A RIEMANN–KEMPF SINGULARITY THEOREM FOR HIGHER RANK BRILL–NOETHER LOCI

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Abstract. Given a vector bundle V over a curve X, we define and study a surjective rational map $\mathrm{Hilb}^d(\mathbb{P}V) \dashrightarrow \mathrm{Quot}^{0,d}(V^*)$ generalising the natural map $\mathrm{Sym}^d X \to \mathrm{Quot}^{0,d}(\mathcal{O}_X)$. We then give a generalisation of the geometric Riemann–Roch theorem to vector bundles of higher rank over X. We use this to give a geometric description of the tangent cone to the Brill–Noether locus $B^r_{r,d}$ at a suitable bundle E with $h^0(E) = r + k$. This gives a generalisation of the Riemann–Kempf singularity theorem. As a corollary, we show that the kth secant variety of the rank one locus of $\mathbb{P}\mathrm{End}\, E$ is contained in the tangent cone.

1. Introduction

Let X be a projective smooth curve of genus $g \geq 2$ and D an effective divisor of degree d on X. By the geometric Riemann–Roch theorem, $\dim |D|$ is exactly the defect of D on the canonical model of X in $|K_X|^*$. If X is general, the line bundle $\mathcal{O}_X(D)$ defines a point of multiplicity $h^0(X, \mathcal{O}_X(D))$ of the Brill–Noether locus $W_d(X)$. The Riemann–Kempf singularity theorem (see [GH94, Chapter 2]) states that the tangent cone to $W_d(X)$ at $\mathcal{O}_X(D)$ is precisely the union of the secants $\mathrm{Span}(D') \subset |K_X|^*$ for $D' \in |D|$. Using this picture, in [KS88], [CS95] and [HM15] new proofs of Torelli's theorem were given using the infinitesimal geometry of Brill–Noether loci.

In recent years, higher-rank analogues of $W_d(X)$ have been the subject of much attention. We denote by $U_X(r,d)$ the moduli space of stable vector bundles of rank r and degree d and consider the *higher-rank Brill–Noether locus*

$$B^k_{r,d} \ = \ \{E \in U_X(r,d) : h^0(X,E) \geq k\}.$$

See [GT09] for a summary of relevant results. Our interest is primarily in the infinitesimal geometry of $B^k_{r,d}$ at singular points. As motivation, we note that in [Pau03] and [HH17] the infinitesimal geometry of generalised theta divisors associated to higher rank vector bundles (examples of *twisted Brill–Noether loci*) was used to prove "Torelli-type" theorems (recovering the curve and the bundle respectively). It seems therefore natural to investigate what can be recovered from the tangent cones $\mathbb{T}_E B^k_{r,d}$ at singular points E. Note that [CT11] gives a comprehensive introduction and many interesting results on the singular loci of higher rank Brill–Noether loci and twisted Brill–Noether loci.

The projectivised tangent space of $U_X(r,d)$ at E is $\mathbb{P}H^1(X, \mathrm{End}\, E)$. It was shown in [HR04] that there is a natural map $\psi \colon \mathbb{P}\mathrm{End}\, E \dashrightarrow \mathbb{P}H^1(X, \mathrm{End}\, E)$, generalising the canonical curve, which is an embedding for general E. In light of this, as a first step towards finding analogues of the above results on line bundles, one can seek generalisations of the geometric Riemann–Roch and Riemann–Kempf theorems for bundles of higher rank, given in terms of the geometry of the scroll $\mathbb{P}\mathrm{End}\, E$. One such generalisation was given in [Hit13] for bundles of Euler characteristic zero, where the tangent cones of

$$B^1_{r,r(g-1)} \;=\; \{E \in U_X(r, r(g-1)) : h^0(X, E) \geq 1\} \;\subset\; U_X(r, r(g-1))$$

another "generalised theta divisor", are described geometrically.

In the present work, we generalise the picture in another way. Returning to the opening example, we note that the sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_X(D)_D \to 0$ realises $\mathcal{O}_X(D)$ as an *elementary transformation* of the trivial bundle. Let V be a vector bundle of rank r and $\pi \colon \mathbb{P}V \to X$ the associated scroll. We consider elementary transformations $0 \to V \to \tilde{V} \to \tau \to 0$; that is, bundles $\tilde{V}$ of rank r containing V as a locally free subsheaf of full rank. If the support of τ is reduced and of degree d, then the choice of $\tilde{V}$ is canonically equivalent to a choice of d points $\nu_1, \ldots, \nu_d$ of $\mathbb{P}V$ belonging respectively to fibres over distinct points $x_1, \ldots x_d$ of X. Generalising the definition of $\mathcal{O}_X(D)$, one can realise the sheaf $\tilde{V}$ as the sheaf of rational sections of V with poles bounded by the x_i and in the direction corresponding to the ν_i. Then $\tilde{V}^*$ is the subsheaf of V^* of sections whose values at x_i belong to the hyperplane determined by ν_i.

Our first goal is to systematise and extend this construction to the case where $\mathrm{Supp}(\tau)$ may be nonreduced. Let Z be a subscheme of $\mathbb{P}V$ of dimension zero and length d. Generalising the operation $D \mapsto \mathcal{O}_X(D)$, we define

$$V_Z \;:=\; \left(\pi_* \left(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1)\right)\right)^* .$$

The association $Z \mapsto V_Z^*$ gives a map $\alpha \colon \mathrm{Hilb}^d(\mathbb{P}V) \dashrightarrow \mathrm{Quot}^{0,d}(V^*)$, generalising the natural map $\mathrm{Sym}^d X \to \mathrm{Quot}^{0,d}(\mathcal{O}_X)$ given by $D \mapsto \mathcal{O}_X(-D)$. In Theorem 2.2 and Theorem 2.7 we show that the restriction of α to the component of $\mathrm{Hilb}^d(\mathbb{P}V)$ containing reduced subschemes is surjective and generically injective.

In contrast to the line bundle case, if $r \geq 2$ then $\deg(V_Z)$ may be strictly less than $\deg V + d$ in special cases. This leads us to the notion of *π-nondefectivity* (Definition 3.4). Much can be said about π-nondefective subschemes, but we limit ourselves here to what is strictly necessary for the present applications.

In § 4, we link the elementary transformations V_Z with the geometry of an image of the scroll $\mathbb{P}V \dashrightarrow \mathbb{P}H^1(X, V)$. This allows us to prove a generalisation of the geometric Riemann–Roch theorem for scrolls (Theorem 4.3). For applications to Brill–Noether loci, it is necessary (and straightforward) to have also a "relative" version of this result, given in terms of the geometry of the rank one locus of $V \otimes E$ (Theorem 4.10). It should be noted that a similar situation is studied in the recent paper [Bri17]; see Remark 4.11 for discussion.

In § 5, we use Theorem 4.10 together with results in [CT11] to give another generalisation of the Riemann–Kempf singularity theorem. Suppose E is a stable, generically generated bundle with $h^0(X, E) = r + n$ which is Petri r-injective (Definition 5.1). Then for generic

$\Lambda \in \mathrm{Gr}(r, H^0(X, E))$, the bundle E is an elementary transformation of $\mathcal{O}_X{}^{\oplus r}$. By the results of the previous sections, there exists $Z_\Lambda \in \mathrm{Hilb}^d(\mathbb{P}E^*)$ such that $\mathcal{O}_X{}^{\oplus r} = (E^*)_{Z_\Lambda}$. This observation leads naturally to a geometric description of the projectivised tangent cone $\mathbb{T}_E B^r_{r,d}$ in terms of the geometry of the rank one locus $\Delta \subseteq \mathbb{P}\mathrm{End}\, E \subset \mathbb{P}H^1(X, \mathrm{End}\, E)$ via the map $\psi \colon \mathbb{P}\mathrm{End}\,(E) \dashrightarrow \mathbb{P}H^1(X, \mathrm{End}\,(E))$ mentioned above. The precise statement, which generalises the Riemann–Kempf theorem, is given in Theorem 5.3. An interesting corollary is that if $h^0(X, E) = r + n$, then $\mathbb{T}_E B^r_{r,d}$ contains the nth secant variety of Δ (Theorem 5.5). This generalises the fact that the tangent cone to the Riemann theta divisor at a point of multiplicity $k + 1$ contains the kth secant variety of the canonical curve.

We hope that these results may be useful in proving new "Torelli-type" statements.

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Notation. We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. If Y is a scheme and $Z \subset Y$ a closed subscheme, we write $\mathcal{I}_Z$ for the ideal sheaf of Z in $\mathcal{O}_Y$.

