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Optimal Portfolio Allocation Under Volatility-Return Regime Switching

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Preface

This study is the final part of the master's degree program in Business & Administration with specialization in Financial Economics at OsloMet. We chose this research topic due to a common interest in finance and personal economic investments. First, we would like to thank our supervisor Einar Belsom. Through inspiring conversations, he has given us good guidance, and useful feedback through the whole semester. The work process has been demanding, especially since we had no experience with MATLAB before this semester. However, we have learned a lot, and the process has been both fun and interesting. We would also like to thank Morten Øvergaard for excellent feedback and help in programming our model in MATLAB.

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Abstract

We study the optimal portfolio allocation when returns, covariances and volatility switch between normal and turbulent regimes. We isolated all the high-volatility events in recent historic return series and produced covariance-matrices corresponding to normal and bear-market regimes. Using these covariance-matrices, along with analysis of the probability of occurrences from these events, led us to produce Monte Carlo simulations with time series of returns used to optimize allocations using investors CRRA-levels, maximizing their utility. Our findings support research showing substantially different allocation when the bear market regimes are taken into account, with heavier weights in lower risk assets, compared to the full-sample covariance-matrix simulation producing much heavier weights in high-risk assets. Our results show that as returns are decreasing with the more conservative allocation, the risk is reduced substantially. Our results also indicate heavy support for the theory of time diversification.

Introduction

If investors could predict the future, they would simply invest in the single highest earning security, providing them with the highest possible return. There would be no need, nor effect in diversification as this would limit future earnings (Markowitz, 1991). We cannot – with certainty – predict the future, so we use diversification strategies to limit overall risk, spreading it over multiple assets and asset classes, lowering the overall returns, limiting risk, to construct efficient portfolios.

When analyzing historical data, investors often assume that returns are generated by a linear process with stable coefficients providing fixed predictive measures. Studies (e.g. Chow, Jacquier, Kritzman, and Lowry (1999), Kritzman and Li (2010) and Ang and Bekaert (2002, 2004)) observing historical data, found several periods where returns and volatility behaved irregularly, implying that the market experienced different regimes, or states of return.

These periods with irregular behavior has been described by Kritzman and Li (2010) as turbulence: *“A condition in which asset prices, given their historical patterns of behavior, behave in an uncharacteristic fashion, including extreme price moves, decoupling of correlated assets, and convergence of uncorrelated assets”* (p. 30). In these periods, often referred to as “Bear market-periods”, the individual securities will experience higher internal correlations and returns will be lower and much more volatile than in normal regime-markets (Ang & Bekaert, 2004). Addressing these observations, Chow et al. (1999) developed a method for extracting outliers, producing separate covariance matrices, corresponding to each regime. The outliers’ sample covariance matrix was produced to obtain more accurate descriptions of the turbulent markets, and their impact on the overall returns.

Through analysis of the historic return from the stock market, Bloom (2009) and Kritzman and Li (2010), argued that most turbulent periods are caused by events, both economic and non-economic, affecting the entire stock market. The turbulent periods do not occur in regularity. This provides no obvious method for us to predict when the next turbulent period will occur.

Previous research states in little regard the specific effect the regimes will have on optimal allocation, nor, how one should deal with these states in terms of simulations. We will study the impact of volatility-return regime switching on optimal portfolio allocation, using the implications and theories from Kritzman and Li (2010), Chow et al. (1999) and Bloom (2009). We will analyze historic time-series of return from two different risky asset-classes, dividing the return into two separate return-series, used to generate separate covariance-matrices, building on the work of Chow et al. (1999). Our method will differ from these previous studies,

as we provide a more practical take on the extraction of outliers', limited to one asset class, labeling all shocks as events.

Using these covariance-matrices, we will be able to simulate new time-series of return, for each regime. Kritzman and Li (2010) used the Markov switching model to reallocate dynamically across event-sensitive portfolios. We will use the Markov Switching model as a tool to simulate the occurrence of shock, providing a new approach to simulating returns, used for the optimal allocation. We will treat the switching between regimes as Markov processes, assuming them to randomly appear based on a probability of occurrences, creating a regime-switching path. Linking the returns from each regime through the probability of occurrences chain, we can create events, simulated based on the impact of those found in the historical data. In total providing us with a simulated return-series which more accurately represent the overall risk coming from these high-volatility regimes.

Using the randomly generated time-series of return, from the regime-switching model, we will run a simulation in order to estimate optimal allocations between risky assets, under the impact of high-volatility regime switching. We then analyze optimal portfolio allocations for different levels of risk aversion and different investment-horizons. Our portfolio will present optimal allocation of assets, adjusted for the impact of high-volatility shocks. Through a static approach, we will be able to see the effects of shocks on overall returns, optimizing for an investor who does not want to trade actively.

The plan of the paper is as follows. In section 1, we will introduce some general background theory of asset allocation, market turbulence and utility maximization based on CRRA-preferences. Next, we will introduce our methodology used as a basis for our test including Monte Carlo simulations and Markov Regime Switching Models. In section 2, we will present the data needed in order to conduct these simulations. We need to categorize and produce data series of returns from two assets classes in different regimes (Normal and Bear), through observation of turbulence in the observation period. The simulations will give us the input foundation for our optimization problem, found in section 3. Here we will optimize using average levels of CRRA to try and conclude with what is optimal portfolio allocation under volatility-return regime switching. Finally, we will conclude on the effects of regime switching on optimal allocations, leaving questions for further research.

Portfolio Allocation and Market Turbulence

We will never fully be able to understand the grand effects of the economic forces, making it impossible to predict the future without doubt or error (Markowitz, 1991). This is why risk is such a salient feature of security investment. Risk is categorized as systematic and unsystematic, so, when diversifying, an investor opts to reduce as much of the unsystematic risk as possible.

Diversification

A fundamental principle in finance is the idea of getting paid to take on risk. The higher risk, thus, higher returns should follow. When tracking these returns, we observe the correlation among securities. When securities do not correlate perfectly, we can reduce risk by investing in multiple securities, reducing the overall risk – by diversification –, limiting overall returns.

A well-diversified portfolio will – in theory – eliminate all the unsystematic risk, leaving the systematic risk, which, even if we could understand the consequences of all the economic conditions, is undiversifiable, as the non-economic influences could entirely change the course of the general prosperity (Markowitz, 1959). The non-economic influences will vary, and can be identified as independent events¹, affecting the capital gains and dividends linked to – virtually – all securities.

Statman (1987) argued that even a small increase in the number of securities invested in will reduce the overall risk substantially, stating that a well-diversified portfolio must include 30-40 different securities to eliminate all unsystematic risk. This builds on the conclusion of Evans and Archer (1968) and Wagner and Lau (1971) who indicated that the power of diversification is exhausted when the number of securities surpasses 10-15. However, Statman's observations on individual stock portfolios proved that their portfolios were not well-diversified.

A survey done by King and Leape (1998), including more than 6.000 U.S. households, indicated that the average U.S. household owned a surprisingly small amount of assets and liabilities providing a lack of diversification. From their studies, they found that the diversification effect occurs even when the portfolio includes assets other than stocks.

Diversified portfolios are observed and measured in terms of efficiency, where an efficient portfolio will be the best allocation alternative given all possible investments. Based on the work of Harry Markowitz, Treynor (1961, 1962), Sharpe (1964), Lintner (1965) and Mossin (1966) introduced the Capital Asset Pricing Model (CAPM). The CAPM – building on

¹ Examples of events can be natural catastrophes, terror, an assassination of presidents, etc. All, which will provide impact to all industries represented in the stock market.

unrealistic assumptions – is one of the most central models for pricing risk². It assumes that all investors are rational and seek to maximize utility through each investment. This assumption gives investors – who are equally rational – the same weights allocated to all risky assets, making this allocation of assets the most rational and efficient portfolio; the market portfolio (Markowitz, 1959). CAPM assumes that all data can be summed up as a linear process with stable coefficients providing fixed smoothed measures, and the investor will be able to borrow at the risk-free rate. This is not so. When using risk measurements as an outlook on returns, the investor cannot assume measures to be fixed and opts to optimize based on other preferences.

Utility

In the analysis of decisions under risk, an important breakthrough was found by the Swiss mathematician, Bernoulli in 1738 (1954). He stated that two people facing the same lottery might value it differently due to their preferences and psychology. The differences are represented by utility, a subject measure of satisfaction for a given individual's preferences (Eeckhoudt, Gollier, & Schlesinger, 2005). Bernoulli argued that individuals do not appreciate objects based on value, they appreciate them based on utility. Individuals would be able to experience different amounts of utility from the same object used in the analysis of uncertainty, called the risk premium. The risk premium can be explained as the amount of wealth an individual is willing to give up in order to avoid a zero-mean gamble³ (Eeckhoudt et al., 2005). For any risk-averse investor, the risk premium (π) is the value that satisfies:

$$E[u(w + Z)] = u(w + E[Z] - \pi(w, Z)), \quad [1]$$

Where $E[Z] = 0$, $U =$ utility, $Z =$ a risky payoff related to the gamble, $W =$ initial wealth, $\pi =$ risk premium. By accepting the risk or by paying the risk premium, the investors will end up with equal utility.

In more recent history, an axiomatic characterization of Bernoulli's Expected Utility Theory (EUT) was presented by Von-Neumann and Morgenstern (1947). Based on Bernoulli's work they developed several axioms that needed to hold if an individual's preferences are to be represented by expected utility. These axioms assume that the investors are rational individuals that seek to maximize their utility. The Von-Neumann-Morgenstern theorem states that an

² CAPM assumes no taxes or transaction costs, that asymmetric information does not exist, and that investors can borrow and lend money at a fixed risk-free rate.

³ A zero-mean gamble is a state where it is neither rational nor irrational to take a gamble, giving the risk-less alternative as optimal as the gamble.

individual faces a set of lotteries, and a binary preference relation will represent the preferences an individual is faced with in respect to different lotteries.

