BRILL–NOETHER LOCI ON MODULI SPACES OF SYMPLECTIC BUNDLES OVER CURVES

ALI BAJRAVANI AND GEORGE H. HITCHING

Abstract. The symplectic Brill–Noether locus $\mathcal{S}^k_{2n,K}$ associated to a curve C parametrises stable rank $2n$ bundles over C with at least k sections and which carry a nondegenerate skewsymmetric bilinear form with values in the canonical bundle. This is a symmetric determinantal variety whose tangent spaces are defined by a symmetrised Petri map. We obtain upper bounds on the dimensions of various components of $\mathcal{S}^k_{2n,K}$. We show the nonemptiness of several $\mathcal{S}^k_{2n,K}$, and in most of these cases also the existence of a component which is generically smooth and of the expected dimension. As an application, for certain values of n and k we exhibit components of excess dimension of the standard Brill–Noether locus $B^k_{2n,2n(g-1)}$ over any curve of genus $g \geq 122$. We obtain similar results for moduli spaces of coherent systems.

1. Introduction

Let C be a projective smooth curve of genus $g \geq 2$ and $\mathcal{U}(r,d)$ the moduli space of stable vector bundles of rank r and degree d over C. A fundamental attribute of $\mathcal{U}(r,d)$ is the stratification by *generalised Brill–Noether loci*

$$B^k_{r,d} \ := \ \{W \in \mathcal{U}(r,d) : h^0(C,W) \geq k\}.$$

This is a determinantal variety whose expected dimension is

$$\beta^k_{r,d} \ := \ r^2(g-1) + 1 - k(k-d+r(g-1)).$$

Moreover, $B^{k+1}_{r,d} \subseteq \mathrm{Sing}(B^k_{r,d})$. If $r = 1$, one obtains the classical Brill–Noether loci on $\mathrm{Pic}^d(C)$, which are traditionally denoted $W^{k-1}_d(C)$. For a generic curve, the $B^k_{1,d}$ behave as regularly as possible: $B^k_{1,d}$ is nonempty of dimension $\beta^k_{1,d}$ if and only if $\beta^k_{1,d} \geq 0$, and furthermore irreducible if this dimension is positive; and $\mathrm{Sing}(B^k_{1,d}) = B^{k+1}_{1,d}$. See [ACGH85] for a full account of this story.

For $r \geq 2$, the situation is more complicated, even for a general curve. In recent years, much attention has been given to determining the components of $B^k_{r,d}$ for $r \geq 2$, together with their dimensions and singular loci. See [GT09] for a survey. Several generalisations have been studied, including coherent systems (see for example [BGMN03] and [Ne11]), generalised theta divisors (see [Be06] for an overview) and more generally twisted Brill–Noether loci (see [Te14] and [HHN18]).

2010 *Mathematics Subject Classification.* 14H60 (14M12).

Key words and phrases. Brill–Noether locus, symplectic vector bundle, determinantal locus.

This is a post-peer-review, pre-copyedit version of an article published in Collectanea Mathematica.
The final authenticated version is available online at: https://doi.org/10.1007/s13348-020-00300-7

A variant of $B^k_{r,d}$ which is of particular relevance for the present work is the *fixed determinant Brill–Noether locus*

$$B^k_{r,L} \ := \ \{W \in \mathcal{U}(r,d) : h^0(C,W) \geq k \text{ and } \det(W) = L\}$$

where L is a fixed line bundle of degree d. Denote by K the canonical bundle T^*_C. The locus $B^k_{2,K}$ has been studied extensively in [BF], [Muk92], [Muk97], [Te04], [Te07], [LNP16] and [Baj19] (see also [GN14]), and we shall return to this below. The loci $B^k_{r,L}$ for other values of r and L are studied in [Os13-1], [Os13-2], [LNS16], [Zh17] and elsewhere.

In the present work, we consider a different generalisation of $B^k_{2,K}$ to higher rank. For any bundle W of rank two, there is a natural skewsymmetric isomorphism $W \xrightarrow{\sim} W^* \otimes \det(W)$. In general, recall that a vector bundle W is said to be *L-valued symplectic* if there is a skewsymmetric isomorphism $W \xrightarrow{\sim} W^* \otimes L$ for some line bundle L; equivalently, if there is a nondegenerate skewsymmetric bilinear form $\omega\colon W \otimes W \to L$. By nondegeneracy, a symplectic bundle must have even rank $2n \geq 2$, and moreover $\det(W) = L^n$. For us, L will always be K. There is a quasiprojective moduli space $\mathcal{MS}(2n,K)$ for stable K-valued symplectic bundles over C, which we discuss in more detail in § 2.1. Our fundamental objects of study are the *symplectic Brill–Noether loci*

$$\mathcal{S}^k_{2n,K} \ := \ \{W \in \mathcal{MS}(2n,K) : h^0(C,W) \geq k\} \ \subseteq \ B^k_{2n,K^n}.$$

It follows from [Muk92, Remark 4.6] that $\mathcal{S}^k_{2n,K}$ is a *symmetric* determinantal variety of expected codimension $\frac{1}{2}k(k+1)$. In § 2.2, we expand upon this remark, showing that $\mathcal{S}^k_{2n,K}$ is étale locally defined by the vanishing of the $(k+1)\times(k+1)$-minors of a symmetric matrix. In § 2.3 we recall a description of the Zariski tangent spaces of $\mathcal{S}^k_{2n,K}$ in terms of a symmetrised Petri map. Adapting well-known results from [ACGH85] to the symplectic case, in §§ 2.4–2.5 we construct a partial desingularisation of $\mathcal{S}^k_{2n,K}$ near a well-behaved singular point W and describe the tangent cone $C_W\mathcal{S}^k_{2n,K}$.

For $2n = 2$, the K-valued symplectic bundles are precisely those of canonical determinant and, as outlined above, $\mathcal{S}^k_{2,K} = B^k_{2,K}$ has been much studied. Our next objective is to answer some of the basic questions of nonemptiness, dimension and smoothness of $\mathcal{S}^k_{2n,K}$ for $2n \geq 4$. In § 3, we prove the following dimension bounds on various components of $\mathcal{S}^k_{2n,K}$, generalising [Baj19, Theorem 3.4] of the first author.

Theorem A. *Let C be any curve of genus $g \geq 2$.*

(a) *(Theorem 3.5) Let X be a closed irreducible sublocus of $\mathcal{S}^k_{2n,K}$ of which a general element W satisfies $H^0(C,W) = H^0(C,L_W)$ for some line subbundle $L_W \subset W$ of degree d. Then for each $W \in X$, we have*

$$\dim X \ \leq \ \dim T_W X \ \leq \ \dim \left(T_{L_W} B^k_{1,d}\right) + n(2n+1)(g-1) - 2nd - 1.$$

(b) *(Theorem 3.7) Let k be an integer satisfying $1 \leq k \leq n(g+1)-1$. Suppose Y is an irreducible component of $\mathcal{S}^k_{2n,K}$ containing a bundle W satisfying $h^0(C,W) = k$ and* *such that the rank of the subbundle of W generated by global sections is r. Then*

$$\dim Y \ \leq \ \dim T_W Y \ \leq \ \min \left\{ n(2n+1)(g-1) - (2k-1), n(2n+1)(g-1) - k - \frac{1}{2}r(r-1) \right\}.$$

In Corollary 3.6, we deduce some conditions on g, n and k for the existence of a *component* X of the form in Theorem A (a).

In § 4 we construct stable symplectic bundles W with prescribed values of $h^0(C, W)$, showing that $\mathcal{S}^k_{2n,K}$ is nonempty in several cases. The approach is a combination of techniques from [Me99] and [CH14]: the W we construct are "almost split" symplectic extensions $0 \to E \to W \to E^* \otimes K \to 0$ where E and $K \otimes E^*$ are stable and have many sections. In § 4.4, we show that if C is Petri, in some cases $\mathcal{S}^k_{2n,K}$ has a component which is smooth and of the expected dimension. To state the results, set

$$k_0 \ := \ \max\{ k \geq 0 : \dim B^k_{1,g-1} \geq 1 \}. \tag{1.1}$$

By Brill–Noether theory, if C is Petri then $k_0 = \lfloor \sqrt{g-1} \rfloor$, where $\lfloor t \rfloor = \max\{ m \in \mathbb{Z} : m \leq t \}$.

Theorem B. *Let C be a curve of genus $g \geq 3$.*

(a) *(Theorem 4.6) For $1 \leq k \leq 2nk_0 - 3$, the locus $\mathcal{S}^k_{2n,K}$ is nonempty.*

(b) *(Theorem 4.9) If C is a general Petri curve, then for $1 \leq k \leq 2nk_0 - 3$ there is a component of $\mathcal{S}^k_{2n,K}$ which is generically smooth and of the expected codimension $\frac{1}{2}k(k+1)$.*

We also briefly mention strictly semistable symplectic bundles in Remark 4.7.

It should be noted that there are significantly stronger results in the rank two case. For $2n = 2$, the bound in Theorem B (a) translates into $4(g-1) \geq (k+3)^2$. For $g \geq 5$, Teixidor [Te04] showed for $4(g-1) \geq k^2 - 1$ that $B^k_{2,K}$ is nonempty and has a component of the expected dimension, with a slightly better result for k even. Furthermore, for $k \geq 8$ and g prime, Lange, Newstead and Park [LNP16] showed that $B^k_{2,K}$ is nonempty for $4g - 4 \geq k^2 - k$. We certainly expect that the bound in Theorem B can be improved for $2n \geq 4$.

In § 5, we give an application of Theorem B to standard Brill–Noether loci $B^k_{2n,2n(g-1)}$. For $r \geq 2$, it was proven in [Te91] that in many cases $B^k_{r,d}$ has a component which is generically smooth and of the expected dimension. However, even for a generic curve, components of larger dimension can appear. Following [CFK18], we call such components *superabundant*. It was noted in [Ne11, § 9] and [BF, §1] that $B^k_{2,K} = \mathcal{S}^k_{2,K}$ in many cases (precisely; for $g < \frac{k(k-1)}{2}$) has expected dimension strictly greater than $\beta^k_{2,2g-2}$, despite the fact that $B^k_{2,K}$ is contained in $B^k_{2,2g-2}$. For $n \geq 2$ it emerges that the expected dimension of $\mathcal{S}^k_{2n,K}$ can also exceed $\beta^k_{2n,2n(g-1)}$ for certain values of g, n and k. We show the following.

Theorem C.

(a) *(Theorem 5.1) Suppose $m \geq 7$ and let C be any curve of genus $g = m^2 + 1$. Then for any $n \geq 1$, the locus $\mathcal{S}^{2nm-3}_{2n,K}$ is nonempty and has dimension greater than $\beta^{2nm-3}_{2n,2n(g-1)}$. In particular $B^{2nm-3}_{2n,2n(g-1)}$ has a superabundant component.*

(b) *(Theorem 5.2) Fix $n \geq 1$ and let C be any curve of genus $g \geq (4n+7)^2+1$. For k_0 as defined in 1.1, the locus $\mathcal{S}^{2nk_0-3}_{2n,K}$ is nonempty and has dimension greater than $\beta^{2nk_0-3}_{2n,2n(g-1)}$. In particular, $B^{2nk_0-3}_{2n,2n(g-1)}$ has a superabundant component.*

In § 5.1, we also obtain similar results for certain moduli spaces of coherent systems, both with and without fixed determinant.

We note that Teixidor [Te04] also obtains superabundant components of $B^k_{2,K} = \mathcal{S}^k_{2,K}$ for certain values of k.

Since Theorem C (b) applies to all curves of genus $g \geq 122$, it gives a systematic way of finding ordinary determinantal varieties of dimension strictly greater than expected, in some ways akin to [HHN18, Proposition 9.1]. We hope that this aspect of the present work may also be of interest outside the context of Brill–Noether theory.

The construction of the locus $\mathcal{S}^k_{2n,K}$ is easily adapted for *K-valued orthogonal* bundles; that is, bundles admitting a *symmetric* K-valued bilinear form (see [Mum71]). However, our methods when applied to orthogonal bundles did not yield superabundant components of any $B^k_{r,d}$; and the argument of Theorem B (b) also fails for orthogonal bundles. Therefore we have restricted our attention for the present to the symplectic case, with the intention of further studying orthogonal Brill–Noether loci in the future.

Acknowledgements. We would like to thank Peter Newstead for helpful comments and for making us aware of several references. We also thank the referee for useful advice and comments. Ali Bajravani was in part supported by a grant from IPM (No. 99140036).

Notation. Throughout, C denotes a smooth projective curve of genus $g \geq 2$ over an algebraically closed field $\mathbb{K}$ of characteristic zero. For a sheaf F over C, we shall often abbreviate $H^i(C,F)$, $h^i(C,F)$ and $\chi(C,F)$ to $H^i(F)$, $h^i(F)$ and $\chi(F)$ respectively. If $A \times B$ is a product, we denote the projections by π_A and π_B.

2. Symplectic Brill–Noether loci

2.1. **Moduli of K-valued symplectic bundles.** Let W be a K-valued symplectic bundle of rank $2n$ over C. By [BG06, § 2], we have $\det(W) = K^n$. If κ is a theta characteristic, then $V := W \otimes \kappa^{-1}$ is $\mathcal{O}_C$-valued symplectic. Thus V is the associated vector bundle of a principal Sp_{2n}-bundle P over C. By a similar argument to that in [Rm81, § 4] (carried out in [Hi05]), the vector bundle V is stable if and only if P is a *regularly stable* principal Sp_{2n}-bundle; that is, stable and satisfying $\mathrm{Aut}\,(P) = Z(\mathrm{Sp}_{2n}) = \mathbb{Z}_2$.

