreducible.)

Let us give a sketch of the strategy for obtaining the formula. We begin with some terminology.

Definition 1.1. *We say that $LQ_e(W)$ has property $\mathcal{P}$ if every component of $LQ_e(W)$ is generically smooth of the expected dimension $D(n, e, \ell)$, and moreover a general point corresponds to a subbundle of W.*

Unlike in the case of $\mathrm{Quot}^{r-k,d-e}(V)$, Lagrangian degeneracy loci on $LQ_e(W)$ do not represent the corresponding Chern classes (see [17, § 4.5]), so we cannot make up a cohomology class on $LQ_e(W)$ whose integral against the fundamental class $[LQ_e(W)]$ gives geometric information like Gromov–Witten invariants. As an alternative, we shall work with 0-cycles, whose degree gives the same information. However, in general $LQ_e(W)$ may exhibit pathologies. Therefore, to begin with we restrict to those W and e for which $LQ_e(W)$ has property $\mathcal{P}$. In this case, given an integer $t \geq 0$ and a monomial $P(\alpha)$ in a set of formal variables $\alpha = (\alpha_1, \ldots, \alpha_n)$, we construct a 0-cycle $\Theta(P(\alpha); t)$ on $LQ_e(W)$ and then we define $N^w_{C,e}(W; \Theta(P(\alpha); t))$ as the degree of $\Theta(P(\alpha); t)$; that is,

$$N^w_{C,e}(W; \Theta(P(\alpha); t)) \;:=\; \int_{LQ_e(W)} \Theta(P(\alpha); t).$$

This extends linearly to any homogeneous polynomial $P(\alpha)$. This number is invariant under a deformation of $LQ_e(W)$, as expected in Gromov–Witten theory.

Next, we extend this to an intersection theory on an arbitrary Lagrangian Quot scheme $LQ_e(W)$. An essential step is to embed $LQ_e(W)$ in $LQ_e(\widetilde{W})$ where $\widetilde{W}$ is a symplectic Hecke transform of W such that $LQ_e(\widetilde{W})$ has property $\mathcal{P}$. Here $LQ_e(W)$ is identified with the intersection of t suitable Lagrangian degeneracy loci on $LQ_e(\widetilde{W})$ corresponding to the maximal length strict partition ρ_n (Corollary 4.8). Then the intersection number $\widetilde{N}^w_{C,e}(W; \Theta(P; 0))$ for $t = 0$, which is our main object of interest, is defined as

$$\widetilde{N}^w_{C,e}(W; \Theta(P; 0)) \;:=\; N^{w+tn}_{C,e}(\widetilde{W}; \Theta(P; t)).$$

Once $\widetilde{N}^w_{C,e}(W; \Theta(P; 0))$ is shown to be independent of the choice of $\widetilde{W}$, it is straightforward to see that the two definitions of intersection number coincide when $LQ_e(W)$ has property $\mathcal{P}$.

We then use this intersection theory to answer the enumerative problem stated at the outset. As the integral $\int_{LQ_e(W)} \Theta_0$ is intractable without further conditions on W, we follow [14] and link $\widetilde{W}$ with the trivial symplectic bundle $\mathcal{O}_C^{2n}$ by another sequence of Hecke transforms. Then this process gives a bridge between the integral and a genus g Gromov–Witten invariant of the Lagrangian Grassmannian $\mathrm{LG}(\mathbb{C}^{2n})$. Using results from [5], [6] and [7], the latter, in turn, can be connected to a genus zero Gromov–Witten invariant of $\mathrm{LG}(\mathbb{C}^{2n})$, whose closed formula is given by a Vafa–Intriligator-type formula. Using results from [5], [6] and [7], the latter can be connected to a genus zero Gromov–Witten invariant of $\mathrm{LG}(\mathbb{C}^{2n})$, whose closed formula is given by a Vafa–Intriligator-type formula.

For $LQ_e(W)$ not enjoying property $\mathcal{P}$, this approach is an alternative to the use of virtual classes in developing an intersection theory, as done in [20] for the usual Quot schemes. It would be interesting to follow the approach of [20] for Lagrangian Quot schemes.

Lastly, let us point out a relation with a quantum field theory. In the case of vector bundles, Marian and Oprea [21] gave a description of a topological quantum field theory (TQFT) studied by Witten in terms of intersection theory on Quot schemes. In particular, they showed that the so-called Verlinde number defined on the moduli of vector bundles is equal to an intersection number on a suitable Quot scheme. Very recently, Goller [12] developed a weighted TQFT to compute intersection numbers explicitly on some Quot schemes. Since all these constructions have counterparts in the symplectic setting, it would be interesting to reveal a connection between our invariants $\widetilde{N}^w_{C,e}(W;P)$ and symplectic Verlinde numbers, and to give a description of weighted TQFT in the symplectic setting.

The paper is organized as follows. In § 2, we review the quantum cohomology of Lagrangian Grassmannians. In § 3, we give basic properties of the Lagrangian Quot scheme $LQ_e(W)$ and discuss property $\mathcal{P}$ and the non-saturated locus. In § 4, we define Lagrangian degeneracy loci on $LQ_e(W)$ and investigate their properties and behavior under Hecke transforms. In § 5, we develop an intersection theory on $LQ_e(W)$ and find relations among intersection numbers. In § 6, we give our main result on enumerating maximal Lagrangian subbundles (Corollary 6.2). At the end, the numbers are explicitly computed for ranks two and four (Corollary 6.3).

Acknowledgements. The first and second authors would like to thank Oslo Metropolitan University for hospitality during a stay in Tønsberg in June 2018, when this work was completed. The first and second authors were supported by Basic Science Research Programs through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1A6A3A11930321 and NRF-2017R1D1A1B03034277 respectively). The third author sincerely thanks Konkuk University, Hanyang University and the Korea Institute of Advanced Study for financial support and hospitality in March 2017. The authors sincerely thank the referee for carefully reading the paper and pointing out the serious error, of which correction is a main task in the revision.

Conventions and notation. Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers. Unless otherwise stated, C is a complex projective smooth curve of genus $g \geq 0$, and W is an L-valued symplectic bundle of rank $2n \geq 2$. We write $\ell := \deg(L)$. The fiber at $p \in C$ of a bundle $V \to C$ is denoted by V_p. We shall most often consider points of Quot schemes as subsheaves. For a bundle V of rank r and degree d, the point in $\mathrm{Quot}^{r-\mathrm{rk}\,(E),d-\deg(E)}(V)$ determined by a subsheaf $E \subseteq V$ will be denoted $[E \to V]$ or simply by E.

2. Quantum cohomology of Lagrangian Grassmannians

In this section, we record some known facts on quantum cohomology of Lagrangian Grassmannians.

2.1. **Notations.** Fix a positive integer n. A *partition* λ is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. The nonzero λ_i are called the *parts* of λ. The number of parts is called the *length* of λ and is denoted $l(\lambda)$. The sum $\sum_{i=1}^n \lambda_i$ is called the *weight* of λ, and denoted $|\lambda|$.

Denote by $\mathcal{R}(n)$ the set of all partitions $(\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \leq n$. A partition λ is called *strict* if $\lambda_1 > \cdots > \lambda_l$ and $\lambda_{l+1} = \cdots = \lambda_n = 0$, where $l = l(\lambda)$. Let $\mathcal{D}(n)$ be the set of all strict partitions $(\lambda_1, \ldots, \lambda_n) \in \mathcal{R}(n)$ such that $\lambda_1 \leq n$. We usually write $(\lambda_1, \ldots, \lambda_l)$ for a (strict) partition $\lambda = (\lambda_1, \ldots, \lambda_l, 0, \ldots, 0)$ of length l, if no confusion should arise. For $\lambda \in \mathcal{D}(n)$, let λ' be the *dual* partition of λ, whose parts complement the parts of λ in the set $\{1, 2, \ldots, n\}$. Set $\rho_n = (n, n-1, \ldots, 1) \in \mathcal{D}(n)$.

Later, we shall also use the following notations to state the Vafa–Intriligator-type formula in § 2.5. For $n = 2m+1$, set

$$\mathcal{T}_n \ := \ \{J = (j_1, \ldots, j_n) \in \mathbb{Z}^n \mid -m \leq j_1 < \cdots < j_n \leq 3m+1\},$$

and for $n = 2m$, set

$$\mathcal{T}_n \ := \ \left\{ J = (j_1, \ldots, j_n) \in \left(\mathbb{Z} + \frac{1}{2}\right)^n \mid -m + \frac{1}{2} \leq j_1 < \cdots < j_n \leq 3m - \frac{1}{2} \right\}.$$

For $J = (j_1, \ldots, j_n) \in \mathcal{T}_n$ and $\zeta := \exp\left(\frac{\pi\sqrt{-1}}{n}\right)$, we write $\zeta^J := (\zeta^{j_1}, \ldots, \zeta^{j_n})$. Define a subset $\mathcal{I}_n$ of $\mathcal{T}_n$ by

$$\mathcal{I}_n \ := \ \left\{ J = (j_1, \ldots, j_n) \in \mathcal{T}_n \mid \zeta^{j_k} \neq -\zeta^{j_l} \text{ for } k \neq l \right\}.$$

Note that $\prod_k \zeta^{j_k} = \pm 1$ for $J = (j_1, \ldots, j_n) \in \mathcal{I}_n$. We put

$$\mathcal{I}_n^e \ := \ \left\{ J \in \mathcal{I}_n \mid \prod\nolimits_k \zeta^{j_k} = 1 \right\}.$$

2.2. **Symmetric polynomials.** Let $X = (x_1, \ldots, x_n)$ be an n-tuple of variables. For $i = 1, \ldots, n$, let $H_i(X)$ (resp. $E_i(X)$) be the i-th complete (resp., elementary) symmetric function in X. Then for any partition λ, the Schur polynomial $S_\lambda(X)$ is defined by

$$S_\lambda(X) \ := \ \det\left[H_{\lambda_i+j-i}(X)\right]_{1\leq i,j\leq n}$$

where $H_0(X) = 1$, and $H_k(X) = 0$ for $k < 0$.

The $\widetilde{Q}$-polynomials of Pragacz and Ratajski [25] are indexed by the elements of $\mathcal{R}(n)$. For $i \geq j$, define

$$\widetilde{Q}_{i,j}(X) \ = \ E_i(X)E_j(X) + 2\sum_{k=1}^{j}(-1)^k E_{i+k}(X)E_{j-k}(X).$$

For any partition λ, not necessarily strict, and for $r = 2\lfloor(l(\lambda)+1)/2\rfloor$, let B_λ be the $r \times r$ skewsymmetric matrix whose (i,j)-th entry is given by $\widetilde{Q}_{\lambda_i,\lambda_j}(X)$ for $i < j$. The $\widetilde{Q}$-polynomial associated to λ is defined by

$$\widetilde{Q}_\lambda(X) \ = \ \mathrm{Pfaff}(B_\lambda).$$

Note that from the definition of $\widetilde{Q}_\lambda(X)$, for $\lambda = (k)$ with $0 \leq k \leq n$ we have $\widetilde{Q}_{(k)}(X) = E_k(X)$. We often write $\widetilde{Q}_k(X)$ for $\widetilde{Q}_{(k)}(X)$.

2.3. **Degeneracy loci of type C.** Let W be a vector bundle of rank $2n$ over a scheme Z, equipped with a symplectic form $\omega\colon W \otimes W \to \mathcal{O}_Z$. Let E be a vector bundle of rank n. Fix a homomorphism of vector bundles $\psi\colon E \to W$ with isotropic image; equivalently, such that the composite $E \to W \to W^\vee \xrightarrow{\psi^t} E^\vee$ is zero, where $W \to W^\vee$ is the isomorphism induced by the symplectic form ω. Assume that W admits a complete flag of isotropic subbundles

$$H_\bullet : \ 0 \ = \ H_0 \ \subset \ H_1 \ \subset \ \cdots \ \subset \ H_n,$$

where $\mathrm{rk}\,(H_k) = k$. For any subbundle $G \subset W$, set

$$G^\perp \ := \ \{w \in W \mid \omega(w \otimes v) = 0 \text{ for all } v \in G\},$$

the orthogonal complement of G with respect to the symplectic form.

Definition 2.1. *The degeneracy locus of type C associated to a strict partition $\lambda \in \mathcal{D}(n)$ is defined as*

$$Z_\lambda(H_\bullet) \ := \ \Big\{ z \in Z \mid \mathrm{rk}\left(E \ \to W/H^\perp_{n+1-\lambda_i}\right)_z \ \leq \ n+1-i-\lambda_i \quad \textit{for each } i \Big\}.$$

Note that $(W/H^\perp_{n+1-\lambda_i})_z \cong \left(H^\vee_{n+1-\lambda_i}\right)_z$.

