FAMILIES OF DETERMINANTAL SCHEMES

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ABSTRACT. Given integers $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$ and $b_1 \leq \cdots \leq b_t$, we denote by $W(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^p(\mathbb{P}^n)$ the locus of good determinantal schemes $X \subset \mathbb{P}^n$ of codimension c defined by the maximal minors of a $t \times (t + c - 1)$ homogeneous matrix with entries homogeneous polynomials of degree $a_j - b_i$. The goal of this paper is to extend and complete the results given by the authors in [12] and determine under weakened numerical assumptions the dimension of $W(\underline{b}; \underline{a})$ as well as whether the closure of $W(\underline{b}; \underline{a})$ is a generically smooth irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^n)$.

1. INTRODUCTION

In this paper, we will deal with good and standard determinantal schemes. A scheme $X \subset \mathbb{P}^n$ of codimension c is called *standard determinantal* if its homogeneous saturated ideal can be generated by the maximal minors of a homogeneous $t \times (t+c-1)$ matrix, and X is said to be *good determinantal* if it is standard determinantal and a generic complete intersection. We denote the Hilbert scheme by $\mathrm{Hilb}^p(\mathbb{P}^n)$. Given integers $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$ and $b_1 \leq \cdots \leq b_t$, we denote by $W(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^p(\mathbb{P}^n)$ (resp. $W_s(\underline{b}; \underline{a})$) the locus of good (resp. standard) determinantal schemes $X \subset \mathbb{P}^n$ of codimension c defined by the maximal minors of a $t \times (t + c - 1)$ homogeneous matrix with entries homogeneous polynomials of degree $a_j - b_i$.

In [11] and [12], we addressed the following 3 crucial problems:

(1) To determine the dimension of $W(\underline{b}; \underline{a})$ in terms of a_j and b_i,
(2) Is the closure of $W(\underline{b}; \underline{a})$ an irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^n)$? and
(3) Is $\mathrm{Hilb}^p(\mathbb{P}^n)$ generically smooth along $W(\underline{b}; \underline{a})$?

In [12] we obtained an upper bound for $\dim W(\underline{b}; \underline{a})$ in terms of a_j and b_i which was achieved in the cases $2 \leq c \leq 5$ and $n - c > 0$ (assuming $char(k) = 0$ if $c = 5$), and in codimension $c > 5$ provided certain numerical conditions are satisfied (See [12], Theorems 3.5 and 4.5; and Corollaries 4.7, 4.10 and 4.14). Concerning problems (2) and (3), we gave in [12] an affirmative answer to both questions in the range $2 \leq c \leq 4$ and $n - c \geq 2$, and in the cases $c \geq 5$ and $n - c \geq 1$ provided certain numerical assumptions are verified (See [12], Corollaries 5.3, 5.7, 5.9 and 5.10. See also [7], [11] for the cases $2 \leq c \leq 3$). Note that since every element of $W(\underline{b}; \underline{a})$ has the same Hilbert function, the assumption $n > c$ is close to being necessary for problem (2). Indeed if $n = c$ the problems (2) and (3) become more natural provided we replace $\mathrm{Hilb}^p(\mathbb{P}^n)$ by the postulation Hilbert scheme, see [10].

In this work we attempt to extend and complete the results of [11] and [12]. Indeed if $a_{t+3} > a_{t-2}$ we almost solve problem (1) in Theorem 3.2 while Theorem 3.4 and Corollary 3.8, for $c > 4$, generalize results of [12] for the problems (2) and (3) substantially. To prove these results we use induction on the codimension by successively deleting the columns of the highest degree and the Eagon-Northcott complex associated to a standard determinantal scheme. We also use the theory of Hilbert flag schemes and the depth of certain mixed determinantal schemes (see Theorem 2.7). We end the paper with two conjectures which are supported by our results and by a huge number of examples computed using Macaulay 2.

Notation: Throughout this paper $\mathbb{P}^n$ is the n-dimensional projective space over an algebraically closed field k, $R = k[x_0, x_1, \ldots, x_n]$ and $\mathfrak{m} = (x_0, \ldots, x_n)$. By $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ we denote the sheaf of local morphisms between coherent $\mathcal{O}_X$-modules while $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ denotes the group of morphisms from $\mathcal{F}$ to $\mathcal{G}$. Moreover we set $\hom(\mathcal{F}, \mathcal{G}) = \dim_k \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ and we correspondingly use small letters for the dimension, as a k-vector space, of similar groups. For any quotient A of R of codimension c, we let $K_A = \mathrm{Ext}_R^c(A, R)(-n - 1)$.

In the sequel, $_\mu\mathrm{Hom}_R(M, N)$ denotes homomorphisms of degree μ of graded R-modules. Moreover, we denote the Hilbert scheme by $\mathrm{Hilb}^p(\mathbb{P}^n)$, p the Hilbert polynomial, and $(X) \in \mathrm{Hilb}^p(\mathbb{P}^n)$ the point which corresponds to the subscheme $X \subset \mathbb{P}^n$ with Hilbert polynomial p. We denote by I_X the saturated homogeneous ideal of $X \subset \mathbb{P}^n$. We say that X is *general* in some irreducible subset $W \subset \mathrm{Hilb}^p(\mathbb{P}^n)$ if (X) belongs to a sufficiently small open subset U of W (small enough so that any $(X) \in U$ has all the openness properties that we want to require).

2. Preliminaries

This section provides the background and basic results on standard determinantal ideals, good determinantal ideals and mixed determinantal ideals needed in the sequel. We refer to [3], [6] [12] and [15] for the details.

Let $\mathcal{A} = (f_{ij})_{i=1,\ldots t}^{j=0,\ldots,t+c-2}$, $\deg f_{ij} = a_j - b_i$, be a $t \times (t + c - 1)$ homogeneous matrix and let

$$(2.1) \qquad \varphi : F = \bigoplus_{i=1}^{t} R(b_i) \longrightarrow G := \bigoplus_{j=0}^{t+c-2} R(a_j)$$

be the graded morphism of free R-modules represented by the transpose, $\mathcal{A}^t$, of $\mathcal{A}$. Let $I(\mathcal{A}) = I_t(\mathcal{A})$ be the ideal of R generated by the maximal minors of $\mathcal{A}$. A codimension c subscheme $X \subset \mathbb{P}^n$ is said to be *standard determinantal* if $I_X = I(\mathcal{A})$ for some homogeneous $t \times (t + c - 1)$ matrix $\mathcal{A}$ as above. Moreover X is *good determinantal* if X is standard determinantal and a generic complete intersection in $\mathbb{P}^n$ ([13], Theorem 3.4). In this paper we suppose $c \geq 2$, $t \geq 2$, $b_1 \leq \ldots \leq b_t$ and $a_0 \leq a_1 \leq \ldots \leq a_{t+c-2}$. (Note that the case $t = 1$ for determinantal schemes corresponds to the well-known complete intersections).