By [Rth96], there is a moduli space $\mathcal{M}(\mathrm{Sp}_{2n})$ for stable principal Sp_{2n}-bundles, which is an irreducible quasiprojective variety of dimension $n(2n+1)(g-1)$, and smooth at all regularly stable points. Moreover, it follows from [Se12, Proposition 2.6 and Theorem 3.2] that the natural map $\mathcal{M}(\mathrm{Sp}_{2n}) \dashrightarrow \mathcal{U}(2n,0)$ is an embedding. Translating by κ, we conclude:

Lemma 2.1. *The moduli space $\mathcal{MS}(2n, K)$ of stable vector bundles of rank $2n$ with K-valued symplectic structure is a smooth irreducible sublocus of $\mathcal{U}(2n, 2n(g-1))$, of dimension $n(2n+1)(g-1)$.*

Furthermore, we recall a description of the tangent spaces of $\mathcal{MS}(2n, K)$. It is well known that first order infinitesimal deformations of a vector bundle $W \to C$ are parametrised by $H^1(\mathrm{End}\, W)$. If $\omega\colon W \to W^* \otimes K$ is a skewsymmetric isomorphism, we have an identification

$$\omega_*\colon H^1(\mathrm{End}\, W) \xrightarrow{\sim} H^1(K \otimes W^* \otimes W^*). \tag{2.1}$$

The following can be shown by a computation similar to that in the proof of [GT09, Proposition 8.1].

Lemma 2.2. *Let (W, ω) be a K-valued symplectic bundle. Then the deformations of W preserving the symplectic structure are parametrised by the subspace $H^1(K \otimes \mathrm{Sym}^2 W^*) \subseteq H^1(K \otimes W^* \otimes W^*)$. In particular, if W is stable, then $T_W \mathcal{MS}(2n, K) \cong H^1(K \otimes \mathrm{Sym}^2 W^*)$.*

2.2. **The scheme structure of symplectic Brill–Noether loci.** As already noted, bundles of rank two and canonical determinant are precisely the K-valued symplectic bundles of rank two. We shall see that the construction of $B^k_{2,K} = \mathcal{S}^k_{2,K}$ in [Muk92] and [Muk97] generalises virtually word for word to higher rank K-valued symplectic bundles.

To construct $\mathcal{S}^k_{2n,K}$ as a scheme, we require a suitable Poincaré bundle equipped with a family of symplectic forms. As $\mathcal{MS}(2n, K) \cong \mathcal{M}(\mathrm{Sp}_{2n})$ and the group Sp_{2n} is not of adjoint type, by [BBNN06] there is no Poincaré bundle over $\mathcal{MS}(2n, K) \times C$. The following lemma shows that Poincaré bundles do exist over small enough étale open subsets of $\mathcal{MS}(2n, K)$.

Lemma 2.3. *There exists an étale open covering $\{U_\alpha\}$ of $\mathcal{MS}(2n, K)$, together with Poincaré bundles $\mathcal{W}_\alpha \to U_\alpha \times C$, each equipped with a family $\omega_\alpha\colon \mathcal{W}_\alpha \otimes \mathcal{W}_\alpha \to \pi_C^* K$ of symplectic forms.*

Proof. As $\mathcal{MS}(2n, K)$ is contained in $\mathcal{U}(2n, 2n(g-1))$, there exists an étale cover $\widetilde{\mathcal{M}} \to \mathcal{MS}(2n, K)$ together with a Poincaré bundle $\mathcal{W} \to \widetilde{\mathcal{M}} \times C$. By stability, for any $W \in \mathcal{MS}(2n, K)$ we have $h^0(K \otimes \wedge^2 W^*) = 1$. Hence by [Ha83, Corollary III.12.9], the sheaf

$$\mathcal{B} \ := \ \left(\pi_{\widetilde{\mathcal{M}}}\right)_* \left(\pi_C^* K \otimes \wedge^2 \mathcal{W}^*\right)$$

is locally free of rank one over $\widetilde{\mathcal{M}}$. Let $\{U_\alpha\}$ be an open covering of $\widetilde{\mathcal{M}}$ such that $\mathcal{B}|_{U_\alpha}$ is trivial for each α. Now if W is a stable vector bundle of slope $g-1$, then any nonzero map $W \to W^* \otimes K$ is an isomorphism. Therefore, any generating section ω_α for $\mathcal{B}|_{U_\alpha}$ defines a family of symplectic structures on $\mathcal{W}_\alpha := \mathcal{W}|_{U_\alpha \times C}$. The lemma follows. □

We proceed to study the *symmetric determinantal* structure of $\mathcal{S}^k_{2n,K}$. The following proposition is an obvious generalisation of [Muk92, Theorem 4.2], and is essentially contained in [Muk92, Remark 4.6]. We give the proof, because the construction will be used further in §§ 2.4–2.5.

Proposition 2.4.

(a) *Scheme-theoretically, $\mathcal{S}^k_{2n,K}$ is étale locally defined by the vanishing of the $(\nu - k + 1) \times (\nu - k + 1)$-minors of a $\nu \times \nu$ symmetric matrix, for some $\nu \geq k$.*
(b) *Each component of $\mathcal{S}^k_{2n,K}$ is of codimension at most $\frac{1}{2}k(k+1)$.*
(c) *The sublocus $\mathcal{S}^{k+1}_{2n,K}$ is contained in $\mathrm{Sing}(\mathcal{S}^k_{2n,K})$.*

Proof. (a) We begin with a slightly more general situation. Let $\mathcal{W} \to S \times C$ be a family of bundles of rank $2n$ over C, and let $\omega \colon \mathcal{W} \otimes \mathcal{W} \to \pi_C^* K$ be a family of K-valued symplectic structures on $\mathcal{W}$. For $k \geq 0$, we define the Brill–Noether locus associated to the family $\mathcal{W}$ set-theoretically as

$$\mathcal{S}^k(\mathcal{W}) \ := \ \{s \in S : h^0(C, \mathcal{W}_s) \geq k\}.$$

Now for any effective divisor D on C, the coherent sheaf

$$\mathcal{F} \ := \ (\pi_S)_* \left(\frac{\mathcal{W} \otimes \pi_C^* \mathcal{O}_C(D)}{\mathcal{W} \otimes \pi_C^* \mathcal{O}_C(-D)} \right), \tag{2.2}$$

is locally free of rank $4n \cdot \deg(D)$ over S. We shall define a symplectic structure on $\mathcal{F}$. We extend ω linearly over $\pi_C^* \mathcal{O}_C$ to a symplectic form

$$\wedge^2 \left(\mathcal{W} \otimes \pi_C^* \mathcal{O}_C(D) \right) \ \to \ \pi_C^* K(2D).$$

Now $\omega_s(\mathcal{W}_s(-D), \mathcal{W}_s(D)) \subseteq K$ for all s. Thus, if t, u are elements of $\mathcal{F}_s = H^0 \left(C, \frac{\mathcal{W}_s(D)}{\mathcal{W}_s(-D)} \right)$ and Res is the residue map, then

$$\sum_{x \in \mathrm{Supp}(D)} \mathrm{Res} \left(\omega_s(t_x, u_x) \right) \ =: \ \overline{\omega}_s(t, u)$$

is a well-defined element of $H^1(K)$. Thus ω descends to a bilinear map

$$\overline{\omega} \colon \wedge^2 \mathcal{F} \ \to \ \mathcal{O}_S \otimes H^1(K) \ = \ \mathcal{O}_S.$$

Moreover, $\overline{\omega}$ is nondegenerate since ω is.

Let us now assume that $\deg(D)$ is large enough that $h^1(C, \mathcal{W}_s(D)) = 0$ for all $s \in S$. Then, as $\mathcal{W}_s \cong \mathcal{W}_s^* \otimes K$, by Serre duality $h^0(C, \mathcal{W}_s(-D)) = 0$ for all $s \in S$ also. Thus the subsheaf

$$\mathcal{L}_1 \ := (\pi_S)_* \left(\frac{\mathcal{W}}{\mathcal{W} \otimes \pi_C^* \mathcal{O}_C(-D)} \right) \ \subset \ \mathcal{F}$$

is locally free of rank $2n \cdot \deg(D)$. As the residue of a regular differential is zero, $\mathcal{L}_1$ is Lagrangian with respect to $\overline{\omega}$.

Furthermore, as $h^1(\mathcal{W}_s(D)) = 0$ for all s, the subsheaf

$$\mathcal{L}_2 \ := \ \mathrm{Im} \left((\pi_S)_* \left(\mathcal{W} \otimes \pi_C^* \mathcal{O}_C(D) \right) \ \to \ \mathcal{F} \right) \ \subset \ \mathcal{F}$$

is also locally free of rank $2n \cdot \deg(D)$. By the residue theorem [Ha83, III.7.14.2], in fact $\mathcal{L}_2$ also defines a Lagrangian subbundle of $\mathcal{F}$. Moreover, it is easy to see that $\mathcal{L}_1|_s \cap \mathcal{L}_2|_s \cong H^0(C, \mathcal{W}_s)$ for each $s \in S$, so

$$\mathcal{S}^k(\mathcal{W}) \ = \ \{s \in S : \dim \left(\mathcal{L}_1|_s \cap \mathcal{L}_2|_s \right) \geq k\}. \tag{2.3}$$

Now let $U \subseteq S$ be an open set over which $\mathcal{F}$ is trivial. Then any choice of Lagrangian subbundle of $\mathcal{F}|_U$ complementary to $\mathcal{L}_1|_U$ defines a local splitting $\mathcal{F}|_U \xrightarrow{\sim} \mathcal{L}_1|_U \oplus \mathcal{L}_1^*|_U$. Perturbing this choice and shrinking U if necessary, we can assume in addition that $\mathcal{L}_1^*|_s \cap \mathcal{L}_2|_s = 0$ for all $s \in U$. Then, as in [Muk97, Examples 1.5 and 1.7], there exists a *symmetric* map $\Sigma_U \colon \mathcal{L}_1|_U \to \mathcal{L}_1^*|_U$ with the property that $\mathcal{L}_2|_U$ is the graph of Σ_U, and for each $s \in U$ moreover

$$\mathrm{Ker}(\Sigma_U|_s) \;=\; \mathcal{L}_1|_s \cap \mathcal{L}_2|_s.$$

It follows by (2.3) that $\mathcal{S}^k(\mathcal{W}) \cap U$ is defined by the condition $\mathrm{rk}\,(\Sigma_U|_s) \leq 2n \cdot \deg(D) - k$, so is cut out by the vanishing of the $(\nu - k + 1) \times (\nu - k + 1)$-minors of a local matrix expression for Σ_U, where $\nu = \mathrm{rk}\,(\mathcal{L}_1) = 2n \cdot \deg(D)$. Clearly, S can be covered by such open sets U.

Now we specialise to $S = U_\alpha$ and $(\mathcal{W}, \omega) = (\mathcal{W}_\alpha, \omega_\alpha)$ as defined in Lemma 2.3. Statement (a) follows as $\mathcal{S}^k_{2n,K}$ is the union of the images of the loci $\mathcal{S}^k(\mathcal{W}_\alpha)$ by an étale map.

Parts (b) and (c) follow from part (a), by general properties of symmetric determinantal loci. (In fact these statements are true for any family $\mathcal{W} \to S \times C$ of K-valued symplectic bundles.) □

Remark 2.5. In [Os13-1], [Os13-2] and [Zh17] the above approach is generalised to the setting of *multiply symplectic Grassmannians* and used to give lower bounds on fixed determinant Brill–Noether loci $B^k_{r,L}$ for special line bundles L.

2.3. **Tangent spaces of symplectic Brill–Noether loci.** Let us now describe the Zariski tangent spaces of $\mathcal{S}^k_{2n,K}$, following the discussion for bundles of rank two in [Te07, § 1]. Firstly, we require a definition. Recall that for any bundle $W \to C$ we have the *Petri map*

$$\mu \colon H^0(W) \otimes H^0(K \otimes W^*) \;\to\; H^0(K \otimes \mathrm{End}\, W).$$

If $\omega \colon W \xrightarrow{\sim} K \otimes W^*$ is an isomorphism, then we obtain an identification of the Petri map with the multiplication map

$$H^0(W) \otimes H^0(W) \;\to\; H^0(W \otimes W), \tag{2.4}$$

If W is simple (for example, stable) then this identification is canonical up to scalar. In this case, we abuse notation slightly and denote the map (2.4) also by μ. Clearly, $\mu\left(\mathrm{Sym}^2 H^0(W)\right) \subseteq H^0(\mathrm{Sym}^2 W)$. Let $\mathrm{sym} \colon H^0(W) \otimes H^0(W) \to \mathrm{Sym}^2 H^0(W)$ be the canonical surjection.

Definition 2.6. Let $W \to C$ be a K-valued symplectic bundle. For any subspace $\Lambda \subseteq H^0(W)$, we write

$$\mu^{\mathrm{s}}_\Lambda \colon \mathrm{sym}(\Lambda \otimes H^0(W)) \;\to\; H^0(\mathrm{Sym}^2 W)$$

for the restriction of (2.4). We abbreviate $\mu^{\mathrm{s}}_{H^0(W)}$ to μ^{s}. Furthermore, for any subspace Π of $H^0(W \otimes W)$ we write

$$\Pi^\perp \;:=\; \{v \in H^1(K \otimes \mathrm{Sym}^2 W^*) : v \cup \Pi = 0\},$$

the orthogonal complement of Π in $H^1(K \otimes \mathrm{Sym}^2 W)$.

Proposition 2.7. *Let W be a simple K-valued symplectic bundle. For any subspace $\Lambda \subseteq H^0(W)$, the space of first-order infinitesimal deformations preserving Λ is exactly* $\mathrm{Im}\,(\mu^{\mathrm{s}}_\Lambda)^\perp$.

Proof. As in the proof of [ACGH85, Proposition IV.4.1], using also the identification (2.1), one shows that the space of first-order infinitesimal deformations of the vector bundle W which preserve the subspace Λ is given by

$$\{v \in H^1(K \otimes W^* \otimes W^*) : v \cup \mu(\Lambda \otimes H^0(W)) = 0\},$$

the orthogonal complement of $\mu\left(\Lambda \otimes H^0(W)\right)$ in the full deformation space $H^1(K \otimes W^* \otimes W^*)$. Thus we must describe the intersection of this space with $H^1(K \otimes \mathrm{Sym}^2 W^*)$.