Degeneracy loci of type A are defined analogously, and their classes can be expressed in terms of the Chern classes of the vector bundles involved (see [11]). For type C, we have a similar expression when ψ is everywhere injective. For F a bundle of rank n and λ a partition, the class $\widetilde{Q}_\lambda(F)$ is defined as $\widetilde{Q}_\lambda(X)$ with the variable x_i specialized to the ith Chern root of F. Recall that if $\lambda = (k)$ where $1 \le k \le n$, then $\widetilde{Q}_\lambda(X) = E_k(X)$. This implies that $\widetilde{Q}_\lambda(F) = c_k(F)$.

As motivation, we quote the following special case of [17, Corollary 4].

Proposition 2.2. *Suppose that Z is Cohen–Macaulay, and that the subbundles $H_1, \ldots, H_n$ are trivial over Z. Assume that $Z_\lambda(H_\bullet)$ is of pure codimension $|\lambda|$. If $\psi\colon E \to W$ defines a Lagrangian subbundle, then in the Chow group $\mathrm{CH}^{|\lambda|}(Z)$, we have $[Z_\lambda(H_\bullet)] \;=\; \widetilde{Q}_\lambda(E^\vee)$.*

Proof. See [17, Corollary 4] and the discussion on [17, p. 1718]. □

Remark 2.3. A few words on Proposition 2.2 are in order. Intersection theory on a scheme Z in general involves a product of cycle classes (or cohmology classes) as an integrand of the integral, and a product of cycle classes is not always possible, unlike of cohomology classes [10, Chap.6]. However if suitable cycles represent corresponding cohomology classes as in Proposition 2.2, then we can obtain a product of cycles via the corresponding cohomology classes. This may make the intersection theory on Z relatively easier. On the other hand, the condition that in Proposition 2.2, $\psi\colon E \to V$ be a vector bundle injection is necessary in general. A counterexample is described in [17, § 4.5] in a case where ψ is not everywhere injective. In fact, as the referee pointed out, this happens in our case. Thus we cannot use a product of cycles, and so shall take an alternative approach to an intersection theory in Section 5.

2.4. **Cohomology of Lagrangian Grassmannians.** Let V be a vector space of dimension $2n$ equipped with a symplectic form $\omega\colon V \otimes V \to \mathbb{C}$. Let $\mathrm{LG}(V)$ be the Lagrangian Grassmannian parametrizing Lagrangian subspaces in V. Over $\mathrm{LG}(V)$, there is a universal exact sequence of bundles

$$0 \;\to\; \mathbf{E} \;\to\; \mathbf{V} \;\to\; \mathbf{E}^\vee \;\to\; 0,$$

where $\mathbf{V} = \mathrm{LG}(V) \times V$. Clearly $\mathbf{V}$ admits a symplectic form induced from V, and the subbundle $\mathbf{E} \subset \mathbf{V}$ is isotropic. Let

$$H_\bullet :\; H_1 \;\subset\; H_2 \;\subset\; \cdots \;\subset\; H_{n-1} \;\subset\; H_n$$

be a complete isotropic flag in V. This induces a complete flag of isotropic subbundles

$$\mathbf{H}_\bullet :\; \mathbf{H}_1 \;\subset\; \mathbf{H}_2 \;\subset\; \cdots \;\subset\; \mathbf{H}_{n-1} \;\subset\; \mathbf{H}_n,$$

in $\mathbf{V}$, where $\mathbf{H}_k := \mathrm{LG}(V) \times H_k$. Then for strict partitions $\lambda \in \mathcal{D}(n)$, the degeneracy loci $Z_\lambda(\mathbf{H}_\bullet)$ are called *Schubert varieties*. By Proposition 2.2, we obtain

$$[Z_\lambda(\mathbf{H}_\bullet)] \;=\; \widetilde{Q}_\lambda(\mathbf{E}^\vee). \tag{2.1}$$

It is well known that the classes $\{\sigma_\lambda := [Z_\lambda(\mathbf{H}_\bullet)] \,|\, \lambda \in \mathcal{D}(n)\}$ form a basis of the Chow group of $\mathrm{LG}(V)$. For $1 \le k \le n$, we have the length 1 partition (k). We write σ_k for the *special Schubert class* $\sigma_{(k)} \in \mathrm{CH}^k(\mathrm{LG}(V))$.

2.5. **A Vafa–Intriligator-type formula.** Fix a symplectic vector space $V = \mathbb{C}^{2n}$, and write $\mathrm{LG}(n)$ for $\mathrm{LG}(\mathbb{C}^{2n})$. In this subsection, we state a Vafa–Intriligator-type formula for $\mathrm{LG}(n)$, which computes the Gromov–Witten invariants. We begin by defining these invariants.

The *degree* of a morphism $f \colon C \to \mathrm{LG}(n)$ is defined as the intersection number

$$\int_{[\mathrm{LG}(n)]} f_*[C] \cdot \sigma_1.$$

Such an f defines a Lagrangian subbundle E_f of the trivial symplectic bundle $\mathcal{O}_C^{\oplus 2n}$, and $\deg(E_f) = -\deg(f)$. The Gromov–Witten invariant is informally defined as follows. For the precise definition, see [26].

Definition 2.4. *Let $p_1, \dots, p_m$ be distinct points of C. Let $\lambda^1, \dots, \lambda^m \in \mathcal{D}(n)$ be strict partitions. Fix $d \in \mathbb{Z}$. We define the* Gromov–Witten invariant $\langle \sigma_{\lambda^1}, \dots, \sigma_{\lambda^m} \rangle_{C,d}$ *as follows. If*

$$\sum_{j=1}^{m} |\lambda^j| \;=\; \frac{1}{2} n(n+1)(1-g) + d(n+1), \tag{2.2}$$

then $\langle \sigma_{\lambda^1}, \dots, \sigma_{\lambda^m} \rangle_{C,d}$ is the number of morphisms $f \colon C \to \mathrm{LG}(n)$ of degree d, such that for each i, we have $f(p_i) \in Z_{\lambda^i}(\gamma_i \cdot \mathbf{H}_\bullet)$ for a general choice of symplectic transformation $\gamma_i \in \mathrm{Sp}_{2n}(\mathbb{C})$.

If (2.2) does not hold, we define $\langle \sigma_{\lambda^1}, \dots, \sigma_{\lambda^m} \rangle_{C,d}$ to be zero.

Now it is well known (see [26, p. 262]) that the Gromov–Witten invariant is independent of the points p_i and the curve C, depending only on the genus g. Thus we write $\langle \sigma_{\lambda^1}, \dots, \sigma_{\lambda^m} \rangle_{g,d}$ for $\langle \sigma_{\lambda^1}, \dots, \sigma_{\lambda^m} \rangle_{C,d}$.

The (small) quantum cohomology ring of $\mathrm{LG}(n)$ is defined via the genus zero three-point Gromov–Witten invariants [18]. Let q be a formal variable of degree $n+1$. The ring $qH^*(\mathrm{LG}(n), \mathbb{Z})$ is isomorphic as a $\mathbb{Z}[q]$-module to $H^*(\mathrm{LG}(n), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$. The multiplication in $qH^*(\mathrm{LG}(n), \mathbb{Z})$ is given by the formula

$$\sigma_\lambda \cdot \sigma_\mu \;=\; \sum_{d \ge 0} \sum_{\nu} \langle \sigma_\lambda, \sigma_\mu, \sigma_{\nu'} \rangle_{0,d}\, \sigma_\nu\, q^d,$$

where ν ranges over all strict partitions with $|\nu| = |\lambda| + |\mu| - (n+1)d$. Note that the specialization of the (complexified) quantum cohomology ring at $q = 1$ is given by

$$qH^*(\mathrm{LG}(n), \mathbb{C})_{q=1} \;:=\; qH^*(\mathrm{LG}(n), \mathbb{C}) \otimes \mathbb{C}[q]/(q-1).$$

As a complex vector space, this is isomorphic to $H^*(\mathrm{LG}(n), \mathbb{C})$.

Now we are ready to give a Vafa–Intriligator-type formula for $\mathrm{LG}(n)$ for an arbitrary genus g.

Proposition 2.5. *Let C be a curve of genus g with m marked points. For strict partitions $\lambda^1, \lambda^2, \ldots, \lambda^m \in \mathcal{D}(n)$ and $d \geq 0$, the genus g Gromov–Witten invariant for* $\mathrm{LG}(n)$ *is computed as*

$$\langle \sigma_{\lambda^1}, \sigma_{\lambda^2}, \ldots, \sigma_{\lambda^m} \rangle_{g,d} \;:=\; 2^{n(g-1)-d} \sum_{J \in \mathcal{I}^e_{n+1}} S_{\rho_n}(\zeta^J)^{g-1} \widetilde{Q}_{\lambda^1}(\zeta^J) \cdots \widetilde{Q}_{\lambda^m}(\zeta^J)$$

whenever $\sum_{i=1}^m |\lambda^i| = \frac{n(n+1)}{2}(1-g) + (n+1)d$, and zero otherwise.

Proof. For $g = 0$, the formula was given in [6]. For an arbitrary g, we have the formula from [5, p. 1263]:

$$\langle \sigma_{\lambda^1}, \sigma_{\lambda^2}, \ldots, \sigma_{\lambda^m} \rangle_{g,d} \;=\; \mathrm{tr}\left([\mathcal{E}^{g-1} \sigma_{\lambda^1} \cdots \sigma_{\lambda^m}] \right), \tag{2.3}$$

where $\mathcal{E}$ is the quantum Euler class (cf. [1]) of $\mathrm{LG}(n)$ in $qH^*(\mathrm{LG}(n), \mathbb{C})_{q=1}$, and $[\sigma]$ denotes the quantum multiplication operator on $qH^*(\mathrm{LG}(n), \mathbb{C})_{q=1}$ determined by σ. Then the formula follows from [7, Theorem 6.6] where the eigenvalues of $[\sigma]$ were computed for an arbitrary $\sigma \in qH^*(\mathrm{LG}(n), \mathbb{C})_{q=1}$. $\square$

3. Lagrangian Quot Schemes

Let $\mathrm{Mor}^d(C, \mathrm{LG}(n))$ be the space of morphisms of degree d from C to $\mathrm{LG}(n)$. Informally, Gromov–Witten invariants of $\mathrm{LG}(n)$ might be thought of as intersection numbers on $\mathrm{Mor}^d(C, \mathrm{LG}(n))$. However, it is necessary to compactify $\mathrm{Mor}^d(C, \mathrm{LG}(n))$ in order to develop an intersection theory. An alternative compactification to Kontsevich's moduli space of stable maps is the Lagrangian Quot scheme, which is practical for computing intersection numbers. In fact, Kresch and Tamvakis [18] used a Lagrangian Quot scheme for $W = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2n}$ to compute the quantum cohomology ring of $\mathrm{LG}(n)$. This may indicate that Lagrangian Quot schemes are important moduli spaces whose intersection theory is of interest. In this section, we describe the Lagrangian Quot schemes of a symplectic bundle over a curve of any genus.

3.1. **Definition and notation.** Let C be a smooth projective curve of genus g, and W a vector bundle of rank r and degree d on C. Let $\mathrm{Quot}^{r-k,d-e}(W)$ be Grothendieck's Quot scheme parameterizing subsheaves $[E \to W]$ of rank k and degree e, or equivalently quotients $[W \to F]$ where F is coherent of rank $r-k$ and degree $d-e$. Let $\mathrm{Quot}^{r-k,d-e}(W)^\circ$ be the open sublocus

$$\left\{ [\psi \colon E \to W] \in \mathrm{Quot}^{r-k,d-e}(W) \mid \psi \text{ is a vector bundle injection} \right\}.$$

Recall that $\mathrm{Quot}^{r-k,d-e}(W)$ is a projective variety, possibly having other components than the closure of $\mathrm{Quot}^{r-k,d-e}(W)^\circ$. If $\pi_C \colon \mathrm{Quot}^{r-k,d-e}(W) \times C \to C$ is the projection, then on $\mathrm{Quot}^{r-k,d-e}(W) \times C$ we have the universal exact sequence of sheaves

$$0 \ \to \ \mathcal{E} \ \to \ \pi_C^* W \ \to \ \mathcal{Q} \to \ 0.$$

Suppose now that $\mathrm{rk}\,(W) = 2n$ and W is equipped with a symplectic form $\omega \colon W \otimes W \to L$, where L is a line bundle of degree ℓ. As ω induces an isomorphism $W \cong W^\vee \otimes L$, in particular $\deg(W) = n\ell$.

The *Lagrangian Quot scheme* $LQ_e(W)$ is the subscheme of $\mathrm{Quot}^{n,n\ell-e}(W)$ consisting of Lagrangian subsheaves. To see that $LQ_e(W)$ is a closed subscheme of $\mathrm{Quot}^{n,n\ell-e}(W)$, consider the map

$$\sigma \colon \mathrm{Quot}^{n,n\ell-e}(W) \ \longrightarrow \ H^0(C, \wedge^2(\mathcal{E}^\vee) \otimes L)$$

sending $[j \colon E \to W]$ to $\omega \circ \wedge^2 j \colon \wedge^2 E \to L$. This σ defines a section of the sheaf $\pi_*(\wedge^2(\mathcal{E}^\vee) \otimes \pi^* L)$, where $\pi \colon \mathrm{Quot}^{n,n\ell-e}(W) \times C \to \mathrm{Quot}^{n,n\ell-e}(W)$ is the projection. The subscheme $LQ_e(W)$ is nothing but the zero locus of σ. (For another argument, see [8, Lemma 2.2].)