Let $W(\underline{b}; \underline{a})$ (resp. $W_s(\underline{b}; \underline{a})$) be the stratum in $\mathrm{Hilb}^p(\mathbb{P}^n)$ consisting of good (resp. standard) determinantal schemes as above. Since our definition does not assume $\mathcal{A}$ to be minimal (i.e. $f_{ij} = 0$ when $b_i = a_j$) for $X = \mathrm{Proj}(R/I_t(\mathcal{A})) \in W(\underline{b}; \underline{a})$ (or $W_s(\underline{b}; \underline{a})$), we

must reconsider Corollary 2.6 of [12] where $\mathcal{A}$ was supposed minimal in the proof (a slight correction to [11] and [12]!). We may, however, use that proof to see that

(2.2) $\quad W(\underline{b};\underline{a}) \neq \emptyset \quad \Leftrightarrow \quad W_s(\underline{b};\underline{a}) \neq \emptyset \quad \Leftrightarrow \quad a_{i-1} \geq b_i \quad$ for all i and $a_{i-1} > b_i \quad$ for some i.

Indeed if we assume the converse of the condition on the right hand side, then either $I_t(\mathcal{A}) \ni 1$ or one of the maximal minors vanishes, i.e. $\operatorname{Proj}(R/I_t(\mathcal{A})) \notin W_s(\underline{b};\underline{a})$. Conversely assuming the right hand side condition (to simplify notations, assume $a_{i-1} \geq b_i$ for $1 \leq i \leq s$ and $a_{i-1} > b_i$ for $s < i \leq t$ for some integer $s < t$), then we may take $\mathcal{A} = \left(\begin{smallmatrix} I & O \\ O & \mathcal{A}' \end{smallmatrix}\right)$ where I is the $s \times s$ identity matrix, O are matrices of zero's and $\mathcal{A}'$, for $t - s > 1$, the $(t-s) \times (t-s+c-1)$ matrix used in [12], Corollary 2.6 to define a good determinantal scheme (if $t - s = 1$ we take the entries of $\mathcal{A}'$ to be a regular sequence). We get $\operatorname{Proj}(R/I_t(\mathcal{A})) \in W(\underline{b};\underline{a})$ and we easily deduce (2.2). Note that by [12], end of p. 2877 and [12], Remark 3.7 we still have that the closures of $W(\underline{b};\underline{a})$ and $W_s(\underline{b};\underline{a})$ in $\operatorname{Hilb}^p(\mathbb{P}^n)$ are equal and irreducible.

Let $A = R/I_X$ be the homogeneous coordinate ring of a standard determinantal scheme. By [3], Theorem 2.20 and [6], Corollaries A2.12 and A2.13 the *Eagon-Northcott complex* yields a minimal free resolution of A

(2.3) $\quad 0 \longrightarrow \wedge^{t+c-1}G^* \otimes S_{c-1}(F) \otimes \wedge^t F \longrightarrow \wedge^{t+c-2}G^* \otimes S_{c-2}(F) \otimes \wedge^t F \longrightarrow \ldots$

$$\longrightarrow \wedge^t G^* \otimes S_0(F) \otimes \wedge^t F \longrightarrow R \longrightarrow A \longrightarrow 0$$

which allows us to deduce that any standard determinantal scheme is arithmetically Cohen-Macaulay (ACM). Moreover if $M_{\mathcal{A}} := \operatorname{coker}(\varphi^*)$ then $K_A(n+1) \cong S_{c-1}M_{\mathcal{A}}(\ell_c)$ where

(2.4) $$\ell_i := \sum_{j=0}^{t+i-2} a_j - \sum_{k=1}^{t} b_k \quad \text{for} \quad 2 \leq i \leq c.$$

Let $\mathcal{B}$ be the matrix obtained by deleting the last column of $\mathcal{A}$, let $B = R/I_B$ be the k-algebra given by the maximal minors of $\mathcal{B}$ and let $M_{\mathcal{B}}$ be the cokernel of $\phi^* = \operatorname{Hom}_R(\phi, R)$ where $\phi : F = \oplus_{i=1}^{t} R(b_i) \to G' := \oplus_{j=0}^{t+c-3} R(a_j)$ is the graded morphism induced by $\mathcal{B}^t$. Recall that if $c > 2$ there is an exact sequence

(2.5) $$0 \longrightarrow B \longrightarrow M_{\mathcal{B}}(a_{t+c-2}) \longrightarrow M_{\mathcal{A}}(a_{t+c-2}) \longrightarrow 0$$

in which $B \longrightarrow M_{\mathcal{B}}(a_{t+c-2})$ is a regular section given by the last column of $\mathcal{A}$. Moreover,

(2.6) $$0 \longrightarrow M_{\mathcal{B}}(a_{t+c-2})^* := \operatorname{Hom}_B(M_{\mathcal{B}}(a_{t+c-2}), B) \longrightarrow B \longrightarrow A \longrightarrow 0$$

is exact by [13] or [11] (e.g. see the text after (3.1) of [11]). Note that the proofs of (2.5)-(2.6) rely heavily on the equality $\operatorname{Ann}(M_{\mathcal{B}}) = I_B$ established in [4]. If $c = 2$ ($\operatorname{codim}_R B = 1$) we have at least $\operatorname{Ann}(M_{\mathcal{B}})I_{t-1}(\mathcal{B}) \subset I_B \subset \operatorname{Ann}(M_{\mathcal{B}})$ by [4]; thus the kernel, $I_{A/B}$, of $B \to A$ satisfies $I_{A/B} = M_{\mathcal{B}}(a_{t+c-2})^*$ (resp. $\tilde{I}_{A/B}|_U = \tilde{M}_{\mathcal{B}}(a_{t+c-2})^*|_U$ where $U = Y - V(I_{t-1}(\mathcal{B}))$) for $c > 2$ (resp. $c = 2$). Due to the Buchsbaum-Rim resolution of $M := M_{\mathcal{A}}$, M is a maximal Cohen-Macaulay A-module, and so is $I_{A/B}$ for $c > 2$ by (2.6).

By successively deleting columns from the right hand side of $\mathcal{A}$, and taking maximal minors, one gets a flag of standard determinantal subschemes

(2.7) $$(\mathbf{X}.) : X = X_c \subset X_{c-1} \subset \ldots \subset X_2 \subset X_1 \subset \mathbb{P}^n$$

where each $X_{i+1} \subset X_i$ (with ideal sheaf $\mathcal{I}_{X_{i+1}/X_i} = \mathcal{I}_i$) is of codimension 1, $X_i \subset \mathbb{P}^n$ is of codimension i $(i = 1, \ldots, c)$ and there exist $\mathcal{O}_{X_i}$-modules $\mathcal{M}_i$ fitting into short exact sequences

$$0 \to \mathcal{O}_{X_i}(-a_{t+i-1}) \to \mathcal{M}_i \to \mathcal{M}_{i+1} \to 0 \ \text{ for } \ 2 \leq i \leq c-1,$$

such that $\mathcal{I}_i(a_{t+i-1})$ is the $\mathcal{O}_{X_i}$-dual of $\mathcal{M}_i$ for $2 \leq i \leq c$ (all this holds also for $i = 1$ provided we restrict the sheaves to $X_1 - V(I_{t-1}(\varphi_1))$ where φ_1 is given by $X_1 = V(I_t(\varphi_1))$). In this context, we let $D_i := R/I_{X_i}$, $I_{D_i} := I_{X_i}$ and $I_i := I_{D_{i+1}}/I_{D_i}$.