The Allais Paradox (Allais, 1979), presented criticism against Von-Neumann & Morgenstern's axiomatic utility theory. Gul (1991) presented the paradox as two problems criticizing the third axiom⁴. The axioms were tested by MacCrimmon (1968), who presented his test subjects with the possibility to reconsider their choices, which were in violation with certain axioms. His findings helped prove that the violation of the axioms were of systematic nature, and not response errors.

Kahneman and Tversky (1979) attempted to take the inconsistencies from the EUT into account, producing the prospect theory as an alternative to the EUT. The prospect theory has been referred to as a breakthrough in the study of behavioral economics. The core idea is that individuals make assessments based on what they may gain or lose as the result of making a choice. Instead of utility depending on the final outcome, the prospect theory depends on gains and losses with respect to a reference point or a certain goal. Their research states that an individual has an S-shaped and asymmetrical utility function for initial wealth, where the function is steeper for losses than gains, suggesting that individuals are loss averse and not only risk averse. When individuals make decisions under risk, Kahneman and Tversky presented the isolation- and reflection effect, affecting the decision at hand. The isolation effect states that people often disregard components that the alternative share, and the reflection effect, which refers to people often having opposite preferences for gambling, differing in the sign of the outcome, both implying that the decision maker might not be completely rational.

Quiggin (1982) found that the theory violated the stochastic dominance, as well as admitting to intransitivity for pairwise choice. This led Kahneman and Tversky to extend their work employing cumulative rather than separable decision weights and extending their theory in several aspects (Kahneman & Tversky, 1992). Their research – applying to the concept of behavioral economics – shows that people are biased when making decisions. Arguing that people who try to understand or explain events tend to employ frames. The frames may introduce inconsistencies and generate anomalous behavior⁵.

⁴ Mathematically, this assumption states that the upper and lower contour sets of a preference relation over lotteries are closed. Along with the other axioms, continuity is needed to ensure that for any gamble in G , there exists some probability such that the decision-maker is indifferent between the “best” and the “worst” outcome.

⁵ Studies of the financial markets (Barberis, Shleifer, & Vishny, 1998; De Bondt & Thaler, 1985) found that investors tended to overreact to past market performance as they increased their risky allocation after markets had gone up and decreasing risky assets after markets had gone down, resulting in substantially reduced overall portfolio performance.

On the basis of EUT, Arrow and Pratt (1965; 1964), assuming that investors seek to optimize their utility, conducted a measure for Absolute Risk Aversion (ARA). The ARA is a measure of the rate at which marginal utility decreases when wealth is increased by one monetary unit. The ARA is the basis of Relative Risk Aversion (RRA)⁶, which see changes when wealth is increased by one percent. When keeping the relative wealth constant, we can measure an individual's level of Constant Relative Risk Aversion (CRRA).

When individuals' behavior is observed, Riley Jr. and Chow (1992) found that there is a difference in how individuals say they want to invest and how they actually invest. They concluded that the result on actual investments showed that investors' risk aversion increased, decreased and remained the same as the initial wealth increased. These same indications were made through the studies of Chiappori and Paiella (2011), who found no change to optimal allocation through changes in wealth, and Sahm (2012) who found no changes in risk aversion relative to changes in wealth. This provides indications that the anomalous behavior will not be relevant for utility maximization, as wealth will not affect the decision maker. The CRRA is presented through the power utility function (Eeckhoudt et al., 2005) – one of the most widely used utility functions – given by:

$$U(x) = \frac{x^\gamma}{\gamma}, \quad \text{where } \gamma = 1 - \lambda,$$

$$U(x) = \frac{x^{1-\lambda}}{1-\lambda}, \quad [2.0]$$

Where $\lambda > 0$, but not $\lambda = 1$, in which case the function takes the form of $U(x) = \ln x$. The (constant) Arrow-Pratt coefficient of relative risk aversion (RRA), if $\gamma < 0$, is equal to

$$RRA(x) = -\frac{U''(x)}{U'(x)}x = 1 - \gamma, \quad [2.1]$$

From this expected final wealth is given as (see e.g. Vigna (2009) for calculations):

$$E(X * (T)) = e^{AT} \left(x_0 + \frac{c}{r} * (1 - e^{-rT}) \right) = \bar{x}_0 e^{\frac{\beta^2 T}{1-\gamma}}, \quad [2.2]$$

⁶ When ARA is multiplied with the wealth, we obtain RRA. This is a measure of how marginal utility changes when wealth is increased by one percent.

And the variance of the final fund is

$$\text{Var}(X * (T)) = (e^{KT} - e^{2AT}) \left(x_0 + \frac{c}{r} (1 - e^{-rT}) \right)^2 = \left(e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1 \right) \left(E(X * (T)) \right)^2, \quad [2.3]$$

Where A and K are given by

$$A = r + \beta^2, \quad K = 2r + 3\beta^2. \quad [2.4]$$

In Samuelson's work, "*The Myth of Time-Diversification*" (1963), he introduced a utility model without the concept of time-diversification, so that the asset allocation is independent for the entire investment horizon.

$$U_T = (\bar{R} * T) - 0.5A(\sigma_1 * \sqrt{T})^2 = T * U_1, \quad [3]$$

Samuelson stated that the utility (U) will increase proportionally as time (T) increases, where A is a parameter corresponding to the investor's risk aversion. He believes that investors should not change their allocation of risky assets on the basis of their investment horizon. This can only be true if:

- i. *Investors have Constant Relative Risk Aversion (CRRA)*
- ii. *Investment returns are independent and normally distributed, indicating that returns follow a random walk*
- iii. *Investors' future wealth is only dependent on their investment portfolios*

Whether risk is defined in terms of variance or standard deviation⁷ will have an effect on whether Samuelson's assumptions will hold or not. Some studies indicate that decreasing risk with investment horizon has something to do with the investors' demographic or economic situation and not the actual attempt to diversify. Kritzman (2015) defined time-diversification as "*the notion that above average returns tend to offset below average returns over long time horizons*" (p. 29). Implying that time-diversification will reduce risk as the securities value may

⁷ An investor measuring risk as variance will experience proportionally increasing risk as time increases. An investor measuring risk in terms of standard deviation will experience decreasing risk as volatility will increase with the square root of time.

increase or decrease over a period of time, supported by evidence. For longer investment periods an investor should allocate larger weights to higher risk securities.

Kritzman and Ritz (1998) refers to three characteristics that return can have:

- i. *Random Walk*
- ii. *Mean-reversion*
- iii. *Mean-aversion*⁸

Mean-reversion is understood as the change in market return in the direction of a reversion level as a reaction to a previous change in the market return (Hillebrand, 2003). All deviations from the mean will cause a reaction and create a return process towards the mean. Hillebrand (2003), states that errors in the perception of mean-reversion expectations can cause stock market crashes. Both before and after the shock of 1987, there was significantly higher mean-reversion, supporting Hillebrand's theory of a mean-reversion disillusion⁹ occurring, leading to a stock market shock. Studies of the U.S. stock market, implied a mean-reversion characterization (Kritzman & Ritz, 1998), supporting the theory of time-diversification. This indicates that Samuelson's theory that utility increases proportionally with time, does not hold.

Optimization

An investor who wants to invest their savings faces the problem of how to allocate their investment. Vigna (2009) presents how an individual would allocate their pension scheme with future contributions to the fund. Assume that the financial market available is a Black-Scholes model (see e.g. Björk (1998)), consisting of a risk-free and a risky asset. Where the risk-free, with the price $B(t)$, follows the dynamics of:

$$dB(t) = rB(t)dt, \quad [4.1]$$

where $r > 0$. The risky asset with the price dynamics $S(t)$, follows a geometric Brownian Motion with drifts $\lambda > 0$, and diffusion $\sigma > 0$:

$$dS(t) = \lambda S(t)dt + \sigma S(t)dW(t), \quad [4.2]$$

⁸ Mean-aversion means that high returns are followed by high returns, and low returns are followed by low returns. Most time-series may not have these characteristics over time, but at certain periods one can imagine that the nominal interest rate level constitutes a trend that gives the return for bonds such a characteristic.

⁹ A mean-reversion disillusion referring to a situation where the stock-price process followed a path that did not properly reflect the true *a-priori* mean-reversion expectations. The process has to be set into a position as if the illusion did not happen. This is a correction in trajectories, not only in the process parameters and hence the switch can be of substantial magnitude. This is the stock-market crash" (Hillebrand, 2003).

$W(t)$ is a standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, with $\mathcal{F}_t = \sigma\{W(s) : s \leq t\}$.

When the investor puts more money in the investment fund, this is assumed to be at a constant rate and are represented with $c \geq 0$. The proportion allocated to the risky asset at time t is denoted by $y(t)$. The fund will at time t , $X(t)$, grow according to the following Stochastic Differential Equation (SDE):

$$\begin{aligned} dX(t) &= \{X(t)[(t)(\lambda - r) + r] + c\}dt + X(t)y(t)\sigma dW(t), \\ X(0) &= x_0 \geq 0. \end{aligned} \quad [4.3]$$

The amount of x_0 is the initial wealth being any non-negative number. The investor enters at time 0 and contributes for the entire period (T). T is a pre-fixed number, representing the investment horizon.

Investors aim to optimize their investments in terms of lowering risk and return, aiming to maximize or minimize based on certain values. A commonly used optimization seeks to optimize using the mean-variance approach, often maximizing the Sharpe-ratio¹⁰. By maximizing the Sharpe-ratio, the individual's risk preference is not taken into account, making the possible investment to be of higher risk than the investor's risk aversion should indicate, giving an uncharacteristic change in the market. In order to achieve optimization that corresponds to the risk aversion one can optimize using the following process:

$$\begin{aligned} & \text{Maximize } (J(y(\cdot))) \equiv E[U(X(T))], \\ & \text{subject to } \begin{cases} y(\cdot) \text{ admissible} \\ X(\cdot), y(\cdot) \text{ Satisfy [4.3].} \end{cases} \end{aligned} \quad [5.1]$$

This optimization problem can be dealt with using classical control theory as found in Yong and Zhou (1999), Øksendal (2013) and Björk (1998). We present a more basic understanding corresponding to the process found in Vigna (2009).

$$J(y(\cdot); t, x) = E_x[U(X(T))], \quad [5.2]$$

¹⁰ The SHARPE-ratio is given by: $S = \frac{R_p - R_f}{\sigma_p}$. Where R_p is the return on the portfolio, R_f is the return of the market and σ_p is the standard deviation of the portfolio.

