AMERICAN JOURNAL of PHYSICS

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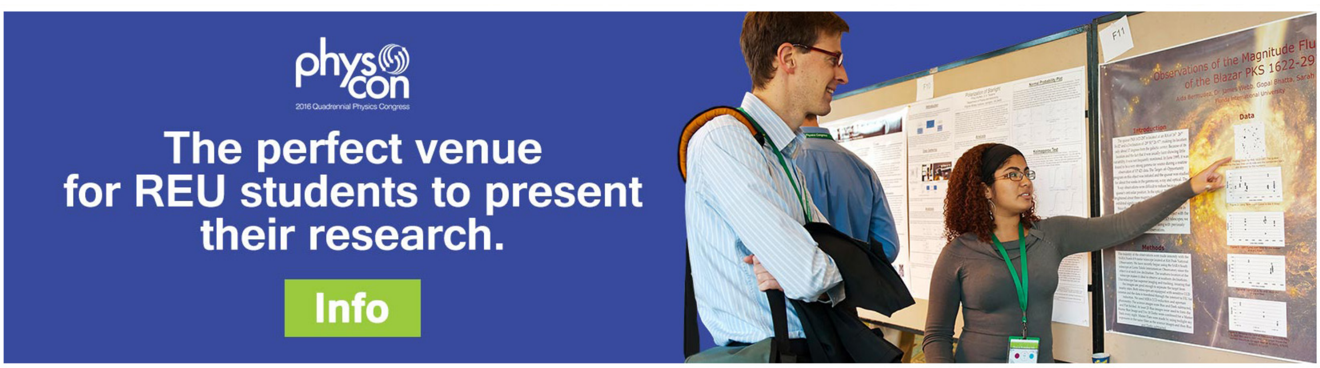
Citation: American Journal of Physics **84**, 419 (2016); doi: 10.1119/1.4942168
View online: http://dx.doi.org/10.1119/1.4942168
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/84/6?ver=pdfcov
Published by the American Association of Physics Teachers



A relativistic trolley paradox

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(Received 25 July 2015; accepted 2 February 2016)

We present an apparent paradox within the special theory of relativity, involving a trolley with relativistic velocity and its rolling wheels. Two solutions are given, both making clear the physical reality of the Lorentz contraction, and that the distance on the rails between each time a specific point on the rim touches the rail is not equal to $2\pi R$, where R is the radius of the wheel, but $2\pi R/\sqrt{1-R^2\Omega^2/c^2}$, where Ω is the angular velocity of the wheels. In one solution, the wheel radius is constant as the velocity of the trolley increases, and in the other the wheels contract in the radial direction. We also explain two surprising facts. First that the shape of a rolling wheel is elliptical in spite of the fact that the upper part of the wheel moves faster than the lower part, and thus is more Lorentz contracted, and second that a Lorentz contracted wheel with relativistic velocity rolls out a larger distance between two successive touches of a point of the wheel on the rails than the length of a circle with the same radius as the wheels.
[http://dx.doi.org/10.1119/1.4942168]

I. INTRODUCTION

The efforts to understand the consequences of the special theory of relativity as applied to rotating systems have a long history.[1] Such efforts have been useful, as analyzing seemingly paradoxical situations usually leads to a more detailed understanding of relativistic kinematics.[2–13]

We here present a new special relativistic "paradox." The situation involves a combination of translational and rotating motion—the translational motion of a trolley and the rotational motion of its wheels. This example makes clear the importance of taking the relativity of simultaneity into account when predicting the behavior of objects moving relativistically. It also sheds light on the physical reality of the Lorentz contraction and provides a method for measuring the increase of the rest length of the circumference of a wheel with constant radius and increasing rotational motion.

II. THE TROLLEY PARADOX

Imagine that an observer on a trolley is spinning the drive wheel to a certain angular velocity Ω, and that this wheel is rolling along the rail without slipping, thus driving the trolley. We denote the rest frame of the trolley by K. Upon reaching the angular velocity Ω, the linear velocity v of the rotating rim of the drive wheel becomes

$$v = R\,\Omega, \tag{1}$$

where R is the radius of the wheel rim (Fig. 1).

Because it is impossible for the wheel rim speed to exceed the speed of light c, the angular velocity of the wheel cannot be higher than c/R. When the angular velocity of rotation of the wheel approaches the value c/R, the translational speed of the rim of the wheel approaches the speed of light c. In the absence of slippage between the rail and the drive wheel, the speed of movement of the rail in reference frame K also tends to the speed of light c.