Remark 2.1. Assume $t \geq 2$, $b_1 \leq \ldots \leq b_t$, $a_0 \leq a_1 \leq \ldots \leq a_{t+c-2}$ and let $\alpha \geq 1$ be an integer. If X is general in $W(\underline{b}; \underline{a})$ and $a_{i-\min(\alpha,t)} - b_i \geq 0$ for $\min(\alpha, t) \leq i \leq t$, then

$$(2.8) \qquad\qquad \mathrm{codim}_{X_j} Sing(X_j) \geq \min\{2\alpha - 1, j+2\} \text{ for all } j = 2, \cdots, c.$$

This follows from the theorem of [5], arguing as in [5], Example 2.1. In particular, if $\alpha \geq 3$, we get for each $i > 0$ that $X_i \hookrightarrow \mathbb{P}^n$ and $X_{i+1} \hookrightarrow X_i$ are local complete intersections (l.c.i.'s) outside some set Z_i of codimension at least $\min(4, i+1)$ in X_{i+1}, cf. the paragraph below. Finally note that it is easy to show (2.8) for $(j, \alpha) = (1, 2)$ provided $a_{i-2} > b_i$ for $2 \leq i \leq t$, cf. [1], (1.10).

Now let $Z \subset X$ (resp. $Z_i \subset X_i$) be some closed subset such that $U := X - Z \hookrightarrow \mathbb{P}^n$ (resp. $U_i := X_i - Z_i \hookrightarrow \mathbb{P}^n$) is an l.c.i. By the fact that the 1st Fitting ideal of M is equal to $I_{t-1}(\varphi)$, we get that $\tilde{M}$ is locally free of rank 1 precisely on $X - V(I_{t-1}(\varphi))$ [2], Lemma 1.4.8. Since the set of non-l.c.i points of $X \hookrightarrow \mathbb{P}^n$ is precisely $V(I_{t-1}(\varphi))$ by e.g. [17], Lemma 1.8, we get that $U \subset X - V(I_{t-1}(\varphi))$ and that $\tilde{M}$ is locally free on U. Indeed $\mathcal{M}_i$ and $\mathcal{I}_{X_i}/\mathcal{I}_{X_i}^2$ are locally free on U_i, as well as on $U_{i-1} \cap X_i$. Note also that since $V(I_{t-1}(\mathcal{B})) \subset V(I_t(\mathcal{A}))$, we may suppose $Z_{i-1} \subset X_i$.

Let us recall the following useful comparison of cohomology groups. If L and N are finitely generated A-modules such that $\mathrm{depth}_{I(Z)} L \geq r+1$ and $\tilde{N}$ is locally free on $U := X - Z$, then the natural map

$$(2.9) \qquad\qquad \mathrm{Ext}_A^i(N, L) \longrightarrow H_*^i(U, \mathcal{H}om_{\mathcal{O}_X}(\tilde{N}, \tilde{L}))$$

is an isomorphism, (resp. an injection) for $i < r$ (resp. $i = r$) cf. [9], exp. VI. Note that we interpret $I(Z)$ as $\mathbf{m}$ if $Z = \emptyset$.

In [12], Conjecture 6.1, we conjectured the dimension of (a non-empty) $W(\underline{b}; \underline{a})$ in terms of the invariant

$$(2.10) \ \ \lambda_c := \sum_{i,j} \binom{a_i - b_j + n}{n} + \sum_{i,j} \binom{b_j - a_i + n}{n} - \sum_{i,j} \binom{a_i - a_j + n}{n} - \sum_{i,j} \binom{b_i - b_j + n}{n} + 1.$$

Here the indices belonging to a_j (resp. b_i) range over $0 \leq j \leq t+c-2$ (resp. $1 \leq i \leq t$) and $\binom{a}{b} = 0$ whenever $a < b$.

Conjecture 2.2. *Given integers $a_0 \leq a_1 \leq \ldots \leq a_{t+c-2}$ and $b_1 \leq \ldots \leq b_t$, we set $\ell_i := \sum_{j=0}^{t+i-2} a_j - \sum_{k=1}^{t} b_k$ and $h_{i-3} := 2a_{t+i-2} - \ell_i + n$, for $i = 3, 4, \ldots, c$. Assume $a_{i-\min([c/2]+1,t)} \geq b_i$ for $\min([c/2] + 1, t) \leq i \leq t$. Then we have*

$$\dim W(\underline{b}; \underline{a}) = \lambda_c + K_3 + K_4 + \ldots + K_c$$

where $K_3 = \binom{h_0}{n}$ *and* $K_4 = \sum_{j=0}^{t+1} \binom{h_1+a_j}{n} - \sum_{i=1}^{t} \binom{h_1+b_i}{n}$ *and in general*

$$K_{i+3} = \sum_{\substack{r+s=i \\ r,s \geq 0}} \sum_{\substack{0 \leq i_1 < ... < i_r \leq t+i \\ 1 \leq j_1 \leq ... \leq j_s \leq t}} (-1)^{i-r} \binom{h_i + a_{i_1} + \cdots + a_{i_r} + b_{j_1} + \cdots + b_{j_s}}{n} \text{ for } 0 \leq i \leq c-3.$$

In [12] we proved that the right hand side in the formula for $\dim W(\underline{b}; \underline{a})$ in Conjecture 2.2 is always an upper bound for $\dim W(\underline{b}; \underline{a})$ ([12], Theorem 3.5) and, moreover, that Conjecture 2.2 holds in the range

(2.11) $\qquad 2 \leq c \leq 5 \text{ and } n - c > 0 \text{ (supposing } char(k) = 0 \text{ if } c = 5),$

cf. [12], Theorems 3.5 and 4.5; and Corollaries 4.7, 4.10 and 4.14. See also [7], [11] for the cases $2 \leq c \leq 3$. In [10], however, the first author gave a counterexample to Conjecture 2.2 for zero-dimensional schemes (see the section of conjectures in this paper).

Remark 2.3. Conjecture 2.2 holds in codimension $c \geq 6$ provided the numerical conditions of [12], Corollary 4.15 are satisfied. By [12], Remarks 4.16, 4.17 and Corollary 4.18 Conjecture 2.2 holds (without assuming $n > c$ or $char(k) = 0$) for any $W(\underline{b}; \underline{a})$ satisfying

$$a_{t+3} > a_{t-1} + a_t + a_{t+1} - a_0 - a_1 \qquad\qquad \text{in codimension } c = 5,$$

$$a_{t+2} > a_{t-1} + a_t - a_0 \qquad\qquad\qquad\qquad \text{for } n = c = 4,$$

$$a_{t+1} > a_{t-1} \text{ and } a_{i-2} > b_i \text{ for } 2 \leq i \leq t \qquad \text{for } n = c = 3.$$

Note that in the case $n = c = 3$ we need $a_{i-2} > b_i$ (not $a_{i-2} \geq b_i$ as assumed in [12]) for $2 \leq i \leq t$ for the proof of [12], Corollary 4.18 to hold, see the last sentence of Remark 2.1.

A goal of this paper is to extend the results summarized in (2.11) and in the above remark. To do this we will need the following result where $\underline{a} = a_0, a_1, ..., a_{t+c-2}$ and $\underline{a}' = a_0, a_1, ..., a_{t+c-3}$.