Hence $LQ_e(W)$ is a compactification of the quasiprojective scheme $LQ_e^\circ(W)$ of Lagrangian subbundles, possibly having components in addition to the closure of $LQ_e^\circ(W)$. For the trivial symplectic bundle $W = \mathcal{O}_C^{\oplus 2n}$ and $e \leq 0$, the subscheme $LQ_e^\circ(W)$ coincides with the space $\mathrm{Mor}^{-e}(C, \mathrm{LG}(n))$ of morphisms of degree $-e$.

3.2. **Property $\mathcal{P}$ on $LQ_e(W)$.** In this subsection, we discuss further the property $\mathcal{P}$ which was defined in § 1. To give a motivating example of an $LQ_e(W)$ having property $\mathcal{P}$, we use the notion of very stability as studied in [4]. A symplectic bundle $W \cong W^\vee \otimes L$ is called *very stable* if the bundle $K_C L^{-1} \otimes \mathrm{Sym}^2 W$ has no nonzero nilpotent sections. The following is proven similarly to [19, Lemma 3.3].

Lemma 3.1. *Let W be a very stable symplectic bundle. Then we have $H^1(C, L \otimes \mathrm{Sym}^2 E^\vee) = 0$ for every Lagrangian subsheaf $E \subset W$.*

By [8, Proposition 2.4], the Zariski tangent space of $LQ_e(W)$ at a point $[E \to W] \in LQ_e^\circ(W)$ is $H^0(C, L \otimes \mathrm{Sym}^2 E^\vee)$. Hence the expected dimension of $LQ_e(W)$ is

$$\chi(C, L \otimes \mathrm{Sym}^2 E^\vee) \;=\; -(n+1)e - \frac{n(n+1)}{2}(g-1-\ell) \;=\; D(n,e,\ell). \tag{3.1}$$

Proposition 3.2. *Let L be a line bundle of degree ℓ and W an L-valued symplectic bundle of rank $2n$ which is general in moduli. As before, we set $e_0 := -\left\lceil \frac{1}{2} n(g-1-\ell) \right\rceil$.*

1. *A maximal Lagrangian subbundle of W has degree e_0, and $LQ_{e_0}(W) = LQ_{e_0}^\circ(W)$.*
2. *If $n(g-1-\ell)$ is even, then $LQ_{e_0}(W)$ is a smooth scheme of dimension zero.*

Proof. The first statement in (1) follows from [9, Theorem 1.4 and Remark 3.6]. For the rest: As the Lagrangian subsheaves parametrized by $LQ_{e_0}(W)$ have maximal degree in W, every point of $LQ_e(W)$ corresponds to a Lagrangian subbundle, for otherwise, the subbundle generated by a subsheaf of degree e_0 would be a Lagrangian subbundle of higher degree. Hence in this case $LQ_e(W) = LQ_e^\circ(W)$.

For (2): By [8, Proposition 2.4], if $H^1(C, L \otimes \mathrm{Sym}^2 E^\vee) = 0$, then $LQ_e(W)$ is smooth of dimension $\chi(C, L \otimes \mathrm{Sym}^2 E^\vee) = 0$ at $[E \to W]$. By Lemma 3.1, this holds for all $[E \to W]$ if W is very stable; and by [4], very stable bundles are dense in moduli. Statement (2) follows. $\square$

In particular, if W is generic and $n(g-1-\ell) \equiv 0 \mod 2$, then $LQ_{e_0}(W)$ has property $\mathcal{P}$. More generally, regarding property $\mathcal{P}$, we cite the main result of [8].

Proposition 3.3. *Let W be a symplectic bundle of degree $n\ell$ over C. Then there exists an integer $e(W)$ such that if $e \leq e(W)$, then $LQ_e(W)$ is an irreducible and generically smooth variety of dimension $D(n,e,\ell)$, of which a general point corresponds to a Lagrangian subbundle. In particular, if $e \leq e(W)$, then $LQ_e(W)$ has property $\mathcal{P}$.*

3.3. **Evaluation maps.** Let W be a symplectic bundle. For each $p \in C$, we have an evaluation map

$$\mathrm{ev}_p \colon LQ_e^\circ(W) \;\to\; \mathrm{LG}(W_p)$$

taking a Lagrangian subbundle to its fiber at p. The map ev_p is defined at $[E \to W]$ if and only if E is saturated at p. Note that ev_p depends on the degree e, but since this will always be clear from the context, to ease notation we simply write ev_p.

The following result assures the surjectivity of the ev_p in cases of interest. Let $e(W)$ be as defined in Proposition 3.3.

Proposition 3.4. *There is an integer $f_0(W) \geq e(W)$ such that for all $e \leq f_0(W)$, the evaluation map $\mathrm{ev}_p \colon LQ_e^\circ(W) \to \mathrm{LG}(W_p)$ is surjective for all $p \in C$.*

Proof. The existence of $f_0(W)$ is a special case of [8, Proposition 4.4]. The inequality $f_0(W) \geq e(W)$ follows from the definition of $e(W)$ at the end of the proof of [8, § 4.2] (note that in [8] the space $LQ_e(W)$ is denoted $LQ_{-e}(W)$ and the subsheaves have degree $-e$ instead of e). $\square$

3.4. **Nonsaturated loci of Lagrangian Quot schemes.** Let W be a symplectic bundle and $E \subset W$ a Lagrangian subsheaf. We denote by $\overline{E}$ the saturation of E in W. This is the sheaf of sections of the subbundle generated by E, or equivalently, the inverse image in W of the torsion subsheaf of W/E. For fixed e and for $r \geq 0$, we write

$$\mathcal{B}_r \ := \ \{[E \to W] \in LQ_e(W) \mid \overline{E}/E \text{ is a torsion sheaf of length } r\}. \tag{3.2}$$

This is a locally closed subscheme of $LQ_e(W)$. The following is clear from the definitions (compare with [2, Theorem 1.4]).

Lemma 3.5. *The association $E \mapsto \overline{E}$ defines a surjective morphism*

$$f_r \colon \mathcal{B}_r \ \to \ LQ_{e+r}^\circ(W).$$

If $F \subset W$ is a Lagrangian subbundle of degree $e+r$, then $f_r^{-1}(F)$ is canonically identified with $\mathrm{Quot}^{0,r}(F)$. In particular, $\mathcal{B}_r \to LQ_{e+r}^\circ(W)$ is topologically a fiber bundle with irreducible fibers of dimension nr.

Notice that $f_0 \colon \mathcal{B}_0 \to LQ_e^\circ(W)$ is the identity map.

4. Degeneracy loci for Lagrangian Quot schemes

In this section we define Lagrangian loci on $LQ_e(W)$ and establish all results necessary for the intersection theory in Section 5.

4.1. **Lagrangian degeneracy loci.** Let W be an L-valued symplectic bundle, and set $\mathbb{X} := LQ_e(W)$. Write $\pi_C \colon \mathbb{X} \times C \to C$ for the projection. There is an exact sequence of sheaves over $\mathbb{X} \times C$ given by

$$0 \ \to \ \mathcal{E} \ \to \ \pi_C^* W \ \to \ \widehat{\mathcal{E}} \tag{4.1}$$

where $\mathcal{E}$ is the universal subsheaf and $\widehat{\mathcal{E}} = \mathcal{E}^\vee \otimes \pi_C^* L$. For $p \in C$, denote by $\mathcal{E}(p)$, $\mathcal{W}(p)$ and $\widehat{\mathcal{E}}(p)$ the restrictions to $\mathbb{X} \times \{p\}$ of $\mathcal{E}$, $\pi_C^* W$ and $\widehat{\mathcal{E}}$ respectively.

Identifying the restrictions $\widehat{\mathcal{E}}(p) \cong \mathcal{E}(p)^\vee$, we obtain

$$0 \ \to \ \mathcal{E}(p) \ \to \ \mathcal{W}(p) \ \to \mathcal{E}(p)^\vee. \tag{4.2}$$

Note that $\mathcal{W}(p) = \mathbb{X} \times W_p$ is a trivial symplectic bundle. Let

$$H_\bullet : \ H_1 \ \subset \ H_2 \ \subset \ \cdots \ \subset \ H_{n-1} \ \subset \ H_n$$

be a complete flag of isotropic subspaces in W_p, and

$$H_n \ = \ H_n^\perp \ \subset \ H_{n-1}^\perp \ \subset \ \cdots \ \subset \ H_2^\perp \ \subset \ H_1^\perp$$

the corresponding coisotropic flag of orthogonal complements. This induces a flag of trivial subbundles

$$\mathcal{H}_{n-k+1}^\perp \ := \ \mathbb{X} \times H_{n-k+1}^\perp : \ 1 \leq k \leq n$$

of $\mathbb{X} \times W_p$. Following [18], we will define Lagrangian degeneracy loci on $\mathbb{X}$. Each Lagrangian subsheaf map $\psi \colon E \to W$ induces a map $E_p \to W_p / H_{n+1-k}^\perp$ for each k. We adapt Definition 2.1 to this case.

Definition 4.1. *For $p \in C$ and $\lambda \in \mathcal{D}(n)$, define $\mathbb{X}_\lambda(H_\bullet; p)$ as*

$$\Big\{ [\psi \colon E \to W] \in \mathbb{X} \mid \mathrm{rk} \left(\mathcal{E}(p) \to \mathcal{W}(p)/\mathcal{H}_{n+1-\lambda_i}^\perp \right)_\psi \ \leq \ n+1-i-\lambda_i, \quad 1 \leq i \leq l(\lambda) \Big\} .$$

Remark 4.2. If $\lambda = (k)$, then $\mathbb{X}_{(k)}(H_\bullet; p)$ is determined by the single isotropic subspace H_{n+1-k} of W_p. Also, for $\rho_n = (n, n-1, \ldots, 1)$, we have

$$\mathbb{X}_{\rho_n}(H_\bullet; p) \ = \ \{ [\psi \colon E \to W] \in \mathbb{X} \mid \psi(E_p) \subseteq H_n \} , \tag{4.3}$$

which depends only on H_n. Henceforth we shall denote $\mathbb{X}_{(k)}(H_\bullet; p)$ and $\mathbb{X}_{\rho_n}(H_\bullet; p)$ by $\mathbb{X}_k(H_{n+1-k}; p)$ and $\mathbb{X}_{\rho_n}(H_n; p)$ respectively. Also, for any λ, we denote by $\mathbb{X}_\lambda^\circ$ the saturated part $\mathbb{X}_\lambda \cap LQ_e^\circ(W)$.

The following lemma will be used in the estimation of various dimensions later.

Lemma 4.3. *Let W be a symplectic bundle of rank $2n$ and $F \subset W$ a Lagrangian subbundle. Fix an integer $r \geq 1$. For $1 \leq k \leq n$, let H be an isotropic subspace of dimension $n+1-k$ in a fiber W_p, such that the composed map $F_p \to W_p \to W_p/H^\perp$ is surjective. Then the locus*

$$\{ [E \to F] \in \mathrm{Quot}^{0,r}(F) \mid \text{ the map } E_p \to F_p \to W_p/H^\perp \text{ is not surjective} \} \tag{4.4}$$

is empty or of codimension at least k in $\mathrm{Quot}^{0,r}(F)$.

Proof. Since $F_p \to W_p/H^\perp$ is surjective, E belongs to (4.4) only if E fails to be saturated at p. On the other hand, since

$$\mathrm{length}(F_p/E_p) \ \leq \ \deg(F/E) \ = \ r,$$

we have $\mathrm{rk}\,(\psi_p \colon E_p \to F_p) \ge n - r$. We shall prove the lemma by showing that for each l satisfying

$$\max\{0, n-r\} \ \le \ l \ \le \ n-1,$$

the intersection of (4.4) with the set

$$\{[E \to F] \in \mathrm{Quot}^{0,r}(F) \mid \mathrm{rk}\,(E_p \to F_p) \ = \ l\} \tag{4.5}$$

has codimension at least k in $\mathrm{Quot}^{0,r}(F)$.

We begin by estimating the dimension of (4.5). A point E of this locus is determined by the choice of $\mathrm{Im}\,(E_p \to F_p) =: \Pi \in \mathrm{Gr}(l, F_p)$ and a point in $\mathrm{Quot}^{0,r-(n-l)}(F_\Pi)$, where F_Π is the elementary transformation

$$0 \ \to \ F_\Pi \ \to \ F \ \to \ F_p/\Pi \ \to \ 0.$$

Thus (4.5) is of dimension at most

$$\begin{aligned} \dim \mathrm{Gr}(l, F_p) + \dim \mathrm{Quot}^{0,r-(n-l)}(F_\Pi) \ &= \ l(n-l) + n(r-(n-l)) \\ &= \ rn - (n-l)^2. \end{aligned}$$

If $l \le n-k$, then $E_p \to W_p/H^\perp$ cannot be surjective, so (4.5) is contained in (4.4). But in this case $k \le n-l \le (n-l)^2$, so (4.5) has codimension at least k in $\mathrm{Quot}^{0,r}(F)$.