Suppose $v \in H^1(K \otimes \mathrm{Sym}^2 W^*)$. Then clearly $v \cup \mu\left(\wedge^2 H^0(W)\right) = 0$, whence

$$v \cup \mu(\sigma) \ = \ v \cup \mu(\mathrm{sym}(\sigma))$$

for all $\sigma \in H^0(W) \otimes H^0(W)$. It follows, as desired, that

$$\mu(\Lambda \otimes H^0(W))^\perp \ = \ \mu \circ \mathrm{sym}(\Lambda \otimes H^0(W))^\perp \ = \ \mathrm{Im}\,(\mu^{\mathrm{s}}_\Lambda)^\perp \ \subseteq \ H^1(K \otimes \mathrm{Sym}^2 W^*). \quad \square$$

Corollary 2.8. *Suppose W is a stable K-valued symplectic bundle with $h^0(W) = k$. Then $\mathcal{S}^k_{2n,K}$ is smooth and of codimension $\frac{1}{2}k(k+1)$ at W if and only if $\mu^{\mathrm{s}} \colon \mathrm{Sym}^2 H^0(W) \to H^0(\mathrm{Sym}^2 W)$ is injective.*

Proof. By Proposition 2.7, we have $T_W \mathcal{S}^k_{2n,K} = \mathrm{Im}\,(\mu^{\mathrm{s}})^\perp$. Now clearly

$$\dim \mathrm{Im}\,(\mu^{\mathrm{s}})^\perp \ = \ \dim \mathcal{MS}(2n, K) - \dim \mathrm{Sym}^2 H^0(W) + \dim \mathrm{Ker}(\mu^{\mathrm{s}}).$$

Since $\mathrm{Sym}^2 H^0(W)$ has dimension $\frac{1}{2}k(k+1)$, we see that $T_W \mathcal{S}^k_{2n,K}$ has the expected codimension if and only if μ^{s} is injective. $\square$

2.4. **Desingularisations of symplectic Brill–Noether loci.** In this subsection, we adapt arguments for determinantal varieties from [ACGH85] to construct a partial desingularisation of (an étale cover of) the symplectic Brill–Noether stratum $\mathcal{S}^k_{2n,K}$, and use it to obtain information on smooth points of lower strata. In the next section, we shall also use the desingularisation to study the tangent cones of $\mathcal{S}^k_{2n,K}$. This approach was used in a similar way in [ACGH85], [CT11] and [HHN18] for the study of, respectively, Brill–Noether loci in $\mathrm{Pic}(C)$, higher rank Brill–Noether loci $B^k_{r,d}$ and twisted Brill–Noether loci $B^k_{n,e}(V)$.

Let W be a stable K-valued symplectic bundle with $h^0(W) \geq k \geq 1$. By Lemma 2.3 and Proposition 2.4 (a), we can find an étale neighbourhood S of W in $\mathcal{MS}(2n, K)$ and a Poincaré bundle $\mathcal{W} \to S \times C$, together with a symmetric map of vector bundles $\Sigma \colon \mathcal{L}_1 \to \mathcal{L}_1^*$ over S such that for each $s \in S$ we have $\mathrm{Ker}\,(\Sigma_s) \cong H^0(\mathcal{W}_s)$, so

$$\mathcal{S}^k_{2n,K} \times_{\mathcal{MS}(2n,K)} S \ = \ \mathcal{S}^k(\mathcal{W}) \ = \ \{s \in S : \dim \mathrm{Ker}(\Sigma|_s) \geq k\},$$

an étale cover of $\mathcal{S}^k_{2n,K}$ near W.

We consider the Grassmann bundle $\mathrm{Gr}(k, \mathcal{L}_1)$ parametrising k-dimensional linear subspaces of fibres of $\mathcal{L}_1$. In analogy with [ACGH85, IV.3], we define

$$SG^k(\mathcal{W}) \ := \ \{\Lambda \in \mathrm{Gr}(k, \mathcal{L}_1) : \Sigma(\Lambda) = 0\}. \tag{2.5}$$

A point of $SG^k(\mathcal{W})$ is a pair $(\mathcal{W}_s, \Lambda)$ where $\mathcal{W}_s$ is a symplectic bundle represented in S and Λ a k-dimensional subspace of $H^0(\mathcal{W}_s)$. Such a pair will be called a *symplectic coherent system.* We write $c\colon SG^k(\mathcal{W}) \to S$ for the projection.

Theorem 2.9. *Let W, S, $\mathcal{W}$ and $\Sigma\colon \mathcal{L}_1 \to \mathcal{L}_1^*$ be as above, and suppose that $\Lambda \subseteq H^0(W)$ is a subspace of dimension k.*

(a) *The tangent space to $SG^k(\mathcal{W})$ at (W, Λ) fits into an exact sequence*

$$0 \ \to \ \mathrm{Hom}(\Lambda, H^0(W)/\Lambda) \ \to \ T_{(W,\Lambda)}SG^k(\mathcal{W}) \ \xrightarrow{c_*} \ T_W\mathcal{MS}(2n, K). \tag{2.6}$$

The image of the differential c_* coincides with $\mathrm{Im}\,(\mu^{\mathrm{s}}_\Lambda)^\perp$ (cf. Definition 2.6).

(b) *The locus $SG^k(\mathcal{W})$ is smooth and of dimension $\dim \mathcal{MS}(2n, K) - \frac{1}{2}k(k+1)$ at (W, Λ) if and only if μ^{s}_Λ is injective.*

(c) *Suppose μ^{s}_Λ is injective for all $\Lambda \in \mathrm{Gr}(k, H^0(W))$. Then $SG^k(\mathcal{W})$ is smooth in a neighbourhood of $c^{-1}(W)$, and $c^{-1}(W)$ is a smooth scheme. In particular, in this case $SG^k(\mathcal{W})$ contains a desingularisation of a neighbourhood of W in $\mathcal{S}^k(\mathcal{W})$. Furthermore, the normal space $N := N_{c^{-1}(W)/SG^k(\mathcal{W})}$ is precisely*

$$\{(\Lambda, v) : v \cup \mathrm{Im}\,(\mu^{\mathrm{s}}_\Lambda) \ = \ 0\} \ \subset \ \mathrm{Gr}(k, H^0(W)) \times H^1(K \otimes \mathrm{Sym}^2 W^*),$$

and the differential $c_*\colon N \to T_W\mathcal{MS}(2n, K)$ is the projection to the second factor.

Proof. (a) By the construction of $SG^k(\mathcal{W})$, we have

$$c^{-1}(W) \ = \ \mathrm{Gr}(k, H^0(W)).$$

Therefore, $\mathrm{Ker}(c_*) \cong T_\Lambda \mathrm{Gr}(k, H^0(W)) \cong \mathrm{Hom}(\Lambda, H^0(W)/\Lambda))$. For the rest: Exactly as in the line bundle case [ACGH85, Proposition IV.4.1 (ii)], the image of c_* is the space of tangent vectors in $T_sS = T_W\mathcal{MS}(2n, K)$ preserving the subspace Λ. By Proposition 2.7, this is exactly $\mathrm{Im}\,(\mu^{\mathrm{s}}_\Lambda)^\perp$.

(b) Note that $\big(\Lambda \otimes H^0(W)\big) \cap \mathrm{Ker}(\mathrm{sym}) = \wedge^2\Lambda$. Therefore,

$$\dim \big(\mathrm{sym}\,\big(\Lambda \otimes H^0(W)\big)\big) \ = \ \dim \big(\Lambda \otimes H^0(W)\big) - \dim \big(\wedge^2\Lambda\big) \ = \ k \cdot h^0(W) - \frac{k(k-1)}{2}.$$

By part (a), the dimension of $T_\Lambda SG^k(\mathcal{W})$ is given by

$$k(h^0(W) - k) + \dim \mathcal{MS}(2n, K) - \dim \big(\mathrm{sym}\,\big(\Lambda \otimes H^0(W)\big)\big) + \dim \ker(\mu^s_\Lambda) \ = \\ \dim \mathcal{MS}(2n, K) - \frac{k(k+1)}{2} + \dim \ker(\mu^s_\Lambda).$$

Part (b) follows. All statements in part (c) are immediate consequences of part (a). ☐

The first application of Theorem 2.9 is very similar to [HHN18, Proposition 3.12]:

Lemma 2.10. *Suppose $\mathcal{S}^k_{2n,K}$ has a component X which is generically smooth of the expected codimension $\frac{1}{2}k(k+1)$. Then for $1 \le \ell \le k$, the component X lies in a component of $\mathcal{S}^\ell_{2n,K}$ which is generically smooth and of the expected codimension $\frac{1}{2}\ell(\ell+1)$.*

Proof. By induction, it suffices to prove this for $\ell = k-1$, where $k \geq 2$. Let W be a smooth point of X, so $h^0(W) = k$ and $\mu^{\mathrm{s}} \colon \mathrm{Sym}^2 H^0(W) \to H^0(\mathrm{Sym}^2 W)$ is injective. Define $SG^{k-1}(\mathcal{W})$ as in (2.5) in an étale neighbourhood of W. By hypothesis and Theorem 2.9 (b), for any $\Lambda \subset H^0(W)$ of dimension $k-1$, the space $SG^{k-1}(\mathcal{W})$ constructed above is smooth and of dimension $\dim \mathcal{MS}(2n, K) - \frac{1}{2}k(k-1)$ at (W, Λ). Thus (W, Λ) lies in a component $\tilde{Y}_{k-1}$ of $SG^{k-1}(\mathcal{W})$ which is generically smooth and of this dimension. Now the inverse image of $\mathcal{S}^k_{2n,K}$ in $\tilde{Y}_{k-1}$ has dimension at most

$$\dim X + \dim \mathrm{Gr}(k-1, k) \ = \ \left(\dim \mathcal{MS}(2n, K) - \frac{k(k-1)}{2} \right) - 1,$$

which is less than $\dim SG^{k-1}(\mathcal{W})$. Therefore, a general $(W', \Lambda') \in \tilde{Y}_{k-1}$ is smooth and satisfies $h^0(W') = k-1$. It follows that the image of $SG^{k-1}(\mathcal{W})$ in $\mathcal{S}^{k-1}_{2n,K}$ lies in a component which is generically smooth and of the expected codimension. The statement follows. □

2.5. **Tangent cones of symplectic Brill–Noether loci.** We shall now describe the tangent cone $C_W \mathcal{S}^k_{2n,K}$ at a "well-behaved" singular point W. We begin by adapting [ACGH85, Lemma, p. 242] for symmetric determinantal varieties. Let A and $\bar{E}$ be vector spaces of dimensions a and $\bar{e}$ respectively, and let $\bar{\phi} \colon \mathrm{Sym}^2 A \to \bar{E}$ be a linear map. As before, write $\mathrm{sym} \colon A \otimes A \to \mathrm{Sym}^2 A$ for the canonical surjection. Let $\{\alpha_1, \ldots, \alpha_a\}$ be a basis of A, and write $x_{ij} := \bar{\phi} \circ \mathrm{sym}(\alpha_i \otimes \alpha_j)$.

Lemma 2.11. *Assume that $\bar{\phi}_\Lambda := \bar{\phi}|_{\mathrm{sym}(\Lambda \otimes A)}$ is injective for each $\Lambda \in \mathrm{Gr}(k, A)$. Set*

$$\bar{I} \ := \ \left\{ (\Lambda, v) \in \mathrm{Gr}(k, A) \times \bar{E}^* : v \in \bar{\phi}\left(\mathrm{sym}(\Lambda \otimes A)\right)^{\perp} \right\}.$$

Let $\bar{p} \colon \mathrm{Gr}(k, A) \times \bar{E}^* \to \bar{E}^*$ denote the projection. Then the following holds.