Proposition 2.4. *Let* $c \geq 3$, *let* $(X) \in W(\underline{b}; \underline{a})$ *and suppose* $\dim W(\underline{b}; \underline{a}') \geq \lambda_{c-1} + K_3 + K_4 + ... + K_{c-1}$ *and* $\operatorname{depth}_{I(Z)} B \geq 2$ *for a general* $Y = \operatorname{Proj}(B) \in W(\underline{b}; \underline{a}')$. *If*

(2.12) $$\quad {}_0\hom_R(I_Y, I_{X/Y}) \leq \sum_{j=0}^{t+c-3} \binom{a_j - a_{t+c-2} + n}{n},$$

then $\dim W(\underline{b}; \underline{a}) = \lambda_c + K_3 + K_4 + ... + K_c$ *and* (2.12) *turns out to be an equality.*

Proof. See [10], Proposition 3.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Let us recall how we proved Remark 2.3 since we want to generalize that approach. Letting $a = a_{t+i-2} - a_{t+i-1}$ we showed in [12] that $\operatorname{Hom}_{D_{i-1}}(I_{i-1}, I_i) \cong D_i(a)$ for $i < c$ using $\operatorname{depth}_{I(Z_{i-1})} D_i \geq 2$. Indeed this follows from (2.9), i.e. from

$$H_*^0(U_{i-1}, \mathcal{H}om(\mathcal{I}_{i-1}, \mathcal{I}_i)) \cong H_*^0(U_{i-1}, \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{I}_{i-1} \otimes_{\mathcal{O}_{X_{i-1}}} \mathcal{O}_{X_i} \otimes \mathcal{I}_i^*, \mathcal{O}_{X_i}))$$

because $\tilde{M}_i|_{U_{i-1}} \cong \mathcal{I}_i^*(-a_{t+i-1})|_{U_{i-1}}$ is locally free, $\tilde{M}_i|_{U_{i-1}} \cong \tilde{M}_{i-1} \otimes \mathcal{O}_{X_i}|_{U_{i-1}}$ and hence $\mathcal{I}_{i-1} \otimes_{\mathcal{O}_{X_{i-1}}} \mathcal{O}_{X_i}|_{U_{i-1}} \cong \mathcal{I}_i(-a)|_{U_{i-1}}$, and note that $U_{i-1} \cap X_i \subset U_i$. Then since

(2.13) $\qquad 0 \to \operatorname{Hom}_R(I_{i-1}, I_i) \to \operatorname{Hom}_R(I_{D_i}, I_i) \to \operatorname{Hom}_R(I_{D_{i-1}}, I_i)$

is exact we were able to show (2.12) for $X = X_c \subset Y = X_{c-1}$ by putting $_0\mathrm{Hom}_R(I_{D_{c-2}}, I_{c-1}) = 0$ (through making the minimal generators of $I_{D_{c-2}}$ and I_{c-1} explicit via (2.3)). Using Proposition 2.4 we obtained Remark 2.3 if the conjecture holds for $W(\underline{b}; \underline{a}') \ni Y := X_{c-1}$.

Let us finish this section by gathering all the results on the depth of mixed determinantal ideals and cogenerated ideals needed in the next section. We start by fixing some notation. Let

$$(2.14) \qquad \mathcal{A} = \begin{pmatrix} f_1^1 & f_2^1 & \cdots & f_q^1 \\ f_1^2 & f_2^2 & \cdots & f_q^2 \\ \vdots & \vdots & \vdots & \vdots \\ f_1^p & f_2^p & \cdots & f_q^p \end{pmatrix}$$

be a homogeneous $p \times q$ matrix with entries homogeneous polynomials $f_i^j \in k[x_0, \cdots, x_n]$ of degree $a_j - b_i$. For any choice $(\underline{\alpha}; \underline{\beta}) = (\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_m)$ of row indexes $1 \leq \alpha_1 < \cdots < \alpha_m \leq p$ and of column indexes $1 \leq \beta_1 < \cdots < \beta_m \leq q$, we denote by $I_{(\underline{\alpha}; \underline{\beta})}(\mathcal{A})$ the *ideal cogenerated by* $(\underline{\alpha}; \underline{\beta})$, i.e. $I_{(\underline{\alpha}, \underline{\beta})}$ is the homogeneous ideal of $k[x_0, \cdots, x_n]$ generated by all $(m + 1) \times (m + 1)$ minors of $\mathcal{A}$, all $i \times i$ minors of the rows $1, \cdots, \alpha_i - 1$ for $i = 1, \cdots, m$ and all $i \times i$ minors of the columns $1, \cdots, \beta_i - 1$ for $i = 1, \cdots, m$.

Example 2.5. (1) If $(\underline{\alpha}; \underline{\beta}) = (1, \cdots, t - 1; 1, \cdots, t - 1)$, then $I_{(\underline{\alpha}; \underline{\beta})}(\mathcal{A})$ is the ideal generated by the $t \times t$ minors of $\mathcal{A}$, i.e., $I_{(\underline{\alpha}; \underline{\beta})}(\mathcal{A}) = I_t(\mathcal{A})$.

(2) Let $\mathcal{A}$ be a homogeneous $p \times q$ matrix and let $\mathcal{A}_i$ be the matrix obtained by deleting the last $i - 1$ columns of $\mathcal{A}$. We assume $p \leq q$ and we fix an integer $m < p$ and $(\underline{\alpha}; \underline{\beta}) = (1, \cdots, m; \beta_1, \cdots, \beta_m)$. Set $j := \min\{i \mid \beta_i > i\}$, $c_1 = 1$ and $c_s = q + 2 - \beta_{m+2-s}$ for $2 \leq s \leq m + 2 - j$. The ideal cogenerated by $(\underline{\alpha}; \underline{\beta}) = (1, \cdots, m; \beta_1, \cdots, \beta_m)$ can be identified with the following mixed determinantal ideal

$$I_{(\underline{\alpha}; \underline{\beta})}(\mathcal{A}) = I_{m+1}(\mathcal{A}_{c_1}) + I_m(\mathcal{A}_{c_2}) + ... + I_j(\mathcal{A}_{c_{m+2-j}}),$$

where $I_\lambda(\mathcal{A}_\varrho)$ denotes the ideal generated by all $\lambda \times \lambda$ minors of $\mathcal{A}_\varrho$.

Theorem 2.6. *Let $\mathcal{A} = (x_{i,j})$ be a $p \times q$ matrix of indeterminates and let $(\underline{\alpha}; \underline{\beta}) = (\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_m)$ be a choice of m row indexes and of m column indexes. Let $I_{(\underline{\alpha}; \underline{\beta})}$ be the ideal cogenerated by $(\underline{\alpha}; \underline{\beta})$. Then, $I_{(\underline{\alpha}; \underline{\beta})}(\mathcal{A})$ is a Cohen-Macaulay ideal and*

$$(2.15) \qquad ht(I_{(\underline{\alpha}; \underline{\beta})}(\mathcal{A})) = pq - (p + q + 1)m + \sum_{i=1}^{m}(\alpha_i + \beta_i).$$

Proof. See [3], Corollary 5.12. □

We are now ready to state the main result of this section and to prove that the above formula for the height of a cogenerated ideal associated to a matrix with entries that are indeterminates also works for a general homogeneous matrix with entries thar are homogeneous polynomials of positive degree. Indeed, we have