Suppose now that on the trolley next to the wheel rim there is a generator of laser light, and on the drive wheel there is a sensor. Each time the sensor passes the laser generator it detects a signal. The laser and sensor act like a clock ticking with a period equal to the time it takes for the sensor to pass the upper position of the wheel on two consecutive occasions.

If the wheel makes f revolutions per second, the frequency of the signals is $f = \Omega/2\pi$. The speed of the wheel rim and of the rail relative to the trolley can in this case be expressed by the frequency f of the signals as $v = R\Omega = 2\pi R f$. The upper limit of the frequency of the signals is $c/2\pi R$.

Consider now the motion of the trolley in reference frame K' rigidly linked to the rail. Inertial observers must agree on their relative velocity, so the speed of the trolley in this reference frame must also be equal to v. In the reference frame K' the frequency of the signals is

$$f' = \gamma^{-1} f = \gamma^{-1}(v/2\pi R) \tag{2}$$

due to the relativistic time dilation, where $\gamma = (1 - v^2/c^2)^{-1/2}$. When the speed of the trolley approaches light speed c, the frequency of the signals tends to zero.

The signal frequency f' tending to zero means termination of the drive wheel rotation in reference frame K'. Consequently, it seems as if the trolley must move along the rails with wheel slipping in reference frame K'. However, the effect of the wheel slipping on the rail is not relative but absolute. Hence, in reference frame K' slipping cannot occur because it disagrees with the absence of slipping in K. So we have arrived at a contradiction, and this is what we have called "the trolley paradox."

We shall consider two solutions of the trolley paradox: one with a constant wheel radius (Sec. III) and one with a contracted wheel radius (Sec. IV). The situations are meant to illustrate special relativistic kinematics and may not be realizable in practice due to properties of realistic materials.

III. SOLUTION OF THE PARADOX WITH CONSTANT RADIUS OF THE WHEELS

The elastic deformation of a rotating disk due to relativistic effects has been investigated by several researchers. Planck[14] pointed out that while it is always the case that the

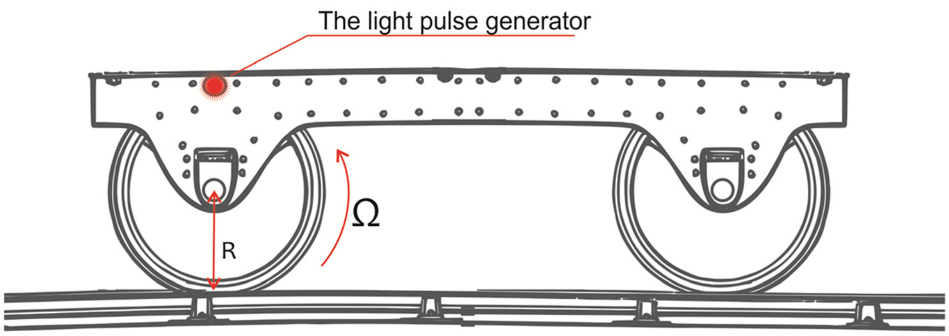

Fig. 1. Relativistic trolley with a light pulse generator.

length of a body as observed with velocity v is Lorentz-contracted with the factor $\sqrt{1-v^2/c^2}$ relative to its length as observed by a co-moving observer, it is not generally true that the length of a body is shortened with the same factor when the body is brought from rest to a velocity v. He then indicated that Ehrenfest's paradox should be investigated as an elasticity problem.

Planck's analysis was followed up by Lorentz[15] and Eddington,[16] who both reached the conclusion that the radius of the rotating disk is reduced from its rest value R to $R' = R(1 - R^2\Omega^2/8c^2)$. Later, Clark[17] gave a more detailed dynamical treatment of the problem and showed that Lorentz and Eddington had overlooked the fact that the speed of propagation of a dilatation cannot exceed the speed of light. Taking this into account, Clark found that for material in which the waves of dilation travel with speed c there is no alteration in the radius of the disk. Further dynamical investigations of this problem have been presented by Cavalleri[18] and McCrea.[19]

In the solution considered in this section, the material will try to Lorentz contract, but is not allowed to do so. Hence there will develop tangential stresses in the material, and for a sufficiently large angular velocity of the wheels, they will break up. The similarity of these stresses to the Dewan-Beran stresses[20,21] associated with translational motion should be noted. But since the present treatment is not concerned with the special relativistic mechanics of elastic media, but with special relativistic kinematics, we shall neglect problems due to material properties here. However, in order to define a situation where this problem does not turn up, we shall consider wheels that are allowed to Lorentz contract in the radial direction in Sec. IV.