Where $E_x = E[\cdot | X(t) = x]$, defines the optimal value function as the supremum of the performance criterion among admissible controls.

$$V(t, x) := \sup_{y(\cdot)} J(y(\cdot); t, x). \quad [5.3]$$

The *Hamilton-Jacobi-Bellman* (HJB) equation including a fundamental theorem of stochastic control theory, which the value associated with the problem must satisfy

$$\sup_y \left[\frac{\partial V}{\partial t} + (x(y(\lambda - r) + r) + c) \frac{\partial V}{\partial x} + \frac{1}{2} x^2 \sigma^2 y^2 \frac{\partial^2 V}{\partial x^2} \right] = 0, \quad [5.4]$$

With boundary conditions

$$V(T, x) = U(x). \quad [5.5]$$

Next, we write the optimal control associated with the problem. This is a function of the partial derivatives to the value function

$$y^*(t, x) = - \frac{\lambda - r}{\sigma^2 x} \frac{V_x}{V_{xx}}, \quad [5.6]$$

Where $V_x = \frac{\partial V}{\partial x}$ and $V_{xx} = \frac{\partial^2 V}{\partial x^2}$ plugs formula [5.6] into the HJB-equation. We find the non-linear partial derivatives equation (PDE)

$$V_t + (rx + c)V_x - \frac{1}{2} \beta^2 \frac{V_x^2}{V_{xx}} = 0, \quad [5.7]$$

With the boundaries set in equation [5.5], solving the PDE we can retrieve what is the optimal control. The most common way to solve a non-linear PDE is guessing the solution by exploiting the natural similarity with the utility function selected.

This approach provides mostly a practical approach to solving the problem of optimal allocations. However, it does not consider the effects of regime-dependent drift and volatility,

as this approach assumes these are fixed, nor does it include a “bond asset”. The “bond asset” would provide an important second risky asset, which is to be seen as crucial for long-term investment. We do not know of any analytical approach to solve the optimization problem under regime switching, making simulation the natural approach.

Turbulence

Turbulent periods – periods with high volatility risk – is described by Kritzman and Li (2010) as “... a condition in which asset prices, given their historical patterns of behavior, behave in an uncharacteristic fashion, including extreme price movements, decoupling of correlating assets, and convergence of uncorrelated assets” (p. 30). As an example they present that “when both U.S. and non-U.S. equities produce returns greater than one standard deviation above their mean, their correlation equals -17%; when both markets produce returns more than one standard deviation below their mean, their correlation rises to +76%” (p. 30)¹¹. This gives reason to assume that all periods should not be treated equally when measuring risk. They further stated that these findings may explain how many investor’s well-diversified portfolios crashed during the financial crisis of 2007-2008, producing massive losses. Just like volatility, turbulence and uncertainty shocks are highly persistent, and one can observe that the risk-reward ratio is substantially lower during turbulent times. These features provide incentives for investors to lower their allocation to risky assets during these times in order to maintain the risk-return trade-off.

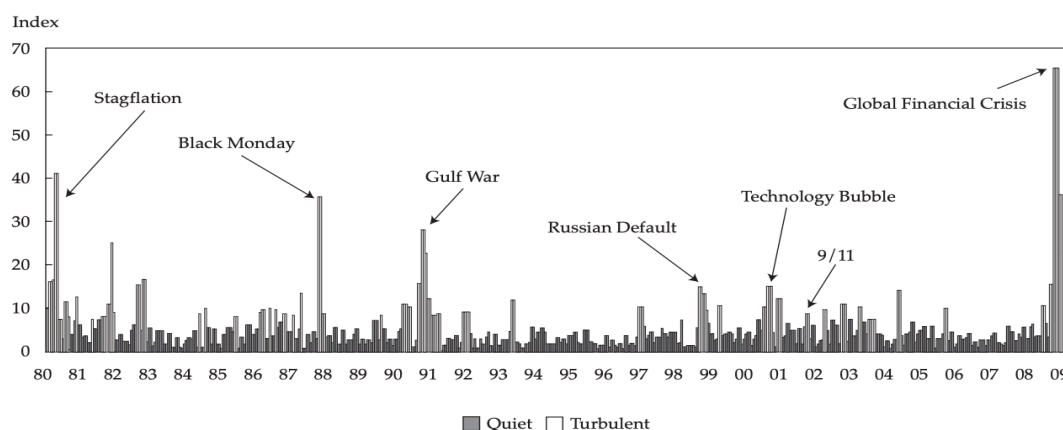


Figure 1 shows how turbulent periods tend to follow large market events (Kritzman & Li, 2010)

Periods with increased risk tend to cluster together, as market participants digest it and react to its cause (Kritzman & Li, 2010). These turbulent periods are a widely known phenomenon known as volatility clustering, described as ARCH-effects: “*Non-homogeneity of*

¹¹ The correlations were based upon monthly returns of the S&P500 Index and MSCI World ex US Index from the period starting January 1970 and ending February 2008.

volatility with highly significant autocorrelation in all measures of volatility despite the insignificant autocorrelation in raw returns” (Lux & Marchesi, 2000, p. 677). The concept of volatility clustering has intrigued many researchers and has served as orientation in the development of stochastic models in finance. Volatility clustering through agent-based models tends to seek explanation through different market participants, described in terms of simple rules (Cont, 2007). From the agent-based model, the concept of market “switching” stands out. The market will switch between high- and low volatility scenarios as can be seen in Figure 1, experiencing *mean-reversion* as the investor will alter the investment based on the current state with different risk preferences, leaving increased periods of risk as an aftermath of the occurrences of financial shocks.

In their study, Kritzman and Li (2010) showed how one can use statistically derived measures of financial turbulence to measure and calculate risk to improve investment performances using turbulence as a filter for exposure to risk. Their results showed greater returns with lower measures of risk, indicating that the full-sample data series will misrepresent a portfolio’s risk attribute during periods of turbulence and financial crisis. Chow et al. (1999) found similar results trying to identify the instabilities of risk parameters. To better give indications of the risk during turbulent times they introduced a procedure for identifying multivariate outliers and used these to estimate a new covariance matrix. They found evidence supporting that the multivariate outliers’ covariance matrix better characterized the riskiness of a portfolio during market turbulence compared to a regular full-sample covariance matrix. Their results showed a more conservative allocation to risky assets, with lower expected return, compared to a full-sample covariance matrix.

Even though the usual trading performances most of the time ends up showing an efficient market, sudden transient phases of destabilization tend to lead into high-volatility or uncertainty phases, with “outbreaks” of volatility that might occur as agents using different techniques surpasses a certain threshold value (Lux & Marchesi, 2000). The agent-based model is used in terms to categorize how a cluster of volatility occurs, not why it occurs. This is simplified in that some people trade based upon volatility, while others believe that the return will follow a fundamental value in the long term.

Data

For the empirical study of historical data, we computed asset prices into total returns for two asset classes, stocks and bonds, measured as *logarithmic* (log) returns. According to Kritzman (1992), *log* returns are better for describing historical returns and the requirements of normality are easier to obtain. We measure volatility as a function of the logarithmic return, assuming that increased returns both positive and negative are indications of increased volatility.

Our primary source of data is historical prices measured in USD extracted from monthly returns in the period: 31.12.1986 through 01.03.2019, providing a total of 776 observations, 388 for each asset class. Our data sample is the *adjusted close price*¹² extracted from Yahoo Finance (2019), which provided the same foundation for all data sources. The sample period of 32 years is chosen to give notable insights of stock and bond market behavior and includes several high-volatility periods, such as Black Monday (1987), the 9/11 terrorist attacks (2001) and the credit crunch (2008).

We used 3 different asset classes for the optimal portfolio problem, two risky assets (stocks and bonds) and one risk-free investment.

Standard & Poor's 500 Index (S&P 500), is a capitalization-weighted index of the 500 largest – publicly traded – U.S. companies by market value. The S&P 500 measures the performance of the U.S. economy through changes in the aggregated market value. It is considered to be one of the most diversified portfolios, widely used as a tracking benchmark for many investment funds. Using this index, we opt to eliminate all the unsystematic risk.

Vanguard Total Bond Market Index Fund (Vanguard) is an open-ended fund incorporated in the United States. Vanguard aims to track the performance of a broad, market-weighted bond index; the Bloomberg Barclays Index. It invests in bonds represented in this index. Reflecting this goal, it allocates approximately 30% in corporate bonds and the remaining 70% in government bonds. Vanguard maintains a dollar-weighted average maturity of 5 to 10 years.

Our *Risk-Free rate of return* is based on historical data for lending and deposit rates from Norwegian banks in the sample period (1987-2019). The data for the risk-free rate measure is extracted from Statistisk Sentralbyrå (2018). We used Norwegian rates as we are trying to map optimal allocation given the average risk aversion rates for the Norwegian population, assuming that Norwegian investors would place their money in Norwegian banks.

¹² The final price traded on the given trading day adjusted for splits and dividends

We observed correspondingly with the study of Bloom (2009) how some high-volatility scenarios – historically – are caused by certain macro- and non-economic events (shown in table 1), causing the stock market returns to move in such degree that it's referred to as a shock. Through our time-series data, we saw indications to how the increase in returns for stocks, caused changes in the same periods for bonds, indicating that they were triggered by the same events¹³. When observing high-volatility returns in bonds, we saw no substantial change in stock returns, indicating that the events causing increased volatility in the bond market do not affect the stock market in a substantial way, indicating degrees of mean-aversion to be present in the bond market. These observations led us to use the stock market as a basis for creating an outlier's sample.