(a) *The scheme $\bar{p}(\bar{I})$ is Cohen–Macaulay, reduced and normal.*
(b) *The ideal of $\bar{p}(\bar{I})$ is generated by the $(a-k+1) \times (a-k+1)$ minors of the symmetric matrix $(x_{ij})_{i,j=1,\ldots,a}$.*
(c) *The degree of $\bar{p}(\bar{I})$ is*

$$\prod_{i=0}^{a-k+1} \frac{\binom{a+i}{a-k-i}}{\binom{2i+1}{i}}.$$

(d) *The morphism $\bar{p}$ maps $\bar{I}$ birationally onto $\bar{p}(\bar{I})$.*

Proof. As this follows very closely the proof of [ACGH85, Lemma, p. 242], we give only a sketch. The injectivity hypothesis implies that $\bar{I}$ is a vector bundle over $\mathrm{Gr}(k, A)$ which is smooth of dimension $\bar{e} - \frac{k(k+1)}{2}$. Let $\bar{J}$ be the subvariety of $\bar{E}^*$ whose ideal is generated by the $(a-k+1) \times (a-k+1)$ minors of the symmetric matrix $(x_{ij})_{i,j=1,\ldots,a}$. As in the proof of loc. cit., we see that $\bar{J}$ is supported exactly on $\bar{p}(\bar{I})$. Hence they coincide scheme-theoretically and $\bar{J}$ is a symmetric determinantal variety of the expected dimension. Thus $\bar{J}$ is Cohen–Macaulay by [Mi08, Theorem 1.2.14]. The proofs of (a), (b) and (d) now follow verbatim those of loc. cit. (i), (ii) and (iv) respectively. As for (c): Note that $\bar{J} = \bar{p}(\bar{I})$ is the pullback of

$$\{M \in \mathrm{Sym}^2\mathbb{K}^a : \dim \mathrm{Ker}(M) \geq k\}$$

by the map $\bar{E}^* \to \mathrm{Sym}^2\mathbb{K}^a$ given by $v \mapsto \Big(x_{ij}(v)\Big)$. As this map is linear and $\bar{J}$ is of the expected codimension, the statement follows directly from [HT84, p. 78]. □

Theorem 2.12. *Suppose $W \in \mathcal{S}^k_{2n,K}$ is such that for all $\Lambda \in \mathrm{Gr}(k, H^0(W))$, the map μ^{s}_Λ is injective. Let $\alpha_1, \ldots, \alpha_{h^0(W)}$ be a basis for $H^0(W)$, and define x_{ij} as above.*

(a) *As sets, we have*

$$C_W\mathcal{S}^k_{2n,K} \;=\; \bigcup_{\Lambda \in \mathrm{Gr}(k,H^0(W))} \mathrm{Im}\,(\mu^s_\Lambda)^\perp.$$

(b) *The tangent cone $C_W\mathcal{S}^k_{2n,K}$ to $\mathcal{S}^k_{2n,K}$ at W is Cohen–Macaulay, reduced and normal.*
(c) *The ideal of $C_W\mathcal{S}^k_{2n,K}$ as a subvariety of $H^1(K \otimes \mathrm{Sym}^2 W^*)$ is generated by the $(h^0(W)-k+1)\times(h^0(W)-k+1)$-minors of the symmetric matrix $(x_{ij})_{i,j=1,\ldots,h^0(W)}$.*
(d) *The multiplicity of $\mathcal{S}^k_{2n,K}$ at W is*

$$\prod_{i=0}^{h^0(W)-k+1} \frac{\binom{h^0(W)+i}{h^0(W)-k-i}}{\binom{2i+1}{i}}.$$

Proof. By Theorem 2.9 (c) and Lemma 2.11 (a) & (d), the hypotheses of [ACGH85, Lemma II.2.1.3, p. 66] are satisfied by the map $\bar{p}\colon \bar{I} \to \bar{E}^*$. Therefore, $\bar{p}(\bar{I})$ coincides scheme-theoretically with $C_W\mathcal{S}^k_{2n,K}$. Part (a) follows immediately from the definition of $\bar{p}$. Parts (b), (c) and (d) follow from Lemma 2.11 (a), (b) and (c) respectively. □

3. Dimension bounds on symplectic Brill–Noether loci

We begin this section with an important result on the structure of bundles with nonvanishing sections.

Lemma 3.1. *Let V be a vector bundle over C with $h^0(V) \geq 1$. Let $B \subset C$ be the subscheme of C along which all sections of V vanish. Its support is the finite set*

$$\{p \in C : s(p) = 0 \text{ for all } s \in H^0(V)\}.$$

If the subbundle $E \subseteq V$ generated by global sections is of rank at least two, then there exists a section of V which is nonzero at all points of $C\backslash\mathrm{Supp}(B)$.

Proof. This is [Baj19, Proposition 1], whose proof is due to Feinberg [Fe] (see [Te92]). □

Corollary 3.2. *Any vector bundle V with $h^0(V) \geq 1$ can be written as an extension $0 \to \mathcal{O}_C(D) \to V \to F \to 0$ where D is effective and $H^0(\mathcal{O}_C(D)) = H^0(V)$ or $h^0(\mathcal{O}_C(D)) = 1$.*

Motivated by Corollary 3.2, we recall [Baj19, Definition 1]:

Definition 3.3. A vector bundle V over C with $h^0(V) \geq 1$ will be said to be *of first type* if V contains a line subbundle L such that $H^0(V) = H^0(L)$. If V contains a line subbundle L with $h^0(L) = 1$, then V is said to be *of second type*. Note that if $h^0(V) = 1$ then V is both of first type and of second type.

The relevance of this for higher rank Brill–Noether loci is illustrated by [CFK18, Theorem 1.1], which states that for $3 \leq \nu \leq \frac{g+8}{4}$, if C is a general ν-gonal curve then $B^2_{2,d}$ has two components, corresponding to the two types in Definition 3.3. In a similar way, we shall see that different dimension bounds apply for components of $\mathcal{S}^k_{2n,K}$ whose generic elements are of different types.

We shall require the following technical lemma in several places.

Lemma 3.4. *Let V be any vector bundle, and let $0 \to M \xrightarrow{\iota} K \otimes V^* \to G \to 0$ be an extension where M has rank one. Consider the induced map*

$$\iota^* \colon K^{-1} \otimes V \otimes V \ \to \ M^{-1} \otimes V.$$

Then the restriction of ι^* to $K^{-1} \otimes \mathrm{Sym}^2 V$ is surjective.

Proof. We dualise the given sequence and tensor by V. Then it is not hard to see that $\mathrm{Ker}\left(\iota^*|_{K^{-1}\otimes \mathrm{Sym}^2 V}\right) \cong K \otimes \mathrm{Sym}^2 G^*$. Thus the image has rank equal to $\mathrm{rk}\, V$, as desired. □

By the Clifford theorem for stable vector bundles [BGN97], for all stable K-valued symplectic bundles W of rank $2n$ we have $h^0(W) \leq n(g+1) - 1$. In what follows, we shall assume $0 \leq k \leq n(g+1) - 1$.

3.1. **Symplectic bundles of first type.**

Theorem 3.5. *Let X be a closed irreducible sublocus of $\mathcal{S}^k_{2n,K}$ of which a general element W satisfies $h^0(W) = h^0(L_W) = k$ for some line subbundle $L_W \subset W$ of degree d. For such W, we have*

$$\dim X \ \leq \ \dim (T_W X) \ \leq \ \dim \left(T_{L_W} B^k_{1,d}\right) + n(2n+1)(g-1) - 2nd - 1.$$

Proof. The inclusion $j \colon L \to W$ induces maps on cohomology

$$j^* \colon H^1(\mathrm{End}\,(W)) \to H^1(\mathrm{Hom}(L, W)) \quad \text{and} \quad j_* \colon H^1(\mathrm{End}\,(L)) \to H^1(\mathrm{Hom}(L, W)).$$

A deformation $\mathbb{W}$ of W induces a given deformation $\mathbb{L}$ of the subbundle L if and only if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r] & \mathbb{W} \arrow[r] & W \arrow[r] & 0 \\ 0 \arrow[r] & L \arrow[r] \arrow[u, "j"] & \mathbb{L} \arrow[r] \arrow[u] & L \arrow[r] \arrow[u, "j"] & 0. \end{tikzcd}$$

This is equivalent to the condition

$$j^*\delta(\mathbb{W}) \ = \ j_*\delta(\mathbb{L}) \text{ in } H^1(\mathrm{Hom}(L, W)), \tag{3.1}$$

where $\delta(\mathbb{W})$ and $\delta(\mathbb{L})$ are the cohomology classes of the extensions defined by the deformations $\mathbb{W}$ and $\mathbb{L}$ respectively. Now L defines a point of $B^k_{1,d}$. The deformation $\mathbb{W}$ corresponds to a tangent direction in $T_W X$ if and only if $\mathbb{W}$ satisfies (3.1) for some $\mathbb{L}$ belonging to $T_L B^k_{1,d} \subseteq H^1(\mathrm{End}\,(L))$. It follows that

$$T_W X \ = \ (j^*)^{-1} j_* \left(T_L B^k_{1,d} \right) . \tag{3.2}$$

Composing with $\omega \colon W \xrightarrow{\sim} K \otimes W^*$, we view j as a map $L \to K \otimes W^*$, and then

$$j^* \colon H^1(K^{-1} \otimes W \otimes W) \ \to \ H^1(L^{-1} \otimes W).$$

By Lemma 3.4, the restriction of j^* to the subspace

$$H^1(K^{-1} \otimes \mathrm{Sym}^2 W) \ \xrightarrow{\sim} \ H^1(K \otimes \mathrm{Sym}^2 W^*) \ = \ T_W \mathcal{MS}(2n, K)$$

remains surjective (the first identification above is given by $\omega \otimes \omega$). By this fact and (3.2), we have

$$\dim(T_W X) \ \le \ \dim(T_L B^k_{1,d}) + h^1(K \otimes \mathrm{Sym}^2 W^*) - h^1(L^{-1} \otimes W). \tag{3.3}$$

Now as W is of first type, there can be at most one independent vector bundle injection $L \to W$, so $h^0(L^{-1} \otimes W) = 1$. Then by Riemann–Roch,

$$h^1(L^{-1} \otimes W) \ = \ 1 - \chi(L^{-1} \otimes W) \ = \ 1 + 2nd.$$

As moreover $h^1(K \otimes \mathrm{Sym}^2 W^*) = n(2n+1)(g-1)$, the theorem follows from (3.3). □

For $k = 1$, Theorem 3.5 together with the codimension condition gives the familiar fact that the set of bundles with sections is a divisor. Moreover, if W is a general bundle with one independent section then this section does not vanish, as if X is a locus as in the theorem with $k = 1$ and $d \ge 1$ then X has codimension at least $(2n-1)d + 1 \ge 2$. More generally, Theorem 3.5 gives the following restrictions on the parameter n for *components* in $\mathcal{S}^k_{2n,K}$ whose general element is of first type.

Corollary 3.6.

(a) *Suppose $n \ge 1$ and $k \ge 2$. Then $\mathcal{S}^k_{2n,K}$ has a component whose generic element W satisfies $H^0(W) = H^0(L_W)$ for a degree d line subbundle only if $8n - 2 \le k$. In particular, for all $n \ge 1$, the generic element of any component of $\mathcal{S}^2_{2n,K}$ is of second type.*

(b) *Suppose $d \ge 1$. Then $\mathcal{S}^k_{2n,K}$ has a component whose generic element W satisfies $H^0(W) = H^0(L_W)$ for a degree d line subbundle L_W only if $n \le \frac{g+4}{16}$.*

Proof. (a) Let W be a general point of a component as in the statement. As any component of $\mathcal{S}^k_{2n,K}$ has codimension at most $\frac{1}{2}k(k+1)$, by Theorem 3.5 we have

$$2nd \ \le \ \dim \left(T_{L_W} B^k_{1,d} \right) + \frac{k(k+1)}{2} - 1, \tag{3.4}$$

By Martens' theorem [ACGH85, p. 191 ff.], and noting that the usual Martens bound is in fact a bound for $\dim\left(T_{L_W} B^k_{1,d}\right)$, we have $\dim\left(T_{L_W} B^k_{1,d}\right) \le d - 2(k-1)$. Thus the above inequality becomes

$$(2n-1)d \ \le \ \frac{k(k+1)}{2} - 2k + 1 \ = \ \frac{(k-1)(k-2)}{2}.$$

By Clifford's theorem [ACGH85, p. 107 ff.] applied to the line bundle L_W, we have $k \le \frac{d}{2}+1$. Using this and the fact that $d \neq 0$ since $k = h^0(L_W) \ge 2$, the above inequality becomes

$$2n - 1 \ \le \ \frac{\frac{d}{2} \cdot (k-2)}{2d} \ = \ \frac{k-2}{4},$$

which gives $8n - 2 \le k$, as desired.

(b) Suppose X is a component as in the statement. As in part (a) we have the inequality (3.4), which yields

$$n \ \le \ \frac{(k-1)(k+2) + 2 \cdot \dim\left(T_{L_W} B^k_{1,d}\right)}{4d}.$$

By Martens' theorem as above, we obtain

$$n \ \le \ \frac{(k-1)(k+2) - 4(k-1) + 2d}{4d} \ = \ \frac{(k-1)(k-2)}{4d} + \frac{1}{2}.$$

The above, by Clifford's theorem, becomes

$$n \ \le \ \frac{\frac{d}{2} \cdot \left(\frac{d-2}{2}\right)}{4d} + \frac{1}{2} \ = \ \frac{d(d-2)}{16d} + \frac{1}{2}.$$

As $d \neq 0$, this simplifies to $n \le \frac{d-2}{16} + \frac{1}{2}$. As W is stable, $d \le g - 2$, whence

$$n \ \le \ \frac{g-4}{16} + \frac{1}{2} \ = \ \frac{g+4}{16}. \qquad \square$$

3.2. **Symplectic bundles of second type.** In [Baj19, Theorem 4], the first author derived a bound on the dimension of the Brill–Noether locus $B^k_{2,K}$ of bundles of rank two and canonical determinant. As noted above, these are precisely the K-valued symplectic bundles of rank two. The following is a generalisation to symplectic bundles of higher rank, whose proof is similar.

Notation. For the remainder of the paper, as we shall only consider symmetric Petri maps, we denote μ^{s} simply by μ to ease notation.

Theorem 3.7. *Let k be an integer satisfying $1 \le k \le n(g+1) - 1$. Suppose Y is an irreducible component of $\mathcal{S}^k_{2n,K}$ containing a bundle W of second type satisfying $h^0(W) = k$ and such that the rank of the subbundle $E \subset W$ generated by global sections is r. Then*

$$\dim(Y) \ \le \ \dim(T_W Y) \ \le \ \min\Big\{ n(2n+1)(g-1) - (2k-1), \\ n(2n+1)(g-1) - k - \frac{1}{2} r(r-1) \Big\}.$$

Proof. Let W be a general element of Y. If $\mu\colon \mathrm{Sym}^2 H^0(W) \to H^0(\mathrm{Sym}^2 W)$ is the Petri map of W, then

$$\dim(T_W Y) \ = \ \dim(\mathcal{MS}(2n, K)) - \frac{1}{2}k(k+1) + \dim \mathrm{Ker}(\mu). \tag{3.5}$$

We shall prove the theorem by finding a bound on $\dim \mathrm{Ker}(\mu)$.