Theorem 2.7. *Fix integers $b_1, \cdots, b_q$ and $a_1, \cdots, a_p$. Let $\mathcal{A} = (f_i^j)_{i=1,\cdots,q}^{j=1,\cdots,p}$ be a general homogeneous $p \times q$ matrix $\mathcal{A}$ with entries that are homogeneous forms of degree $a_j - b_i$, and assume that $a_j > b_i$ for all j, i. Let $(\underline{\alpha}; \underline{\beta}) = (\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_m)$ be any choice of m*

row indexes and of m column indexes, and suppose $n+1 \geq pq-(p+q+1)m+\sum_{i=1}^{m}(\alpha_i+\beta_i)$. Then $I_{(\underline{\alpha};\underline{\beta})}(\mathcal{A})$ is a Cohen-Macaulay ideal and

$$ht(I_{(\underline{\alpha};\underline{\beta})}(\mathcal{A})) = pq - (p+q+1)m + \sum_{i=1}^{m}(\alpha_i + \beta_i).$$

Proof. We clearly have $ht(I_{(\underline{\alpha};\underline{\beta})}(\mathcal{A})) \leq pq - (p+q+1)m + \sum_{i=1}^{m}(\alpha_i + \beta_i)$ and it will be enough to construct an example of homogeneous $p \times q$ matrix $\mathcal{M}$ with entries homogeneous forms $f_i^j \in k[x_0, \cdots, x_n]$ such that the ideal $I_{(\underline{\alpha};\underline{\beta})}(\mathcal{M})$ cogenerated by $(\underline{\alpha}; \underline{\beta})$ is Cohen-Macaulay and $ht(I_{(\underline{\alpha};\underline{\beta})}(\mathcal{M})) = pq - (p+q+1)m + \sum_{i=1}^{m}(\alpha_i + \beta_i)$. Therefore, let $n + 1 \geq pq - (p+q+1)m + \sum_{i=1}^{m}(\alpha_i + \beta_i)$. We will distinguish 2 cases:

<u>Case 1.</u> Assume $a_j - b_i = 1$ for all i, j (i.e. the entries of the matrix $\mathcal{A}$ are linear forms). By Theorem 2.6 for any choice of p, q and $(\underline{\alpha}; \underline{\beta})$, we have the matrix $\overline{\mathcal{A}} = (x_{i,j})$ of indeterminates and the ideal $I_{(\underline{\alpha};\underline{\beta})}(\overline{\mathcal{A}}) \subset S := k[x_{1,1} \cdots, x_{p,q}]$ cogenerated by $(\underline{\alpha}; \underline{\beta})$ is Cohen-Macaulay of height

$$ht(I_{(\underline{\alpha};\underline{\beta})}(\overline{\mathcal{A}})) = pq - (p+q+1)m + \sum_{i=1}^{m}(\alpha_i + \beta_i).$$

We choose $pq - n - 1$ general linear forms $\ell_1, \cdots, \ell_{pq-n-1} \in S = k[x_{1,1} \cdots, x_{p,q}]$ and we set $S/(\ell_1, \cdots, \ell_{pq-n-1}) \cong k[x_0, x_1, \cdots, x_n] =: R$. Let us call $I \subset R$ the ideal of R isomorphic to the ideal $I_{(\underline{\alpha};\underline{\beta})}(\overline{\mathcal{A}})/(\ell_1, \cdots, \ell_{pq-n-1})$ of $S/(\ell_1, \cdots, \ell_{pq-n-1})$. Note that I is a Cohen-Macaulay ideal of height $ht(I) = pq - (p+q+1)m + \sum_{i=1}^{m}(\alpha_i + \beta_i)$ and, in addition, I is nothing but the ideal $I_{(\underline{\alpha};\underline{\beta})}(\mathcal{M})$ where $\mathcal{M} = (m_i^j)$ is a $p \times q$ homogeneous matrix with entries linear forms in $k[x_0, x_1, \cdots, x_n]$ obtained from $\overline{\mathcal{A}} = (x_{i,j})$ by substituting using the equations $\ell_1, \cdots, \ell_{pq-n-1}$, which proves what we want.

<u>Case 2.</u> Assume $a_j - b_i \geq 1$ for all i, j. In this case, it is enough to raise the entry m_i^j of the above matrix $\mathcal{M}$ to the power $a_j - b_i$. $\square$

Remark 2.8. Fix integers $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$ and $b_1 \leq ... \leq b_t$. Let $X \subset \mathbb{P}^n$, $(X) \in W(\underline{b}; \underline{a})$ be a general determinantal ideal associated to a $t \times (t+c-1)$ matrix $\mathcal{A}$ represented by a graded morphism as in (2.1). Let

$$(\mathbf{X}.) : X = X_c \subset X_{c-1} \subset ... \subset X_2 \subset X_1 \subset \mathbb{P}^n$$

be the flag of standard determinantal subschemes that we obtain by successively deleting columns from the right hand side of $\mathcal{A}$ and let φ_i be the graded morphism associated to the matrix which defines X_i. Assume $a_0 > b_t$ and $c \geq 3$. Applying Theorem 2.7, we get the following formula

(2.16) $$\dim R/(I_{t-1}(\varphi_1) + I_t(\varphi_{c-1})) = \dim D_{c-1} - 2 .$$

Even more, if $\dim D_{c-1} \geq 3$ and $c \geq 4$ we have the equalities

(2.17) $$\dim R/(I_{t-1}(\varphi_i) + I_t(\varphi_{c-1})) = \dim D_{c-1} - i - 1$$

for $1 \leq i \leq 2$ which will play an important role in the next section.

Remark 2.9. If $t = 2$ we can prove (2.16) directly as follows. Let $\mathcal{A} = [\mathcal{C}, \mathcal{D}, v]$ where $\mathcal{C}$ is a 2 by 2 general enough matrix in the variables x_0, x_1, x_2, x_3 (e.g. with rows $(x_0^{a_0-b_1}, x_1^{a_1-b_1})$ and $(x_2^{a_0-b_2}, x_3^{a_1-b_2})$), v is some column and $\mathcal{D}$ is a 2 by $c-2$ matrix whose first row is $(x_4^{a_2-b_1}, x_5^{a_3-b_1}, ..., x_c^{a_{c-2}-b_1}, 0)$ and whose second row is $(0, x_4^{a_3-b_2}, x_5^{a_4-b_2}, ..., x_c^{a_{c-1}-b_2})$. Note that φ_1 corresponds to $\mathcal{C}$ and φ_{c-1} to $[\mathcal{C}, \mathcal{D}]$. Since $\mathcal{C}$ is general, we get $\mathrm{codim}_R R/I_{t-1}(\varphi_1) = 4$, i.e. the radical of $I_{t-1}(\varphi_1)$ is (x_0, x_1, x_2, x_3). Since it is clear that the radical of the ideal generated by the maximal minors of $\mathcal{D}$ is $(x_4, x_5, ..., x_c)$, it follows that the scheme $X_{c-1} = \mathrm{Proj}(D_{c-1}) \in W(\underline{b}; \underline{a}')$ given by the maximal minors of $[\mathcal{C}, \mathcal{D}]$ satisfies (2.16). The general element X of $W(\underline{b}; \underline{a})$ will therefore have a flag where X_{c-1} satisfies (2.16).

3. Smoothness and dimension of the determinantal locus

This section is the heart of the paper and contains the results which generalize quite a lot of our previous contributions to problems (1)-(3) stated in the introduction.