Major Stock-Market Volatility Shocks

Event	Max Volatility	First Volatility	Type
Black Monday	November 1987	October 1987	Economic
Gulf War I	October 1990	September 1990	War
Asian Crisis	November 1997	November 1997	Economic
Russian, LTCM default	September 1998	September 1998	Economic
9/11 Terrorist Attack	September 2001	September 2001	Terror
Worldcom and Enron	September 2002	July 2002	Economic
Gulf War II	February 2003	February 2003	War
Credit crunch	October 2008	August 2007	Economic

Table 1 shows the cause of the major stock-market volatility shocks. Inspired by Bloom (2009)

When determining the outliers, we saw that as the large historic high-volatility- and stock market shock events, shown in Table 1, have proven to be caused by – mainly – non-economic factors. We have no guaranteed way of forecasting the shocks, other than to assume them as randomly distributed, based on probability of occurments, from historic measures. Bloom suggested that shocks are periods where returns move more than 1.65 standard deviations from the mean – in either direction – selected as the 10% one-tailed significance level, over a period of 1 month¹⁴. Financial turbulence often coincides with excessive risk aversion, illiquidity, and devaluation of risky assets. This gives us a basis to determine two

¹³ Events are not necessarily happenings but referred to as the reasoning for the occurrence of the high-volatility periods.

¹⁴ We do not aim to explain the factors causing the volatility or stock market shocks, nor do we separate them. Observing the logarithmic returns from the stock market, claiming that all anomalies are caused by events; economic or non-economic.

separate covariance matrices created from a normal and an outliers sample of returns, to better determine the effect of high-volatility periods in correspondence with the theories presented in Chow et al. (1999).

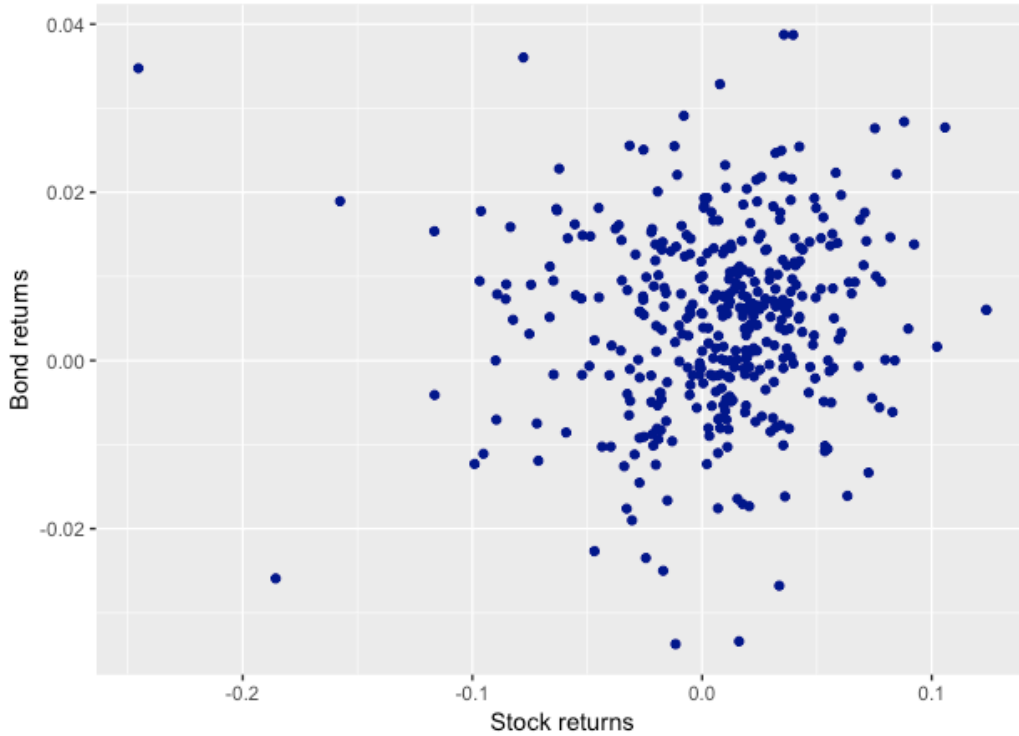


Figure 2 showing independent log-returns before divided into states.

We calculated each logarithmic return's standard deviation from the mean in order to identify the volatility shocks, producing a scatter plot (Figure 2) for illustration, to construct two different covariance matrices. One for each "state"; Normal and Shock. Where any level of d_t above 1.65, would indicate a shock.

$$d_t = \frac{y_t - \mu}{\sigma}, \quad [6]$$

d_t is a measure for the standard deviation of the individual return in period t . y_t is a measure of the independent assets return, μ represent the mean asset return for the sample period and σ is a measure of the overall standard deviation.

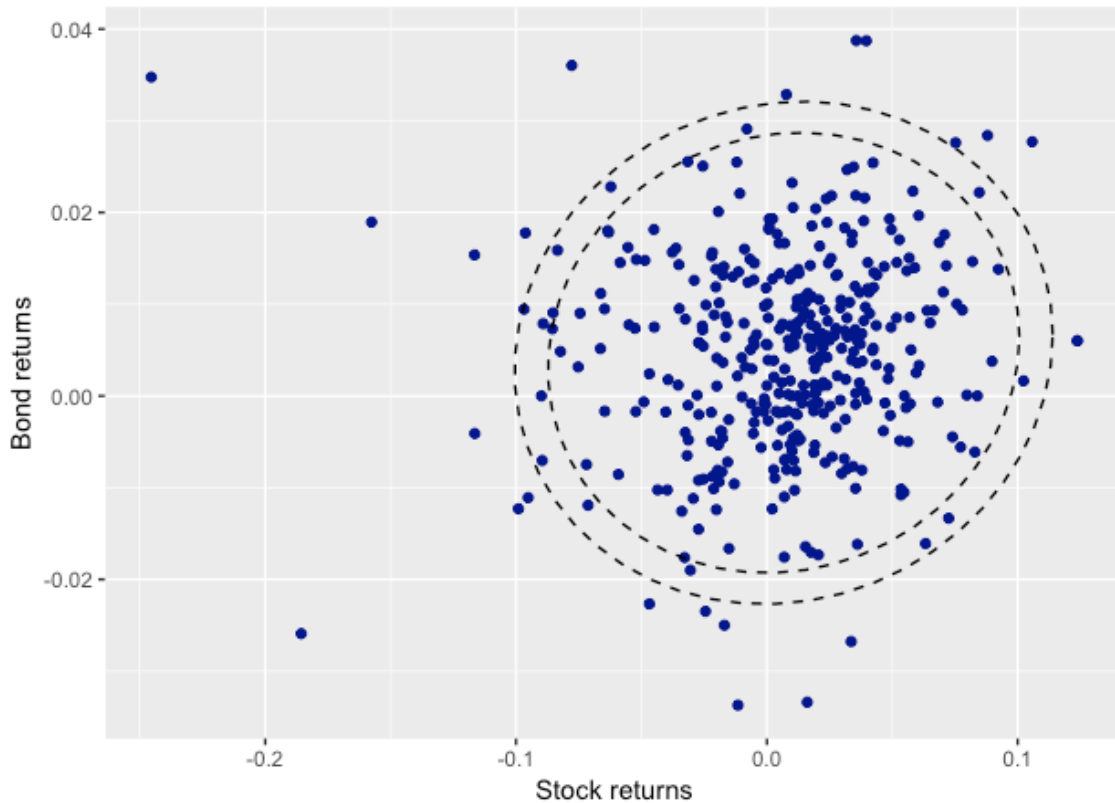


Figure 3 illustrates a scatter plot of turbulence or shock measures from the full-sample data series of historical returns. Showing an ellipse representing the 90% and 95% significance level.

From these calculations, we extracted all observation corresponding to stock market shocks, outside of the 90% ellipse, where returns exceeded 1.65 standard deviations from the mean, as illustrated in figure 3. We replaced all extracted variables with mean-values from our full-sample data series. The remaining data became our *Normal state* covariance-matrix used for the first Monte Carlo simulation. From the removed *outliers* we gathered daily returns to create our *shock state* data series. We obtained daily data to see the effects in a larger scale than the actual event happening. To avoid jumps in the data set we replaced all returns where the time period exceeded one day with the full-sample mean return. This data became the foundation for our *Shock state* covariance-matrix used for the second simulation. Table 2 shows that the shock-state covariance-matrix differ substantially from the full-sample observations suggesting that internal correlation between securities varies in turbulent periods.

Variance-Covariance Normal State

	S&P 500	VanGuard
S&P 500	8.20E-04	3.7839E-05
VanGuard	3.7839E-05	1.08E-04

Variance-Covariance Shock State

	S&P 500	VanGuard
S&P 500	5.21E-04	1.336E-05
VanGuard	1.336E-05	2.04E-05

Table 2 showing covariance matrices for both regimes

To perform the Monte Carlo Markov Chain (MCMC) simulation, we created a probability matrix used as input measure. On the basis of the 10% one-tailed significance measure, we found 26 high-volatility periods in the S&P 500 data series where the preceding period was of *normal* state. This gave us a probability measure of 6.70% of a high-volatility state occurring. We further calculated the probability of a *shock* state occurring for more than one period, finding 11 periods of high-volatility preceded by high-volatility, giving a measure of 42% for the *shocks* to be persistent. We observed periods were high-volatility scenarios lasted for more than two months, giving implications that the length of a shock, is also at random. These final calculations gave us the input data shown in table 3, used in the simulation.¹⁵

Normal State

	S&P 500	VanGuard
Monthly Return	1.05%	0.46%
Standard Deviation	2.87%	1.04%
Annual Return	13.37%	5.63%
Annualized SD	9.93%	3.61%

Shock State

	S&P 500	VanGuard
Monthly Return	- 3.02%	1.32%
Standard Deviation	10.42%	2.06%
Annual Return	-30.81%	17.08%
Annualized SD	36.10%	7.15%

Correlation Normal State

	S&P 500	VanGuard
S&P 500	1.0000	0.1269
VanGuard	0.1269	1.0000

¹⁵ We set no upper limit for the overall duration of the simulated shocks.

	S&P 500	VanGuard
S&P 500	1.0000	-0.1306
VanGuard	-0.1306	1.0000

Table 3 shows input data observed from historical observations, calculated to use in simulation

As mentioned, the studies of Riley Jr. and Chow (1992), Chiappori and Paiella (2011), and Sahm (2012), showed that there was no observable change in risk preferences with increasing wealth. This supported our theory that CRRA-utility maximization was the best measure for our optimization problem. We will assume that the investor opts to statically invest, leaving an investment in period 0 and leave the allocation constant for the entire investment period.