2. Elementary transformations and finite subschemes of scrolls

Let V be a vector bundle of rank r over a projective smooth curve X, and denote by $\pi \colon \mathbb{P}V \to X$ the corresponding scroll. The latter is naturally isomorphic to the Quot scheme $\mathrm{Quot}^{0,1}(\mathbb{P}V^*)$ parametrising elementary transformations of the form $0 \to \tilde{V}^* \to V^* \to \tau \to 0$, where τ is a skyscraper sheaf of length 1; equivalently, elementary transformations of the form $0 \to V \to \tilde{V} \to \tau \to 0$. More generally, in [Tyu74] a tower of projective bundles was constructed parametrising such $\tilde{V}$ for τ of degree $d \geq 1$ (see also [CH10, Lemma 4.2]). Here we study an alternative way of parametrising these elementary transformations. Our approach has some features in common with that of [BGL94], whose (r, n)-*divisors* are $\mathcal{O}_X{}^{\oplus r}$-valued principal parts in the sense below.

Let $Z \subset \mathbb{P}V$ be a subscheme of dimension zero and length d, corresponding to a point of the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{P}V)$. Tensoring the sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}V} \to \mathcal{O}_Z \to 0$ by $\mathcal{O}_{\mathbb{P}V}(1)$ and taking direct images, we obtain an exact sequence

$$0 \to \pi_* \left(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1) \right) \to V^* \to \pi_* \left(\mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1) \right) \to \cdots \tag{2.1}$$

Set $V_Z := \left(\pi_* \left(\mathcal{O}_{\mathbb{P}V}(1) \otimes \mathcal{I}_Z \right) \right)^*$. Then $0 \to V \to V_Z \to V_Z/V \to 0$ is an elementary transformation. We write $\tau_Z := V_Z/V$.

Remark 2.1. We will switch freely between elementary transformations of the form $V \subset \tilde{V}$ and $\tilde{V}^* \subset V^*$, depending on what is more convenient at a given time. It will be necessary to distinguish between the statement that $V_Z \cong \tilde{V}$ as vector bundles and the stronger statement that $V_Z^* \subset V^*$ and $\tilde{V}^* \subset V^*$ define the same point of the Quot scheme $\mathrm{Quot}^{0,d}(V^*)$. To this end, we will occasionally abuse language and speak of an elementary transformation of the form $0 \to V \to \tilde{V} \to \tau \to 0$ as "an element of $\mathrm{Quot}^{0,d}(V^*)$".

If $r = 1$ then Z is a divisor of degree d on $\mathbb{P}V = X$, and then $V_Z^* = V^* \otimes \mathcal{O}_X(-Z)$ as points of $\mathrm{Quot}^{0,d}(V^*)$. For $r \geq 2$ we only have the inequality $\deg(V_Z) \leq \deg(V) + d$, which may be strict. This will be discussed further in § 3.1. For now, let us give the main result of the present section.

Theorem 2.2. *Let $0 \to V \to \tilde{V} \to \tau \to 0$ be an elementary transformation where τ has degree $d \geq 1$.*

(a) *There exists $Z \in \mathrm{Hilb}^d(\mathbb{P}V)$ such that $\tilde{V}^* = V_Z^*$ as elements of $\mathrm{Quot}^{0,d}(V^*)$.*
(b) *If τ has reduced support on X, then Z is reduced and uniquely determined.*

The proof of this theorem and its refinement Theorem 2.7 will occupy the remainder of the section. As there are several ingredients, let us first give an overview. Firstly, note that (b) is almost obvious: In this case, $\tilde{V}$ is determined by specifying a line $\mathrm{Ker}\left(V|_x \to \tilde{V}|_x\right)$ for each $x \in \mathrm{Supp}(\tau)$, so we obtain naturally d points of $\mathbb{P}V$.

If τ has nonreduced support, then it emerges that a certain choice of $\mathbb{K}$-basis of $H^0(X, \tau)$ determines a scheme Z such that $\tilde{V} = V_Z$ as elements of $\mathrm{Quot}^{0,d}(V^*)$. As in the proof of [Hit13, Theorem 3.1], we view sections of $V^* \to X$ as sections of $\mathcal{O}_{\mathbb{P}V}(1) \to \mathbb{P}V$, and show how certain linear conditions defined by the chosen basis elements of $H^0(X, \tau)$ determine a suitable subscheme Z.

Now to the details. Firstly, let us describe $\tilde{V}^*$ more explicitly. We recall that any locally free sheaf V on X has the flasque resolution

$$0 \to V \to \underline{\mathrm{Rat}}\,(V) \to \underline{\mathrm{Prin}}\,(V) \to 0,$$

where $\underline{\mathrm{Rat}}\,(V)$ is the sheaf of rational sections of V, and $\underline{\mathrm{Prin}}\,(V) = \underline{\mathrm{Rat}}\,(V)/V$ the sheaf of V-valued principal parts[1]. The natural $\mathcal{O}_X$-bilinear pairing

$$\underline{\mathrm{Rat}}\,(V) \times \underline{\mathrm{Rat}}\,(V^*) \to \underline{\mathrm{Rat}}\,(\mathcal{O}_X)$$

induces a well-defined $\mathcal{O}_X$-linear map

$$\langle \cdot, \cdot \rangle \colon \underline{\mathrm{Prin}}\,(V) \times V^* \to \underline{\mathrm{Prin}}\,(\mathcal{O}_X).$$

The map $f \mapsto \langle \cdot, f \rangle$ in turn defines an $\mathcal{O}_X$-module homomorphism

$$V^* \to \mathrm{Hom}_{\mathcal{O}_X}\left(\underline{\mathrm{Prin}}\,(V), \underline{\mathrm{Prin}}\,(\mathcal{O}_X)\right).$$

Restricting to the subsheaf $\tau = \tilde{V}/V \subset \underline{\mathrm{Prin}}\,(V) = \underline{\mathrm{Rat}}\,(V)/V$, we obtain a map

$$V^* \to \mathrm{Hom}_{\mathcal{O}_X}(\tau, \underline{\mathrm{Prin}}\,(\mathcal{O}_X)). \tag{2.2}$$

Proposition 2.3. *The subsheaf $\tilde{V}^*$ of V^* is the kernel of (2.2). Equivalently,*

$$\tilde{V}^* \;=\; \{f \in V^* : \langle p, f\rangle \text{ is zero in } \underline{\mathrm{Prin}}\,(\mathcal{O}_X) \text{ for all } p \in \tau\}.$$

[1]In the literature there is an ambiguity in terminology: The *jet sheaf* parametrising germs of local sections of V to order k is sometimes called the "sheaf of kth order principal parts of V". This is a different object from our $\underline{\mathrm{Prin}}\,(V)$.

Proof. A section f of V^* defines a regular section of $\tilde{V}^*$ if and only if $\langle \tilde{v}, f \rangle$ is a regular function for all $\tilde{v} \in \tilde{V}$. Since f is regular, this is equivalent to saying that the principal part $\langle p, f \rangle$ vanishes for all $p \in \tilde{V}/V = \tau$. □

Proposition 2.4. *Suppose $x \in \mathrm{Supp}(\tau)$. Let z be a uniformiser at x. Then there exist local sections $v_1, \dots, v_s$ of V near x which are linearly independent at x, and positive integers $k_1, \dots, k_s$ such that τ_x is spanned over $\mathcal{O}_X$ by the principal parts*

$$\frac{v_1}{z^{k_1}}, \ \dots, \ \frac{v_s}{z^{k_s}}.$$

In particular, $s \leq r$.

Proof. Let $p_1, \dots, p_s$ be a set of generators for τ_x over $\mathcal{O}_{X,x}$. For $1 \leq j \leq s$, we write k_j for the order of the pole of p_j, and reorder so that $k_1 \geq k_2 \geq \cdots \geq k_s$. Then for each j, there exists a local section $v_j \in V_x \backslash m_x V_x$ such that $p_j = \frac{v_j}{z^{k_j}}$. (The section v_j is well-defined modulo $m_x^{k_j} V_x$.) We claim that after reducing s and the k_j if necessary, we may assume that the v_j are linearly independent at x.