Proposition 3.1. *With notation as in Remark 2.8, let $c \geq 3$ and suppose (2.16) (this holds if $a_0 > b_t$). If $a_{t+c-2} > a_{t-2}$ then (2.12) holds for $X := X_c \subset Y := X_{c-1}$, i.e.*

$$_0\mathrm{hom}_R(I_{D_{c-1}}, I_{D_c/D_{c-1}}) \leq \sum_{j=0}^{t+c-3} \binom{a_j - a_{t+c-2} + n}{n}.$$

In particular we get $\dim W(\underline{b}; \underline{a}) = \lambda_c + K_3 + K_4 + ... + K_c$ *provided* $\dim W(\underline{b}; \underline{a}') = \lambda_{c-1} + K_3 + K_4 + ... + K_{c-1}$ *where* $\underline{a}' = a_0, a_1, ..., a_{t+c-3}$.

Proof. We *claim* that $\mathrm{depth}_{I(Z_j)} D_{c-1} \geq 2$ for all j satisfying $0 < j < c-1$. Indeed by the discussion right after Remark 2.1 we see that we may take $I(Z_j)$ to be the ideal $I_{t-1}(\varphi_j) \subset R$ of submaximal minors of the matrix which defines X_j, and $I(Z_j)D_{c-1}$ to be $(I_{t-1}(\varphi_j) + I_t(\varphi_{c-1}))D_{c-1}$. It follows that $\mathrm{depth}_{I_{t-1}(\varphi_j)} D_{c-1} \geq 2$ for $j = 1$, i.e. $\dim D_{c-1} - \dim D_{c-1}/(I_{t-1}(\varphi_1) + I_t(\varphi_{c-1}))D_{c-1} \geq 2$, implies the claim. Hence we conclude the proof of the claim by (2.16).

For every j, $0 < j < c-1$, put $a := a_{t+j-1} - a_{t+c-2}$. We *claim* that

$$\mathrm{Hom}_{D_j}(I_j, I_{c-1}) \cong D_{c-1}(a) .$$

To prove this claim we remark that $\mathrm{depth}_{I(Z_j)} I_{c-1} \geq 2$ since I_{c-1} is maximally CM. Using that $\mathcal{I}_j$ is locally free on U_j and the arguments in the text before (2.13), see the text accompanying (2.7) for $j = 1$, we get

$$\mathcal{I}_j \otimes_{\mathcal{O}_{X_j}} \mathcal{O}_{X_{c-1}}|_{U_j} \cong \mathcal{I}_j \otimes_{\mathcal{O}_{X_j}} \mathcal{O}_{X_{j+1}} \otimes ... \otimes_{\mathcal{O}_{X_{c-2}}} \mathcal{O}_{X_{c-1}}|_{U_j} \cong \mathcal{I}_{c-1}(-a)|_{U_j}.$$

It follows that $\mathcal{H}om_{\mathcal{O}_{X_j}}(\mathcal{I}_j, \mathcal{I}_{c-1})(-a) \cong \mathcal{H}om_{\mathcal{O}_{X_{c-1}}}(\mathcal{I}_{c-1}, \mathcal{I}_{c-1}) \cong \mathcal{O}_{X_{c-1}}$ are isomorphic as sheaves on $U_j \cap X_{c-1}$, i.e. we get that $H^0_*(U_j, \mathcal{H}om(\mathcal{I}_j, \mathcal{I}_{c-1})) \cong H^0_*(U_j, \mathcal{O}_{X_{c-1}})(a)$ and hence the claim from $\mathrm{depth}_{I(Z_j)} D_{c-1} = \mathrm{depth}_{I(Z_j)} I_{c-1} \geq 2$ and (2.9).

Now we repeatedly use the exact sequence

(3.1) $0 \rightarrow D_{c-1}(a) \cong \mathrm{Hom}_{D_j}(I_j, I_{c-1}) \rightarrow \mathrm{Hom}_R(I_{D_{j+1}}, I_{c-1}) \rightarrow \mathrm{Hom}_R(I_{D_j}, I_{c-1}) \rightarrow$

for $j = c-2, c-3, ..., 1$. Since $a_{t+j-1} \leq a_{t+c-2}$, we have $\dim D_{c-1}(a)_0 = \binom{a+n}{n}$. It follows that

$$_0\hom(I_{D_{c-1}}, I_{c-1}) \leq {}_0\hom(I_{D_1}, I_{c-1}) + \sum_{i=t}^{t+c-3} \binom{a_i - a_{t+c-2} + n}{n}$$

where we have replaced a_{t+j-1} by a_i in which case $1 \leq j \leq c-2$ corresponds to $t \leq i \leq t+c-3$.

It remains to prove that $_0\hom(I_{D_1}, I_{c-1}) \leq \sum_{i=0}^{t-1} \binom{a_i - a_{t+c-2}+n}{n}$ since we then by Proposition 2.4 get the dimension formula. Using that $X_1 = \text{Proj}(D_1)$ is a hypersurface of degree ℓ_1, we find $_0\hom(I_{D_1}, I_{c-1}) \cong \dim I_{c-1}(\ell_1)_0$ where $\ell_k = \sum_{j=0}^{t+k-2} a_j - \sum_{i=1}^{t} b_i$. Now we have to make the degrees of the minimal generators of I_{c-1} explicit. Taking a close look at (2.3), we see that a minimal generator f of $I_{c-1} \cong I_{D_c}/I_{D_{c-1}}$ of the smallest possible degree has degree $s(I_{c-1}) := \ell_c - \sum_{j=t-1}^{t+c-3} a_j$ because $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$. Since $\ell_1 - s(I_{c-1}) = a_{t-1} - a_{t+c-2} \leq 0$ by the definition of ℓ_k, we get either $\dim I_{c-1}(\ell_1)_0 = 0$ or $a_{t-1} = a_{t+c-2}$. In the latter case the assumption $a_{t+c-2} > a_{t-2}$ implies that the degrees of all minimal generators, except for f, are strictly greater than $s(I_{c-1})$, i.e. we get $\dim I_{c-1}(\ell_1)_0 = \binom{a_{t-1} - a_{t+c-2}+n}{n}$ and we are done. $\square$

By repeatedly using Proposition 3.1 we get the

Theorem 3.2. *Let $X \subset \mathbb{P}^n$, $(X) \in W(\underline{b}; \underline{a})$, be a general determinantal scheme and suppose $a_0 > b_t$. Moreover if $c \geq 6$ we suppose $a_{t+3} > a_{t-2}$ (or $a_{t+4} > a_{t-2}$ provided chark $= 0$) and if $3 \leq c \leq 5$ we suppose $a_{t+c-2} > a_{t-2}$. Then we have*

$$\dim W(\underline{b}; \underline{a}) = \lambda_c + K_3 + K_4 + \cdots + K_c .$$

Proof. If $3 \leq c \leq 6$ (*chark* $= 0$ if $c = 6$) and $a_{t+c-2} > a_{t-2}$ we use (2.11) to find $\dim W(\underline{b}; \underline{a}')$ and we conclude the proof by Proposition 3.1.

If $c \geq 6$ and $a_{t+3} > a_{t-2}$ (resp. $a_{t+4} > a_{t-2}$ if $c \geq 7$) we repeatedly use Proposition 3.1 to reduce to the case $c = 5$ (resp. $c = 6$) and we conclude by the first part of the proof (note that the assumption $a_{t+c-2} > a_{t-2}$ of Proposition 3.1 is satisfied in this induction). $\square$

Remark 3.3. We expect that the assumption $a_0 > b_t$ can be weakened in Theorem 3.2, as well as in (2.16). At least it does for $c = 3$ provided we assume $a_{i-2} > b_i$ for $2 \leq i \leq t$. Indeed since $I_t(\varphi_2) \subset I_{t-1}(\varphi_1)$ we first show (2.16) using Remark 2.1. Then the proof above applies to conclude as in Theorem 3.2 provided $a_{t+1} > a_{t-2}$, cf. Remark 2.3.