As W is of second type, we may fix an exact sequence $0 \to \mathcal{O}_C(D) \to W \xrightarrow{q} F \to 0$, where D is an effective divisor with $h^0(\mathcal{O}_C(D)) = 1$. Now we have an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sym}^2 H^0(\mathcal{O}_C(D)) \arrow[r, "i"] \arrow[d, "\mu_1"'] & \mathrm{Sym}^2 H^0(W) \arrow[r, "j"] \arrow[d, "\mu"'] & \frac{\mathrm{Sym}^2 H^0(W)}{\mathrm{Sym}^2 H^0(\mathcal{O}_C(D))} \arrow[r] \arrow[d, "\mu_2"'] & 0 \\
0 \arrow[r] & H^0(\mathcal{O}_C(2D)) \arrow[r] & H^0(\mathrm{Sym}^2 W) \arrow[r] & H^0\left(\frac{\mathrm{Sym}^2 W}{\mathcal{O}_C(2D)}\right). &
\end{tikzcd}$$

As $h^0(\mathcal{O}_C(D)) = 1$, clearly μ_1 is injective. Thus, by the Snake Lemma,

$$\dim \mathrm{Ker}(\mu) \ \le \ \dim \mathrm{Ker}(\mu_2). \tag{3.6}$$

Next, write V for the image of $q\colon H^0(W) \to H^0(F)$. There is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & V \otimes H^0(\mathcal{O}_C(D)) \arrow[r, "\iota"] \arrow[d, "\gamma"] & \frac{\mathrm{Sym}^2 H^0(W)}{\mathrm{Sym}^2 H^0(\mathcal{O}_C(D))} \arrow[r] \arrow[d, "\mu_2"'] & \mathrm{Sym}^2 V \arrow[r] \arrow[d, "\mu_3"'] & 0 \\
0 \arrow[r] & H^0(F(D)) \arrow[r] & H^0\left(\frac{\mathrm{Sym}^2 W}{\mathcal{O}_C(2D)}\right) \arrow[r] & H^0(\mathrm{Sym}^2 F) &
\end{tikzcd}$$

Here γ is the multiplication map on sections, and ι is induced by $\mathrm{sym}\colon H^0(W) \otimes H^0(\mathcal{O}_C(D)) \to \mathrm{Sym}^2 H^0(W)$. As D is effective and $h^0(\mathcal{O}_C(D)) = 1$, the map γ is injective. Hence by the Snake Lemma and (3.6) we have

$$\dim \mathrm{Ker}(\mu) \ \le \ \dim \mathrm{Ker}(\mu_3). \tag{3.7}$$

Therefore by Lemma 3.8 below, $\dim \mathrm{Ker}(\mu)$ is bounded above by

$$\min\left\{\frac{1}{2}k(k-1) - \frac{1}{2}r(r-1), \frac{1}{2}k(k-1) - (k-1)\right\} \ = \ \min\left\{\frac{1}{2}k(k+1) - \left(k + \frac{1}{2}r(r-1)\right), \frac{1}{2}k(k+1) - (2k-1)\right\}$$

The theorem now follows from (3.5). □

Lemma 3.8. *Let F be any vector bundle, and V a nonzero subspace of $H^0(F)$. Let E be the subbundle of F generated by V, and write $m := \mathrm{rk}\,(E)$. Let $\mu_3\colon \mathrm{Sym}^2 V \to H^0(\mathrm{Sym}^2 F)$ be the restriction of the symmetric Petri map of F. Then*

$$\dim \mathrm{Im}\,(\mu_3) \ \ge \ \max\left\{\frac{1}{2}m(m+1), \dim(V)\right\}.$$

Proof. Let $\Lambda \subseteq V$ be a subspace of dimension m which generically generates E. Then for generic $p \in C$, the composed map

$$\mathrm{Sym}^2\Lambda \xrightarrow{\mu_3|_{\mathrm{Sym}^2\Lambda}} H^0(\mathrm{Sym}^2 F) \xrightarrow{\mathrm{ev}} \mathrm{Sym}^2 F|_p$$

is an isomorphism onto $\mathrm{Sym}^2 E|_p \subseteq \mathrm{Sym}^2 F|_p$. Thus $\dim \mathrm{Im}\,(\mu_3) \geq \mathrm{rk}\,(\mathrm{Sym}^2 E) = \frac{1}{2}m(m+1)$.

For the rest: Choose any nonzero $t \in V$, and write L for the line subbundle generated by t. There is a commutative diagram

$$\begin{tikzcd}
\mathbb{K}\cdot t \otimes V \arrow[r, hook] \arrow[dr, hook] & H^0(L)\otimes V \arrow[r, "\mathrm{sym}"] \arrow[d] & \mathrm{Sym}^2 V \arrow[d, "\mu_3"] \\
 & H^0(L\otimes F) \arrow[r, hook, "\Sigma"] & H^0(\mathrm{Sym}^2 F),
\end{tikzcd}$$

where $\Sigma\colon F\otimes F \to \mathrm{Sym}^2 F$ is the canonical surjection. Since $\dim(\mathbb{K}\cdot t) = 1$, the top row is injective. On the other hand, since L has rank one, $(L\otimes F)\cap \wedge^2 F = 0$. Thus Σ is induced by an injective bundle map, and so is injective. By commutativity, the restriction of μ_3 to $\mathrm{sym}(\mathbb{K}\cdot t\otimes V)$ is injective. Thus $\dim \mathrm{Im}\,(\mu_3) \geq \dim(V)$. $\square$

Remark 3.9. We mention some special cases. If $h^0(W) = 1$, then W is both of first and of second type, and Theorems 3.5 and 3.7 both confirm that $\mathcal{S}^1_{2n,K}$ is a generically smooth reduced divisor. More generally, if $k = r$, then W belongs to a unique component of $\mathcal{S}^k_{2n,K}$ which is generically smooth and of the expected dimension.

4. Nonemptiness of symplectic Brill–Noether loci

In this section, we shall prove nonemptiness of $\mathcal{S}^k_{2n,K}$ for certain values of g, n and k. We use a combination of techniques from [Me99] and [CH14]. In §§ 4.1 and 4.2 we recall or prove the necessary ingredients, and then proceed to the questions of nonemptiness and smoothness of $\mathcal{S}^k_{2n,K}$.

4.1. Mercat's construction.

Here we recall and further analyse the bundles constructed in [Me99, p. 76] as elementary transformations of sums of line bundles. Let C be any curve of genus $g \geq 3$. As in the introduction, set

$$k_0 \ := \ \max\{k \geq 0 : \beta^k_{1,g-1} > 0\}.$$

Fix $n \geq 1$. By definition of k_0, the Brill–Noether locus $B^{k_0}_{1,g-1}$ is of positive dimension. Let $L_1, \ldots, L_n$ be general elements of $B^{k_0}_{1,g-1}$, in particular such that

$$L_1, \ldots, L_n, KL_1^{-1}, \ldots, KL_n^{-1}$$

are mutually nonisomorphic. Choose any point $x \in C$. Let E be an elementary transformation

$$0 \ \to \ E \ \to \ \bigoplus_{i=1}^n L_i \ \to \ \mathcal{O}_x \ \to \ 0 \tag{4.1}$$

which is general in the sense that no $L_i|_x$ is contained in E. One checks using [Me99, p. 79] that such an E is stable. Hence $K \otimes E^*$ is also stable, so any proper subbundle has slope at most $g-1$. In fact we shall require the following stronger statement.

Lemma 4.1. *Suppose $n \geq 2$. Let E be as in (4.1).*

(a) *Any slope $g-1$ subbundle of $K \otimes E^*$ contains a line subbundle of degree $g-1$.*
(b) *The bundle $K \otimes E^*$ contains a finite number of line subbundles of degree $g-1$.*

Proof. We use induction on n. Firstly, suppose $n = 2$. For part (a), there is nothing to prove. Recall that the Segre invariant $s_1(K \otimes E^*)$ is defined as

$$\min\{\deg(K \otimes E^*) - 2\deg(M) : M \text{ a line subbundle of } K \otimes E^*\}.$$

As KL_i^{-1} is clearly a maximal line subbundle of $K \otimes E^*$, we have $s_1(K \otimes E^*) = 1$. Then statement (b) follows from [LN83, Proposition 4.2].

Now suppose $n \geq 3$. We have a diagram

$$\begin{tikzcd}
0 \arrow[r] & E_n \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & L_n \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \bigoplus_{i=1}^{n-1} L_i \arrow[r] \arrow[d] & \bigoplus_{i=1}^{n} L_i \arrow[r] \arrow[d] & L_n \arrow[r] & 0 \\
& \mathcal{O}_x \arrow[r, "="] & \mathcal{O}_x & &
\end{tikzcd}$$

where E_n has rank $n-1$ and degree $(n-1)(g-1)-1$. Since no L_i is contained in E, in particular no L_i is contained in E_n. Thus, by induction we may assume that statements (a) and (b) hold for $K \otimes E_n^*$.

We now prove part (a). Suppose F is a slope $g-1$ subbundle of $K \otimes E^*$. We have a diagram of sheaves

(4.2)
$$\begin{tikzcd}
0 \arrow[r] & KL_n^{-1} \arrow[r] & K \otimes E^* \arrow[r] & K \otimes E_n^* \arrow[r] & 0 \\
0 \arrow[r] & F_1 \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & F_2 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where F_1 is the sheaf-theoretic intersection of F and KL_n^{-1}. If $F_1 \neq 0$ then $F_1 = KL_n^{-1}$ and we are done. If $F_1 = 0$ then $F \cong F_2$ is a slope $g-1$ subsheaf of $K \otimes E_n^*$. Since the latter is stable of slope $g - 1 + \frac{1}{\mathrm{rk}\,(E_n)}$, in fact F_2 must be saturated; that is, a subbundle. By induction, $F \cong F_2$ contains a line subbundle of degree $g-1$. This proves (a).

As for (b): By the top row of (4.2), any degree $g-1$ line subbundle $M \subset K \otimes E^*$ is either KL_n^{-1} or is a subbundle of $K \otimes E_n^*$. By induction, we may assume there are at most finitely many degree $g-1$ subbundles of $K \otimes E_n^*$. For a fixed such subbundle M, the set of liftings of M to $K \otimes E^*$ is a pseudotorsor over $H^0(\mathrm{Hom}(M, KL_n^{-1}))$. Since the L_i are chosen generally from the positive dimensional locus $B^{k_0}_{1,g-1}$, perturbing L_n if necessary we can assume that $KL_n^{-1} \not\cong M$, so $h^0(\mathrm{Hom}(M, KL_n^{-1})) = 0$. Statement (b) follows. □

4.2. **Symplectic extensions.** In this subsection we shall recall a method for constructing symplectic bundles as extensions, together with a geometric criterion for liftings in such extensions.

Criterion 4.2. *Let C be a curve, and let E be a simple vector bundle over C. An extension*

$$0 \ \to \ E \ \to \ W \ \to \ K \otimes E^* \ \to \ 0 \tag{4.3}$$

admits a K-valued symplectic form with respect to which E is isotropic if and only if the extension class $\delta(W)$ belongs to $H^1(C, K^{-1} \otimes \mathrm{Sym}^2 E)$.

Proof. This is a special case of [Hi07, Criterion 2.1]. □

Let us now recall some geometric objects living naturally in the projectivised extension space $\mathbb{P}H^1(K^{-1} \otimes E \otimes E)$. Let V be any vector bundle over C with $h^1(V) \neq 0$. Write $\pi \colon \mathbb{P}V \to C$ for the projection. Via Serre duality and the projection formula, there is a canonical identification

$$\mathbb{P}H^1(V) \ \xrightarrow{\sim} \ |\mathcal{O}_{\mathbb{P}V}(1) \otimes \pi^* K|^*.$$

Hence there is a natural map $\psi \colon \mathbb{P}V \dashrightarrow \mathbb{P}H^1(V)$ with nondegenerate image. Let us recall a useful way to realise this map fibrewise.

Lemma 4.3. *On a fibre $\mathbb{P}V|_y$, the map ψ can be identified with the projectivised coboundary map of the sequence*

$$H^0(V) \ \to \ H^0(C, V(y)) \ \to \ V(y)|_y \ \to \ H^1(V) \to \cdots .$$

Proof. This follows by direct calculation, or from the discussion on [CH10, pp. 469–470]. □

Now set $V = K^{-1} \otimes E \otimes E$. We shall recall a result from [CH10] relating the geometry of $\psi(\mathbb{P}(E \otimes E))$ and liftings of subsheaves of $K \otimes E^*$ to extensions of the form (4.3), in the spirit of [LN83, Proposition 1.1]. Let $e_1, \ldots, e_m$ be points of E lying over distinct points $y_1, \ldots, y_m \in C$. These define an elementary transformation

$$0 \ \to \ F_{e_1,\ldots,e_m} \ \to \ K \otimes E^* \ \to \ \bigoplus_{l=1}^m \mathcal{O}_{y_l} \ \to \ 0.$$

Proposition 4.4. *With E and $F := F_{e_1,\ldots,e_m}$ as above, let $0 \to E \to W \to K \otimes E^* \to 0$ be an extension of class $\delta(W) \in \mathbb{P}H^1(K^{-1} \otimes E \otimes E)$. Then F lifts to W if and only if $\delta(W)$ belongs to the secant spanned by $\psi(e_1 \otimes f_1), \ldots, \psi(e_m \otimes f_m)$ for some nonzero $f_1 \in E|_{y_1}, \ldots, f_m \in E|_{y_m}$.*

Proof. Let $\beta \colon H^1(K^{-1} \otimes E \otimes E) \to H^1(F^* \otimes E)$ be the induced map on cohomology. Then F lifts to an extension W if and only if $\delta(W) \in \mathrm{Ker}(\beta)$. By [CH10, Lemma 4.3 (ii)], the space $\mathrm{Ker}(\beta)$ is exactly the span of the projective linear spaces $\psi\left(\mathbb{P}(\mathbb{K} \cdot e_l \otimes K^{-1} \otimes E)\right)$ for $1 \le l \le m$. (Note that the assumption on the degrees in [CH10] is made solely to ensure that ψ be an embedding, which we do not require in the present situation.) □

Next, as in [CH14, § 2.2], composing ψ with the relative Segre embedding, we obtain a map

$$\psi_{\mathrm{s}} \colon \mathbb{P}E \ \hookrightarrow \ \mathbb{P}(\mathrm{Sym}^2 E) \ \dashrightarrow \ \mathbb{P}H^1(K^{-1} \otimes \mathrm{Sym}^2 E) \tag{4.4}$$

with nondegenerate image. Note that $\psi_{\mathrm{s}}(e) = \psi(e \otimes e)$. We remark that ψ_{s} is the map associated to

$$|\mathcal{O}_{\mathbb{P}E}(2) \otimes \pi^* K^2|^* \ \cong \ \mathbb{P}H^0(K^2 \otimes \mathrm{Sym}^2 E^*)^* \ \cong \ \mathbb{P}H^1(K^{-1} \otimes \mathrm{Sym}^2 E).$$

4.3. **The construction.** Suppose $g \geq 3$ and $n \geq 1$. Let $L_1, \ldots, L_n$ and E be as defined in § 4.1. Let e_1, e_2 be general points of $\mathbb{P}E$ lying over distinct $y_1, y_2 \in C$ respectively. Let

$$0 \ \to \ E \ \to \ W \ \to \ K \otimes E^* \ \to \ 0 \tag{4.5}$$

be a nontrivial extension such that $\delta(W)$ is a general point of the line spanned by $\psi_{\mathrm{s}}(e_1)$ and $\psi_{\mathrm{s}}(e_2)$. As $\delta(W) \in H^1(K^{-1} \otimes \mathrm{Sym}^2 E)$, by Criterion 4.2 there is a K-valued symplectic structure on W.