Suppose there is a nontrivial $\mathbb{K}$-linear combination $\sum_{j=1}^{s} a_j v_j$ whose value at x is zero; that is, which belongs to $m_x V_x$. Let $l \in \{1, \dots, s\}$ be the largest index such that $a_l \neq 0$. Since $k_1 \geq k_2 \geq \cdots \geq k_s$, the function $z^{k_j - k_l}$ belongs to $\mathcal{O}_{X,x}$ for each $j < l$. Set

$$p_l' \ := \ p_l + \sum_{j=1}^{l-1} z^{k_j - k_l} \frac{a_j}{a_l} p_j. \tag{2.3}$$

A computation shows that

$$p_l' \ = \ \frac{1}{a_l} \cdot \frac{\left(\sum_{j=1}^{l} a_j v_j \right)}{z^{k_l}}.$$

This has a pole of order at most $k_l - 1$ at x, since the numerator belongs to $m_x V_x$. In light of (2.3), the set

$$p_1, \ \dots, \ p_{l-1}, \ p_l', \ p_{l+1}, \ \dots, \ p_s$$

also generates τ_x over $\mathcal{O}_X$. After performing a finite number of operations of this kind (possibly annihilating some of the p_j), we arrive at a generating set $p_1, \dots, p_s$ of principal parts whose leading coefficients define linearly independent points of $V|_x$. In particular, $s \leq r$. □

If $s < r$, we complete the partial frame $v_1, \dots, v_s$ at x to a frame $v_1, \dots, v_r$. Write $f_1, \dots, f_r$ for the dual frame of V^* near x. Note that if $k_1 \geq 2$, these frames carry infinitesimal information also.

Corollary 2.5. *The locally free sheaf $\tilde{V}^*$ is spanned near x by*

$$z^{k_1} f_1, \ z^{k_2} f_2, \ \dots, \ z^{k_s} f_s, \ f_{s+1}, \ \dots, \ f_r.$$

In particular, $\deg \tau_x = k_1 + \cdots + k_s$.

Proof. This follows from Propositions 2.3 and 2.4. □

Each v_j determines uniquely a nonzero point of $V|_x$, given by the image of v_j in the fibre $V_x/m_x V_x$. Hence v_j also determines a point of $\mathbb{P}V|_x$, which we denote by ν_j.

Remark 2.6. If length $(\tau_x) = 1$ then the generator $p_1 \in V(x)/V$ is unique up to scalar multiple, and the point ν_1 is uniquely determined. However, if length $(\tau_x) \geq 2$ then the choice of generators $p_1, \ldots, p_s$, and hence the points ν_j are in general not unique. For example, consider the elementary transformation $0 \to V \to V(x) \to V(x)/V \to 0$. Any frame $\{v_i\}$ for V near x determines a generating set $\{\frac{v_i}{z}\}$ for τ and points $\nu_1, \ldots, \nu_r$ which span $\mathbb{P}V|_x$. Furthermore, a general choice of two distinct frames will define different spanning sets for $\mathbb{P}V|_x$. Note also that the number s of generators of τ_x over $\mathcal{O}_X$ (as distinct from over $\mathbb{K}$) is not determined by length (τ_x). For example, if $\tau_x \cong \mathcal{O}_x^{\oplus 2}$ then two generators are required over $\mathcal{O}_X$, whereas if $\tau_x \cong \mathcal{O}_{2x}$ then one suffices.

Proof of Theorem 2.2. Firstly, suppose that τ is supported at a single point $x \in X$. For $1 \leq j \leq s$, define an elementary transformation W_j of V^* by

$$W_j = \mathrm{Ker}\left(\frac{v_j}{z^{k_j}} \colon V^* \to \underline{\mathrm{Prin}}\,(\mathcal{O}_X) \right)$$

where the v_j are as in Proposition 2.4. Clearly V^*/W_j is supported at x, and W_j is spanned near x by

$$f_1, \ \ldots, \ f_{j-1}, \ z^{k_j} f_j, \ \ldots, \ f_s, \ f_{s+1}, \ \ldots, \ f_r, \tag{2.4}$$

where the f_i are as defined before Corollary 2.5. (Note that $\tilde{V}^* = \cap_{j=1}^{s} W_j$.)

Since π is flat, the functor π^* is exact. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^* W_j & \longrightarrow & \pi^* V^* & \longrightarrow & \pi^*(V^*/W_j) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_j & \longrightarrow & \mathcal{O}_{\mathbb{P}V}(1) & \longrightarrow & \mathcal{T}_j & \longrightarrow & 0 \end{array} \tag{2.5}$$

where $\mathcal{F}_j$ is the image of the composed map $\pi^* W_j \to \pi^* V^* \to \mathcal{O}_{\mathbb{P}V}(1)$, and $\mathcal{T}_j$ is the quotient. We will sometimes abuse notation and denote both $\pi^* z$ and its image in (2.5) simply by z; and similarly for $f_1, \ldots, f_r$. From (2.4), it follows that $\mathcal{F}_j \to \mathcal{O}_{\mathbb{P}V}(1)$ is an isomorphism away from ν_j. Furthermore, a $\mathbb{K}$-basis for $\mathcal{T}_j$ is given by the images of

$$f_j, \ z f_j, \ \ldots, \ z^{k_j - 1} f_j.$$

In particular, length $\mathcal{T}_j = k_j$.

Tensoring the lower row of (2.5) with the invertible sheaf $\mathcal{O}_{\mathbb{P}V}(-1)$, we obtain

$$0 \to \mathcal{I}_j \to \mathcal{O}_{\mathbb{P}V} \to \mathcal{T}_j \otimes \mathcal{O}_{\mathbb{P}V}(-1) \to 0$$

where $\mathcal{I}_j := \mathcal{F}_i \otimes \mathcal{O}_{\mathbb{P}V}(-1)$ is an ideal sheaf. We write Z_j for the subscheme of $\mathbb{P}V$ defined by $\mathcal{I}_j$. Then $\mathcal{T}_j \otimes \mathcal{O}_{\mathbb{P}V}(-1)$ is naturally identified with $\mathcal{O}_{Z_j}$, whence we see that Z_j has length k_j.

We now define

$$\mathcal{I} \ := \ \bigcap_{j=1}^{s} \mathcal{I}_j .$$

As the support of each Z_j is the isolated point ν_j, we see that $\mathcal{I}$ is the ideal sheaf of $Z_1 \cup \cdots \cup Z_s =: Z$, a subscheme supported along $\{\nu_1, \ldots, \nu_s\}$ of $\mathbb{P}V$. This Z has dimension zero and length $k_1 + \cdots + k_s$. For each j, the stalk of $\mathcal{I}$ at ν_j coincides with that of $\mathcal{I}_j$, and is generated by

$$z^{k_j}, \frac{f_i}{f_j} \ : \ 1 \leq i \leq r; \ i \neq j. \tag{2.6}$$

Correspondingly, the stalk of $\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}V}(1)$ at ν_j is generated by the images of

$$f_1, \ldots, f_{j-1}, z^{k_j} f_j, f_{j+1}, \ldots, f_r. \tag{2.7}$$

We now show that $\pi_* \left(\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}V}(1) \right) = \widetilde{V}^*$. This is essentially book-keeping. Clearly the two sheaves are equal away from x. Let $U \subseteq X$ be a neighbourhood of x over which V is trivial. We have

$$\mathcal{O}_{\mathbb{P}V} \left(\pi^{-1} U \right) \ \cong \ \mathcal{O}_{X,U} \otimes \Gamma \left(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r} \right) \ \cong \ \mathcal{O}_{X,U}.$$

Thus a section t of $\mathcal{O}_{\mathbb{P}V}(1)$ over $\pi^{-1}U$ is of the form $h_1 \cdot f_1 + \cdots + h_r \cdot f_r$ where each $h_j \in \mathcal{O}_{X,U}$ and the f_j are as above. By (2.7), such a t belongs to $\Gamma(\pi^{-1}U, \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}V}(1))$ if and only if $h_j \in z^{k_j} \cdot \mathcal{O}_{X,U}$ for each j. By Corollary 2.5, this is equivalent to the statement that t, viewed as a section of $V^* \to X$, takes its values in $\tilde{V}^*$. Thus $\pi_* \left(\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}V}(1) \right) = \tilde{V}^*$. This proves (a) in case τ is supported at a single point x.

More generally, if τ is supported at two or more points, the construction above yields an ideal sheaf $\mathcal{I}(x)$ and a zero-dimensional subscheme $Z(x)$ of length d_x for each $x \in \mathrm{Supp}(\tau)$. We write

$$Z \ := \bigcup_{x \in \mathrm{Supp}(\tau)} Z(x) \quad \text{and} \quad \mathcal{I}_Z \ := \bigcap_{x \in \mathrm{Supp}\tau} \mathcal{I}(x).$$

By the local argument above applied to each $x \in \mathrm{Supp}(\tau)$, we have $\pi_* \left(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1) \right) = \widetilde{V}^*$. This completes the proof of (a) in general.