Under the assumption of CRRA-preferences (represented by λ), Aarbu and Schroyen (2009) found – through a survey of a representative sample of the Norwegian population – that the average risk aversion coefficient is $\lambda = 3.7$. This estimate is substantially lower than in other, similar, studies of CRRA-preferences. Kimball, Sahm, and Shapiro (2008) found an average coefficient of $\lambda = 8.2$. This was measured using a representative sample of the U.S. population, based on the work of Barsky, Kimball, Juster, and Shapiro (1997) who found $\lambda \approx 8.0$, as a representative measure of the U.S. population. Using both these studies, Sahm (2012) concluded that average risk aversion is more in the area of $\lambda = 9.6$. Aarbu and Schroyen (2009) argued that their low average value might be caused by the Norwegian welfare state, causing insurance against substantial risk, reducing the background risk. Through a study of Norwegian insurance customers, Haga and Rivenæs (2016) indicated that the average Norwegian values are more in line with the average U.S. values, presented at $\lambda = 9.024$. The difference might be explained by Sahm (2012) who found that worsening macro-economic conditions results in increased risk-aversion. Making an argument that Aarbu and Schroyen’s measure is on the lower side as the survey was conducted pre-financial crisis.

Simulation Model

The problem we are faced with contains great uncertainty in terms of forecasting future values. We conducted a Monte Carlo simulation as an alternative to replace the uncertain variables with average numbers. The problem consists of a large number of generated variables, which provides a substantial number of trials and errors, leading us to conclude that our problem cannot be solved analytically, but has to be solved using a numerical simulated approach.

Based on certain specifications, or limitations, the random numbers in them self, are more in the region of *pseudo-random*, indicating that they are imitating – in many different respects – the behavior of independent observations with a specified distribution (McLeish, 2005). A prerequisite for using a Monte Carlo simulation is that the time series follows a stochastic process. A stochastic process indicates that under certain conditions a time series will appear randomly as a single number is not determined by the previous one. Though, based on historical data we can say that if certain conditions had been different in the past, the outcome of the stochastic process would have been different (Wooldridge, 2013).

In the 1950s a computer was put to analyze economic time series, on the background of economists believing that tracing several economic variables' change over time would clarify and help them predict the behavior of the economy through “boom” and “bust” periods (Bodie, Kane, & Marcus, 2014). Kendall (1953) found that he could not identify any predictable patterns in stock price movements and that prices seemed to follow a random walk. And it became apparent that random price movements indicated a well-functioning or efficient market, not an irrational one (Bodie et al., 2014).

The stock market is known to follow a Markov process, indicating that only the current value of a variable is relevant for predicting the future (Hull, 2012). The essence of the efficient market hypothesis (Fama, 1970), which in its weak form, states that one cannot predict the future return by analyzing the past. There are no patterns or dependencies to observe in historical prices so that in the long run there cannot be excess return only from the analysis of historical data. The history of the variable and the way that the present has emerged from the past are irrelevant. Our Monte Carlo simulation is used to determine the end (terminal) value of a data series, randomly generated from our calculations based on historical measures of expected returns, standard deviations, and correlation among assets.

Following the Monte Carlo simulation and the theory of Markov processes, a Markov Chain Monte Carlo, or a Markov Regime Switching Model, is a sequence generator of (discrete) random variables $X_1, X_2 \dots X_n$, which will take the integer value between 1,2, ... N referred to as “states”. The number of states is pre-arranged, and the model can contain an infinite number of states (McLeish, 2005).

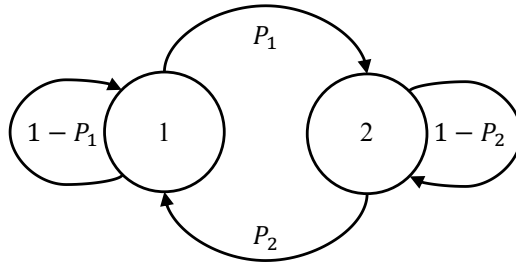


Figure 4 Showing a Discrete-time series Markov Chain, (Source: Own production).

For our simulation, the number of states is prefixed and presented through a probability matrix, which shows a description of the probability of moving between different states, illustrated in figure 3. We will produce an *ergodic* chain, meaning that the state previously occupied is irrelevant for the next move: “1 → 2 → 2 → 1 → 1”. Stating that the next move will only be dependent on state occupied at the given moment, J . J_{+1} will – when drawn – forget the previous occupied state giving that the probability of the current state is only dependent on J .

For the main test, we performed 100.000 simulations in order to get the most accurate measures for our optimization problem.¹⁶ We chose the number of simulations in order to minimize the variance of the final end result, giving deviations in the 4th decimal. By running 100.000 simulations we hoped to eliminate all other effects rather than the change of variables in our final results. For the optimization we have put no limit on lending, giving that it is likely that investors with lower degrees of risk aversion are willing to borrow money, leveraging their investment, in order to increase the expected return, presented with the possibility.

We used the *rng* (random number generator) and the *portsim* function built into MATLAB and generated 120 periods of independent returns. This left us with 2 arrays of 120-by-2-by-100.000 of simulated returns, one for each state.

We used our generated probability of occurrences-matrix [6] as input for the MCMC, to generate a random path of states

$$P = \begin{bmatrix} 0.9330 & 0.0670 \\ 0.5800 & 0.4200 \end{bmatrix}, \quad [6]$$

¹⁶ The entire code can be seen in Appendix B

The MCMC-simulation left us with a matrix of 120-by-100.000 consisting of the numbers 1 and 2. The number one is representing the *Normal* state, whilst the number two is representing the *Shock* state.

We linked the two 120-by-2-by-100.000 arrays of simulated returns with the 120-by-100.000 matrix, creating a new 120-by-2-by-100.000 array consisting of a new assembled series of returns constructed from both states, based on the probability of occurrences. We summed returns gaining terminal values of expected return for stocks and bonds, creating a new matrix of 100.000-by-2. This matrix provided us with enough simulated data to perform our optimization. The input parameters for the optimization consisted of *Initial wealth* set to 1 as the invested amount will have no real effect of the CRRA preferences of the investor. Compared to the practical approach presented by Vigna (2009), we simulate a static investment approach, with no contributions other than the initial investment. *Number of Simulations* matching input matrix. *Scenario Table*, the newly created 120-by-100.000 matrix of terminal returns. *Risk-free rate of return* calculated as historic mean and computed into a 120-period return to match the end values of simulated returns. *Lending rate* calculated as historic mean and computed into a 120-period return to match the end values of simulated returns. *Risk aversion* a parameter consistent with the findings of Aarbu and Schroyen (2009) and set to $\lambda = 3.7$.¹⁷ For the simulation we determined all possible outcomes as equally probable, leaving the final value as an average of all simulated terminal values.

Our final result consists of two values: Allocation to stocks and bonds. Leaving risk-free investment as a product of:

$$W_{rf} = 1 - (W_s + W_b) \quad [7]$$

Where W_{rf} is the weights allocated to risk-free assets, W_s represent the stock allocation and W_b is the weight allocated to bonds. From these three weights, we can conclude with optimal allocation for given values of CRRA and calculate the simulated expected return and standard deviation of the portfolio.

Our main input data are based on a lot of uncertain assumption leaving it appropriate to test our model in light of changes to these parameters. Through our sensitivity analysis, we changed one parameter at the time tracking the effect of this parameters on optimal allocation. Investment horizon and risk aversion were altered simultaneously. To see the effect of the

¹⁷ The risk aversion coefficient will be tested for further values, leaving 3.7 as our main value.

individual investor's risk aversion we changed the CRRA-coefficient to coincide with other research. We used measures found by Aarbu and Schroyen (2009) $\lambda = 3.7$, testing for +/- one standard deviations of 2.2, leaving measures of $\lambda = 1.5$ and $\lambda = 5.9$. Kimball et al. (2008) with the parameter $\lambda = 8.0$, as a measure for the U.S. population, $\lambda \approx 8.2$ found by Barsky et al. (1997), $\lambda = 9.6$, from the studies of Sahm (2012), and lastly the study of Haga and Rivenæs (2016) – providing a measure of the Norwegian population closer to the U.S. findings – at $\lambda = 9.024$.

The deposit and lending rates have – historically – been through the same fluctuating periods as the risky assets. To better understand how the risk-free rates, impact the optimal allocation we altered the rates by +/- 1% annually giving altered input data as shown in table 4.

	Deposit	Lending
Historical	4,29 %	7,60 %
+1%	5,29 %	8,60 %
-1%	3,29 %	6,60 %

Table 4 showing risk-free deposit and lending rates at +/- 1% alterations

We kept the investment horizon at 10 years (120 periods) to test for time diversification, as we presumed that shocks would be neutralized after a 10-year period. To see the actual effects of time diversification we ran the optimization problem with altering investment horizons at 12, 36, and 60 periods. Corresponding to 1, 3, and 5-years. To run these tests, we also had to alter the deposit and lending rates as they were computed to match the simulated returns of the investment horizon. We used the historic average measure and computed it to match the investment horizons.

For the last alteration, we changed the input data to coincide with the theory of using only one covariance-matrix constructed from the full-sample historical data. providing input data as can be seen in table 5

Full-sample covariance-matrix

	S&P 500	VanGuard
Monthly Return	0.64%	0.47%
Standard Deviation	4.33%	1.12%
Annual Return	7.92%	5.64%
Annualized SD	15.00%	3.87%

Table 5 Showing input data for the full-sample estimation

This simulation allows us to see the grand effects of the separate covariance-matrix approach and get a better understanding of high-volatility periods, for all risk aversion parameters and investment horizons.¹⁸

Results

An overview of all main results can be found in Appendix A.