Proposition 4.5. *The bundle W is stable as a vector bundle.*

Proof. The following uses ideas from [CH14, § 3] and [HP15, Lemma 7]. As every proper subbundle of $K \otimes E^*$ has slope at most $g - 1$, and the extension W is nontrivial, it is not hard to see that any subbundle of W has slope at most $g - 1$. Thus we need only to exclude the existence of a subbundle of slope $g - 1$.

Furthermore, for any proper subbundle $F \subset W_1$, we have a short exact sequence $0 \to F^\perp \to W \to F^* \otimes K \to 0$ where $F^\perp$ is the orthogonal complement of F with respect to the bilinear form. An easy computation shows that

$$\mu(F^\perp) \ = \ (g-1) + \frac{\mathrm{rk}\,(F)}{2n - \mathrm{rk}\,(F)} \left(\mu(F) - (g-1) \right).$$

Hence $\mu(F) \geq (g-1)$ if and only if $\mu(F^\perp) \geq (g-1)$. As $\mathrm{rk}\,(F^\perp) = 2n - \mathrm{rk}\,(F)$, to prove stability of W it suffices to exclude the existence of subbundles of slope $g - 1$ and rank at most n.

Let $F \subset W$ be a subbundle of rank at most n. Then there is a sheaf diagram

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \arrow[r] & W \arrow[r] & K \otimes E^* \arrow[r] & 0,
\end{tikzcd}$$

where F_1 is a subbundle of E and F_2 a subsheaf of $K \otimes E^*$. For $j = 1, 2$ write $r_j := \mathrm{rk}\,(F_j)$. If $r_1 > 0$, then $r_2 < n$. As $\mu(F_2) < g - 1 + \frac{1}{n}$, in fact $\mu(F_2) \leq g - 1$, whence

$$\mu(F) \ \leq \ \frac{r_1 \cdot \mu(E) + r_2 \cdot (g-1)}{r_1 + r_2} \ < \ g - 1.$$

Thus we may assume that $r_1 = 0$ and $F \cong F_2$ is a subsheaf of $K \otimes E^*$.

If $r_2 < n$, then by Lemma 4.1 (a) we may assume $n \geq 2$ and $r_2 = 1$. Let $\iota \colon M \to K \otimes E^*$ be a line subbundle of degree $g - 1$. Then ι lifts to a map $M \to W$ if and only if

$$\delta(W) \ \in \ \mathrm{Ker}\left(\iota^* \colon H^1(C, K^{-1} \otimes E \otimes E) \ \to \ H^1(C, M^{-1} \otimes E))\right).$$

As $H^1(C, M^{-1} \otimes E))$ is nonzero, by Lemma 3.4, the restriction of ι^* to $H^1(K^{-1} \otimes \mathrm{Sym}^2 E)$ is nonzero. Furthermore, by Lemma 4.1 (b), there are only finitely many possibilities for ι. We conclude that the locus of extensions in $H^1(K^{-1} \otimes \mathrm{Sym}^2 E)$ admitting a lifting of some such $\iota \colon M \to K \otimes E^*$ is a finite union of proper linear subspaces. Since

$$\psi_{\mathrm{s}}(\mathbb{P}E) \ \subset \ \mathbb{P}H^1(K^{-1} \otimes \mathrm{Sym}^2 E) \ \cong |\mathcal{O}_{\mathbb{P}E}(2) \otimes \pi^* K^2|^*$$

is nondegenerate and $\delta(W)$ is a general point of a general 2-secant to $\psi_{\mathrm{s}}(\mathbb{P}E)$, we may assume that $\delta(W)$ does not belong to any of these proper linear subspaces.

Finally, we must exclude a lifting of some F_2 of rank $r_2 = n \geq 1$; that is, an elementary transformation $0 \to F_2 \to K \otimes E^* \to \mathcal{O}_y \to 0$. By Proposition 4.4, such a lifting exists only if $\delta(W)$ belongs to $\psi(\Delta)$, where

$$\Delta \ := \mathbb{P}E \times_C \mathbb{P}(K^{-1} \otimes E)$$

is the rank one locus of $\mathbb{P}(K^{-1} \otimes E \otimes E)$.

Now $h^0(K^{-1} \otimes \mathrm{Sym}^2 E) = 0$ since E is stable of slope $< g - 1$. Hence by Riemann–Roch,

$$h^1(K^{-1} \otimes \mathrm{Sym}^2 E) \ = \ \frac{1}{2} n(n+1)(g-1) + n + 1.$$

One checks easily that for $g \geq 3$, this is greater than $\dim(\mathbb{P}E) + 1 = n + 1$, so $\psi_{\mathrm{s}}(\mathbb{P}E)$ is a proper subvariety of $\mathbb{P}H^1(K^{-1} \otimes \mathrm{Sym}^2 E)$. It follows that the secant variety $\mathrm{Sec}^2(\psi_{\mathrm{s}}(\mathbb{P}E))$ strictly contains $\psi_{\mathrm{s}}(\mathbb{P}E)$. Hence, since the points e_1, e_2 were chosen generally and $\delta(W)$ is general in the line $\overline{\psi_{\mathrm{s}}(e_1)\psi_{\mathrm{s}}(e_2)}$, we may assume $\delta(W) \notin \psi_{\mathrm{s}}(\mathbb{P}E)$. Thus $\delta(W)$ belongs to $\psi(\Delta)$ only if $\psi(e \otimes f) \in \mathbb{P}H^1(K^{-1} \otimes \mathrm{Sym}^2 E)$ for some independent e, f in some fibre $E|_y$. In view of Lemma 4.3 and the diagram

$$\begin{tikzcd} H^0(K^{-1} \otimes E \otimes E(y)) \arrow[r] & K^{-1} \otimes E \otimes E(y)|_y \arrow[r] & H^1(K^{-1} \otimes E \otimes E) \\ & & H^0(K^{-1} \otimes \mathrm{Sym}^2 E) \arrow[u] \end{tikzcd}$$

this happens if and only if there is a global section α of $K^{-1} \otimes E \otimes E(y)$ with value $\frac{1}{2}(e \otimes f - f \otimes e)$ at y. We claim that such an α can exist for at most finitely many y. Since $K^{-1} \otimes E \otimes E(y)$ is a subsheaf of $\bigoplus_{i,j} K^{-1} L_i L_j(y)$, it suffices to show for almost all $y \in C$ that $h^0(K^{-1} L_i L_j(y)) = 0$; equivalently, that $h^1(K^{-1} L_i L_j(y)) = g - 2$. By Serre duality, this is in turn equivalent to $h^0(K^2 L_i^{-1} L_j^{-1}(-y)) = g - 2$. But since $L_i L_j \neq K$, we have $h^0(K^2 L_i^{-1} L_j^{-1}) = g - 1$, and so $h^0(K^2 L_i^{-1} L_j^{-1}(-y)) = g - 2$ for almost all $y \in C$, as required.

Therefore, writing Δ' for the complement of the relative diagonal $\mathbb{P}E \subset \Delta$, the intersection of $\psi(\Delta')$ with $H^1(K^{-1} \otimes \mathrm{Sym}^2 E)$ is contained in at most a finite number of fibres $\Delta'|_y$.

As the linear span of $\psi(\Delta'|_y)$ is $\psi(\mathbb{P}(K^{-1} \otimes E \otimes E)|_y)$, we conclude that the locus of extensions in $H^1(K^{-1} \otimes \mathrm{Sym}^2 E)$ lying over $\psi(\Delta')$ is contained in a finite union of linear subspaces of dimension at most n^2. Again, one computes using $g \geq 3$ that $h^1(K^{-1} \otimes \mathrm{Sym}^2 E) > n^2$. Thus the locus of symplectic extensions (4.3) admitting a lifting of an elementary transformation $F_2 \subset K \otimes E^*$ with $\deg \left(\frac{K \otimes E^*}{F_2} \right) = 1$ is contained in a finite union of proper linear subspaces. As above, by nondegeneracy of $\psi_{\mathrm{s}}(\mathbb{P}E)$ we can assume that W does not admit such a lifting. This completes the proof that W is stable as a vector bundle. $\Box$

Theorem 4.6. *Let C be a curve of genus $g \geq 3$, and let k_0 be as defined in (1.1). For each $n \geq 1$ and for $0 \leq k \leq 2nk_0 - 3$, the locus $\mathcal{S}^k_{2n,K}$ has a component which is nonempty and of codimension at most $\frac{1}{2}k(k+1)$.*

Proof. Let W be the K-valued symplectic bundle constructed in (4.5), which is stable by Proposition 4.5. By Proposition 4.4, the elementary transformation

$$0 \ \to \ F_{e_1,e_2} \ \to K \otimes E^* \ \xrightarrow{e_1,e_2} \ \mathcal{O}_{y_1} \oplus \mathcal{O}_{y_2} \ \to \ 0$$

lifts to a subsheaf F of W (which is in fact a subbundle, as W is stable). Since e_1 and e_2 are general and $K \otimes E^*$ is generically generated, we may assume $h^0(F) = nk_0 - 2$. Hence $h^0(W) \geq h^0(E) + h^0(F) = 2nk_0 - 3$ and W defines a point of $\mathcal{S}^k_{2n,K}$. In particular, $\mathcal{S}^k_{2n,K}$ is nonempty. By Proposition 2.2 (b), each component is of codimension at most $\frac{1}{2}k(k+1)$. $\Box$

Remark 4.7. If one allows strictly semistable symplectic bundles, it is easy to give examples of K-valued symplectic bundles with larger h^0 over any curve. Set

$$k_1 \ := \ \max\{h^0(L) : L \in \mathrm{Pic}^{g-1}(C)\}.$$

Let $L_1, \ldots, L_n$ be (not necessarily pairwise nonisomorphic) line bundles of degree $g-1$ with $h^0(L_i) \geq k_1$. Then the direct sum

$$W \ := \ \bigoplus \left(L_i \oplus K L_i^{-1} \right)$$

endowed with the sum of the standard skewsymmetric forms on the $L_i \oplus K L_i^{-1}$ is semistable (but not stable) K-valued symplectic of rank $2n$ with $h^0(W) = 2nk_1 > 2nk_0 - 3$.

4.4. **Smoothness.** Now we shall prove that if C is a general Petri curve, the component of $\mathcal{S}^k_{2n,K}$ whose existence was shown above is smooth and of the expected codimension $\frac{1}{2}k(k+1)$. We shall require the following lemma, whose proof is straightforward.

Lemma 4.8. *Let V be a vector bundle. Suppose $F_1, \ldots, F_m$ are sheaves such that $\bigoplus_{i=1}^m F_i$ is a subsheaf of V with $H^0(V) = \bigoplus_{i=1}^m H^0(F_i)$. Suppose that the multiplication maps*

$$H^0(F_i) \otimes H^0(F_j) \ \to \ H^0(F_i \otimes F_j) \quad \textit{and} \quad \mathrm{Sym}^2 H^0(F_i) \ \to \ H^0(\mathrm{Sym}^2 F_i)$$

are injective for $1 \leq i \leq j \leq m$. Then the Petri map $\mathrm{Sym}^2 H^0(V) \to H^0(\mathrm{Sym}^2 V)$ is injective.

Theorem 4.9. *Let C be a general Petri curve of genus $g \geq 3$. Then for $n \geq 2$ and $k \leq 2nk_0 - 3$, the locus $\mathcal{S}^k_{2n,K}$ has a component which is generically smooth and of the expected dimension.*

Remark 4.10. Note that the Petri assumption implies that $k_0 = \lfloor \sqrt{g-1} \rfloor$.

Proof of Theorem 4.9. Recall the K-valued symplectic bundle W constructed in (4.5), which by Proposition 4.5 defines a point of $\mathcal{S}^{2nk_0-3}_{2n,K}$. By Corollary 2.8 and Lemma 2.10, the statement will follow if we can show that $\mu\colon \mathrm{Sym}^2 H^0(W) \to H^0(\mathrm{Sym}^2 W)$ is injective.