Suppose on the other hand that $l \ge n-k+1$. Noting that

$$\mathrm{Ker}\left(\psi(E_p) \to W_p/H^\perp\right) \ = \ \psi(E_p) \cap \left(H^\perp \cap F_p\right),$$

we see that E belongs to (4.4) if and only if

$$\dim\left(\psi(E_p) \cap H^\perp \cap F_p\right) \ \ge \ l-n+k \ = \ \dim \psi(E_p) - \dim W_p/H^\perp + 1.$$

This is a Schubert condition on $\psi(E_p) \in \mathrm{Gr}(l, F_p)$, of codimension $l-n+k$. Thus for $l \ge n-k+1$, the intersection of (4.5) with (4.4) is of codimension

$$(n-l)^2 + (k+l-n) \ = \ k + (n-l)(n-l-1) \ \ge \ k$$

in $\mathrm{Quot}^{0,r}(F)$. (Notice that we have equality if $l = n-1$.) This completes the proof. $\square$

Recall now the locus $\mathcal{B}_r$ defined in (3.2).

Proposition 4.4. *Let $p \in C$ be a point and let $H \subset W_p$ be an isotropic subspace of dimension $n+1-k$, where $1 \le k \le n$.*

1. *For general $\gamma \in \mathrm{Sp}(W_p)$, the intersection $\mathbb{X}_k(\gamma \cdot H; p) \cap \mathcal{B}_r$ is either empty or of codimension at least k in $\mathcal{B}_r$.*
2. *Suppose that $e \ge e(W)$, so $\mathbb{X} = LQ_e(W)$ has property $\mathcal{P}$ and the evaluation map $\mathrm{ev}_p \colon LQ_e^\circ(W) \to \mathrm{LG}(W_p)$ is surjective. Then if $\gamma \in \mathrm{Sp}(W_p)$ is general, $\mathbb{X}_k(\gamma \cdot H; p)$ is nonempty and generically smooth of codimension k in $\mathbb{X}$, and a generic element is saturated.*

Proof. (1) We adapt the approach of [2, Theorem 1.4]. For a fixed $r \geq 0$, we define the degeneracy locus

$$\mathbb{Y}_k^\circ(H;p) \ := \ \left\{ [F \to W] \in LQ_{e+r}^\circ(W) \mid \mathrm{rk}\,(F_p \to W_p/H^\perp) \leq n-k \right\}.$$

This is the preimage by $\mathrm{ev}_p \colon LQ_{e+r}^\circ(W) \to \mathrm{LG}(W_p)$ of the degeneracy locus

$$Z_k(H) \ = \ \{\Lambda \in \mathrm{LG}\,(W_p) \ | \ \dim(\Lambda \cap H^\perp) \geq k\}$$

(cf. Definition 2.1), which has codimension k. Now $\mathrm{ev}_p|_{LQ_{e+r}^\circ(W)}$ is a morphism, so by [15, Theorem 2 (i)], for a general $\gamma \in \mathrm{Sp}(W_p)$, the locus $\mathbb{Y}_k^\circ(\gamma \cdot H;p)$ is either empty or of codimension k in $LQ_{e+r}^\circ(W)$.

Consider now the set

$$\{E \in \mathbb{X}_k(\gamma \cdot H;p) \cap \mathcal{B}_r \mid \mathrm{rk}\,(\overline{E}_p \to W_p/H^\perp) \ \leq \ n-k\}.$$

This is precisely $f_r^{-1}\left(\mathbb{Y}_k^\circ(\gamma \cdot H;p)\right)$, where $f_r \colon \mathcal{B}_r \to LQ_{e+r}^\circ(W)$ is as defined in § 3.4. By Lemma 3.5 and the last paragraph, $f_r^{-1}\left(\mathbb{Y}_k^\circ(\gamma \cdot H;p)\right)$ is either empty or has codimension k in $\mathcal{B}_r$.

(For the remainder of the proof, we do not use the assumption that γ is general.) It remains to treat the situation where $\overline{E}_p \to W_p/H^\perp$ is surjective. In this case, E can belong to $\mathbb{X}_k(H;p)$ only if E fails to be saturated at p; in particular, $r \geq 1$. By Lemma 4.3, for fixed $F \in LQ_e(W)_{e+r}^\circ(W)$, all components of the locus

$$\{[E \to F] \in \mathrm{Quot}^{0,r}(F) | \text{ the map } E_p \to F_p \to W_p/H^\perp \text{ is not surjective}\}$$

are of codimension at least k in $\mathrm{Quot}^{0,r}(F) \cong f_r^{-1}(F)$. Letting F vary in $LQ_{e+r}^\circ(W)$, we see that the locus of $E \in \mathbb{X}_k(p;H)$ with $\overline{E}_p \to W/H^\perp$ surjective is of codimension at least k in $\mathcal{B}_r$. Thus (1) is proven.

(2) Since by hypothesis $\mathrm{ev}_p \colon LQ_e^\circ(W) \to \mathrm{LG}(W_p)$ is surjective, for general $\gamma \in \mathrm{Sp}(W_p)$ the locus $\mathbb{X}_k^\circ(\gamma \cdot H;p)$ is nonempty and of codimension k by [15, Theorem 2 (i)]. By part (1), for each $r \geq 0$ the intersection $\mathbb{X}_k(\gamma \cdot H;p) \cap \mathcal{B}_r$ has codimension at least k in $\mathcal{B}_r$. As $LQ_e(W)$ has property $\mathcal{P}$, for $r \geq 1$ this intersection has codimension strictly greater than k in $\mathbb{X}$. Thus a general point of $\mathbb{X}_k(\gamma \cdot H;p)$ is saturated.

For smoothness: Since $LQ_e(W)$ has property $\mathcal{P}$, it is generically smooth. Let $\mathbb{X}_{\mathrm{sm}}^\circ$ be the smooth and saturated part of $LQ_e(W)$, a dense open subset. Let $Z_k(\gamma \cdot H)_{\mathrm{sm}}$ be the smooth part of $Z_k(\gamma \cdot H;p)$. Consider the diagram

$$\begin{tikzcd}
\mathbb{X}_{\mathrm{sm}}^\circ \cap \mathbb{X}_k(\gamma \cdot H;p) \arrow[r, "\sim"] & \mathbb{X}_{\mathrm{sm}}^\circ \times_{\mathrm{LG}(W_p)} Z_k(\gamma \cdot H)_{\mathrm{sm}} \arrow[r] \arrow[d] & Z_k(\gamma \cdot H)_{\mathrm{sm}} \arrow[d, hook] \\
& \mathbb{X}_{\mathrm{sm}}^\circ \arrow[r, "\mathrm{ev}_p"] & \mathrm{LG}(W_p)
\end{tikzcd}$$

The top left intersection is contained in $\mathbb{X}_k(\gamma \cdot H;p)$, and by [15, Theorem 2 (i) & (ii)], for general γ is smooth and of codimension k. In a similar way, by [15, Theorem 2 (i)], for general γ the intersection of $\mathbb{X}_k(\gamma \cdot H; p)$ with the singular part of $\mathbb{X}^\circ$ is of codimension strictly greater than k in $\mathbb{X}^\circ$. Hence a general point of $\mathbb{X}_k(\gamma \cdot H; p)$ is smooth. $\square$

We now prove an analogue of Proposition 4.4 for the partition ρ_n. This will be used in dealing with symplectic Hecke transforms.

Proposition 4.5. *Let Λ be a Lagrangian subspace of W_q for a point $q \in C$.*

1. *For each $r \geq 0$, for general $\eta \in \mathrm{Sp}(W_q)$, the intersection $\mathbb{X}_{\rho_n}(\eta \cdot \Lambda; q) \cap \mathcal{B}_r$ is either empty or of codimension at least $\frac{1}{2}n(n+1)$ in $\mathcal{B}_r$.*
2. *Suppose that $e \geq e(W)$, so $\mathrm{ev}_q \colon LQ_e^\circ(W) \to \mathrm{LG}(W_q)$ is surjective and $\mathbb{X} = LQ_e(W)$ has property $\mathcal{P}$. Then if $\eta \in \mathrm{Sp}(W_q)$ is general, $\mathbb{X}_{\rho_n}(\eta \cdot \Lambda; q)$ is nonempty and generically smooth of codimension $\frac{1}{2}n(n+1)$ in $\mathbb{X}$, and a generic element is saturated.*

Proof. (1) For $0 \leq m \leq n$, consider the locus

$$Z_{\rho_m}(\Lambda) \ := \ \{\Pi \in \mathrm{LG}(W_q) : \dim(\Pi \cap \Lambda) \geq m\}.$$

Any $\Pi \in Z_{\rho_m}(\Lambda)$ fits into an exact sequence $0 \to \Pi_1 \to \Pi \to \Pi_2 \to 0$, where Π_1 is an m-dimensional subspace of Λ and Π_2 a Lagrangian subspace of the $2(n-m)$-dimensional symplectic vector space $\Pi_1^\perp / \Pi_1$. Hence the codimension of $Z_{\rho_m}(\Lambda)$ in $\mathrm{LG}(W_p)$ is at least

$$\dim \mathrm{LG}(W_p) - \dim \mathrm{Gr}(m, \Lambda) - \dim \mathrm{LG}(n-m) \ = \ \frac{1}{2}m(m+1).$$

Now for any $E \in \mathbb{X}_{\rho_n}(\Lambda; q) \cap \mathcal{B}_r$, there is a commutative diagram

$$\begin{tikzcd} E_q \arrow[r] & \overline{E}_q \cap \Lambda \arrow[r] \arrow[d] & \Lambda \arrow[d] \\ & \overline{E}_q \arrow[r] & W_q \end{tikzcd}.$$

If $\dim(\overline{E}_q \cap \Lambda) = m$, then $\overline{E} \in \mathrm{ev}_q^{-1}\left(Z_{\rho_m}(\eta \cdot \Lambda)\right)$. (Note that $r \geq n-m$, so $m \geq n-r$.) Moreover, letting $\overline{E}_\Lambda$ be the elementary transformation

$$0 \ \to \ \overline{E}_\Lambda \ \to \ \overline{E} \ \to \ \overline{E}_q / (\overline{E}_q \cap \Lambda) \ \to \ 0,$$

we have

$$[E \to \overline{E}] \ \in \ \mathrm{Im} \left(\mathrm{Quot}^{0, r-(n-m)}(\overline{E}_\Lambda) \ \hookrightarrow \mathrm{Quot}^{0,r}(\overline{E}) \right). \tag{4.6}$$

Now by [15, Theorem 2 (i)] and the first paragraph, for general $\eta \in \mathrm{Sp}(W_q)$, the locus

(4.7)

$$\{[F \to W] \in LQ_{e+r}^\circ(W) \mid \dim\left(F_p \cap (\eta \cdot \Lambda)\right) = m\} \ \subseteq \ \mathrm{ev}_q^{-1}\left(Z_{\rho_m}(\eta \cdot \Lambda)\right)$$

is either empty or has codimension $\frac{1}{2}m(m+1)$ in $LQ^{\circ}_{e+r}(W)$. Furthermore, for each F in this locus, setting $F_{\eta\cdot\Lambda} = \mathrm{Ker}(F \to F_q/(F_q \cap \eta \cdot \Lambda))$ as above, the locus $\mathrm{Quot}^{0,r-(n-m)}(F_{\eta\cdot\Lambda})$ has codimension

$$nr - n(r-(n-m)) \ = \ n(n-m)$$

in $\mathrm{Quot}^{0,r}(F) \cong f_r^{-1}(F)$. As (4.7) and (4.6) are conditions purely on the base and the fibers of f_r respectively, in view of Lemma 3.5 we may add the codimensions and conclude that for $\min\{0, n-r\} \leq m \leq n$, the locus

$$\{[E \to W] \in \mathbb{X}_{\rho_n}(\eta \cdot \Lambda; q) \cap \mathcal{B}_r \mid \dim(\overline{E}_q \cap \Lambda) = m\}$$

is of codimension

$$\frac{1}{2}m(m+1) + n^2 - nm \ = \ \frac{1}{2}n(n+1) + \frac{1}{2}(n-m)(n-m-1)$$

in $\mathcal{B}_r$. As $n \geq m$, this is at least $\frac{1}{2}n(n+1)$, as desired. (Notice also that we have equality for $m = n$ and $m = n-1$.)

Part (2) can be proven similarly to Proposition 4.4 (2). $\square$

4.2. **The Hecke transform.** In this subsection, given a vector bundle V and a divisor D on C, we denote $V \otimes \mathcal{O}_C(D)$ by $V(D)$.