(b) If τ has reduced support, then $d_x = 1$ for each $x \in \mathrm{Supp}(\tau)$. By Remark 2.6, the intersection of Z with the fibre over x consists of a single, uniquely determined, reduced point. The statement follows. □

Hilbert schemes of points. Let us now give a refinement of Theorem 2.2. The scheme $\mathrm{Hilb}^d(\mathbb{P}V)$ is connected since $\mathbb{P}V$ is, but by [CEVV09] it is not irreducible for $r \geq 4$ and $e \geq 8$. We denote by $\mathrm{Hilb}^d(\mathbb{P}V)_0$ the irreducible component containing reduced subschemes.

Theorem 2.7.

(a) *The association* $Z \ \mapsto \ \pi_* \left(\mathcal{O}_{\mathbb{P}V}(1) \otimes \mathcal{I}_Z \right)$ *defines a rational map*

$$\alpha \colon \mathrm{Hilb}^d(\mathbb{P}V) \ \dashrightarrow \ \mathrm{Quot}^{0,d}(V^*)$$

whose restriction to $\mathrm{Hilb}^d(\mathbb{P}V)_0$ *is surjective.*

(b) *The restriction of* α *to the subset*

$$\{Z \in \mathrm{Hilb}^d(\mathbb{P}V)_0 : \pi(Z) \text{ is reduced}\} \ \subset \ \mathrm{Sym}^d \mathbb{P}V \backslash \Delta$$

is bijective. In particular, the restriction of α *to* $\mathrm{Hilb}^d(\mathbb{P}V)_0$ *is a birational equivalence.*

(c) *No other component of* $\mathrm{Hilb}^d(\mathbb{P}V)$ *dominates* $\mathrm{Quot}^{0,d}(V^*)$*.*

Proof. (a) It is straightforward to globalise the construction $V_Z^* = \pi_* \left(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1)\right)$ and obtain a family of elementary transformations of V^* of degree $-\deg V - d$, parametrised by the locus

$$\{Z \in \mathrm{Hilb}^d(\mathbb{P}V) : \deg(V_Z) = \deg V + d\}.$$

As the rank and degree are constant and X has dimension 1, this family is flat. The existence of α then follows from the universal property of Quot schemes.

To conclude, by Theorem 2.2 it will suffice to show that if $Z = Z_1 \cup \cdots \cup Z_s$ is a nonreduced scheme of length $d \geq 2$ arising from the construction in the proof of Theorem 2.2, then Z is smoothable. Clearly Z is smoothable if and only if each Z_j is smoothable. Thus it will suffice to prove the smoothability of a Z supported at a single point $\nu_1 \in \mathbb{P}V|_x$ with length $k \geq 2$. In this case, with the setup of Theorem 2.2, we have

$$\mathcal{I}_Z \ = \ \left(\pi^* z^k, \frac{\pi^* f_2}{\pi^* f_1}, \dots, \frac{\pi^* f_r}{\pi^* f_1} \right)$$

where $f_1, \dots, f_r$ is a frame for V^* on an open subset $U \subseteq X$, and $\mathcal{O}_Z$ is generated by the images of $1, \pi^* z, \dots, \pi^* z^{k-1}$.

Now since $\pi_* \mathcal{O}_Z$ is supported at x and generated by $1, z, \dots, z^{k-1}$, clearly the map $\mathcal{O}_X \to \pi_* \mathcal{O}_Z$ is surjective. Therefore, $\pi \colon \mathbb{P}V \to X$ restricts to a closed embedding $Z \hookrightarrow X$. Thus Z is curvilinear and hence smoothable.

Part (b) follows from Theorem 2.2 (b). For the rest; clearly V_Z/V has reduced support only if $Z \in \mathrm{Hilb}^d(\mathbb{P}V)_0$. As quotients with reduced support are dense in $\mathrm{Quot}^{0,d}(V^*)$, we obtain (c). □

In the next section, we will describe the indeterminacy locus of α in more detail.

3. Defective secants

Here we recall some facts about defective secants, which will be used in several contexts. Let Y be a variety equipped with a line bundle $\mathcal{L}$ and a map $\psi \colon Y \dashrightarrow |\mathcal{L}|^*$ (not necessarily base point free). If $Z \subseteq Y$ is a subscheme, then $\mathrm{Span}\left(\psi(Z)\right)$ is the projective linear subspace

$$\mathbb{P}\mathrm{Ker}\left(H^0(Y, \mathcal{L})^* \to H^0\left(Y, \mathcal{I}_Z \otimes \mathcal{L}\right)^*\right). \tag{3.1}$$

When the map ψ is clear from the context, we will simply write $\mathrm{Span}(Z)$. If Z is of dimension zero and reduced, then $\mathrm{Span}(Z)$ is the secant spanned by the images of the points of Z. In general, $\mathrm{Span}(Z)$ is a subspace of the span of the union of certain osculating spaces to $\psi(Y)$ at points of $\mathrm{Supp}(Z)$.

Definition 3.1. Suppose $Z \subset Y$ has dimension zero. We recall that the *defect* of $\psi(Z)$ is defined by

$$\mathrm{def}\,(\psi(Z)) \; := \; \mathrm{length}\, Z - 1 - \dim(\mathrm{Span}(\psi(Z))).$$

Again, if the context is clear, we will simply write $\mathrm{def}(Z)$. We say that Z is *nondefective* if $\mathrm{def}(Z) = 0$, and *defective* otherwise. If $\mathrm{Span}(Z)$ is empty, we define $\dim(\mathrm{Span}(Z)) = -1$.

Remark 3.2. The map ψ is base point free if and only if all $y \in Y$ are nondefective, and an embedding if and only if all $Z \in \mathrm{Hilb}^2(Y)$ are nondefective.

Remark 3.3. Recall that the nth secant variety $\mathrm{Sec}^k(Y)$ of a nondegenerate variety $Y \subset \mathbb{P}^N$ is the Zariski closure of the union of the linear spans of all subsets of k points of Y. In general, Y is said to be "secant defective" if some $\mathrm{Sec}^k Y$ has less than the expected dimension. The above definition of a defective scheme of dimension zero is a special case of this.

Note that if Z has dimension zero, then nondefectivity of Z is equivalent to the surjectivity of the restriction map in (3.1). We will use this observation to study the indeterminacy locus of the map $\alpha \colon \mathrm{Hilb}^d(\mathbb{P}V) \dashrightarrow \mathrm{Quot}^{0,d}(V^*)$ defined in the previous section.

3.1. **Relatively nondefective subschemes.** We return to the situation of Theorem 2.7. The map $\alpha \colon \mathrm{Hilb}^d(\mathbb{P}V) \dashrightarrow \mathrm{Quot}^{0,d}(V^*)$ is defined at Z if and only if $\deg V_Z = \deg V + d$. This is clearly the case for generic $Z \in \mathrm{Hilb}^d(\mathbb{P}V)_0$, in particular if $\mathrm{Supp}(\pi(Z))$ consists of d distinct points of X. However, it is not true if for example Z is a union of $r+1$ points $\nu_1, \ldots, \nu_{r+1}$ in general position in a fibre $\mathbb{P}V|_x$. Here $V_Z = V \otimes \mathcal{O}_X(x)$ has degree $\deg V + r < \deg V + \mathrm{length}\, Z$, as the evaluation map

$$\pi_*(\mathcal{O}_{\mathbb{P}V}(1))|_x \;=\; V^*|_x \;\to\; \pi_*(\mathcal{O}_{\mathbb{P}V}(1) \otimes \mathcal{O}_Z) \;=\; \bigoplus_{i=1}^{r+1} \nu_i^*$$

is not surjective. This motivates a definition.

Definition 3.4. Let $Z \subset \mathbb{P}V$ be a subscheme of dimension zero. Recall that we have defined $\tau_Z := V_Z/V$. The subscheme Z will be called *π-nondefective* if the following equivalent conditions obtain:

- the map $V^* \to \pi_*(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1))$ in (2.1) is surjective;
- the map $V^*/V_Z^* \to \pi_*(\mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1))$ is an isomorphism;
- $\deg V_Z = \deg V + \mathrm{length}\,(Z)$; equivalently $\deg(\tau_Z) = \mathrm{length}\,(Z)$.

Otherwise, Z will be called *π-defective*.

Geometrically speaking, Z is π-defective if and only if for some $x \in X$ the image of $Z \cap \mathbb{P}V|_x \to \mathbb{P}^{r-1}$ defined by the restriction of $\mathcal{O}_{\mathbb{P}V}(1)$ is defective. This means that Z is secant defective in $|\mathcal{O}_{\mathbb{P}V}(1) \otimes \pi^* L|^*$ for *any* $L \in \mathrm{Pic}(X)$.

Remark 3.5. (a) The scheme Z constructed in the proof of Theorem 2.2 is clearly π-nondefective.