Our main simulation, as found in table 6, using CRRA-levels of 3.7, and an investment horizon of 10 years, resulted in allocations of 27.3% in stocks and 72.7% in bonds, leaving the allocation to risk-free assets at 0%. In table 7 we show simulated returns for both asset classes, for all investment horizons. This main simulation produced a total stock return of 74.31% over the course of 10 years, providing annual returns of 5.71%. Bonds returned 66.10% over the course of 10 years, with annual returns of 5.21%. The optimal allocation would yield 68.34% in returns over the 10-year investment horizon, with an annually compounded return of 5.34%, presented in table 8.

CRRA	1 year		3 years		5 years		10 years	
	Stocks	Bonds	Stocks	Bonds	Stocks	Bonds	Stocks	Bonds
1,5	4,9 %	95,1 %	30,5 %	69,5 %	41,5 %	58,5 %	51,9 %	48,1 %
3,7	7,2 %	92,8 %	18,8 %	81,2 %	21,5 %	78,5 %	27,3 %	72,7 %
5,9	7,8 %	92,2 %	14,6 %	85,4 %	16,9 %	83,1 %	20,3 %	79,7 %
8,0	8,0 %	92,0 %	13,1 %	86,9 %	14,7 %	85,3 %	17,2 %	82,8 %
8,2	8,2 %	91,8 %	13,0 %	87,0 %	14,3 %	85,7 %	17,0 %	83,0 %
9,024	8,0 %	92,0 %	12,4 %	87,6 %	13,9 %	86,1 %	16,0 %	84,0 %
9,6	8,1 %	91,9 %	11,9 %	88,1 %	13,4 %	86,6 %	15,5 %	84,5 %

Table 6 showing optimal allocation for all CRRA-values and investment horizons

All optimizations are static, giving an investment in period 0, with no opportunity for alterations of allocations during the investment period. In table 6 we present optimal allocations from altering the investment horizon, and the risk aversion coefficients. We simulated optimal allocation using investment horizons of 1, 3, 5 and 10-years, corresponding to CRRA-values of

¹⁸ For this simulation, we experienced some numerical issues with our model. Leaving all terminal values to be equal, though the intermediate values were of random characteristics. We changed the normal and shock state input data to have equal input. Assuming that all scenarios are equal, we altered the probability matrix to $P = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$, leaving alterations between different states every period. This was done to gain randomly generated returns. And to get plausible allocations of assets.

1.5, 3.7, 5.9, 8.0, 8.2, 9.024, 9.6. The simulated return from stocks and bonds can be seen in table 7.

	1-year	3-year	3-year a	5-year	5-year a	10-year	10-year a
Stocks	6,64 %	21,80 %	6,79 %	36,88 %	6,48 %	74,31 %	5,71 %
Bonds	6,78 %	19,95 %	6,25 %	33,15 %	5,89 %	66,10 %	5,21 %

Table 7 showing returns for all risky asset classes. "a" is annually compounded returns.

The total return from the optimally allocated portfolios is presented in table 8, showing returns for all investment horizons and all levels of CRRA. All annual returns are annually compounded and do not represent any single year of returns in terms of simulated values.

CRRA	1-year	3-year	3-year a	5-year	5-year a	10-year	10-year a
1,5	6,77 %	20,51 %	6,42 %	34,70 %	6,14 %	70,36 %	5,47 %
3,7	6,77 %	20,30 %	6,35 %	33,95 %	6,02 %	68,34 %	5,34 %
5,9	6,77 %	20,22 %	6,33 %	33,78 %	5,99 %	67,77 %	5,31 %
8,0	6,77 %	20,19 %	6,32 %	33,70 %	5,98 %	67,51 %	5,29 %
8,2	6,77 %	20,19 %	6,32 %	33,68 %	5,98 %	67,49 %	5,29 %
9,024	6,77 %	20,18 %	6,32 %	33,67 %	5,97 %	67,41 %	5,29 %
9,6	6,77 %	20,17 %	6,32 %	33,65 %	5,97 %	67,37 %	5,28 %

Table 8 Showing returns on optimal allocations for all measures of CRRA and investment horizons

When altering the deposit and lending rates by +/- 1% annually, we simulated keeping all initial variables constant gaining results for optimal allocation given CRRA=3.7, and investment horizon of 120 periods (10 years).

CRRA=3.7	Stocks	Bonds	Risk-free
+1%	14.20%	-52.39%	138.19%
-1%	27.30%	72.70%	0 %

Table 9 showing optimal allocation after altering in risk-free rates

The results shown in table 9, indicate no change in optimal allocation when decreasing the rates by 1% annually. Increasing the annual rates by 1% the results changed indicating an optimal allocation of 14.20% in stocks, a short-selling of 52.39% in bonds whilst 138.19% should be invested in risk-free assets. As presented in table 10, this allocation would have

yielded a return of 69.07% over the course of 10 years, providing an annually compounded return of 5.39%.

CRRA 3,7	10-years	
	Regime-allocation	
Risk-free rate	Historic	+1%
Annually portfolio return	5,34 %	5,39 %
Portfolio return	68,34 %	69,07 %

Table 10 Showing the annual and overall returns for historical and altered risk-free rate

In the final test, we altered the simulation to the use of a single covariance matrix based on the full-sample data set. This simulation was conducted for all levels of CRRA and all investment horizons. Table 11 shows the allocation for a full-sample covariance matrix simulation, stating that 76% should be allocated to stocks and 24% allocated to bonds for the 10-year period with CRRA of 3.7.

CRRA	1 Years		3 Years		5 Years		10 Years	
	Stocks	Bonds	Stocks	Bonds	Stocks	Bonds	Stocks	Bonds
1,5	123 %	-23 %	135 %	-35 %	141 %	-41 %	129 %	-29 %
3,7	54 %	46 %	59 %	41 %	65 %	35 %	76 %	24 %
5,9	36 %	64 %	40 %	60 %	43 %	57 %	51 %	49 %
8	28 %	72 %	30 %	70 %	32 %	68 %	38 %	62 %
8,2	28 %	72 %	29 %	71 %	32 %	68 %	38 %	62 %
9,024	25 %	75 %	28 %	72 %	29 %	71 %	34 %	66 %
9,6	24 %	76 %	26 %	74 %	28 %	72 %	33 %	67 %

Table 11 shows optimal allocation using a single-covariance matrix based on the full-sample data set. There are no constraints on lending

Table 12 shows the allocated portfolios annual standard deviations calculated using:

$$\sigma_P = \sqrt{W_s^2 \sigma_s^2 + W_b^2 \sigma_b^2 + 2W_s W_b Cov_{s,b}} \quad [8]$$

Where W_s represents the weight allocated to stocks, W_b represents the weight allocated to bonds, σ_s is the stocks standard deviation, σ_b is the standard deviation from the bonds, and $Cov_{s,b}$ is the covariance between stocks and bonds.

CRRA	1-year		3-years		5-years		10-years	
	Regime	Full-sample	Regime	Full-sample	Regime	Full-sample	Regime	Full-sample
1,5	4,05 %	18,44 %	5,52 %	20,24 %	6,84 %	21,13 %	8,24 %	19,36 %
3,7	4,04 %	8,30 %	4,46 %	9,02 %	4,66 %	9,80 %	5,18 %	11,44 %
5,9	4,04 %	6,00 %	4,22 %	6,41 %	4,34 %	6,77 %	4,57 %	7,84 %
8	4,04 %	5,07 %	4,16 %	5,25 %	4,22 %	5,52 %	4,36 %	6,20 %
8,2	4,04 %	5,03 %	4,15 %	5,18 %	4,21 %	5,42 %	4,34 %	6,14 %
9,024	4,04 %	4,73 %	4,13 %	5,02 %	4,19 %	5,20 %	4,29 %	5,73 %
9,6	4,04 %	4,65 %	4,11 %	4,80 %	4,17 %	5,00 %	4,26 %	5,54 %

Table 12 Showing annual standard deviations for portfolios with full-sample and regime covariance matrices

Discussion

Looking at the optimal allocations, we found substantial differences when isolating high-volatility periods, compared to a full sample simulation. The difference can be seen both in terms of allocation and returns. We observe through table 13 that for all scenarios there is no investment allocated to the risk-free asset. This indicates that the risk-free asset provides subpar returns, compared to both risky assets, even when the regime-switching is initiated.

CRRA	Separate covariance matrices		Full-sample historic mean	
	Stocks	Bonds	Stocks	Bonds
1,5	51.88 %	48.12 %	129.30 %	-29.30 %
3,7	27.28 %	72.72 %	76.22 %	23.78 %
5,9	20.30 %	79.70 %	50.80 %	49.20 %
8	17.16 %	82.84 %	38.16 %	61.84 %
8,2	16.96 %	83.04 %	37.68 %	62.32 %
9,024	16.01 %	83.99 %	34.21 %	65.79 %
9,6	15.49 %	84.51 %	32.62 %	67.38 %

Table 13 Showing allocations coming from the full-sample matrix and the regime-switching model for a 10-year investment horizon

As we see in table 14, a CRRA-level of 3.7, an investment horizon of 10-years, and a full-sample covariance matrix the investor should optimally invest 76.2% in stocks and 23.8% in bonds, yielding a portfolio return of 6.25% annually.¹⁹ In comparison through isolated covariance-matrices, the allocations are much more conservative with 27.3% in stocks and 72.7% in bonds yielding an annual return of 5.34%. These findings correspond with those of Chow et al. (1999) who stated that using separate covariance-matrices would provide much more conservative allocations and lower returns. The overall portfolio risk – given these

¹⁹ All returns, allocations and descriptive results are presented in appendix A

allocations – measures 5.18% annually for separate covariance-matrices and 11.44% using a full-sample covariance-matrix, as can be seen in table 11. Showing almost a twice as high standard deviation with an increase of only .22% in annual returns, presented in table 14.

CRRA 3,7	10-years	
	Regime-switching	Full-sample
Annually portfolio return	5,34 %	5,56 %

Table 14 Showing annual returns for the regime- and full-sample covariance matrices

These findings give support to the theory that a covariance-matrix extracted from high-volatility events better represents the risks associated with turbulent periods compared to a full-sample covariance matrix, providing portfolios with lower risk, with smaller deviations in terms of return.