Let W be a bundle with symplectic form $\omega\colon W \otimes W \to L$. Fix $p \in C$ and choose a subspace $\Lambda \subset W_p$. Let W_Λ be the Hecke transform of W, which is defined as the kernel of the composition map $W \to W_p \to W_p/\Lambda$. Then we have the exact sequence of sheaves

$$0 \to \ W_\Lambda \ \to \ W \to W_p/\Lambda \ \to \ 0. \tag{4.8}$$

By [3, Proposition 2.2], if Λ is a Lagrangian subspace of W_p, then W_Λ is bundle of degree $\deg(W) - n$ admitting the symplectic form

$$\omega_\Lambda\colon W_\Lambda \otimes W_\Lambda \ \to \ L(-p)$$

and fitting into the commutative diagram

$$\begin{tikzcd} W_\Lambda \otimes W_\Lambda \arrow[r] \arrow[d, "\omega_\Lambda"'] & W \otimes W \arrow[d, "\omega"] \\ L(-p) \arrow[r] & L. \end{tikzcd} \tag{4.9}$$

Dualizing (4.8), we obtain a sequence

$$0 \to W^\vee \to (W_\Lambda)^\vee \to \mathbb{C}^n \otimes \mathcal{O}_p \to 0.$$

Here $\mathcal{O}_p$ is a skyscraper sheaf of length one supported at p. Using the isomorphisms $W \cong W^\vee \otimes L$ and $W_\Lambda \cong (W_\Lambda)^\vee \otimes L(-p)$, we obtain a sequence

$$0 \to W \to W^\Lambda \to \mathbb{C}^n \otimes \mathcal{O}_p \to 0, \tag{4.10}$$

where $W^\Lambda := W_\Lambda(p)$ is an $L(p)$-valued symplectic bundle. In this way, to each Lagrangian subspace $\Lambda \subset W_p$ we can associate a symplectic bundle W^Λ fitting into (4.10). Since $\Lambda^\vee \subset (W_\Lambda)_p^\vee$, we may regard $\Lambda^\vee$ as a Lagrangian subspace of $(W^\Lambda)_p$. If $E \subset W$ is a subsheaf, then E can be viewed as a subsheaf of W^Λ via the inclusion $W \subset W^\Lambda$. Furthermore, if $E \subset W$ is a Lagrangian subsheaf, so is $E \subset W^\Lambda$. Hence there is a well-defined morphism

$$\Psi^\Lambda \colon LQ_e(W) \ \to \ LQ_e(W^\Lambda).$$

One can check that Ψ^Λ is an embedding. Furthermore, $\Psi^\Lambda([E \to W])$ belongs to $LQ_e^\circ(W^\Lambda)$ if and only if $[E \to W] \in LQ_e^\circ(W)$ and $E_p \cap \Lambda = 0$ in W_p.

Proposition 4.6. *Fix $p \in C$ and a Lagrangian subspace $\Lambda \subset W_p$. Then the image of Ψ^Λ coincides with the Lagrangian degeneracy locus $\mathbb{X}_{\rho_n}(\Lambda^\vee; p) \subseteq LQ_e(W^\Lambda)$.*

Proof. By definition, $[E \to W^\Lambda]$ belongs to $\mathbb{X}_{\rho_n}(\Lambda^\vee; p)$ if and only if the map $E_p \to (W^\Lambda)_p$ factorizes via $\Lambda^\vee \subset (W^\Lambda)_p$. This is equivalent to $E \to W^\Lambda$ lifting to a degree e Lagrangian subsheaf of W. □

Now choose distinct points $q_1, \ldots, q_t \in C$ and Lagrangian subspaces $\Lambda_1, \ldots, \Lambda_t$ in $W_{q_1}, \ldots, W_{q_t}$ respectively. Let $\widetilde{W}$ be the symplectic bundle obtained from a sequence of t Hecke transforms associated to $\Lambda_1, \ldots, \Lambda_t$. Then $\widetilde{W}$ fits into the sequence

$$0 \ \to \ W \ \to \ \widetilde{W} \ \to \ \bigoplus_{j=1}^{t} \left(\mathbb{C}^n \otimes \mathcal{O}_{q_j} \right) \ \to \ 0. \tag{4.11}$$

Then $\deg(\widetilde{W}) = \deg(W) + tn$, and as in the case above with $t = 1$, there is an embedding $LQ_e(W) \subset LQ_e(\widetilde{W})$.

Lemma 4.7. *Let W be any symplectic bundle with $LQ_e(W)$ nonempty. There exists an integer $t_0(W)$ such that if $\widetilde{W}$ is the Hecke transform defined by a general choice of $t \geq t_0(W)$ points $q_1, \ldots, q_t \in C$ and Lagrangian subspaces $\Lambda_i \subset W_{q_i}$, then the Lagrangian Quot scheme $LQ_e(\widetilde{W})$ has property $\mathcal{P}$.*

Proof. By Proposition 3.3, there exists $m_0 \geq 0$ such that $LQ_{e-mn}(W)$ has property $\mathcal{P}$ for all $m \geq m_0$. Let D be a reduced effective divisor of degree $m \geq m_0$. Then $W(D)$ is an $L(2D)$-valued symplectic bundle, and

$$LQ_{e-mn}(W) \ \cong \ LQ_e(W(D))$$

via the map $[E \to W] \mapsto [E(D) \mapsto W(D)]$.

Now $W(D)$ is a symplectic Hecke transformation of W. Precisely, $W(D)$ is obtained from W by transforming along m pairs of complementary Lagrangian subspaces of W, one pair from each point of $\mathrm{Supp}(D)$. Clearly $W(D)$ can be deformed to the Hecke transform defined by a general choice of $2m$ Lagrangian subspaces of distinct fibers of W. Therefore, as property $\mathcal{P}$ is open in families, a general Hecke transform $\widetilde{W}$ with $\deg\left(\widetilde{W}/W\right) = 2mn$ has property $\mathcal{P}$.

Similarly, let W^Λ be any Hecke transform of W along a single Lagrangian subspace Λ. Applying the above argument to W^Λ, there exists $m_1 \geq 0$ such that if $m \geq m_1$ and $\widetilde{W^\Lambda}$ is a general Hecke transform of W^Λ with $\deg\left(\widetilde{W^\Lambda}/W^\Lambda\right) = 2mn$, the scheme $LQ_e(\widetilde{W^\Lambda})$ has property $\mathcal{P}$. But such a $\widetilde{W^\Lambda}$ is also a Hecke transform of W along $2m+1$ Lagrangian subspaces (including Λ).

Thus, for $t \geq t_0(W) := \max\{2m_0, 2m_1+1\}$, if $\widetilde{W}$ is the Hecke transform along a general choice of t Lagrangian subspaces, then $LQ_e(\widetilde{W})$ has property $\mathcal{P}$. □

Corollary 4.8. *Let W be any symplectic bundle over C and $\widetilde{W}$ be the Hecke transform in (4.11). Assume $LQ_e(W)$ is not empty. Then, as subschemes of $LQ_e(\widetilde{W})$, we have*

$$LQ_e(W) \; = \; \bigcap_{i=1}^{t} \mathbb{X}_{\rho_n}(\Lambda_i^\vee; q_i),$$

where we view $\Lambda_i^\vee$ as a Lagrangian subspace of $\widetilde{W}_{q_i}$.

Proof. The equality of subschemes follows by applying Proposition 4.6 repeatedly. □

We shall also require the following corollary later.

Corollary 4.9. *Let e be a fixed integer. Then there is a number $\widetilde{w}(e)$ such that for a general symplectic W of degree $\widetilde{w} \geq \widetilde{w}(e)$, the Lagrangian Quot scheme $LQ_e(W)$ has property $\mathcal{P}$.*

Proof. Clearly we can find a symplectic bundle W_0 of large enough degree w_0 such that $LQ_e(W_0)$ is nonempty. By Lemma 4.7, for $t \geq t_0(W_0)$ there exists a symplectic bundle $\widetilde{W_0}$ of degree $w_0 + tn$ such that $LQ_e(\widetilde{W_0})$ has property $\mathcal{P}$. As property $\mathcal{P}$ is open in families of symplectic bundles, we may set $\widetilde{w}(e) := w_0 + nt_0(W_0)$. □

5. Intersection theory on $LQ_e(W)$

We shall now develop an intersection theory on $LQ_e(W)$. We firstly define intersection numbers on a Lagrangian Quot scheme $LQ_e(W)$ having property $\mathcal{P}$, and then extend this to an arbitrary Lagrangian Quot scheme $LQ_e(W)$. Note that as pointed out in Remark 2.3, we cannot generate desired 0-cycles for the integral through a product of cycles on $LQ_e(W)$. However we obtain them by directly taking an intersection of Lagrangian loci in Proposition 5.1.

5.1. **Intersection theory on $LQ_e(W)$ having property $\mathcal{P}$.** Recall that for a fixed g, we have defined

$$D(n,e,\ell) \ := \ -(n+1)e - \frac{n(n+1)}{2}(g-\ell-1),$$

the expected dimension of $LQ_e(W)$ for a symplectic bundle W of rank $2n$ and degree $n\ell$ over a curve of genus g. For a nonnegative integer t, write

$$D_t(n,e,\ell) \ := \ D(n,e,\ell) - \frac{n(n+1)}{2}\cdot t.$$

The next proposition, whose proof will be given in § 5.5, plays a key role in developing the intersection theory on $LQ_e(W)$.

Proposition 5.1. *Let W be any symplectic bundle of degree $w = n\ell$. Assume that $e \leq e(W)$, so $LQ_e(W)$ has property $\mathcal{P}$ and $\mathrm{ev}_x \colon LQ_e^\circ(W) \to \mathrm{LG}(W_x)$ is surjective for all $x \in C$. Let $t, k_1, \ldots, k_s$ be integers with $t \geq 0$ and $1 \leq k_i \leq n$, satisfying $\sum_{i=1}^s k_i = D_t(n,e,\ell)$. Let $p_1, \ldots, p_s, q_1, \ldots, q_t$ be distinct points of C. For each p_i, let $H_i \subset W_{p_i}$ be an isotropic subspace of dimension $n+1-k_i$. For each q_j, let $\Lambda_j \subset W_{q_j}$ be a Lagrangian subspace. Then for a general choice of $\gamma_i \in \mathrm{Sp}(W_{p_i})$ and $\eta_j \in \mathrm{Sp}(W_{q_j})$, the following holds.*

(1) *The intersection*

$$\left(\bigcap_{i=1}^{s} \mathbb{X}_{k_i}(\gamma_i \cdot H_i; p_i)\right) \ \cap \ \left(\bigcap_{j=1}^{t} \mathbb{X}_{\rho_n}(\eta_j \cdot \Lambda_j; q_j)\right) \tag{5.1}$$

is a 0-dimensional subscheme of $LQ_e(W)$.

(2) *The intersection (5.1) is equal to*

$$\left(\bigcap_{i=1}^{s} \mathbb{X}^\circ_{k_i}(\gamma_i \cdot H_i; p_i)\right) \ \cap \ \left(\bigcap_{j=1}^{t} \mathbb{X}^\circ_{\rho_n}(\eta_j \cdot \Lambda_j; q_j)\right). \tag{5.2}$$

In particular, each point of the intersection corresponds to a saturated subsheaf.

(3) *The intersection (5.2) is reduced.*

For $1 \leq i \leq n$, let α_i be a formal variable of weight i, and set $\alpha := (\alpha_1, \ldots, \alpha_n)$. Let $P(\alpha)$ be a homogeneous polynomial in $\alpha_1, \ldots, \alpha_n$.

Definition 5.2. *Assume $\mathbb{X} = LQ_e(W)$ has property $\mathcal{P}$. Let t be a nonnegative integer and $P = P(\alpha) = \prod_{i=1}^{s} \alpha_{k_i}$ be a monomial with $\deg P(\alpha) = D_t(n, e, \ell)$. Define $\Theta(P; t)$ as the 0-cycle determined by the intersection (5.1).*

Notice that $\Theta(P; t)$ depends on the choice of reference points p_i and q_j, and the subspaces H_i and Λ_j.

Definition 5.3. *Suppose that $\mathbb{X} = LQ_e(W)$ has property $\mathcal{P}$. Let $t \geq 0$ be a nonnegative integer and $P(\alpha)$ a homogenous polynomial such that $\deg P(\alpha) = D_t(n, e, \ell)$. Define $N^w_{C,e}(\Theta(P, t); W)$ as follows. If $P(\alpha)$ is a monomial $\prod_{i=1}^{s} \alpha_{k_i}$, then*

$$N^w_{C,e}(\Theta(P, t); W) \ := \ \int_{\mathbb{X}} \Theta(P; t).$$

Then $N^w_{C,e}(\Theta(P, t); W)$ is defined for any homogeneous polynomial $P(\alpha)$ of degree $D_t(n, e, \ell)$ by linearity of the integral. By convention, we set $N^w_{C,e}(\Theta(P, t); W) = 0$ if $\deg P(\alpha) \neq D_t(n, e, \ell)$.