(b) As $R^1\pi_*\mathcal{O}_{\mathbb{P}V}(1) = 0$, a subscheme Z is π-nondefective if and only if

$$R^1\pi_*(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1)) \ = \ 0.$$

(c) By definition, the locus of π-*defective* subschemes in $\mathrm{Hilb}^d(\mathbb{P}V)$ is exactly the indeterminacy locus of $\alpha\colon \mathrm{Hilb}^d(\mathbb{P}V) \dashrightarrow \mathrm{Quot}^{0,d}(V^*)$.

(d) By either (b) or (c) and the universal property of the Hilbert scheme, it follows that π-nondefectivity is an open property in (flat) families of zero-dimensional subschemes of length d of $\mathbb{P}V$.

4. Geometric Riemann–Roch for scrolls

Let $V \to X$ be any vector bundle with $h^1(X,V) \geq 1$, and $\pi\colon \mathbb{P}V \to X$ the corresponding scroll. We begin by describing a map $\mathbb{P}V \dashrightarrow \mathbb{P}H^1(X,V)$. This is a slight generalisation of a construction in [HR04, § 3], also used in various guises in [CH10], [Hit13], [Bri17] and elsewhere.

By Serre duality and the projection formula, there are identifications

$$H^1(X,V) \ \cong \ H^0(X, K_X \otimes V^*)^* \ = \ H^0(\mathbb{P}V, \pi^*K_X \otimes \mathcal{O}_{\mathbb{P}V}(1))^* \tag{4.1}$$

By standard algebraic geometry, we obtain a map $\psi\colon \mathbb{P}V \dashrightarrow \mathbb{P}H^1(X,V)$. By the proof of [HR04, Theorem 3.1], generalising a standard fact on line bundles, we have:

Proposition 4.1. *The map ψ is an embedding if and only if for all effective degree two divisors $x+y$ on X we have*

$$h^0(X, K_X(-x-y) \otimes V^*) \ = \ h^0(X, K_X \otimes V^*) - 2r;$$

equivalently, if $h^0(X, V(x+y)) = h^0(X,V)$ for all $x+y$.

The following key result describes a link between the geometry of $\psi(\mathbb{P}V)$ and the elementary transformations V_Z.

Proposition 4.2. *Let Z be a subscheme of $\mathbb{P}V$ (not necessarily π-nondefective). Then* $\mathrm{Span}(Z)$ *is the projectivisation of the image of the coboundary map ∂_Z of the sequence*

$$0 \to H^0(X,V) \to H^0(X,V_Z) \to H^0(X,\tau_Z) \xrightarrow{\partial_Z} H^1(X,V) \to H^1(X,V_Z) \to 0. \tag{4.2}$$

Proof. By Serre duality, the map $H^1(X,V) \to H^1(X,V_Z)$ is identified with

$$H^0(X, K_X \otimes V^*)^* \ \to \ H^0(X, K_X \otimes V_Z^*)^*.$$

By the projection formula and since $V_Z^* = \pi_*(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1))$, this becomes in turn

$$H^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(1) \otimes \pi^*K_X)^* \ \to \ H^0(\mathbb{P}V, \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}V}(1) \otimes \pi^*K_X)^*.$$

The statement now follows from (3.1). □

We give next a generalisation of the geometric Riemann–Roch theorem.

Theorem 4.3 (Geometric Riemann–Roch for scrolls)**.** *Let* $0 \to V \to \tilde{V} \to \tau \to 0$ *be an elementary transformation. Then for any* π*-nondefective* Z *such that* $V_Z \cong \tilde{V}$ *as vector bundles, we have* $h^0(X, \tilde{V}) - h^0(X, V) = \mathrm{def}(Z)$*.*

Note that by Theorem 2.2, such a Z always exists.

Proof. Let Z be a zero-dimensional π-nondefective subscheme of $\mathbb{P}V$ such that the vector bundles $\tilde{V}$ and V_Z are isomorphic. We have:

$$\begin{aligned} \dim(\mathrm{Span}(Z)) &= \dim(\mathrm{Im}\,(\partial_Z)) - 1 \text{ by Proposition 4.2} \\ &= h^0(X, \tau_Z) - (h^0(X, \tilde{V}) - h^0(X, V)) - 1 \text{ by exactness and since } V_Z \cong \tilde{V} \\ &= \mathrm{length}\,(Z) - (h^0(X, \tilde{V}) - h^0(X, V)) - 1 \text{ by } \pi\text{-nondefectivity.} \end{aligned}$$

Therefore, $(\mathrm{length}\,(Z) - 1) - \dim(\mathrm{Span}(Z)) = h^0(X, \tilde{V}) - h^0(X, V)$. As the expression on the left is exactly $\mathrm{def}(Z)$, the statement follows. □

Note that Z is generally not unique; for example, distinct linearly equivalent effective divisors define isomorphic line bundles. If V is a line bundle, the condition that $V_Z^* = \tilde{V}^*$ in $\mathrm{Quot}^{0,d}(V^*)$ uniquely determines Z. This is no longer true for $r \geq 2$ (Remark 2.6). We can however give a necessary geometric condition for the equality $V_{Z'}^* = V_Z^*$ in $\mathrm{Quot}^{0,d}(V^*)$.

Proposition 4.4. *Suppose* Z *and* Z' *are such that* V_Z^* *and* $V_{Z'}^*$ *define the same point of* $\mathrm{Quot}^{0,d}(V^*)$*. Then* $\mathrm{Span}(Z) = \mathrm{Span}(Z')$*.*

Proof. By hypothesis and by definition of the Quot scheme, $V_Z^* = V_{Z'}^*$ as subsheaves of V^*. A diagram chasing argument shows that

$$\mathrm{Ker}\left(H^1(X, V) \to H^1(X, V_Z)\right) = \mathrm{Ker}\left(H^1(X, V) \to H^1(X, V_{Z'})\right). \tag{4.3}$$

But by Proposition 4.2 and exactness, for any zero-dimensional $Z \subset \mathbb{P}V$ we have

$$\mathrm{Span}(Z) = \mathbb{P}\mathrm{Im}\left(\Gamma(\tau_Z) \to H^1(X, V)\right) = \mathbb{P}\mathrm{Ker}\left(H^1(X, V) \to H^1(X, V_Z)\right).$$

Putting this together with (4.3), we see that $\mathrm{Span}(Z) = \mathrm{Span}(Z')$. □

Remark 4.5.
(a) Proposition 4.4 holds even if Z and Z' are not π-nondefective, or if they have different lengths. For example, suppose V has rank 2 and Z is any finite reduced subscheme of length $d \geq 2$ of a fibre $\mathbb{P}V|_x$. Then $\mathrm{Span}(Z)$ is the fibre $\mathbb{P}V|_x$ and V_Z is isomorphic to $V \otimes \mathcal{O}_X(x)$, independently of $\mathrm{length}\,(Z)$.
(b) The converse of the proposition does not hold. For example, if $h^1(X, V) = 1$ then ψ is constant.

Remark 4.6. Suppose $V = \mathcal{O}_X$, so $\pi\colon \mathbb{P}V \to X$ is the identity map, and

$$\psi\colon \mathbb{P}\mathcal{O}_X = X \to \mathbb{P}H^1(X, \mathcal{O}_X) = |K_X|^*$$

is the canonical map. Then $\mathrm{Hilb}^d X = \mathrm{Sym}^d X$ parametrises effective divisors of degree d on X. Trivially, all such Z are π-nondefective, and we obtain the usual geometric Riemann–Roch theorem.

Before making the next remark, and in view of the situation to be studied in § 5, for completeness we recall the definition of a stable vector bundle.

Definition 4.7. A vector bundle $V \to X$ is said to be *stable* if for each proper subbundle $W \subset V$ we have $\frac{\deg(W)}{\mathrm{rk}\,(W)} < \frac{\deg(V)}{\mathrm{rk}\,(V)}$.

As is well known, stability is an open condition on families of vector bundles of fixed rank and degree.