We observed through table 11 that there were 0% allocated to risk-free investments, indicating that stocks and bonds outperformed the risk-adjusted return from risk-free assets through all simulated scenarios. This made us question whether or not our historical measures were correct. We saw that increasing the annual rates by 1%, allocations changed (table 9). Our model optimally suggested a short-selling of 52.39% in bonds, a 14.20% investment in stocks, and 138.19% allocated to risk-free assets. This allocation (see table 10) would return 0.73% more over the course of 10 years. Recent years have provided unusually low returns for deposit rates, raising a question of our calculations. When comparing our findings to the studies of Das and Uppal (2004), we saw that our results coincided. They found through their study of jump-diffusion between international equity indexes, that 0% became the optimal allocation to risk-free assets. We can argue that interest rates are forecasted to increase in the foreseeable future, moving towards our historical measures.

Looking at the results of table 6, we see that allocations to high-risk assets increase as investment horizon increases. For an investor with CRRA-levels of 3.7, the investor should optimally invest 7.2% in stocks and 92.8% in bonds for the 1-year investment period. As time increases, the investor should increase the investment to higher risk assets leaving an allocation of 27.3% in higher risk assets with a 10-year horizon. These findings are consistent through all simulations²⁰, and are strongly in support of time diversification, presented by Kritzman and Ritz (1998), indicating some form of mean-reversion tendencies. Implying that throughout the investment period, the assets will both increase and decrease, leaving time to take some of the

²⁰ For the full-sample covariance simulations the CRRA-level of 1.5 gave some results where the increase is consistent through all increases except for the increase from 5 to 10 years.

risk. Showing indications that high-volatility periods have occurred, supporting that time diversification is present.

We find that through the optimization, the allocation to risky assets seems reasonable. Pointing in the direction that the levels of risk aversion found by Aarbu and Schroyen (2009) might be consistent with the average Norwegian investor, as our findings show no signs that the average investor would take on high levels of risk. These implications follow the findings from using a full-sample covariance matrix, where – though riskier – the investor allocates within reasonable limits.

Some of our simulations showed suggestions for leveraging the investment. When using Bernoulli's utility theory (Bernoulli, 1954) one would assume that all investors are rational and utility maximizing, and would be willing to leverage investments. Analyzing in light of the prospect theory (Kahneman & Tversky, 1992), investors would be seen as loss averse and not only risk-averse – per se. This would most likely have given different results, especially in terms of leveraging the investment, and risk preferences, probably leading to more conservative investments.

A result that for us became somewhat unclear was the allocation for the 1-year horizon using separate covariance-matrices, which provided an increase in high-risk assets as CRRA-levels increased. This indicates increased risk as investors become more risk-averse. The rational explanation could be that there is some form of diversification effect as bonds and stocks do not correlate perfectly.

Conclusion

We analyzed historic abilities of turbulence and used measures of financial risk and volatility to create separate covariance-matrices based on volatility-return performance. Using the derived data, we performed Monte Carlo simulations producing time-series of returns for different risk regimes, isolating the high-volatility scenario impact. Using the derived data, we conducted a Markov Regime Switching model generating possibilities of entering high-volatility scenarios over the course of the investment period. The new simulated time series of returns including high-volatility scenarios gave intel used to create optimal allocations of risky assets, maximizing expected utility through measures of CRRA. Our results showed portfolios heavily weighted in bonds, with heavier weights in stocks as the investment horizon increased. These findings showed heavy support for the theory of time diversification stating that over time risky assets will both increase and decrease, leaving time to take some of the risk. Further,

our findings showed that the implementation of separate covariance-matrices gave vastly different investment policies, producing much more conservative investments when isolating high-volatility periods, gaining less returns, but with substantially lower risk.

Our simulation used a two-regime switching model, based on the analysis that the main events affected both assets through the stock market. Our “shock-states” are transitory, making the “normal-state” based on regular noise. By increasing the number of “state-variables”, based on analysis of e.g. the bond market, risk-free rates, interest-rate curves, one might capture the grand effects of events happening in the overall market to an even greater extent.

Further, analyzing the persistence and duration of shocks might increase the credibility linked to the simulation, providing even more accurate analysis of the impact of shocks. When enhancing the impact of high-volatility regimes, one might assume higher tail-risk, as volatility has a bigger impact on the return-series. Our findings remain premised on our assumptions of CRRA-levels in investors. There is a growing acceptance for the notion that investor’s utility-curve are S-shaped and better represented by this bilinear curve. This notion indicates that investors are more in the line of loss averse and not only risk averse. The prospect theory argues that investor implements certain “frames” to their decision making, disliking risk, they would – during high-volatility periods – alter their investment being more loss averse during some periods. We acknowledge these findings as more relevant used with dynamic investments, allowing investors to “re-optimize” at each change of regime.

Increasing the number of states, analyzing the persistence of shocks and leaving investors with the ability to re-optimize allocations as events occur, might provide a more accurate model linked to the loss-averse aspect of the prospect theory. We leave this for further studies.

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APPENDIX A – Showing all simulated output data.

	1 year		3 year		5 year		10 year	
	Regime		Full sample		Regime		Full sample	
	Stocks	Bonds	Stocks	Bonds	Stocks	Bonds	Stocks	Bonds
CRA 1.5								
Allocated weights	0,0490	0,9510	1,2320	-0,2320	0,3053	0,6947	1,3514	-0,3514
Return	6,64 %	6,78 %	7,65 %	5,64 %	6,79 %	6,25 %	7,16 %	5,34 %
Portfolio Return annu	6,77 %		8,12 %		6,42 %		7,80 %	
Total Portfolio Return	6,77 %		8,12 %		20,51 %		25,22 %	
Annual Standard Dev	15,4 %	4,19 %	14,9 %	3,87 %	15,4 %	4,19 %	14,9 %	3,87 %
Portfolio SD annual	4,05 %		18,44 %		5,52 %		20,24 %	
Portfolio Covariance	-0,0000281		0,0000347		-0,0000281		0,0000347	
CRA 3.7								
Allocation	0,0720	0,9280	0,5415	0,4585	0,1879	0,8121	0,5938	0,4062
Return	6,64 %	6,78 %	7,65 %	5,64 %	6,79 %	6,25 %	7,16 %	5,34 %
Portfolio Return annu	6,77 %		6,73 %		6,35 %		6,42 %	
Total Portfolio Return	6,77 %		6,73 %		20,30 %		20,56 %	
Annual Standard Dev	15,4 %	4,19 %	14,9 %	3,87 %	15,4 %	4,19 %	14,9 %	3,87 %
Portfolio SD annual	4,04 %		8,30 %		4,46 %		9,02 %	
Portfolio Covariance	-0,0000281		0,0000347		-0,0000281		0,0000347	
CRA 5.9								
Allocation	0,078187	0,921813	0,364889	0,635111	0,146235	0,853765	0,398692	0,601308
Return	6,64 %	6,78 %	7,65 %	5,64 %	6,79 %	6,25 %	7,16 %	5,34 %
Portfolio Return annu	6,77 %		6,37 %		6,33 %		6,07 %	
Total Portfolio Return	6,77 %		6,37 %		20,22 %		19,36 %	
Annual Standard Dev	15,4 %	4,19 %	14,9 %	3,87 %	15,4 %	4,19 %	14,9 %	3,87 %
Portfolio SD annual	4,04 %		6,00 %		4,22 %		6,41 %	
Portfolio Covariance	-0,0000281		0,0000347		-0,0000281		0,0000347	
CRA 8.0								
Allocation	0,079563	0,920437	0,282751	0,717249	0,130798	0,869202	0,300	0,700
Return	6,64 %	6,78 %	7,65 %	5,64 %	6,79 %	6,25 %	7,16 %	5,34 %
Portfolio Return annu	6,77 %		6,21 %		6,32 %		5,89 %	
Total Portfolio Return	6,77 %		6,21 %		20,19 %		18,75 %	
Annual Standard Dev	15,4 %	4,19 %	14,9 %	3,87 %	15,4 %	4,19 %	14,9 %	3,87 %
Portfolio SD annual	4,04 %		5,07 %		4,16 %		5,25 %	
Portfolio Covariance	-0,0000281		0,0000347		-0,0000281		0,0000347	
CRA 10.0								
Allocation	0,079563	0,920437	0,282751	0,717249	0,130798	0,869202	0,300	0,700
Return	6,64 %	6,78 %	7,65 %	5,64 %	6,79 %	6,25 %	7,16 %	5,34 %
Portfolio Return annu	6,77 %		6,21 %		6,32 %		5,89 %	
Total Portfolio Return	6,77 %		6,21 %		20,19 %		18,75 %	
Annual Standard Dev	15,4 %	4,19 %	14,9 %	3,87 %	15,4 %	4,19 %	14,9 %	3,87 %
Portfolio SD annual	4,04 %		5,07 %		4,16 %		5,25 %	
Portfolio Covariance	-0,0000281		0,0000347		-0,0000281		0,0000347	