We are ultimately interested in the integral of cycles $\Theta(P, 0)$, i.e., for $t = 0$ and the case with $t > 0$ plays an auxiliary role in our theory (cf. Definition 5.9). We use a simpler notation for $t = 0$:

Notation 5.4. *For simplicity, in the case where $t = 0$ we write $N^w_{C,e}(P; W)$ for $N^w_{C,e}(\Theta(P, 0); W)$.*

Remark 5.5. By [10, Proposition 10.2], the number $N^w_{C,e}(\Theta(P; t); W)$ is well-defined in the sense that it is independent of the chosen reference points p_i and q_j and the subspaces H_i and Λ_j provided that the intersection (5.1) remains 0-dimensional; and by Proposition 5.1, this is the case when the H_i and Λ_j are chosen generally in their respective fibers.

5.2. **Invariance of intersection number under deformations.** We show now that the intersection number $N^w_{C,e}(\Theta(P; t); W)$ is invariant in a family of Lagrangian Quot schemes with property $\mathcal{P}$. Let $\mathcal{C} \to B$ be a family of smooth projective curves over an irreducible curve B and $\mathscr{L} \to \mathcal{C}$ a line bundle of relative degree ℓ. Let $\mathscr{W}$ be a vector bundle over $\mathcal{C}$ such that $\mathscr{W}_b$ is an $\mathscr{L}_b$-valued symplectic bundle over $\mathcal{C}_b$ for each $b \in B$. The family $\mathscr{W} \to \mathcal{C}$ gives rise to a family $\phi \colon \widetilde{\mathbb{X}} = LQ_e(\mathscr{W}) \to B$ of Lagrangian Quot schemes parameterized by B.

Let $\widetilde{\mathcal{E}}$ be a universal bundle over $LQ_e(\mathscr{W}) \times_B \mathcal{C}$. Now we define a relative version of the Lagrangian degeneracy locus on $\widetilde{\mathbb{X}}$ associated to a strict partition $\lambda \in \mathcal{D}(n)$. This can be done locally on B.

To do this, we replace points $p, q \in C$ and subspaces $H \in W_p$ and $\Lambda \subset W_q$ with local sections corresponding to these. For simplicity, we proceed with

$\lambda = \rho_n$, the general case being similar. Let $\tilde{q}\colon U \to \mathcal{C}$ be a local section of the family $\mathcal{C} \to B$. Let $\mathscr{W}_{\tilde{q}}$ be the restriction of $\mathscr{W}$ to $\operatorname{Im}\tilde{q} \subset \mathcal{C}$. Denote by $\widetilde{\mathcal{E}}(\tilde{q})$ the restriction of $\widetilde{\mathcal{E}}$ to $\widetilde{\mathbb{X}}|_U = LQ_e(\mathscr{W}|_U) \times_U \operatorname{Im}\tilde{q}$. Shrinking U if necessary, we may choose a section $\widetilde{\Lambda}$ of the Lagrangian Grassmannian bundle $\mathrm{LG}(\mathscr{W}_{\tilde{q}}) \to U$. Then the Lagrangian degeneracy locus $\widetilde{\mathbb{X}}_{\rho_n}(\widetilde{\Lambda};\tilde{q})$, which is parameterized by U, is defined as in (4.3) or Definition 4.1 by

$$\widetilde{\mathbb{X}}_{\rho_n}(\widetilde{\Lambda};\tilde{q}) \;:=\; \{[\psi\colon E \to \mathscr{W}_b] \in LQ_e(\mathscr{W}|_U) \mid \psi(E_{\tilde{q}(b)}) \subseteq \widetilde{\Lambda}(b)\}.$$

Now suppose $t \geq 0$, and let $P(\alpha) = \prod_{i=1}^{s} \alpha_{k_i}$ be a monomial with $\deg P(\alpha) = D_t(n,e,\ell)$. Then, by the above description, for each $b \in B$ there is a neighborhood U of b in B such that the relative version of (5.1) is defined over U. Furthermore, by shrinking U, if the (chosen) reference sections are general, this family over U is a one-dimensional subscheme of $\widetilde{\mathbb{X}}|_U$ (recall that $\dim B = 1$) of which every component dominates U. Thus, for each $b_1 \in U$, it determines a 0-cycle $\Theta(P;t))$ of $LQ_e(\mathscr{W}_{b_1})$.

Proposition 5.6. *Let $\mathcal{C} \to B$ and $\mathscr{W}$ be as above. Suppose that $LQ_e(\mathscr{W}_b)$ has property $\mathcal{P}$ for each $b \in B$. For a nonnegative integer t and a homogeneous polynomial $P(\alpha)$ of $\deg P(\alpha) = D_t(n,e,\ell)$, the intersection number $N^w_{\mathcal{C}_b,e}(\Theta(P;t);\mathscr{W}_b)$ is independent of $b \in B$.*

Proof. It suffices to prove the proposition for a monomial $P(\alpha) = \prod_{i=1}^{s} \alpha_{k_i}$. Take an open covering $\{U_\iota \mid \iota \in I\}$ of B such that for each $\iota \in I$, the family version of the intersection (5.1) is defined over U_ι. Then, once we take general reference sections over U_ι, by invariance of degree of a family of 0-cycles shown in [10, Proposition 10.2], the intersection number $N^w_{\mathcal{C}_b,e}(\Theta(P;t);\mathscr{W}_b)$ is independent of $b \in U_\iota$. If $U_\iota \cap U_\varepsilon$ is not empty, then by the definition, it is obvious that $N^w_{\mathcal{C}_b,e}(\Theta(P;t);\mathscr{W}_b)$ is independent of $b \in U_\iota \cap U_\varepsilon$. Thus the proposition is immediate. □

5.3. **Definition of intersection numbers on an arbitrary Lagrangian Quot scheme.** We shall now extend our definition of intersection number to arbitrary Lagrangian Quot schemes, not necessarily enjoying property $\mathcal{P}$.

For a symplectic bundle W and a symplectic Hecke transform $\widetilde{W}$, we write $\mathbb{X} := LQ_e(W)$ and $\mathbb{Y} := LQ_e(\widetilde{W})$.

Lemma 5.7. *Let W be a symplectic bundle and $\widetilde{W}$ a general Hecke transform of W as in (4.11). Let $r_1, \ldots, r_u$ be distinct points of C distinct from the p_i and q_j, and for $1 \leq l \leq u$ let Π_l be a Lagrangian subspace* *of $W_{u_l} \cong \widetilde{W}_{u_l}$. Then there is a set bijection*

$$(5.3)\quad \bigcap_{i=1}^{s} \mathbb{X}_{k_i}(\gamma_i \cdot H_i; p_i) \cap \bigcap_{l=1}^{u} \mathbb{X}_{\rho_n}(\Pi_l; r_l) \xrightarrow{\sim}$$

$$\left(\bigcap_{i=1}^{s} \mathbb{Y}_{k_i}(\gamma_i \cdot H_i; p_i) \right) \cap \bigcap_{l=1}^{u} \mathbb{Y}_{\rho_n}(\Pi_l; r_l) \cap \left(\bigcap_{j=1}^{t} \mathbb{Y}_{\rho_n}(\Lambda_j^\vee; q_j) \right).$$

In particular, if $LQ_e(W)$ and $LQ_e(\widetilde{W})$ have property $\mathcal{P}$, then for any homogenous polynomial $P(\alpha)$ of degree $D_u(e,n,\ell)$, we have

$$(5.4)\qquad N_{C,e}^{w+un}(\Theta(P(\alpha);u);W) \;=\; N_{C,e}^{w+(u+t)n}\left(\Theta(P(\alpha);u+t);\widetilde{W}\right).$$

Proof. By Proposition 4.6, the map $[E \to W] \mapsto [E \to W \to \widetilde{W}]$ defines a one-to-one correspondence between

$$\{\text{Lagrangian subsheaves of } W \text{ of degree } e\}$$

and

$$\{\text{Lagrangian subsheaves } \tilde{E} \subset \widetilde{W} \text{ of degree } e \text{ satisfying}$$
$$\mathrm{Im}\left(\tilde{E}_{q_j} \to \widetilde{W}_{q_j}\right) \subseteq \Lambda_j^\vee \text{ for } 1 \le j \le t\}.$$

Furthermore, as the points $p_1, \dots, p_s, q_1, \dots, q_t$ and $r_1, \dots, r_u$ are distinct, for $1 \le i \le s$ and $1 \le l \le u$ each map of fibers $W_{p_i} \to \widetilde{W}_{p_i}$ and $W_{r_l} \to \widetilde{W}_{r_l}$ is an isomorphism. Therefore, an element $[E \to W] \in LQ_e(W)$ satisfies

$$\mathrm{rk}\left(E_{p_i} \;\to\; W_{p_i}/\gamma_i \cdot H_i^\perp\right) \;\le\; k_i - 1$$

if and only if the corresponding element $[E \to W \to \widetilde{W}]$ satisfies

$$\mathrm{rk}\left(E_{p_i} \;\to\; \widetilde{W}_{p_i}/\gamma_i \cdot H_i^\perp\right) \;\le\; k_i - 1.$$

Similarly, $\mathrm{Im}\,(E_{r_l} \to W_{r_l}) \subseteq \Pi_l$ if and only if $\mathrm{Im}\left(E_{r_l} \to \widetilde{W}_{r_l}\right) \subseteq \Pi_l$. Combining these observations, we conclude the existence of the bijection (5.3).

Now suppose $LQ_e(W)$ has property $\mathcal{P}$. Then for general γ_i, by Proposition 5.1 (c) the left hand side of (5.3) is a finite and reduced scheme of length $N_{C,e}^{w+un}(\Theta(P(\alpha);u);W)$. As the map $LQ_e(W) \to LQ_e(\widetilde{W})$ is an embedding, the right hand side is also reduced of length $N_{C,e}^{w+un}(\Theta(P(\alpha);u);W)$ in $LQ_e(\widetilde{W})$.

By Proposition 5.6, since the right hand intersection in (5.3) has dimension zero, its degree is constant under small deformations of the $\Lambda_j^\vee$ in $\mathrm{LG}(\widetilde{W})$. By definition, this degree is equal to $N_{C,e}^{w+(u+t)n}(\Theta(P(\alpha);u+t);\widetilde{W})$ for the monomial $P(\alpha) = \prod_{i=1}^{s} \alpha_{k_i}$. Thus (5.4) holds for $P(\alpha) = \prod_{i=1}^{s} \alpha_{k_i}$. By linearity, (5.4) holds for any homogeneous polynomial $P(\alpha)$. $\square$

Remark 5.8. Furthermore, by Proposition 5.1 (b) all elements $[E \to W]$ of the left hand side of (5.3) define saturated subsheaves. Since the Lagrangian subspaces Λ_j were chosen generally, deforming them if necessary we can assume that for each of these finitely many E we have $E_{q_j} \cap \Lambda_j = 0$. Thus $[E \to \widetilde{W}]$ is also saturated for all such E.

Motivated by Lemma 5.7, we make a definition.

Definition 5.9. *Let W be a symplectic bundle of degree w, and suppose $LQ_e(W)$ is nonempty. For $t \gg 0$, let $\widetilde{W}$ be a general symplectic Hecke transform of W with $\deg(\widetilde{W}/W) = tn$, so that $LQ_e(\widetilde{W})$ has property $\mathcal{P}$ by Lemma 4.7. We define*

$$\widetilde{N}^w_{C,e}(P(\alpha); W) \ := \ N^{w+tn}_{C,e}\left(\Theta(P(\alpha); t); \widetilde{W}\right).$$

Lemma 5.10. *The number $\widetilde{N}^w_{C,e}(P(\alpha); W)$ is well-defined and depends only on g, e and w once the polynomial $P(\alpha)$ is specified. More precisely,*

1. *It does not depend on the chosen Hecke transform $\widetilde{W}$.*
2. *Let $\mathscr{W} \to \mathcal{C} \to B$ be a family of symplectic bundles parametrized by a connected curve B, such that $LQ_e(\mathscr{W}_b)$ is nonempty for all $b \in B$. Then $\widetilde{N}^w_{\mathcal{C}_b,e}(P(\alpha); \mathscr{W}_b)$ is constant with respect to $b \in B$. (In particular, it is invariant even for not necessarily flat families of Lagrangian Quot schemes.)*

Proof. (1) Let $\widetilde{w}(e)$ be as defined in Corollary 4.9. Choose two different general Hecke transforms $\widetilde{W_1}$ and $\widetilde{W_2}$ of W, of degree at least $\widetilde{w}(e)$. We may assume that the Hecke transforms are obtained at distinct points $p_1, \dots, p_{t_1}$ and $q_1, \dots, q_{t_2}$, respectively. We can take a Hecke transform of $\widetilde{W_1}$ at appropriate general Lagrangian subspaces of $(\widetilde{W_1})_{q_i} = W_{q_i}$ for $1 \leq i \leq t_2$, and also a Hecke transform of $\widetilde{W_2}$ at suitable general Lagrangian subspaces of $(\widetilde{W_2})_{p_j} = W_{p_j}$ for $1 \leq j \leq t_1$ to obtain a symplectic bundle $\widetilde{W_3}$ which is a common Hecke transform of $\widetilde{W_1}$ and $\widetilde{W_2}$. By generality of the choices and by Corollary 4.9, all the intermediate Hecke transforms $\widetilde{W_1} \subset W' \subset \widetilde{W_3}$ and $\widetilde{W_2} \subset W'' \subset \widetilde{W_3}$ may be assumed to have property $\mathcal{P}$. Hence we may apply Lemma 5.7 to obtain the desired equality

$$N^{w+t_1 n}_{C,e}(\Theta(P(\alpha); t_1); \widetilde{W_1}) \ = \ N^{w+t_2 n}_{C,e}(\Theta(P(\alpha); t_2); \widetilde{W_2}),$$

using the fact that $\widetilde{W_3}$ is the common Hecke transform.