Remark 4.8. (A generalised Abel–Jacobi map) For $d \geq 1$, let $\alpha_d \colon \mathrm{Sym}^d X \to \mathrm{Pic}^d(X)$ be the Abel–Jacobi map $D \mapsto \mathcal{O}_X(D)$. As was classically known, $\alpha_d^{-1}(L)$ is exactly the linear series $|L|$. More generally; fix a bundle V of rank r and degree $e - d$. Sending a length d subscheme $Z \subset \mathbb{P}V$ to the moduli point of V_Z in $U_X(r, e)$ defines a rational map

$$\alpha_{V,d} \colon \mathrm{Hilb}^d(\mathbb{P}V) \dashrightarrow U_X(r, e),$$

generalising the Abel–Jacobi map. In particular, if $V = \mathcal{O}_X{}^{\oplus r}$ then, as in the rank one case, the image of $\alpha_{V,d}$ is exactly the locus of stable, generically generated bundles. Moreover, by Theorem 4.3, the preimage of

$$\{E \in U_X(r, d) : E \text{ is generically generated and } h^0(X, E) \geq r + 1\} \subseteq B^{r+1}_{r,d}$$

is exactly the locus

$$\left\{ Z \in \mathrm{Hilb}^d(\mathcal{O}_X{}^{\oplus r}) : \psi(Z) \text{ is defective in } \mathbb{P}H^1(\mathcal{O}_X{}^{\oplus r}) \text{ and } \left(\mathcal{O}_X{}^{\oplus r}\right)_Z \text{ is stable} \right\}.$$

For $r \geq 2$, the situation is complicated by the requirement of stability (but see [BBN15]) and the presence of nontrivial automorphisms of $X \times \mathbb{P}^r$. Nonetheless, viewing sums of points on X as *zero-dimensional subschemes* (as opposed to *codimension one subschemes*), Theorems 2.7 (a) and 4.3 give a natural generalisation of the picture for line bundles and linear series on X to bundles of higher rank.

A relative version. For applications to Brill–Noether loci, we will need a more general version of Theorem 4.3. Let V and F be vector bundles over X. For any $Z \in \mathrm{Hilb}^d(\mathbb{P}V)$, we have an exact sequence

$$0 \to V \otimes F \to V_Z \otimes F \to \tau_Z \otimes F \to 0.$$

Inside the scroll $\mathbb{P}(V \otimes F)$ we have the rank one locus $\Delta := \mathbb{P}V \times_X \mathbb{P}F$, also called the *decomposable locus*. There is a commutative diagram

(4.4)
$$\begin{array}{ccc} \Delta & \xrightarrow{\widetilde{\omega}} & \mathbb{P}V \\ \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}F & \xrightarrow{\omega} & X. \end{array}$$

Since $\mathbb{P}F \to X$ is flat, for any closed subscheme $Z \subset \mathbb{P}V$ we have $\mathcal{I}_{Z \times_X \mathbb{P}F} = \widetilde{\omega}^* \mathcal{I}_Z$. Thus if $\mathcal{L} \to \Delta$ is a line bundle and $\psi \colon \Delta \dashrightarrow |\mathcal{L}|^*$ the associated map, in view of (3.1) we obtain

$$\mathrm{Span}(\psi(Z \times_X \mathbb{P}F)) = \mathbb{P}\mathrm{Ker}\left(H^0(\Delta, \mathcal{L})^* \to H^0(\Delta, \widetilde{\omega}^* \mathcal{I}_Z \otimes \mathcal{L})^*\right). \tag{4.5}$$

Note that Δ is just the projective bundle $\mathbb{P}(\pi^* F) \to \mathbb{P}V$. If $Z \in \mathrm{Hilb}^d(\mathbb{P}V)$ is reduced, $Z \times_X \mathbb{P}F$ is a union of d fibres of $\mathbb{P}F$. In general, the expected dimension of $\mathrm{Span}(Z \times_X \mathbb{P}F)$ is $\mathrm{rk}\,(F) \cdot \mathrm{length}\,(Z) - 1$.

Definition 4.9. For V, Z, F and ψ as above, the *defect* of $\psi(Z \times_X \mathbb{P}F)$ is

$$\mathrm{def}(Z \times_X \mathbb{P}F) \ := \ (\mathrm{rk}\,(F) \cdot \mathrm{length}\,(Z) - 1) - \dim(\mathrm{Span}(Z \times_X \mathbb{P}F)).$$

Now we can generalise the previous results of this section. We write $\mathcal{L}$ for the line bundle

$$(\widetilde{\pi}^* \omega^* K_X) \otimes (\widetilde{\pi}^* \mathcal{O}_{\mathbb{P}F}(1)) \otimes (\widetilde{\omega}^* \mathcal{O}_{\mathbb{P}V}(1)) \ \to \ \Delta.$$

Theorem 4.10.

(a) *There is a natural identification* $H^1(X, V \otimes F) \ \xrightarrow{\sim} \ H^0(\Delta, \mathcal{L})^*$. *In particular, there is a natural map* $\Delta \dashrightarrow \mathbb{P}H^1(X, V \otimes F)$, *which we again denote* ψ.

(b) *Via the above identification,* $\mathrm{Span}(Z \times_X \mathbb{P}F)$ *coincides with the projectivised image of the coboundary map* ∂_Z *in the sequence*

$$(4.6) \quad 0 \ \to \ H^0(X, V \otimes F) \ \to \ H^0(X, V_Z \otimes F) \ \to \ H^0(X, \tau_Z \otimes F) \xrightarrow{\partial_Z} H^1(X, V \otimes F) \ \to \ H^1(X, V_Z \otimes F) \ \to \ 0.$$

(c) *(Relative generalised geometric Riemann–Roch) For any* π*-nondefective* Z *such that* $V_Z \cong \tilde{V}$ *as vector bundles, we have*

$$h^0(X, \tilde{V} \otimes F) - h^0(X, V \otimes F) \ = \ \mathrm{def}\,(\psi\,(Z \times_X \mathbb{P}F))\,.$$

(d) *Suppose* Z *and* Z' *are such that* V_Z^* *and* $V_{Z'}^*$ *define the same point of* $\mathrm{Quot}^{0,d}(V^*)$. *Then* $\mathrm{Span}(Z \times_X \mathbb{P}F) = \mathrm{Span}(Z' \times_X \mathbb{P}F)$.

Proof. (a) This is a technical but straightforward computation. By Serre duality and repeated use of the projection formula, we get an identification

$$(4.7) \qquad H^1(X, V \otimes F) \ \xrightarrow{\sim} \ H^0\left(\mathbb{P}V, \pi^* \left\{\omega_* \left(\omega^* K_X \otimes \mathcal{O}_{\mathbb{P}F}(1)\right)\right\} \otimes \mathcal{O}_{\mathbb{P}V}(1)\right)^*.$$

As $\pi \colon \mathbb{P}V \to X$ is flat, by [Har77, Proposition III.9.3] there is a canonical isomorphism

$$\pi^*(\omega_* \mathcal{S}) \ \xrightarrow{\sim} \ \widetilde{\omega}_*(\widetilde{\pi}^* \mathcal{S})$$

for any coherent sheaf $\mathcal{S}$ on $\mathbb{P}F$. Setting $\mathcal{S} = \omega^* K_X \otimes \mathcal{O}_{\mathbb{P}F}(1)$ in (4.7), we obtain

$$H^0\left(\mathbb{P}V, \widetilde{\omega}_* \left\{\widetilde{\pi}^* \left(\omega^* K_X \otimes \mathcal{O}_{\mathbb{P}F}(1)\right)\right\} \otimes \mathcal{O}_{\mathbb{P}V}(1)\right)^*.$$

Using the projection formula, we see easily that this is identified with

$$H^0\left(\Delta, (\widetilde{\pi}^* \omega^* K_X) \otimes (\widetilde{\pi}^* \mathcal{O}_{\mathbb{P}F}(1)) \otimes (\widetilde{\omega}^* \mathcal{O}_{\mathbb{P}V}(1))\right)^*,$$

which is exactly $H^0(\Delta, \mathcal{L})^*$.

(b) A calculation similar to that in (a) shows that via the identification in (a), the subspace

$$H^1(X, V_Z \otimes F)^* \ \cong \ H^0(X, K_X \otimes F^* \otimes V_Z^*) \ \subset \ H^0(X, K_X \otimes F^* \otimes V^*)$$

is identified with $H^0(\Delta, \mathcal{L} \otimes \widetilde{\omega}^* \mathcal{I}_Z) \subseteq H^0(\Delta, \mathcal{L})$. Then the statement follows from (4.5).