	1 year		3 year		5 year		10 year			
	Regime	Full sample	Regime	Full sample	Regime	Full sample	Regime	Full sample		
	Stocks	Bonds	Stocks	Bonds	Stocks	Bonds	Stocks	Bonds		
CRRRA 8.2	Allocation	0,081515 0,918485	0,27852 0,72148	0,130323 0,869677	0,293427 0,706573	0,142796 0,857204	0,315408 0,684592	0,170 0,830	0,377 0,623	
	Return	6,64 % 6,78 %	7,65 % 5,64 %	6,79 % 6,25 %	7,16 % 5,34 %	6,48 % 5,89 %	6,71 % 5,09 %	5,71 % 5,21 %	5,87 % 4,57 %	
	Portfolio Return annually	6,77 %	6,20 %	6,32 %	5,88 %	5,98 %	5,60 %	5,29 %	5,06 %	
	Total Portfolio Return	6,77 %	6,20 %	20,19 %	18,71 %	33,68 %	31,40 %	67,49 %	64,11 %	
	Standard Deviation	15,4 % 4,19 %	14,9 % 3,87 %	15,4 % 4,19 %	14,9 % 3,87 %	15,4 % 4,19 %	14,9 % 3,87 %	15,4 % 4,19 %	14,9 % 3,87 %	
	Portfolio SD annual	4,04 %	5,03 %	4,15 %	5,18 %	4,21 %	5,42 %	4,34 %	6,14 %	
	Portfolio Covariance	-0,0000281	0,0000347	-0,0000281	0,0000347	-0,0000281	0,0000347	-0,0000281	0,0000347	
	CRRRA 9.024	Allocation	0,080455 0,919545	0,248545 0,751455	0,1243 0,8757	0,277462 0,722538	0,139339 0,860661	0,294653 0,705347	0,160 0,840	0,342 0,658
	Return	6,64 % 6,78 %	7,65 % 5,64 %	6,79 % 6,25 %	7,16 % 5,34 %	6,48 % 5,89 %	6,71 % 5,09 %	5,71 % 5,21 %	5,87 % 4,57 %	
	Portfolio Return annually	6,77 %	6,14 %	6,32 %	5,85 %	5,97 %	5,57 %	5,29 %	5,02 %	
Total Portfolio Return	6,77 %	6,14 %	20,18 %	18,61 %	33,67 %	31,19 %	67,41 %	63,40 %		
Standard Deviation	15,4 % 4,19 %	14,9 % 3,87 %	15,4 % 4,19 %	14,9 % 3,87 %	15,4 % 4,19 %	14,9 % 3,87 %	15,4 % 4,19 %	14,9 % 3,87 %		
Portfolio SD annual	4,04 %	4,73 %	4,13 %	5,02 %	4,19 %	5,20 %	4,29 %	5,73 %		
Portfolio Covariance	-0,0000281	0,0000347	-0,0000281	0,0000347	-0,0000281	0,0000347	-0,0000281	0,0000347		
CRRRA 9.6	Allocation	0,081 0,919	0,240 0,760	0,119 0,881	0,256 0,744	0,134 0,866	0,276 0,724	0,155 0,845	0,326 0,674	
Return	6,64 % 6,78 %	7,65 % 5,64 %	6,79 % 6,25 %	7,16 % 5,34 %	6,48 % 5,89 %	6,71 % 5,09 %	5,71 % 5,21 %	5,87 % 4,57 %		
Portfolio Return annually	6,77 %	6,12 %	6,32 %	5,81 %	5,97 %	5,54 %	5,28 %	5,00 %		
Total Portfolio Return	6,77 %	6,12 %	20,17 %	18,48 %	33,65 %	31,00 %	67,37 %	63,08 %		
Standard Deviation	15,4 % 4,19 %	14,9 % 3,87 %	15,4 % 4,19 %	14,9 % 3,87 %	15,4 % 4,19 %	14,9 % 3,87 %	15,4 % 4,19 %	14,9 % 3,87 %		
Portfolio SD annual	4,04 %	4,65 %	4,11 %	4,80 %	4,17 %	5,00 %	4,26 %	5,54 %		
Portfolio Covariance	-0,0000281	0,0000347	-0,0000281	0,0000347	-0,0000281	0,0000347	-0,0000281	0,0000347		

Appendix B

Simulation model ~ Code

```
format long

%% Global values

numberOfDraws      = 120; %Sets number of simulated periods
numberOfSimulations = 100000; %Sets number of simulations

%% Monte Carlo normal state

expectedReturn      = [1.05  0.46]/100; %Historic input data
standardDeviations  = [2.87  1.04]/100; %Historic input data
correlationMatrix   = [1      0.1269      %Historic input data
                      0.1269      1];

monteCarloNormal = MonteCarloCorrelated (expectedReturn, ...
standardDeviations, correlationMatrix, numberOfDraws, numberOfSimulations);

expectedReturn      = [-3.02  1.32]/100; %Historic input data
standardDeviations  = [10.42  2.06]/100; %Historic input data
correlationMatrix   = [1      -0.1306      %Historic input data
                      -0.1306      1];

monteCarloShock = MonteCarloCorrelated (expectedReturn, ...
standardDeviations, correlationMatrix, numberOfDraws, numberOfSimulations);

%% Markov Chain

probabilityMatrix = [... %Historic calculated probability
    0.9330 0.0670
    0.5800 0.4200
];
%probabilityMatrix = [... %Alternative Probability for sensitivity
% 0.0000 1.0000
% 1.0000 0.0000
% ];

markovChains = zeros(numberOfSimulations, numberOfDraws);
for i=1:numberOfSimulations
    markovChains(i,1:end) = ((MarkovChain(probabilityMatrix,
numberOfDraws)).');
end

%% Link chain to monte carlo
values = {monteCarloNormal, monteCarloShock};
markovValues = CorrelateMarkovToValues(values, markovChains);

%% Find average end value

averages = AverageSimulatedReturn(markovValues);
assetMean = mean(averages);
```

```

%% Average standard deviation

stdDeviations = StandardDeviation(markovValues);
stdDeviationsAverages = AverageSimulatedReturn(stdDeviations);
assetVar = mean(stdDeviationsAverages);

%% Asset Covariance

assetCovar = AvgAssetCovar(markovValues);

%% CRRA Optimization

sumMarkov      = sum(markovValues, 1);
sumMarkovValues = squeeze(sumMarkov)';
SCENARIOTABELL = sumMarkovValues;
RISKAVERSIONPARAMETER = 1.5; % Sets the degree of risk aversion in CRRA
SCENARIONUMBER = numberOfSimulations; % Sets the number of scenarios
%used for bond and stock returns
RISKFREERATE = 0.0428; % Sets the return on the risk-free asset
%for the total period
LENDINGRATE = 0.0759; % Sets the lending rate for the total
%period
INITIALWEALTH = 1; % Sets initial wealth - high value (>>1) will
%create numerical problems!

[wts,utility,eksit,out] = maxCRRAutility(INITIALWEALTH, SCENARIONUMBER, ...
sumMarkovValues, RISKFREERATE, LENDINGRATE, RISKAVERSIONPARAMETER)

```

Appendix B1

Monte Carlo for correlated variables

```

function [...
    returnStockBonds... %
] = ...
MonteCarloCorrelated(...
    expectedReturn,... % Historically estimated
    standardDeviations,... % Historically estimated
    correlationMatrix,... % Historically estimated
    numberOfDraws,... % Pre-determined
    numberOfSimulations... % Pre-determined
)
%MONTECARLOCORRELATED Draw random samples based on correlation matrix

returnIntervals = 1;

expectedCovariance = corr2cov(standardDeviations, correlationMatrix);

rng('shuffle'); % set random number generator, default to get same result
%every time
returnStockBonds = portsim (expectedReturn, expectedCovariance, ...
numberOfDraws,returnIntervals, numberOfSimulations, 'Exact');

end

```

Appendix B2

Markov Chain Monte Carlo

```
function [dmarkovChain] = MarkovChain(...
    probabilityMatrix,...
    numberOfDraws)
%MarkovChain Randomly drawn path

mc = dtmc(probabilityMatrix);
dmarkovChain = simulate(mc, numberOfDraws);
dmarkovChain = dmarkovChain (2:end);
% mc = dtmc(probabilityMatrix,'StateNames',arrayOfStateNames);

end
```

Appendix B3

Correlate Markov Chain to Random values

```
function [probabilityChain] = CorrelateMarkovToValues(...
    values, ... % List of data matrices, time
    ...% in rows {normal, shock}
    ...% These must be 3 dimensional...
    ...%
    MarkovChain ... % N possible states
)
numberOfMatrixRows = size(values{1}(1:end,1,1), 1);
numberOfMatrixColumns = size(values{1}(1,:,1), 2);
numberOfMatrixLayers = size(values{1}(1,1,1:end), 3); % simulations
numberOfSimulations = numberOfMatrixLayers;
numberOfDataSets = length(values);

probabilityChain = zeros(numberOfMatrixRows, numberOfMatrixColumns,...
    numberOfMatrixLayers);

for simulation=1:numberOfSimulations;
    for timestep=1:numberOfMatrixRows;
        for dataType=1:numberOfMatrixColumns
            markovState = MarkovChain(simulation, timestep);
            probabilityChain(timestep,dataType,simulation) =...
                values{markovState}(timestep,dataType,simulation);
        end
    end
end

end
```


Appendix B4

Calculate average of simulated values

```
function [averages] = AverageSimulatedReturn(simulatedValues)

%AverageSimulatedReturn Calculated average return from simulated variables
% Kolonne 1 og 2 er Stocks og Bonds. Tidsserie fra rader. Data stammer fra
% normal og sjokktilstand, antall simuleringer langs 3. akse.

matrixSize = size(simulatedValues);
summations = zeros(matrixSize(2), matrixSize(3));
for simulation = 1:matrixSize(3)
    for type = 1:matrixSize(2)
        summations(type, simulation) = sum(simulatedValues(:,type,
simulation));
    end
end

averages = zeros(matrixSize(2),1);

for type = 1:matrixSize(2)
    averages(type) = sum(summations(type,:))/ matrixSize(3);
end

end
```

Appendix B5

Calculate asset covariance from average values

```
function [covarM] = AvgAssetCovar( ...
                                markovValues)

%UNTITLED3 Summary of this function goes here
% Detailed explanation goes here

[rows, cols, numberOfSimulations] = size(markovValues);

assetCov3d = zeros(cols,cols, numberOfSimulations);
for simulation = 1:numberOfSimulations
    assetCov3d(:, :, simulation) = cov(markovValues(:, :, simulation));
end

covarM = zeros(cols, cols);
for row = 1:cols
    for col=1:cols
        covarM(row, col) = sum(assetCov3d(row, col,
:))/numberOfSimulations;
    end
end
```

Appendix B6

Optimization function

```
function
[x,fval,exitflag,output]=maxCRRUtility(initialwealth,scenarionumber,scenar
iotable,riskfreereturn,lendingrate,riskaversionparameter)
fun=@(x) -
CRRExpectedUtility(initialwealth,x(1),x(2),scenariotable,scenarionumber,ri
skfreereturn,lendingrate,riskaversionparameter);
x0=[0.5,0.5];
options = optimset('TolFun',0.0000001,'TolX',0.0000001);
[x,fval,exitflag,output] = fminsearch(fun,x0,options);
end
```