(2) For a given $b_0 \in B$, by Lemma 4.7, there exists $t \gg 0$ such that if $\widetilde{\mathscr{W}_{b_0}}$ is a general Hecke transform along t Lagrangian subspaces of $\mathscr{W}_{b_0}$, then $LQ_e\left(\widetilde{\mathscr{W}_{b_0}}\right)$ has property $\mathcal{P}$. By openness of property $\mathcal{P}$, there exists an open subset U of the component of B containing b_0, such that for each $b \in U$ and

for a general symplectic Hecke transformation of $\mathscr{W}_b$ of degree $w + tn$, the scheme $LQ_e\left(\widetilde{\mathscr{W}_b}\right)$ has property $\mathcal{P}$.

Thus, shrinking U if necessary, we may choose a family of degree $w + tn$ Hecke transforms $\widetilde{\mathscr{W}|_U} \to \mathcal{C}|_U \to U$, all having property $\mathcal{P}$. By Proposition 5.6, we see that

$$(5.5) \qquad N^{w+tn}_{\mathcal{C}_b,e}\left(\Theta(P(\alpha);t);\widetilde{\mathscr{W}_b}\right) \text{ is constant with respect to } b \in U.$$

Now let b' be any other point of B. As each component of B is a quasi-projective curve, we can find a finite connected chain of open subsets $U = U_0, U_1, \dots, U_\nu$ of components of B with $b_0 \in U_0$ and $b' \in U_\nu$, equipped with families of Hecke transforms

$$\widetilde{\mathscr{W}_j} \ \to \ \mathcal{C}|_{U_j} \ \to \ U_j$$

of $\mathscr{W}|_{U_j}$ of degree $w + t_j n$ as above such that $LQ_e\left(\widetilde{\mathscr{W}_j}|_b\right)$ has property $\mathcal{P}$ for each $b \in U_j$. Now the the numbers t_j may be different, but for $b \in U_j \cap U_k$, by part (1) we have equality

$$N^{w+t_j n}_{\mathcal{C}_b,e}\left(\Theta(P(\alpha);t_j);\widetilde{\mathscr{W}_j}|_b\right) \ = \ N^{w+t_k n}_{\mathcal{C}_b,e}\left(\Theta(P(\alpha);t_k);\widetilde{\mathscr{W}_k}|_b\right).$$

By definition of $\widetilde{N}^w_{C,e}(P;W)$ and by (5.5) it follows that $\widetilde{N}^w_{\mathcal{C}_b,e}(P(\alpha);\mathscr{W}_b)$ is constant with respect to $b \in B$. □

If $LQ_e(W)$ has property $\mathcal{P}$, then in computing $\widetilde{N}^w_{C,e}(P(\alpha);W)$ we can take $\widetilde{W} = W$. Thus we obtain:

Proposition 5.11. *Let W be any symplectic bundle of degree w such that $LQ_e(W)$ has property $\mathcal{P}$. Then we have*

$$\widetilde{N}^w_{C,e}(P(\alpha);W) \ = \ N^w_{C,e}(P(\alpha);W).$$

In particular, the two definitions of intersection number coincide.

We shall shortly see that if $LQ_e(W)$ has property $\mathcal{P}$, then $N^w_{C,e}(P;W)$ enumerates Lagrangian subbundles of W satisfying a certain condition.

5.4. **Relations between intersection numbers.** Here we study the behavior of the numbers $\widetilde{N}^w_{C,e}(P;W)$ under various transformations. Let W be an L-valued symplectic bundle of degree w over C. Let $\widehat{L}$ be a line bundle of degree $\hat{\ell}$ over C. Then $W \otimes \widehat{L}$ is an $L \otimes \widehat{L}^2$-valued symplectic bundle of degree $w + 2n\hat{\ell}$.

Proposition 5.12. *Let W and $\widehat{L}$ be as above. Then*

$$\widetilde{N}^w_{C,e}(P(\alpha);W) \ = \ \widetilde{N}^{w+2n\hat{\ell}}_{C,e+n\hat{\ell}}(P(\alpha);W \otimes \widehat{L}).$$

Proof. The proposition is immediate from the fact, already used in Lemma 4.7, that the association

$$[E \to W] \ \mapsto \ \left[(E \otimes \widehat{L}) \to (W \otimes \widehat{L})\right]$$

defines an isomorphism $LQ_e(W) \xrightarrow{\sim} LQ_{e+n\hat{\ell}}(W \otimes \widehat{L})$. □

Proposition 5.13. *Let W be an arbitrary symplectic bundle of degree w, and assume $LQ_e(W)$ is nonempty. Then for any integer $k \geq 0$, we have*

$$\widetilde{N}^w_{C,e}(P(\alpha); W) \ = \ \widetilde{N}^w_{C,e-nk}(\Theta(P(\alpha); 2k); W). \tag{5.6}$$

Proof. Firstly, by the definition of $\widetilde{N}^w_{C,e}(P(\alpha); W)$, for large enough $m \gg 2k$ the left hand side of (5.6) can be written as

$$\widetilde{N}^w_{C,e}(P(\alpha); W) \ = \ N^{w+mn}_{C,e}(\Theta(P(\alpha); m); \widetilde{W}) \tag{5.7}$$

for a general Hecke transform $W \subset \widetilde{W}$ with $\deg(\widetilde{W}) = w + mn$.

Now set $h := m - 2k$. Since h is sufficiently large, the right hand side of (5.6) can be written as

$$\widetilde{N}^w_{C,e-nk}(\Theta(P(\alpha); 2k); W) \ = \ N^{w+hn}_{C,e-nk}(\Theta(P(\alpha); 2k+h); \widetilde{W}_1) \tag{5.8}$$

for a general Hecke transform $W \subset \widetilde{W}_1$ with $\deg(\widetilde{W}_1) = w + nh$. Let $\widehat{L}$ be a line bundle of degree k. Then by Proposition 5.12, the right hand side of (5.8) can in turn be written as

(5.9)
$$\begin{aligned} N^{w+hn}_{C,e-nk}(\Theta(P(\alpha); 2k+h); \widetilde{W}_1) \ &= \ N^{w+hn+2nk}_{C,e}(\Theta(P(\alpha); 2k+h); \widetilde{W}_1 \otimes \widehat{L}) \\ &= \ N^{w+mn}_{C,e}(\Theta(P(\alpha); m); \widetilde{W}_1 \otimes \widehat{L}) \end{aligned}$$

since $m = h + 2k$. As both $LQ_e(\widetilde{W})$ and $LQ_e(\widetilde{W}_1 \otimes \widehat{L})$ have property $\mathcal{P}$, by Lemma 5.10 (2) the right hand sides of (5.7) and (5.9) coincide. □

5.5. **Proof of Proposition 5.1.** In this subsection, we give a proof of Proposition 5.1.

Proof of Proposition 5.1. Firstly, we claim that for a general choice of γ_i and η_j, the locus (5.2) has dimension zero. To see this, note that

$$\prod_{i=1}^{s} \mathrm{Sp}(W_{p_i}) \times \prod_{j=1}^{t} \mathrm{Sp}(W_{q_j}) \text{ acts transitively on } \prod_{i=1}^{s} \mathrm{LG}(W_{p_i}) \times \prod_{j=1}^{t} \mathrm{LG}(W_{q_j}).$$

Now each $Z_{k_i}(H_i)$ has codimension k_i in $\mathrm{LG}(W_{p_i})$, and each $Z_{\rho_n}(\Lambda_j)$ has codimension $\frac{1}{2}n(n+1)$ in $\mathrm{LG}(W_{q_j})$. Hence by [15, Theorem 2 (i)], if the γ_i

and η_j are general, then (5.2) is empty or of codimension

$$\sum_{i=1}^{s_1} k_i + t \cdot \frac{1}{2} n(n+1) \ = \ D(n, e, \ell)$$

in $LQ_e^\circ(W)$. The claim follows since $LQ_e(W)$ has property $\mathcal{P}$.

In view of the claim, statements (1) and (2) would follow if we show that (5.1) is contained in the saturated part $LQ_e^\circ(W)$. So let us consider elements $[E \to W]$ belonging to (5.1) such that E is nonsaturated, so that $\overline{E}/E$ is a torsion sheaf of degree $r \geq 1$. For any such E, for $1 \leq i \leq s$, we have maps $\overline{E}_{p_i} \to \frac{W_{p_i}}{(\gamma_i \cdot H_i)^\perp}$. Without loss of generality, we may assume that for some $s_1 \in \{0, \ldots, s\}$ these maps are not surjective for $0 \leq i \leq s_1$, and are surjective for $s_1 + 1 \leq i \leq s$. (The case $s_1 = s$ (resp., $s_1 = 0$) corresponds trivially to none (resp., all) being surjective.)

For $1 \leq i \leq s_1$ the point $[E \to W]$ belongs to

$$f_r^{-1} \left(\mathbb{Y}_{k_i}^\circ(\gamma_i \cdot H_i; p_i) \right), \tag{5.10}$$

where f_r is as defined in § 3.4, and

$$\mathbb{Y}_{k_i}^\circ(\gamma_i \cdot H_i; p_i) \ = \ \mathrm{ev}_{p_i}^{-1} \left(Z_{k_i}(\gamma_i \cdot H_i) \right) \ = \\ \{ F \in LQ_{e+r}^\circ(W) \mid F_{p_i} \to W_{p_i}/(\gamma_i \cdot H_i^\perp) \text{ is not surjective} \}$$

is as defined in the proof of Proposition 4.4. By an argument similar to that in the first paragraph, we see that $\bigcap_{i=1}^{s_1} \mathbb{Y}_{k_i}^\circ(\gamma_i \cdot H_i; p_i)$ is empty or of codimension $\sum_{i=1}^{s_1} k_i$ in each component of $LQ_{e+r}^\circ(W)$ (note that the latter may not be equidimensional).

Furthermore, for $s_1 + 1 \leq i \leq s$, by Lemma 4.3, for each $F \in LQ_{e+r}^\circ(W)$ the set

$$\{ [E \to W] \in f_r^{-1}(F) \mid E_{p_i} \to F_{p_i} \to W_{p_i}/H_i^\perp \text{ is not surjective} \} \tag{5.11}$$

is of codimension at least k_i on $f_r^{-1}(F) \cong \mathrm{Quot}^{0,r}(F)$.

Next, let $m_1, \ldots, m_t$ be elements of $\{0, \ldots, n\}$ satisfying

$$r \ \geq \ \sum_{j=1}^{t} (n - m_j).$$

Consider the set

$$\{ F \in LQ_{e+r}^\circ(W) \mid \dim(F_{q_j} \cap \eta_j \cdot \Lambda_j) = m_j \text{ for } 1 \leq j \leq t \} \\ \subseteq \ \bigcap_{j=1}^{t} \mathrm{ev}_{q_j}^{-1} \left(Z_{\rho_{m_j}}(\eta_j \cdot \Lambda_j) \right). \tag{5.12}$$

Recall from the proof of Proposition 4.5 that $Z_{\rho_{m_j}}(\Lambda_j)$ is of codimension $\frac{1}{2} m_j(m_j + 1)$ in $\mathrm{LG}(W_{q_j})$. By [15, Theorem 2 (i)], for general $\eta_1, \ldots, \eta_t$ the set (5.12) is empty or of codimension $\frac{1}{2}\sum_{j=1}^{t} m_j(m_j+1)$ in each component of $LQ^{\circ}_{e+r}(W)$.

Next, given such an F, we consider the elementary transformation

$$0 \ \to \ F_{(\eta_j \cdot \Lambda_j)} \ \to \ F \ \to \ F_{q_j}/(F_{q_j} \cap \eta_j \cdot \Lambda_j) \ \to \ 0.$$

Then, as in the proof of Proposition 4.5, an element $E \in f_r^{-1}(F) \cong \mathrm{Quot}^{0,r}(F)$ belongs to all $\mathbb{X}_{\rho_n}(\eta_j \cdot \Lambda_j; q_j)$ if and only if $E \subset F'$, where $F' \subseteq F$ is the elementary transformation

$$F' \ = \ \bigcap_{j=1}^{t} F_{(\eta_j \cdot \Lambda_j)};$$

more precisely, $[E \to F]$ defines an element of

$$\text{(5.13)} \qquad \mathrm{Im}\left(\mathrm{Quot}^{0, r - \sum_{j=1}^{t}(n-m_j)}(F') \ \hookrightarrow \ \mathrm{Quot}^{0,r}(F)\right),$$

a locus of codimension $nr - n(r - \sum_{j=1}^{t}(n-m_j)) \ = \ \sum_{j=1}^{t} n(n-m_j)$ in $f_r^{-1}(F)$.