The following argument is modelled upon the proof of [HHN18, Lemma 7.2]. Let $p \in C$ be a point which is not a base point for any KL_i^{-1}, so $h^0(L_i(p)) = h^0(L_i)$ for $1 \leq i \leq n$. For each i, we have a commutative diagram

$$\begin{tikzcd} H^0(L_i) \otimes H^0(KL_i^{-1}) \arrow[r] \arrow[d, "\wr"] & H^0(K) \arrow[d, "\wr"] \\ H^0(L_i(p)) \otimes H^0(KL_i^{-1}) \arrow[r] & H^0(K(p)). \end{tikzcd}$$

Now let U be the open subset of $B^{k_0}_{1,g-1}$ over which $h^0(L) = h^0(L(p)) = k_0$. (Note that since C is Petri, $B^{k_0}_{1,g-1}$ is irreducible by [HHN18, Remark 4.2].) Let $\mathcal{A}$ and $\mathcal{B}$ be vector bundles over $U \times U$ whose fibres at (L, N) are $H^0(L(p)) \otimes H^0(KN^{-1})$ and $H^0(KLN^{-1}(p))$ respectively. These have rank k_0^2 and g respectively. Let $\tilde{\mu}\colon \mathcal{A} \to \mathcal{B}$ be the globalised Petri map. Since C is Petri, the composed map

$$H^0(L_i) \otimes H^0(KL_i^{-1}) \ \to \ H^0(K) \ \to \ H^0(K(p))$$

is injective for all L_i. Hence $\tilde{\mu}$ is injective on an open subset of $U \times U$. Deforming the L_i if necessary, we may assume that the multiplication maps

$$(4.6) \quad H^0(L_i) \otimes H^0(L_j) \ \to \ H^0(L_i L_j) \quad \text{and} \quad H^0(L_i) \otimes H^0(KL_j^{-1}) \ \to \ H^0(KL_i L_j^{-1}) \\ \text{and} \quad H^0(KL_i^{-1}) \otimes H^0(KL_j^{-1}) \ \to H^0(K^2 L_i^{-1} L_j^{-1})$$

are injective for all i, j.

Furthermore, as C is now assumed general in moduli and the L_i were chosen generally in the positive-dimensional locus $B^{k_0}_{1,g-1}$, by [Bal12, Theorem 1] the symmetric Petri maps

$$(4.7) \qquad \mathrm{Sym}^2 H^0(L_i) \ \to \ H^0(L_i^2) \quad \text{and} \quad \mathrm{Sym}^2 H^0(KL_i^{-1}) \ \to \ H^0(K^2 L_i^{-2})$$

are injective for all i.

Next, from the proof of Proposition 4.5 we recall the subbundle $F \subset W$ lifting from the elementary transformation $F_{e_1,e_2} \subset K \otimes E^*$. We claim that $H^0(W) = H^0(E) \oplus H^0(F)$. Clearly $H^0(E) \oplus H^0(F) \subseteq H^0(W)$. For the reverse inclusion:

For $1 \leq \ell \leq 2$, let $\widehat{e_\ell} \in E(y_\ell)|_{y_\ell}$ be a point lying over the image of e_ℓ via the canonical isomorphism $\mathbb{P}E \xrightarrow{\sim} \mathbb{P}(E(y_\ell))$, and let κ_ℓ be a generator of $K^{-1}|_{y_\ell}$. Then, since $\delta(W)$ was chosen to be a general point of the secant $\overline{\psi(e_1)\psi(e_2)}$, by Lemma 4.3 we can write $\delta(W)$ as the image in $H^1(K^{-1} \otimes E \otimes E)$ of a point

$$\nabla \ := \ (\lambda_1 \kappa_1 \otimes e_1 \otimes \widehat{e_1}, \lambda_2 \kappa_2 \otimes e_2 \otimes \widehat{e_2}) \ \in \ K^{-1} \otimes E \otimes E(y_1 + y_2)|_{y_1+y_2}$$

for nonzero scalars λ_1, λ_2. Then there is a commutative diagram

$$\begin{tikzcd}
H^0(K \otimes E^*) \otimes \mathbb{K} \cdot \nabla \arrow[d, hook] \arrow[dr, "\varepsilon"] & \\
H^0(K \otimes E^*) \otimes \left(K^{-1} \otimes E \otimes E(y_1 + y_2)|_{y_1+y_2}\right) \arrow[r, "\varepsilon_1"] \arrow[d] & E(y_1 + y_2)|_{y_1+y_2} \arrow[d] \\
H^0(K \otimes E^*) \otimes H^1(K^{-1} \otimes E \otimes E) \arrow[r, "\cup"] & H^1(E)
\end{tikzcd}$$

where the lower vertical arrows are induced by coboundary maps, and ε and ε_1 are induced by evaluation of sections.

Now since $H^0(K \otimes E^*) \cong \bigoplus_{i=1}^n H^0(KL_i^{-1})$ and each $h^0(KL_i^{-1}) \geq 2$, after perturbing e_1 and e_2 if necessary, we can find sections $t_1, t_2 \in H^0(K \otimes E^*)$ such that $t_\ell(\kappa_m \otimes e_m) = \delta_{\ell,m}$, where $\delta_{\ell,m}$ is the Kronecker delta. It follows that the image of ε is spanned by $\widehat{e_1}$ and $\widehat{e_2}$. Then by commutativity and in view of Lemma 4.3 (with $V = E$), the projectivised image of $\cup\delta(W)$ is spanned by the images of e_1 and e_2 in $\mathbb{P}H^1(E) = |\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* K|^*$. Perturbing $e_1$and e_2 again if necessary, we may assume that these images span a $\mathbb{P}^1$. We conclude that $\cup\delta(W)$ has rank 2, whence $h^0(W) = 2k_0 - 3$ and $H^0(W) = H^0(E) \oplus H^0(F)$ as desired. As

$$H^0(E) \ \subset \ \bigoplus_i H^0(L_i) \quad \text{and} \quad H^0(F) \ \subset \ \bigoplus_j H^0(KL_j^{-1}),$$

by injectivity of the maps in (4.6) and (4.7) and by Lemma 4.8, we obtain the injectivity of $\mu \colon \mathrm{Sym}^2 H^0(W) \to H^0(\mathrm{Sym}^2 W)$. This completes the proof. □

Remark 4.11. Recall that the scheme $\mathcal{S}^k_{2n,K}$ has expected dimension

$$\beta^k_{2n,s}(K) \ := \ n(2n+1)(g-1) - \frac{1}{2}k(k+1).$$

In the case $2n = 2$, Bertram and Feinberg conjectured in [BF], that if the expected dimension

$$\beta^k_{2,s}(K) \ = \ 3g - 3 - \frac{1}{2}k(k+1)$$

is nonnegative, then $\mathcal{S}^k_{2,K} = B^k(2, K)$ would be nonempty. They further predicted that on a general curve, $\mathcal{S}^k_{2,K}$ would be nonempty only if $\beta^k_{2,s}(K) \geq 0$. Mukai states this conjecture as a problem in [Muk92, Problem 4.11] and [Muk97, Problem 4.8].

Teixidor i Bigas proves in [Te07, Theorem 1.1] that on a general curve, if $k = 2k_1$, then $\mathcal{S}^k_{2,K}$ is nonempty for $g \geq k_1^2$ if $k_1 > 2$, for $g \geq 5$ if $k_1 = 2$, and for $g \geq 3$ if $k_1 = 1$. Moreover, under these conditions, it has a component of the expected dimension $\beta^k_{2n,s}(K)$. In the case $k = 2k_1 + 1$, she proves that $\mathcal{S}^k_{2,K}$ is nonempty when $g \geq (k_1)^2 + k_1 + 1$ and has a component of the right dimension.

Lange, Newstead and Park [LNS16] proved that if C is a general curve of odd prime genus g and if $g-1 \geq \max\{2k-1, \frac{1}{4}k(k-1)\}$, then $\mathcal{S}^k_{2,K}$ is nonempty.

The above Theorems 4.6 and 4.9 push forward the Bertram–Feinberg–Mukai conjecture and extend it when $n \geq 2$, covering also the issue of smoothness in many cases for Petri curves. Note that Theorem 4.6 does not need any genericity condition; however, a sharp bound for k in Theorems 4.6 and 4.9 will however require further studies.

5. Superabundant components of Brill–Noether loci

The usual Brill–Noether locus $B^k_{r,d}$ has expected dimension

$$\beta^k_{r,d} \;=\; \dim \mathcal{U}(r,d) - k(k-d+r(g-1)).$$

As outlined in the introduction, examples of components of excess dimension are relevant both to Brill–Noether theory and the study of determinantal varieties. Building on the observation [Ne11, § 9] that $B^k_{2,K}$ can have larger expected dimension than the locus $B^k_{2,2g-2}$ containing it, we shall now show for infinitely many n and g the existence of superabundant components of $B^k_{2n,2n(g-1)}$ for any curve of genus g.

The expected dimension of $\mathcal{S}^k_{2n,K}$ exceeds that of $B^k_{2n,2n(g-1)}$ if and only if

$$\dim \mathcal{MS}(2n,K) - \frac{1}{2}k(k+1) \;>\; \dim \mathcal{U}(2n, 2n(g-1)) - k(k-d+r(g-1)),$$

which is equivalent to

$$\frac{1}{2}k(k-1) \;>\; n(2n-1)(g-1)+1. \tag{5.1}$$

Thus if $\mathcal{S}^k_{2n,K}$ is nonempty for a value of k satisfying this inequality, there exists a superabundant component of $B^k_{2n,2n(g-1)}$. We shall give examples using Theorem 4.6. Firstly, for certain values of g, one can obtain statements for all n.

Theorem 5.1. *Suppose $m \geq 7$ and let C be any curve of genus $g = m^2+1$. Then for any $n \geq 1$, the locus $\mathcal{S}^{2nm-3}_{2n,K}$ is nonempty and has dimension greater than $\beta^{2nm-3}_{2n,2n(g-1)}$. In particular, $B^{2nm-3}_{2n,2n(g-1)}$ has a superabundant component.*

Proof. As before, set $k_0 := \max\{k \geq 0 : \dim B^k_{1,g-1} \geq 1\}$. Then $k_0 \geq \lfloor \sqrt{g-1} \rfloor = m$ (with equality if C is Petri). Hence the bundle W defined in (4.5) defines a point of $\mathcal{S}^{2nm-3}_{2n,K}$. For $k = 2nm-3$, the inequality (5.1) becomes

$$\frac{(2nm-3)(2nm-4)}{2} \;>\; n(2n-1)m^2+1.$$

The n^2-terms cancel, and the inequality reduces to $nm^2 - 7nm + 5 \;>\; 0$. One checks easily that this holds for all $n \geq 1$ when $m \geq 7$. $\square$

With the same approach, if we fix n, then we can obtain a statement for any curve of large enough genus. For a fixed g, we set $k_1 := \lfloor \sqrt{g-1} \rfloor$. (If C is Petri then $k_1 = k_0$.)

Theorem 5.2. *Fix $n \geq 1$ and let C be any curve of genus $g \geq (4n+7)^2+1$. Then $\mathcal{S}_{2n,K}^{2nk_1-3}$ is nonempty and has dimension greater than $\beta_{2n,2n(g-1)}^{2nk_1-3}$. In particular, for fixed $n \geq 1$, there are infinitely many g such that for some k depending on g, the locus $B_{2n,2n(g-1)}^{k}$ has a superabundant component for any curve C of genus g.*

Proof. Let W be as above. As $k_1^2 \leq g-1$ but $(k_1+1)^2 \geq g$, we have

$$\sqrt{g}-1 \ \leq \ k_1 \ \leq \ \sqrt{g-1}. \tag{5.2}$$

Now let us check inequality (5.1) for $k = h^0(W) = 2nk_1 - 3$; explicitly, that

$$\frac{(2nk_1-3)(2nk_1-4)}{2} \ > \ n(2n-1)(g-1)+1,$$

that is,

$$2n^2k_1^2 - 7nk_1 + 5 \ > \ 2n^2(g-1) - n(g-1). \tag{5.3}$$

Rewriting the left side as $2n^2(k_1^2+2k_1) - 4n^2k_1 - 7nk_1 + 6$ and noting that $k_1^2+2k_1 \geq g-1$ by the left hand inequality in (5.2), we see that (5.3) would follow from the inequality

$$-4n^2k_1 - 7nk_1 + 5 \ > \ -n(g-1),$$

that is, $(g-1) + \frac{5}{n} > k_1(4n+7)$. As $k_1 \leq \sqrt{g-1}$ by (5.2), this would follow from

$$\sqrt{g-1}\left(1 + \frac{5}{n(g-1)}\right) \ > \ 4n+7.$$

This follows from the hypothesis $g \geq (4n+7)^2 + 1$. □

Setting $n = 1$, the above theorem shows in particular:

Corollary 5.3. *For any curve of genus $g \geq 122$, there exist Brill–Noether loci with superabundant components.*

Remark 5.4. The bundle W is not a smooth point of the component of $B_{2n,2n(g-1)}^{2nk_1-3}$. The usual Petri map is identified with the multiplication $H^0(W) \otimes H^0(W) \to H^0(W \otimes W)$. Since W has at least one line subbundle L_1 with at least two independent sections, the restriction of this map to $\wedge^2 H^0(W)$ has nonzero kernel containing $\wedge^2 H^0(L_1)$. Note moreover that we have only shown that $\mathcal{S}_{2n,K}^{2nk_1-3}$ has a component contained in a superabundant component of $B_{2n,2n(g-1)}^{2nk_1-3}$; the latter component could in general have even larger dimension.

Remark 5.5. In [CFK18], the authors show that in rank two for a general ν-gonal curve, the superabundant components of $B_{2,d}^k$ are all of first type (cf. Definition 3.3). However, W is generically generated, since E is generically generated and the subspace $H^0(F)$ lifting from $H^0(K \otimes E^*)$ generically generates F. This is another aspect in which the higher rank case differs from the rank two case.