If the condition (2.17) is satisfied, then we can prove the following result for $\overline{W(\underline{b}; \underline{a})}$ to be a generically smooth irreducible component.

Theorem 3.4. *Let $X \subset \mathbb{P}^n$, $(X) \in W(\underline{b}; \underline{a})$, be a general determinantal scheme of dimension $n - c \geq 1$, let $c > 2$ and let $X = X_c \subset X_{c-1} \subset ... \subset X_2 \subset \mathbb{P}^n$, $X_i = \text{Proj}(D_i)$, be the flag obtained by successively deleting columns from the right hand side. If $a_0 > b_t$,*

$$_0\text{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_2) = 0 \quad and \quad {}_0\text{Ext}^1_{D_3}(I_{D_3}/I^2_{D_3}, I_i) = 0 \text{ for } i = 3, ..., c-1 ,$$

then $\overline{W(\underline{b}; \underline{a})}$ is a generically smooth irreducible component of the Hilbert scheme $\text{Hilb}^p(\mathbb{P}^n)$.

Remark 3.5. If $n - c \geq 2$ then we have $_0\text{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_2) = 0$ provided $c = 3$, and $_0\text{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_2) = {}_0\text{Ext}^1_{D_3}(I_{D_3}/I^2_{D_3}, I_3) = 0$ provided $c = 4$ by [12], (5.4)-(5.8). Hence

the conclusion of Theorem 3.4 holds provided $n - c \geq 2$ and $3 \leq c \leq 4$. Moreover if $n - c \geq 1$ and $c > 4$ then both Ext-groups above still vanish by [12] and we may in this case replace the assumption of Theorem 3.4 given by the displayed formula with

$$_0\mathrm{Ext}^1_{D_3}(I_{D_3}/I^2_{D_3}, I_i) = 0 \text{ for } i = 4, ..., c - 1 .$$

Note that in the case $n - c \geq 1$ and $c = 2$, the conclusion of Theorem 3.4 holds and moreover, $\mathrm{Hilb}^p(\mathbb{P}^n)$ is smooth at any $(X) \in W(\underline{b}; \underline{a})$ by [7].

Remark 3.6. For $c > 2$ one knows that $\mathrm{Hilb}^p(\mathbb{P}^n)$ is not always smooth at any $(X) \in W(\underline{b}; \underline{a})$ [14]. Indeed, since $W(\underline{b}; \underline{a})$ is irreducible, it is not difficult to find singular points of $\mathrm{Hilb}^p(\mathbb{P}^n)$ by first computing its tangent space dimension, $h^0(\mathcal{N}_X)$, at a general $(X) \in W(\underline{b}; \underline{a})$, using Macaulay 2. Then by experimenting with special choices of $(X_0) \in W(\underline{b}; \underline{a})$ one may find $h^0(\mathcal{N}_X) < h^0(\mathcal{N}_{X_0})$ which means that $\mathrm{Hilb}^p(\mathbb{P}^n)$ is singular at (X_0), see [16] which even computes the obstructions of deformations, using Singular, in a related case.

Proof. Due to Theorem 5.1 of [12] we must show that

$$(3.2) \qquad\qquad _0\mathrm{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i) = 0 \text{ for } i = 2, ..., c - 1.$$

By assumption we need to prove the vanishing (3.2) for $i = 4, ..., c - 1$ and $c > 4$. By induction on c it suffices to show it for $i = c - 1$, $c \geq 5$. Hence it suffices to see that there exist injections

$$(3.3) \qquad _0\mathrm{Ext}^1_{D_{j+1}}(I_{D_{j+1}}/I^2_{D_{j+1}}, I_{c-1}) \hookrightarrow {}_0\mathrm{Ext}^1_{D_j}(I_{D_j}/I^2_{D_j}, I_{c-1}) \text{ for } j = 2, ..., c - 2.$$

By (2.17) and the arguments in the first paragraph of the proof of Proposition 3.1 we may suppose $\mathrm{depth}_{I(Z_j)} D_{c-1} \geq 3$ for all j satisfying $1 < j < c - 1$.

We *claim* that the left-exact sequence (3.1) is also right-exact, i.e. that the rightmost map of the Hom-groups is surjective for $1 < j < c - 1$. To show this, it certainly suffices to prove $\mathrm{Ext}^1_R(I_j, I_{c-1}) = 0$. However, by paying closer attention to the modules of (3.1) we shall see that also $\mathrm{Ext}^1_{D_j}(I_j, I_{c-1}) = 0$ for $1 < j < c - 1$ suffices for proving the claim. Indeed if we apply $(-) \otimes_R D_j$ to $0 \to I_{D_j} \to I_{D_{j+1}} \to I_j \to 0$ we get the right-exact sequence $I_{D_j} \otimes_R D_j \to I_{D_{j+1}} \otimes_R D_j \to I_j \to 0$ where $I_{D_{j+1}} \otimes_R D_j \cong I_{D_{j+1}}/I_{D_j} \cdot I_{D_{j+1}}$ and since $I_{D_j} \otimes_R D_{j+1} \cong I_{D_j}/I_{D_j} \cdot I_{D_{j+1}} = \ker(I_{D_{j+1}}/I_{D_j} \cdot I_{D_{j+1}} \to I_j)$ we obtain the exact sequence

$$0 \longrightarrow I_{D_j} \otimes_R D_{j+1} \longrightarrow I_{D_{j+1}} \otimes_R D_j \longrightarrow I_j \longrightarrow 0$$

to which we apply $\mathrm{Hom}_{D_j}(-, I_{c-1})$. Then we get exactly (3.1) continued to the right by $\mathrm{Ext}^1_{D_j}(I_j, I_{c-1})$. Thus the vanishing of $\mathrm{Ext}^1_{D_j}(I_j, I_{c-1})$ implies that (3.1) is right-exact.

To see that $\mathrm{Ext}^1_{D_j}(I_j, I_{c-1}) = 0$ we use the isomorphism $\mathcal{H}om_{\mathcal{O}_{X_j}}(\mathcal{I}_j, \mathcal{I}_{c-1})(-a)|_{U_j \cap X_{c-1}} \cong \mathcal{O}_{X_{c-1}}|_{U_j \cap X_{c-1}}$ which we obtained in the proof of Proposition 3.1. By (2.9) it follows that

$$\mathrm{Ext}^1_{D_j}(I_j, I_{c-1}) \cong H^1_*(U_j, \mathcal{H}om(\mathcal{I}_j, \mathcal{I}_{c-1})) \cong H^1_*(U_j, \mathcal{O}_{X_{c-1}}(a)) = 0$$

because we have $\mathrm{depth}_{I(Z_j)} D_{c-1} \geq 3$ and hence $\mathrm{depth}_{I(Z_j)} I_{c-1} \geq 3$ (I_{c-1} is maximally CM). This proves the claim.