Parts (c) and (d) can be proven exactly as Theorem 4.3 and Proposition 4.4 respectively. □

Remark 4.11. The idea behind Theorem 4.10 (b)–(c) is present in a recent work of Brivio [Bri17]. Let E be a general bundle in $U_X(r, r(2g-1))$ and $D = x_1 + \cdots + x_g$ an effective divisor of degree g. The condition of interest in [Bri17] is that $h^0(X, E(-D)) = 1$. As $\chi(X, E(-D)) = 0$, this is equivalent to $h^0(X, K_X \otimes E^*(D)) = 1$. These spaces appear in the cohomology sequence

$$\begin{aligned} 0 \to H^0(X, K_X \otimes E^*) \to H^0(X, K_X \otimes E^*(D)) \to H^0(X, K_X \otimes E^*(D)|_D) \\ \to H^0(X, E)^* \to H^0(X, E(-D))^* \to 0. \end{aligned}$$

This is exactly (4.6) with $V = K_X(-D)$ and $Z = D$, and $F = E^*$. Then by Theorem 4.10 (b), we can interpret the coboundary map as the restriction of the *tautological model* of [Bri17] to the fibres of $\mathbb{P}(K_X \otimes E^*) \cong \mathbb{P}E^*$ along D. (Note that $\mathbb{P}E^*$ is denoted by $\mathbb{P}E$ in [Bri17].) Then Theorem 4.10 (c) gives another proof that $h^0(X, E(-D)) = h^0(X, K_X \otimes E^*(D)) = 1$ if and only if the fibres $\mathbb{P}E^*|_{x_1}, \dots, \mathbb{P}E^*|_{x_g}$ span a space of dimension one less than expected; equivalently, that there exist

$$(\eta_1, \dots \eta_g) \in \mathbb{P}E^*|_{x_1} \times \cdots \times \mathbb{P}E^*|_{x_g}$$

which are linearly dependent in $\mathbb{P}H^0(X, E)^*$. (Compare with [Bri17, Lemma 5.1 and Proposition 7.2]).

5. Tangent cones of higher rank Brill–Noether loci

Suppose $L \to X$ is an effective line bundle of degree $d \leq g$. Then L defines a point of the Brill–Noether locus

$$W_d = \{L \in \mathrm{Pic}^d(X) : h^0(X, L) \geq 1\}.$$

The projectivised tangent cone $\mathbb{T}_L W_d$ at L belongs to

$$\mathbb{P}T_L \mathrm{Pic}^d(X) = \mathbb{P}H^1(X, \mathcal{O}_X) = |K_X|^*.$$

The *Riemann–Kempf singularity theorem* (see [GH94, Chapter 2]) states that

$$\mathbb{T}_L W_d = \bigcup_{L \in |D|} \mathrm{Span}\left(\phi_{K_X}(D)\right)$$

where ϕ_{K_X} is the canonical map. We will generalise this picture to bundles of higher rank, using Theorems 2.7 and 4.3.

5.1. **Higher rank Brill–Noether loci.** Here we recall briefly some essential facts, referring the reader to [GT09] and [CT11] for details. The moduli space $U_X(r,d)$ of stable bundles of rank r and degree d over X is an irreducible quasi-projective variety of dimension $r^2(g-1)+1$. The *Brill–Noether locus* $B^k_{r,d}$ is defined set-theoretically by

$$B^k_{r,d} \ = \ \{E \in U_X(r,d) : h^0(X,E) \geq k\}.$$

This is a determinantal subvariety of $U_X(r,d)$, with expected codimension $k(k-d+r(g-1))$, when this is nonnegative. Suppose $E \in U_X(r,d)$ satisfies $h^0(X,E) = k$. Then the Zariski tangent space $T_E B^k_{r,d}$ is exactly $\mathrm{Im}\,(\mu)^\perp$, where μ is the *Petri map*

$$H^0(X,E) \otimes H^0(X, K_X \otimes E^*) \ \to \ H^0(X, K_X \otimes \mathrm{End}\, E).$$

Equivalently, via Serre duality we have

$$T_E B^k_{r,d} \ = \ \mathrm{Ker}\left(\cup \colon H^1(X, \mathrm{End}\, E) \to \mathrm{Hom}\left(H^0(X,E), H^1(X,E)\right)\right).$$

Thus $B^k_{r,d}$ is smooth and of the expected dimension at E if and only if μ is injective; equivalently, if $\cup$ is surjective.

More generally, suppose $h^0(X,E) = k \geq m \geq 1$. We denote $\mathrm{Gr}(m, H^0(X,E))$ simply by Gr to ease notation. Write $\mathcal{U}$ for the universal bundle over Gr, and consider the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{Gr}} \otimes H^1(X, \mathrm{End}\, E) & \xrightarrow{\ \cup\ } & \mathcal{O}_{\mathrm{Gr}} \otimes \mathrm{Hom}\left(H^0(X,E), H^1(X,E)\right) . \\ & \searrow^{\tilde{\cup}} & \downarrow \\ & & \mathrm{Hom}\left(\mathcal{U}, \mathcal{O}_{\mathrm{Gr}} \otimes H^1(X,E)\right) \end{array}$$

Definition 5.1. (cf. [CT11, § 2]) Let E be as above. If the restricted Petri map

$$\mu_\Lambda \ := \ \mu|_{\Lambda \otimes H^0(X, K_X \otimes E^*)}$$

is injective for all subspaces $\Lambda \in \mathrm{Gr}(m, H^0(X,E))$, then E is said to be *Petri m-injective*. Equivalently, E is Petri m-injective if and only if

$$\tilde{\cup}|_\Lambda : H^1(X, \mathrm{End}\, E) \ \to \ \mathrm{Hom}\left(\Lambda, H^1(X,E)\right) \tag{5.1}$$

is surjective for all $\Lambda \in \mathrm{Gr}$.

If E is Petri m-injective, then $\mathrm{Ker}(\tilde{\cup})$ is a vector subbundle of $\mathcal{O}_{\mathrm{Gr}} \otimes H^1(X, \mathrm{End}\, E)$. By [CT11, Theorem 2.4 (4)], the projectivised tangent cone $\mathbb{T}_E B^m_{r,d}$ is given by

$$\mathbb{T}_E B^m_{r,d} \ = \ \bigcup_{\Lambda \in \mathrm{Gr}} \mathrm{Im}\,(\mu_\Lambda)^\perp \ = \ \bigcup_{\Lambda \in \mathrm{Gr}} \mathrm{Ker}(\tilde{\cup}|_\Lambda),$$

the second equality following from Serre duality as above. Thus, in summary we obtain:

Lemma 5.2. *Suppose E is a stable bundle with $h^0(X,E) = k \geq m \geq 1$ which is Petri m-injective. Then $\mathbb{T}_E B^m_{r,d}$ is the image of the scroll*

$$S \ := \ \mathbb{P}\mathrm{Ker}(\tilde{\cup}) \ \subseteq \ \mathrm{Gr} \times \mathbb{P}H^1(X, \mathrm{End}\, E) \tag{5.2}$$

by the projection to $\mathbb{P}H^1(X, \operatorname{End} E)$*. In particular, the tangent cone is closed and irreducible.*

Note that as all fibres of S contain the subspace

$$\mathbb{P}\mathrm{Ker}\left(\cup\colon H^1(X, \operatorname{End} E) \to \operatorname{Hom}(H^0(X,E), H^1(X,E))\right) \; = \; T_E B^k_{r,d},$$

the map $S \to \mathbb{P}H^1(X, \operatorname{End} E)$ is an embedding only if $\cup$ is injective; equivalently, if μ is surjective.

5.2. **Riemann–Kempf for generically generated bundles.** Throughout this section, E will be a stable bundle with $h^0(X,E) = k \geq r$ and which is generically generated and Petri r-injective. Setting $V = E^*$ and $F = E$, we consider the map $\psi\colon \mathbb{P}\mathrm{End}\, E \dashrightarrow \mathbb{P}H^1(X, \operatorname{End} E)$ as defined in § 4. Although we will not require this fact, we note that by [HR04, Theorem 3.1], this is an embedding for general E and X if $g \geq 5$.

We now take $m = r$ and write $\mathrm{Gr} := \mathrm{Gr}(r, H^0(X,E))$. Set

$$U \; := \; \{\Lambda \in \mathrm{Gr} : \mathrm{ev}_\Lambda \colon \mathcal{O}_X \otimes \Lambda \to E \text{ is generically injective}\}.$$

This is an open subset of Gr, which by hypothesis is dense. For each $\Lambda \in U$, transposing the evaluation map $\mathcal{O}_X \otimes \Lambda \to E$, we obtain

$$0 \to E^* \to \mathcal{O}_X \otimes \Lambda^* \to \tau \to 0$$

where τ is a torsion sheaf of degree d, supported along a divisor in $|\det E|$. Hence, by Theorem 2.2, there exists $Z_\Lambda \in \mathrm{Hilb}^d(\mathbb{P}E^*)$ such that the elementary transformation $\mathcal{O}_X \otimes \Lambda^*$ coincides with $(E^*)_{Z_\Lambda}$ as points of $\mathrm{Quot}^{0,d}(E)$ (cf. Remark 2.1). Moreover, there is a commutative diagram

$$\begin{array}{ccc} & H^1(X, \operatorname{End} E) & \\ \swarrow & & \searrow^{\tilde{\cup}|_\Lambda} \\ H^1(X, \Lambda^* \otimes E) & \xrightarrow[\cup]{\sim} & \operatorname{Hom}(\Lambda, H^1(X,E)) \end{array},$$

whence $\mathrm{Ker}(\tilde{\cup}|_\Lambda) = \mathrm{Ker}\left(H^1(X, \operatorname{End} E) \to H^1(X, \Lambda^* \otimes E)\right)$. Therefore, by exactness and by Theorem 4.10 (b), with

$$V \; = \; E^* \quad \text{and} \quad V_Z \; = \; (E^*)_{Z_\Lambda} \; = \; \mathcal{O}_X \otimes \Lambda^* \quad \text{and} \quad F \; = \; E,$$

we have

$$\mathbb{P}\mathrm{Ker}(\tilde{\cup}|_\Lambda) \; = \; \mathrm{Span}(Z \times_X \mathbb{P}E). \tag{5.3}$$

Note that although Z_Λ may not be unique, $\mathrm{Span}(Z \times_X \mathbb{P}E)$ is independent of the choice of Z_Λ by Theorem 4.10 (d).