Now (5.10) and (5.12) are conditions purely on the base of $f_r \colon \mathcal{B}_r \to LQ^{\circ}_{e+r}(W)$. By [15, Theorem 2 (i)], since the γ_i and η_j are general, the intersection of the loci defined on LQ°_{e+r} by (5.10) and (5.12) is either empty or of the expected codimension on each component of $LQ^{\circ}_{e+r}(W)$.

Next, (5.11) and (5.13) are conditions purely on the fibers of $f_r \colon \mathcal{B}_r \to LQ^{\circ}_{e+r}(W)$. As the points p_i and q_j are all distinct, the loci defined by these conditions intersect properly in each fiber of f_r. (Note that this is true for arbitrary γ_i and η_j.)

Therefore, to compute the codimension of (5.2) in $\mathcal{B}_r$ for general γ_i and η_j, we can add the codimensions of the sets (5.10), (5.11), (5.12) and (5.13). We obtain a locus in $\mathcal{B}_r$ which is empty or of codimension at least

$$\sum_{i=1}^{s} k_i + \frac{1}{2}\sum_{j=1}^{t} m_j(m_j+1) + \sum_{j=1}^{t} n(n-m_j) \ = $$
$$\sum_{i=1}^{s} k_i + t \cdot \frac{1}{2} n(n+1) + \frac{1}{2}\sum_{j=1}^{t}(n-m_j)(n-m_j-1) \ \geq \ D(n,e,\ell).$$

But since $LQ_e(W)$ has property $\mathcal{P}$, no $\mathcal{B}_r$ is dense. Thus the intersection of (5.2) with the nonsaturated locus is empty for general γ_i and η_j, as desired. This completes the proof of (1) and (2).

As for (3): By [15, Theorem 2 (ii)], the intersection

$$\text{(5.14)} \qquad \left(\bigcap_{i=1}^{s} \mathbb{X}^{\circ}_{k_i}(\gamma_i \cdot H_i; p_i)_{\mathrm{sm}}\right) \ \cap \ \left(\bigcap_{j=1}^{t} \mathbb{X}^{\circ}_{\rho_n}(\eta_j \cdot \Lambda_j; q_j)_{\mathrm{sm}}\right)$$

is smooth. Moreover, by Proposition 4.4 (2) and 4.5 (2) respectively, the intersections

$$\mathrm{Sing}\left(\mathbb{X}^\circ_{k_{i_0}}(\gamma_{i_0}\cdot H_{i_0})\right)\cap\left(\bigcap_{i\neq i_0}\mathbb{X}^\circ_{k_i}(\gamma_i\cdot H_i;p_i)\right)\ \cap\ \left(\bigcap_{j}\mathbb{X}^\circ_{\rho_n}(\eta_j\cdot\Lambda_j;q_j)\right)$$

and

$$\mathrm{Sing}\left(\mathbb{X}^\circ_{\rho_n}(\eta_{j_0}\cdot\eta_{j_0}\cdot\Lambda_{j_0})\right)\cap\left(\bigcap_{i}\mathbb{X}^\circ_{k_i}(\gamma_i\cdot H_i;p_i)\right)\ \cap\ \left(\bigcap_{j\neq j_0}\mathbb{X}^\circ_{\rho_n}(\eta_j\cdot\Lambda_j;q_j)\right)$$

have expected codimension strictly greater than $D(n,e,\ell)$. Hence they are empty by [15, Theorem 2 (i)]. Therefore, (5.14) coincides with (5.2), and hence also with (5.1) by part (2). This proves (3). □

5.6. **A relation to Gromov–Witten invariants of the Lagrangian Grassmannian.** Kresch and Tamvakis [18] used intersection theory on $\mathrm{LQ}_e(\mathcal{O}_C^{\oplus 2n})$ to work out the (small) quantum cohomology of $\mathrm{LG}(n)$, which gives all genus zero 3-point GW invariants and so n-point GW invariants of the type in Definition 2.4. (Note that n-point GW invariants of this type do not coincide with the ordinary GW invariants unless $n=3$, and are determined by 3-point GW invariants of this type.) Similarly, we show that the intersection number $\widetilde{N}^0_{C,e}(P(\alpha);\mathcal{O}_C^{\oplus 2n})$ is equal to the corresponding GW invariants of $\mathrm{LG}(n)$.

Proposition 5.14. *Let C be a smooth projective curve of genus g. Suppose $\lambda^1,\dots,\lambda^m\in\mathcal{D}(n)$ are strict partitions such that $\sum_{i=1}^m|\lambda^i|=D(n,e,0)$. Set $P(\alpha)=\prod_{i=1}^m\widetilde{Q}_{\lambda^i}(\alpha)$. Then we have the equality*

$$\widetilde{N}^0_{C,e}(P(\alpha);\mathcal{O}_C^{\oplus 2n})\ =\ \langle\sigma_{\lambda^1},\dots,\sigma_{\lambda^m}\rangle_{g,|e|}\,. \tag{5.15}$$

Proof. If $P(\alpha)=\prod_{j=1}^s\alpha_{k_j}$ is a monomial in $\alpha_1,\dots,\alpha_n$ of weighted degree $D(n,e,0)$, then using (2) and (3) of Proposition 5.1, we easily obtain

$$\widetilde{N}^0_{C,e}(P(\alpha);\mathcal{O}_C^{\oplus 2n})\ =\ \langle\sigma_{k_1},\dots,\sigma_{k_s}\rangle_{g,|e|}\,. \tag{5.16}$$

On the other hand, the Vafa–Intriligator-type formula shows that the Gromov–Witten invariant $\langle\sigma_{\lambda^1},\dots,\sigma_{\lambda^m}\rangle_{g,d}$ only depends on the product $\prod_{i=1}^m\sigma_{\lambda^i}$ of the arguments. Since $\sigma_1,\dots,\sigma_n$ generate $\mathrm{CH}^*(\mathrm{LG}(n))$, the class $P(\sigma):=\prod_{i=1}^m\sigma_{\lambda^i}$ can be written as a sum of monomials in $\sigma_1,\dots,\sigma_n$. But since each σ_i corresponds to α_i and hence $P(\sigma)$ to $P(\alpha)$, the desired equality follows from the linearity of both sides of the equality (5.16). □

Proposition 5.14 together with Proposition 5.13 yields the following recursive relation among Gromov–Witten invariants of $\mathrm{LG}(n)$.

Corollary 5.15. *Let $n > 0$ and $g, d \geq 0$ be given. Suppose $\sum_{i=1}^{m} |\lambda^i| = (n+1)d - \frac{n(n+1)}{2}(g-1)$. Then for any $k \geq 0$, we have*

$$\langle \sigma_{\lambda^1}, \ldots, \sigma_{\lambda^m} \rangle_{g,d} \;=\; \left\langle \sigma_{\rho_n}^{2k}, \sigma_{\lambda^1}, \ldots, \sigma_{\lambda^m} \right\rangle_{g,d+kn}.$$

6. Main results

From the discussion in the previous sections, we conclude:

Theorem 6.1. *Let C be a smooth projective curve of genus g and W a symplectic bundle over C of degree $w = n\ell$. Then for a polynomial $P(\alpha)$ of degree $D(n, e, \ell)$, the number $\widetilde{N}^w_{g,e}(P(\alpha); W)$ is computed by*

$$\widetilde{N}^w_{C,e}(P(\alpha); W) \;=\; \begin{cases} A \sum_{J \in \mathcal{I}^e_{n+1}} \left\{ S_{\rho_n}(\zeta^J) \right\}^{g-1} P(\zeta^J) & \text{if } \ell = 2m, \\ A \sum_{J \in \mathcal{I}^e_{n+1}} \left\{ S_{\rho_n}(\zeta^J) \right\}^{g-1} \widetilde{Q}_{\rho_n}(\zeta^J) P(\zeta^J) & \text{if } \ell = 2m-1, \end{cases}$$

where $A := 2^{n(g-1)+e-mn}$ and $P(\zeta^J) := P\left(E_1(\zeta^J), \ldots, E_n(\zeta^J)\right)$.

Proof. For the case $w = 2mn$, we take a line bundle Ξ on C of degree $-m$, so that $\widetilde{W} := W \otimes \Xi$ is a symplectic bundle of degree 0 over C. Then by Proposition 5.12, we have

$$\widetilde{N}^w_{C,e}(P(\alpha); W) \;=\; \widetilde{N}^0_{C,e-mn}(P(\alpha); \widetilde{W}).$$

Thus the result follows from Propositions 2.5 and 5.14.

If $w = (2m-1)n$, by Lemma 5.7 we have

$$\widetilde{N}^w_{C,e}(P(\alpha); W) \;=\; \widetilde{N}^{w+n}_{C,e}(\widetilde{Q}_{\rho_n}(\alpha) P(\alpha); W^H)$$

for some Hecke transform W^H of degree $w + n$. Since $w + n = 2mn$, we are reduced to the previous case. □

Assume W is general (for example, very stable). Let P be the constant polynomial 1, so that $\Theta(1, 0)$ is the fundamental cycle of $LQ_e(W)$. If $e = e_0$ and $n(\ell - g + 1)$ is even, where e_0 was defined in Proposition 1, then $LQ_e(W)$ is zero dimensional and hence $\widetilde{N}^w_{C,e}(1; W)$ is precisely the number of maximal Lagrangian subbundles of W. Recall from Lemma 5.10 that in this case $\widetilde{N}^w_{C,e}(1; W)$ depends only on the genus of C, so we denote it by $N(g, n, \ell, e)$. The following is immediate from Theorem 6.1.

Corollary 6.2. *Let W be a general stable symplectic bundle over C of rank $2n$ and degree $w = n\ell$, where $n(\ell - g + 1)$ is even. Let $e = \frac{1}{2}n(\ell - g + 1)$. Then the number $N(g, n, \ell, e)$ of maximal Lagrangian subbundles is given by*

$$N(g, n, \ell, e) \;=\; \begin{cases} B_1 \sum_{J \in \mathcal{I}^e_{n+1}} \left\{ S_{\rho_n}(\zeta^J) \right\}^{g-1} & \text{if } \ell = 2m, \\ B_2 \sum_{J \in \mathcal{I}^e_{n+1}} \left\{ S_{\rho_n}(\zeta^J) \right\}^{g-1} \widetilde{Q}_{\rho_n}(\zeta^J) & \text{if } \ell = 2m-1, \end{cases}$$

where $B_1 = \sqrt{2}^{n(g-1)}$ *and* $B_2 = \sqrt{2}^{n(g-2)}$.

Using this formula, we compute by hand the number of maximal Lagrangian subbundles of a general W of rank $2n \leq 4$.

Corollary 6.3. *For* $g \geq 2$ *and* $e = \frac{1}{2}n(\ell - g + 1)$*, we have the following.*

(1) $n = 1$, $\ell \not\equiv g \mod 2$*:* $N(g, 1, \ell, e) = 2^g$*.*
(2) $n = 2$, g *even,* ℓ *odd:* $N(g, 2, -1, -g) = 2^{g-1}(3^g + 1)$*.*
(3) $n = 2$, g *even,* ℓ *even:* $N(g, 2, 0, -g+1) = 2^{g-1}(3^g - 1)$*.*
(4) $n = 2$, g *odd,* ℓ *odd:* $N(g, 2, -1, -g) = 2^{g-1}(3^g - 1)$*.*
(5) $n = 2$, g *odd,* ℓ *even:* $N(g, 2, 0, -g+1) = 2^{g-1}(3^g + 1)$*.*

Remark 6.4. In (1), the number 2^g coincides with the number of maximal line subbundles of a general rank 2 vector bundle obtained in [27] and [22]. This can be explained by the fact that any rank 2 vector bundle V has a symplectic structure given by $V \cong V^\vee \otimes \det(V)$, and any line subbundle is Lagrangian.

Remark 6.5. By Holla [14, Theorem 4.2], if $g = 2$, the number of maximal rank 2 subbundles of a general rank 4 vector bundle V is 24 (resp., 40), if $\deg(V) \equiv 2 \mod 4$ (resp., $\deg(V) \equiv 0 \mod 4$). These can be compared with the numbers 20 and 16 given by (2) and (3) respectively.

It should be noted that [13, Theorem 2], in our language, states incorrectly that $N(2, 2, 0, -1) = 24$. This is due to a mistake in the geometric argument on [13, p. 270]. The correct statement of [13, Theorem 2] is that the moduli map Φ is surjective and generically finite of degree 20.

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