5.1. **Superabundant components of moduli of coherent systems.** Coherent systems on C were briefly mentioned in § 2.4. We recall now some more facts, referring the reader to [Br09] for more information and references; and to [BGMN03] for the connection to Brill–Noether theory. For a coherent system (W, Λ) of type (r, d, k) on C and a real number α, recall that the *α-slope* of (W, Λ) is defined to be the real number

$$\mu_\alpha(W, \Lambda) := \frac{d}{r} + \alpha\frac{k}{r}.$$

The coherent system (W, Λ) is called *α-stable* if for any coherent subsystem (V, Π) of (W, Λ) one has $\mu_\alpha(V, \Pi) < \mu_\alpha(W, \Lambda)$. For any real number $\alpha > 0$, there exists a moduli space $G(\alpha; r, d, k)$ parametrising α-stable coherent systems, which has expected dimension

$$\beta^k_{r,d} \ = \ r^2(g-1) + 1 - k(k - d + r(g-1)).$$

Furthermore there is an increasing finite sequence of real numbers $0 = \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_\ell$ with the property that if α and α' belong to the open interval (α_i, α_{i+1}) then $G(\alpha; r, d, k) \cong G(\alpha'; r, d, k)$. The numbers α_i are called *critical values* for the type (r, d, k).

For any $L \in \mathrm{Pic}^d(C)$, we may also consider the closed sublocus

$$G(\alpha; r, L, k) \ := \ \{(W, \Lambda) \in G(\alpha; r, d, k) : \det W \cong L\}.$$

It is clear that every component of $G(\alpha; r, L, k)$ has dimension at least $\beta^k_{r,d} - g$. However, in [GN14], the authors show that in several cases this is not sharp, and conjecture in [GN14, § 2] that every component of $G(\alpha; r, L, k)$ has dimension at least

$$\beta^k_{r,d} - g + \binom{k}{2} \cdot h^1(L) \ =: \ \gamma^k_{r,L}. \tag{5.4}$$

We have the following result on superabundant components of moduli of coherent systems.

Theorem 5.6. *Let C be a general curve of genus $g \ge 3$, so that $k_0 = \lfloor \sqrt{g-1} \rfloor$, and W be the K-valued symplectic bundle constructed in (4.5). Set $k = 2nk_0 - 3$. Let α_1 be the smallest positive critical value for the type $(2n, 2n(g-1), k)$, and suppose $0 < \alpha < \alpha_1$.*

(a) *The coherent system $(W, H^0(W))$ is of type $(2n, 2n(g-1), k)$ and α-stable.*

(b) *The fixed determinant locus $G(\alpha; 2n, K^n, k)$, and hence also the full moduli space $G(\alpha; 2n, 2n(g-1), k)$, contains a component of dimension at least*

$$n(2n+1)(g-1) - \frac{1}{2}k(k+1).$$

(c) *Suppose $m \ge 7$ and $g = m^2+1$, so $k = 2nm-3$. Then for any $n \ge 1$, the component of $G(\alpha; 2n, 2n(g-1), 2nm-3)$ referred to in (b) is superabundant. Moreover, $G(\alpha; 2n, K^n, 2nm-3)$ has a component of dimension larger than $\gamma^{2nm-3}_{2n,K^n} + g$ (cf. (5.4)).*

(d) *Fix $n \ge 1$ and $g \ge (4n+7)^2+1$. Then the component of $G(\alpha; 2n, 2n(g-1), 2nk_0-3)$ referred to in (b) is superabundant. Moreover, $G(\alpha; 2n, K^n, 2nk_0-3)$ has a component of dimension larger than $\gamma^{2nk_0-3}_{2n,K^n} + g$.*

Proof. (a) By the proof of Theorem 4.9, we have $h^0(W) = k$, so $(W, H^0(W))$ is of type $(2n, 2n(g-1), k)$. For α-stability (see also [KN95]): By Proposition 4.5, the bundle W is stable. In particular, if V is a proper subbundle of rank r, then $\mu(V) \le \mu(W) - \frac{1}{2nr}$. It is then easy to check that the coherent system $(W, H^0(W))$ is α-stable for $0 < \alpha < \frac{1}{2nk}$. Since $G(\alpha; r, d, k) \cong G(\alpha'; r, d, k)$ for any α, α' in the interval $(0, \alpha_1)$, the coherent system $(W, H^0(W))$ is α-stable for $0 < \alpha < \alpha_1$.

(b) Denote by X the component of $\mathcal{S}^k_{2n,K}$ containing W. By part (a), for generic $W' \in X$ the coherent system $(W', H^0(W'))$ is α-stable, so there is a map

$$X \dashrightarrow G(\alpha; 2n, 2n(g-1), 2nk_0 - 3)$$

given by $W' \mapsto (W', H^0(W'))$. Clearly this is generically injective. In particular, the moduli space $G(\alpha; 2n, 2n(g-1), 2nk_0 - 3)$ has a component of dimension at least $n(2n + 1)(g-1) - \frac{1}{2}k(k+1)$. Moreover, as any K-valued symplectic bundle has determinant K^n, the image of X is contained in the fixed determinant locus $G(\alpha; 2n, K^n, k)$.

Finally, as $G(\alpha; 2n, 2n(g-1), k)$ has the same expected dimension as $B^k_{2n,2n(g-1)}$, parts (c) and (d) follow from the computations in the proofs of Theorems 5.1 and 5.2. □

References

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris. Geometry of algebraic curves, vol. I. *Grundlehren der Mathematischen Wissenschaften* 267. New York: Springer-Verlag, 1985.

[Baj19] A. Bajravani. Bounds on the dimension of the Brill–Noether schemes of rank two bundles. *Ann. Mat. Pura Appl. (4)* **199** (1), 345–354, 2019.

[BBNN06] V. Balaji, I. Biswas, D. S. Nagaraj, and P. E. Newstead. Universal families on moduli spaces of principal bundles on curves. *Int. Math. Res. Not.* 2006, no. 2, 16 pp.

[Bal12] E. Ballico. Embeddings of general curves in projective spaces: the range of the quadrics. *Lith. Math. J.* **52** (2), 134–137, 2012.

[Be06] A. Beauville. Vector bundles on curves and theta functions. In *Moduli spaces and arithmetic geometry. Papers of the 13th International Research Institute of the Mathematical Society of Japan, Kyoto, Japan, September 8–15, 2004.* Tokyo: Mathematical Society of Japan, 2006.

[BF] A. Bertram and B. Feinberg. On stable rank two bundles with canonical determinant and many sections. In: Algebraic Geometry (Catania, 1993/Barcelona 1994), *Lecture Notes in Pure and Appl. Math.* **200**, 259–269. New York: Dekker, 1998.

[BG06] I. Biswas and T. L. Gómez. Hecke correspondence for symplectic bundles with application to the Picard bundles. Int. J. Math. **17** (1), 45–63, 2006.

[Br09] S. B. Bradlow. Coherent systems: a brief survey. With an appendix by H. Lange. In *Moduli spaces and vector bundles. A tribute to Peter Newstead*, pp. 229–264. Cambridge: Cambridge University Press, 2009.

[BGMN03] S. B. Bradlow, O. García-Prada, V. Muñoz and P. E. Newstead. Coherent systems and Brill–Noether theory. *Int. J. Math.* **14** (7), 683–733, 2003.

[BGN97] L. Brambila-Paz, I. Grzegorczyk and P. E. Newstead. Geography of Brill–Noether loci for small slopes. *J. Algebr. Geom.*, **6** (4), 645–669, 1997.

[CT11] S. Casalaina-Martin and M. Teixidor i Bigas. Singularities of Brill–Noether loci for vector bundles on a curve. *Math. Nachr.* **284** (14–15), 1846–1871, 2011.

[CH10] I. Choe and G. H. Hitching. Secant varieties and Hirschowitz bound on vector bundles over a curve. *Manuscr. Math.* **133** (3-4), 465–477, 2010.

[CH14] I. Choe and G. H. Hitching. A stratification on the moduli spaces of symplectic and orthogonal bundles over a curve. *Int. J. Math.* **25** (5), 27 pp., 2014.

[CFK18] Y. Choi, F. Flamini and S. Kim. Brill–Noether loci of rank 2 vector bundles on a general ν-gonal curve. *Proc. Am. Math. Soc.* **146** (8), 3233–3248, 2018.

[Fe] B. Feinberg. On the Dimension and Irreducibility of Brill–Noether Loci. Manuscript.

[GN14] I. Grzegorczyk and P. E. Newstead. On coherent systems with fixed determinant. *Int. J. Math.* **25** (5), 11 pp., 2014.

[GT09] I. Grzegorczyk and M. Teixidor i Bigas. Brill–Noether theory for stable vector bundles. In *Moduli spaces and vector bundles*, London Math. Soc. Lecture Note Ser. 359, 29–50. Cambridge: Cambridge Univ. Press, 2009.

[HT84] J. Harris and L. W. Tu. On symmetric and skew-symmetric determinantal varieties. *Topology* **23**, 71–84, 1984.

[Ha83] R. Hartshorne. Algebraic geometry. Corr. 3rd printing. Grad. Texts Math. 52, New York: Springer, 1983.

[Hi05] G. H. Hitching. Moduli of symplectic bundles over curves. Ph. D. dissertation, University of Durham, England, 2005.

[Hi07] G. H. Hitching. Subbundles of symplectic and orthogonal vector bundles over curves. *Math. Nachr.* **280** (13–14), 1510–1517, 2007.

[HHN18] G. H. Hitching, M. Hoff and P. E. Newstead. Nonemptiness and smoothness of twisted Brill–Noether loci. *Ann. Mat. Pura Appl.*, to appear.

[HP15] G. H. Hitching and C. Pauly. Theta divisors of stable vector bundles may be nonreduced. *Geom. Dedicata* **177**, 257–273, 2015.

[KN95] A. D. King and P. E. Newstead. Moduli of Brill–Noether pairs on algebraic curves. *Int. J. Math.*, **6** (5), 733–748, 1995.

[LN83] H. Lange and M. S. Narasimhan. Maximal subbundles of rank two vector bundles on curves. Math. Ann. **266**, 55–72, 1983.

[LNP16] H. Lange, P. E. Newstead and S. S. Park. Non-emptiness of Brill–Noether loci in $M(2, K)$. *Comm. in Algebra* **44** (2), 746–767, 2016.

[LNS16] H. Lange, P. E. Newstead and V. Strehl. Non-emptiness of Brill–Noether loci in $M(2, L)$. *Int. J. Math.*, **26**, no. 12, 2015 (1550108, 26 pages).

[Me99] V. Mercat. Le problème de Brill–Noether et le théorème de Teixidor. *Manuscr. Math.* **98** (1), 75–85, 1999.

[Mi08] R. M. Miró-Roig. Determinantal ideals. *Prog. Math.*, vol. 264. Basel: Birkhäuser, 2008.

[Muk92] S. Mukai. Vector bundles and Brill–Noether theory. In: *Current topics in Complex Algebraic Geometry* (Berkeley, CA, 1992/93), 145-–158, Math. Sci. Res. Inst. Publ. 28. Cambridge: Cambridge Univ. Press, 1995.

[Muk97] S. Mukai. Non-Abelian Brill–Noether theory and Fano 3-folds. Sugako Expositions 14, 124–153, 2001.

[Mum71] D. Mumford. Theta characteristics of an algebraic curve. *Ann. Sci. Éc. Norm. Supér.* **4**, 181–192, 1971.

[NR75] M. S. Narasimhan and S. Ramanan. Deformations of the moduli space of vector bundles over an algebraic curve. *Ann. Math. (2)*, **101**, 391–417, 1975.

[Ne11] P. E. Newstead. Existence of α-stable coherent systems on algebraic curves. In *Grassmannians, moduli spaces and vector bundles*, vol. 14 of *Clay Math. Proc.*, pp. 121–139. Amer. Math. Soc., Providence, RI, 2011.

[Os13-1] B. Osserman. Special determinants in higher-rank Brill–Noether theory. *Int. J. Math.* **24** (11), 17 pp., 2013.

[Os13-2] B. Osserman. Brill–Noether loci with fixed determinant in rank 2. *Int. J. Math.* **24** (13), 24 pp., 2013.

[Rm81] S. Ramanan. Orthogonal and spin bundles over hyperelliptic curves. *Proc. Indian Acad. Sci., Math. Sci.* **90**, 151–166, 1981.

[Rth96] A. Ramanathan. Moduli for principal bundles over algebraic curves I & II. *Proc. Indian Acad. Sci., Math. Sci.*, **106** (3), 301–328 and (4), 421–449, 1996.

[Se12] O. Serman. Orthogonal and symplectic bundles on curves and quiver representations. In *Geometric methods in representation theory II. Selected papers based on the presentations at the summer school, Grenoble, France, June 16 – July 4, 2008*, pp. 393–418. Paris: Société Mathématique de France, 2012.

[Te91] M. Teixidor i Bigas. Brill–Noether theory for stable vector bundles. *Duke Math. J.* **62** (2), 385–400, 1991.

[Te92] M. Teixidor i Bigas. On the Gieseker–Petri map for rank 2 vector bundles. *Manuscr. Math.*, **75** (4), 375–382, 1992.

[Te04] M. Teixidor i Bigas. Rank two vector bundles with canonical determinant. *Math. Nach.*, **265** (1), 100–106, 2004.

[Te07] M. Teixidor i Bigas. Petri map for rank two bundles with canonical determinant. *Compos. Math.* **144** (3), 705–720, 2008.

[Te14] M. Teixidor i Bigas. Injectivity of the Petri map for twisted Brill–Noether loci. *Manuscr. Math.* **145** (3–4), 389–397, 2014.

[Zh17] N. Zhang. Expected dimension of higher rank Brill–Noether loci. *Proc. AMS.* **145**, 3735–3746, 2017.

(A. Bajravani) Department of Mathematics, Faculty of Basic Sciences, Azarbaijan Shahid Madani University, Tabriz, I. R. Iran., P. O. Box: 53751-71379

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746. Tehran, Iran

Email address: `bajravani@azaruniv.ac.ir`

(G. H. Hitching) Oslo Metropolitan University, Postboks 4, St. Olavs plass, 0130 Oslo, Norway.

Email address: `gehahi@oslomet.no`