We consider a useful special case. Set

$$U_1 \; := \; \{\Lambda \in U : \mathrm{Supp}(\tau_\Lambda) \text{ is reduced}\}.$$

Clearly this is an open subset of U. By an argument using Bertini's theorem, the locus U_1 is nonempty for example if E is globally generated. If $\Lambda \in U_1$, then Z_Λ is reduced and uniquely determined, by Theorem 2.2 (b). Explicitly, $Z_\Lambda = \{\nu_1, \ldots, \nu_d\}$ where $\nu_i = \mathrm{Ker}(E^*|_{x_i} \to \mathbb{K}^r)$ for distinct points $x_1, \ldots, x_d \in X$. Equivalently, $\nu_i \in E^*|_{x_i}$ defines the hyperplane Im $(\Lambda \to E|_{x_i})$. In this case,

$$Z_\Lambda \times_X \mathbb{P}E \;=\; \bigcup_{i=1}^{d} \mathbb{P}(\nu_i \otimes E|_{x_i}) \tag{5.4}$$

Theorem 5.3 (Generalised Riemann–Kempf singularity)**.** *Suppose E is a stable bundle of degree $d < rg$ with $h^0(X,E) = k \geq r$ which is Petri r-injective and generically generated.*

(a) *For any $\Lambda \in U$, we have* $\mathrm{def}(Z_\Lambda \times_X \mathbb{P}E) \;=\; kr - 1$.
(b) *The tangent cone to $B^r_{r,d}$ at E is the Zariski closure of*

$$\bigcup_{\Lambda \in U} \mathrm{Span}\,(Z_\Lambda \times_X \mathbb{P}E)\,. \tag{5.5}$$

(c) *Suppose U_1 is nonempty (for example, if E is globally generated). Then the tangent cone to $B^r_{r,d}$ at E is the Zariski closure of*

$$\bigcup_{\Lambda \in U_1} \mathrm{Span}\left(\bigcup_{\nu \in Z_\Lambda} \mathbb{P}\left(\nu \otimes E|_{\pi(\nu)}\right) \right). \tag{5.6}$$

Note that the hypothesis $d < rg$ ensures that $B^r_{r,d}$ is a proper sublocus of $U_X(r,d)$.

Proof. (a) By Theorem 4.10 (c), we have

$$\mathrm{def}\,(Z_\Lambda \times_X \mathbb{P}E) \;=\; h^0(X, \Lambda^* \otimes E) - h^0(X, E^* \otimes E) \;= kr - 1.$$

(b) By (5.3), we see that the locus (5.5) is the image of the dense open subset $S|_U$, where S is the scroll defined in (5.2). By Petri r-injectivity and Lemma 5.2, the tangent cone is closed and irreducible. Hence by a topological argument it is exactly the closure of (5.5).

(c) In view of (5.4), the locus (5.6) is the image of the dense open subset $S|_{U_1}$. As in part (b), the closure of (5.6) is the tangent cone. □

Remark 5.4. The hypothesis of Petri r-injectivity is only required in the proofs of (b) and (c). In general, the loci (5.5) and (5.6) are contained in $\mathbb{T}_E B^r_{r,d}$.

5.3. **Secant varieties.** In [CT11, § 5] it is shown that the tangent cones to certain generalised theta divisors contain secant varieties of the curve X. Here we deduce a similar statement for $\mathbb{T}_E B^k_{r,d}$ using Theorem 5.3.

Theorem 5.5. *Suppose E is a generically generated stable bundle of degree $d < rg$ with $h^0(X,E) = r + n$ for some $n \geq 1$. Then $\mathbb{T}_E B^r_{r,d}$ contains* $\mathrm{Sec}^n \Delta$.

Proof. Let $\nu_1 \otimes e_1, \ldots, \nu_n \otimes e_n$ be a collection of n points of $\Delta \subset \mathbb{P}(E^* \otimes E)$ lying over distinct $x_1, \ldots, x_n \in X$ respectively. Since E is generically generated, the condition that a section s satisfy $s(x_i) \in \mathrm{Ker}(\nu_i)$ for $1 \leq i \leq n$ determines a linear subspace of codimension at most

n in $H^0(X, E)$. Thus there exists $\Lambda \in \mathrm{Gr}$ such that Z_Λ contains the points $\nu_1, \ldots, \nu_n$. (If the ν_i are general enough, Λ is unique.) By Theorem 5.3 (b), the tangent cone contains

$$\mathrm{Span}\,(Z \times_X \mathbb{P}E) \;=\; \mathrm{Span}\left(\bigcup_{i=1}^{n} \mathbb{P}(\nu_i \otimes E|_{x_i}) \right).$$

hence in particular the secant spanned by $\nu_1 \otimes e_1, \ldots, \nu_n \otimes e_n$. Since the ν_i were chosen generally, $\mathbb{T}_E B^r_{r,d}$ contains a dense subset of $\mathrm{Sec}^n(\Delta)$, and hence all of $\mathrm{Sec}^n(\Delta)$ since $\mathbb{T}_E B^r_{r,d}$ is closed. □

5.4. **Existence of good singular points.** It is nontrivial to establish that there exist bundles E satisfying the hypotheses of Theorem 5.3. For $k = r + 1$, sufficient conditions on d and g are given in [BBN08, Proposition 6.6].

If we relax the condition of Petri r-injectivity and only ask that E be stable and generically generated, then Theorem 5.5 is still valid. Such E can be constructed using a method in [Mer99], which we recall for the reader's convenience. Let $L_1, \ldots, L_r$ be mutually nonisomorphic line bundles of degree $e < d/r$. Consider elementary transformations

$$0 \;\to\; L_1 \oplus \cdots \oplus L_r \;\to\; E \;\to\; \tau \;\to\; 0 \tag{5.7}$$

where τ is a torsion sheaf of degree $f \geq 1$. By [Mer99, Théorème A-5], a general such E is stable for any choice of $L_1, \ldots, L_r$.

We require that E also be generically generated and satisfy $h^0(X, E) = r + n$ for some $n \geq 1$. As above, we assume $d < rg$. Taking L_i to be effective of degree $e < d/r$ and τ of degree $f = d - re$, we obtain a generically generated stable E of degree d. If $h^0(X, L_i) \geq 2$ for at least one i, then E is a singular point of $B^r_{r,d}$ and satisfies the hypotheses of Theorem 5.5. Note that in this case $d \geq r \cdot \mathrm{gon}(X) + 1$.

Remark 5.6. In fact, it is possible to show the following. Let X be a general curve of genus g. Suppose the following are satisfied:

(i) $1 \leq e \leq g - 1$
(ii) $g - m(m - e + g - 1) \geq 1$
(iii) $d := re + f < rg$

Then for $r \leq k \leq rm$, there exist stable, generically generated E of rank r and degree d with $h^0(X, E) = k$ and $\mu \colon H^0(X, E) \otimes H^0(X, K_X \otimes E^*) \to H^0(X, K_X \otimes \mathrm{End}\, E)$ injective.

Idea of proof. In view of (ii), by the Brill–Noether theory of line bundles, we may choose mutually nonisomorphic $L_1, \ldots, L_r$ of degree e with $h^0(X, L_i) = m$. Using (i) and (iii), the construction (5.7) yields a stable, generically generated E of degree d with $h^0(X, E) = rm$. Then an argument similar to that in [Hir87, Théorème 1.2], using the generality of X, shows that a general such E is in fact Petri injective. The theory of χ-like stratifications [Hir88] then shows that E can be deformed to a stable, generically generated, Petri injective E' with $h^0(X, E) = k$ for $r \leq k \leq rm - 1$. □

As a more general statement will be proven in a forthcoming paper, we omit the details.

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