Now we can rewrite the exact sequence (3.1) as

$$(3.4) \qquad 0 \to D_{c-1}(a) \to \mathrm{Hom}_{D_{j+1}}(I_{D_{j+1}}/I^2_{D_{j+1}}, I_{c-1}) \to \mathrm{Hom}_{D_j}(I_{D_j}/I^2_{D_j}, I_{c-1}) \to 0 .$$

Sheafifying, restricting to $U_j \cap X_{c-1}$ (note that $\mathcal{I}_{D_{j+1}}$ is also locally free on $U_j \cap X_{c-1}$) and taking cohomology, we get

$$\to H^1(U_j, \mathcal{O}_{X_{c-1}}(a)) \to H^1(U_j, \mathcal{H}om(\mathcal{I}_{X_{j+1}}/\mathcal{I}^2_{X_{j+1}}, \mathcal{I}_{c-1})) \to H^1(U_j, \mathcal{H}om(\mathcal{I}_{X_j}/\mathcal{I}^2_{X_j}, \mathcal{I}_{c-1})) \to$$

Since $\operatorname{depth}_{I(Z_j)} D_{c-1} \geq 3$, the two latter H^1-groups are by (2.9) isomorphic to the $_0\text{Ext}^1$-groups quoted in (3.3) and since $H^1(U_j, \mathcal{O}_{X_{c-1}}(a)) = 0$ we are done. $\qquad\square$

Remark 3.7. Since (2.17) holds also for $i = 3$ provided $\dim D_{c-1} \geq 4$, $c \geq 5$ and $a_0 > b_t$ by Theorem 2.7, we may continue the proof above to see that the injections (3.3) are isomorphisms for $j \geq 3$. Hence if X is general and $n - c \geq 2$, then

$$_0\text{Ext}^1_{D_3}(I_{D_3}/I^2_{D_3}, I_{c-1}) \cong {}_0\text{Ext}^1_{D_{c-1}}(I_{D_{c-1}}/I^2_{D_{c-1}}, I_{c-1}) \ .$$

Here the leftmost Ext^1-group is computed much faster by Macaulay 2 than the rightmost one. We also get an injection in (3.3) for $j = 2$, but now it is not necessarily an isomorphism.

Corollary 3.8. *Let $n - c \geq 1$, $c \geq 5$ and suppose $a_0 > b_t$ and $\quad a_{t+3} > a_{t-1} + a_t - b_1$. Then $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\text{Hilb}^p(\mathbb{P}^n)$ of dimension $\lambda_c + K_3 + ... + K_c$.*

Proof. We get $\dim \overline{W(\underline{b};\underline{a})}$ from Theorem 3.2. Hence by (3.3) and Remark 3.5 it suffices to show that $_0\text{Ext}^1_R(I_{D_2}, I_i) = 0$ for $4 \leq i \leq c - 1$ since $_0\text{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_i)$ is a subgroup of $_0\text{Ext}^1_R(I_{D_2}, I_i)$. First let $i = c - 1$. By the Eagon-Northcott resolution (2.3) we see that the largest possible degree of a relation for I_{D_2} is $\ell_2 - b_1$ and the smallest possible degree of a generator of $I_{c-1} \cong I_{D_c}/I_{D_{c-1}}$ is $\ell_c - \sum_{j=t-1}^{t+c-3} a_j$. Since $\ell_c = \ell_2 + \sum_{j=t+1}^{t+c-2} a_j$, we get $_0\text{Ext}^1_R(I_{D_2}, I_{c-1}) = 0$ from

$$\ell_2 - b_1 < \ell_2 + \sum_{j=t+1}^{t+c-2} a_j - \sum_{j=t-1}^{t+c-3} a_j = \ell_2 - a_{t-1} - a_t + a_{t+c-2} \ ,$$

i.e. from $a_{t+c-2} > a_{t-1} + a_t - b_1$. Since we need the vanishing of $_0\text{Ext}^1_R(I_{D_2}, I_i)$ for any $i = 4, 5, ..., c-1$, we must suppose $a_{t+3} > a_{t-1} + a_t - b_1$ and hence we get the corollary. $\qquad\square$

Remark 3.9. Note that if $c = 3$ (resp. $c = 4$) we can argue as above to see that the conclusions of Corollary 3.8 hold provided $a_{t+1} > a_{t-1} + a_t - b_1$ (resp. $a_{t+2} > a_{t-1} + a_t - b_1$). This is, however, proved in [12], Corollary 5.10. For $c \geq 5$, Corollary 3.8 generalizes the corresponding result [12], Corollary 5.9 quite a lot.

4. Conjectures

In [10] the first author discovered a counterexample to Conjecture 2.2 for every c in the range $n = c \geq 3$. Indeed the vanishing all 2×2 minors of a general $2 \times (c + 1)$ matrix of linear entries defines a reduced scheme of $c + 1$ different points in $\mathbb{P}^c$. The conjectured dimension of $W(0, 0; 1, 1, ..., 1)$ is $c(c + 1) + c - 2$ while its actual dimension is at most $c(c + 1)$.

On the other hand Theorem 3.2 is quite close to proving Conjecture 2.2. The crucial assumption in Theorem 3.2 is the inequality $a_{t+c-2} > a_{t-2}$ (or $a_{t+3} > a_{t-2}$ if $c > 5$). Since we, in addition to proving Theorem 3.2, have computed quite a lot of examples where we have $a_{t+c-2} = a_{t-2}$ and $a_{i-\min([c/2]+1,t)} > b_i$, and each time, except for the counterexample, obtained (2.12) and hence the conjecture, we now want to slightly change Conjecture 2.2 to

Conjecture 4.1. *Given integers $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$ and $b_1 \leq ... \leq b_t$, we assume $a_{i-\min([c/2]+1,t)} \geq b_i$ provided $n > c$ and $a_{i-\min([c/2]+1,t)} > b_i$ provided $n = c$ for $\min([c/2]+1,t) \leq i \leq t$. Except for the family $W(0,0;1,1,...,1)$ of zero-dimensional schemes above we have, for $W(\underline{b};\underline{a}) \neq \emptyset$, that*

$$\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + K_4 + ... + K_c .$$

Indeed in the situation of Proposition 2.4 we even *expect* (2.12) to hold! This will imply Conjecture 4.1 provided the conjecture holds for $W(\underline{b};\underline{a}')$. Note that the conclusion of the conjecture is true provided $n - c \geq 1$ and $2 \leq c \leq 5$ ($char(k) = 0$ if $c = 5$) by [12].

Finally we will state a conjecture related to the problems (2) and (3) of the Introduction:

Conjecture 4.2. *Given integers $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$ and $b_1 \leq ... \leq b_t$, we suppose $n - c \geq 2$, $c \geq 5$ and $a_0 > b_t$. Then $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of the Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^n)$.*

Indeed due to the results of this paper and many examples computed by Macaulay 2 in the range $a_0 > b_t$ we even *expect* the groups $_0\mathrm{Ext}^1_{D_3}(I_{D_3}/I^2_{D_3}, I_i)$ for $i = 4, ..., c-1$ of Theorem 3.4 to vanish! This will imply Conjecture 4.2. The conclusion of the conjecture may even be true for $0 \leq n - c \leq 1$ (we have no counterexample), but in this range we have verified that the Ext^1-groups above do not always vanish. Note that the conclusion of Conjecture 4.2 is true provided $n - c \geq 2$ and $2 \leq c \leq 4$ ([7], [11], [12]).

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