In order to get an understanding of what happens to the wheels when the trolley is accelerated to a relativistic speed, we shall first consider the acceleration process. If one wants to know what happens to an object, for example, whether it will break up under some circumstances, one should analyse what happens to the object in its rest frame.

A wheel with constant radius will stretch in the tangential direction due to the acceleration program when the angular velocity of the wheels increases.[22] Imagine that the acceleration is due to a series of (tangential) blows. This acceleration program is such that all elements on the rim of a wheel get blows that are simultaneous in the rest frame K of the trolley. Hence, these blows are not simultaneous in the rest frames of the elements of the rim. It was shown in Ref. 23 that a point at the front end of an element is accelerated a little earlier than a point at the rear end of the element. Therefore, the element will stretch due to the blows.

This stretching introduces a difficulty when we are talking about the wheels. We will thus need to use the term "rest length" of an element on the rim of a wheel. But this "rest length" is not the length of an element when the wheel is not rotating. Instead, "rest length" will mean the length of an element of the *rotating* wheel as measured by an inertial observer instantaneously at rest relative to the element.

The solution of the trolley paradox is hidden in the relativistic kinematics as applied to a rolling wheel.[24,25] Let us consider one of the wheels of the trolley. We introduce an x-axis along the horizontal rail and a y-axis in the vertical direction. Consider a point P that is at the origin of the coordinate system and at the bottom of the wheel at time $t = 0$. In frame K it moves along a circular path. At a point of time t, it has a position

$$x = R\,\sin(\Omega t), \quad y = R[1-\cos(\Omega t)]. \tag{3}$$

The Lorentz transformation to K' is

$$t' = \gamma[t - (v/c^2)x], \quad x' = \gamma(x - vt), \quad y' = y, \tag{4}$$

where $v = R\Omega$. Note that the trolley moves in the negative x'-direction in K'. We shall also need the inverse time transformation

$$t = \gamma[t' + (v/c^2)x']. \tag{5}$$

Inserting Eqs. (3) and (5) into Eq. (4) and using that $1 + \gamma^2R^2\Omega^2/c^2 = \gamma^2$, we find implicit equations for the coordinates of point P as a function of time in K'

$$x' = R\left\{\gamma^{-1}\sin\left[\gamma\,\Omega\left(t' + \frac{R\Omega}{c^2}x'\right)\right] - \Omega t'\right\},$$
$$y' = R\left\{1 - \cos\left[\gamma\,\Omega\left(t' + \frac{R\Omega}{c^2}x'\right)\right]\right\}. \tag{6}$$

The equation for the corresponding nonrelativistic cycloid is

$$x'_N = R[\sin(\Omega t') - \Omega t'], \quad y'_N = R[1 - \cos(\Omega t')]. \tag{7}$$

We shall now find the distance l' between the points on the rail where point P on the wheel touches the rail. This distance is given by the value of t' obtained by setting $y' = 0$ and $x' = -l'$ in the second expression of Eq. (6), giving

$$\gamma\,\Omega\left(t' - \frac{R\,\Omega}{c^2}l'\right) = 2\pi. \tag{8}$$

Substituting this into the first expression in Eq. (6), with $x' = -l'$ gives

$$l' = \gamma 2\pi R. \tag{9}$$

This result has also been obtained in a very simple way by Vøyenli[26] as the "rolled out" circumference of a rolling wheel.

It follows from Eqs. (1) and (2) that the frequency of the "wheel clock" as measured in K' is related to the angular velocity of the wheel as measured in K by

$$f' = \gamma^{-1}\Omega/2\pi. \tag{10}$$

Because one revolution of the radius corresponds to an angular increase of 2π both in K and in K', the angular velocity in K' is related to the frequency in K' by

$$\Omega' = 2\pi f'. \tag{11}$$

Equations (10) and (11) lead to

$$\Omega' = \Omega\gamma^{-1}, \tag{12}$$

and if we let $\Omega_c \equiv c/R$, then

$$\Omega' = \Omega\sqrt{1-(\Omega/\Omega_c)^2}. \tag{13}$$

The above equation is shown graphically for $\Omega_c =$ constant in Fig. 2. As observed in K' the angular velocity of the wheels increases from zero, when the trolley is at rest, to a maximal value $\Omega'_{\max} = \Omega_c/2$, when $\Omega = \Omega_c/\sqrt{2}$. Increasing the velocity of the trolley further towards c, the angular velocity of the wheels decreases towards zero at $R\Omega = c$.

In K' the velocity of the trolley is

$$v' = \gamma R\Omega' = R\Omega = v, \tag{14}$$

in agreement with the requirement that the velocity of the trolley as observed in the rest frame of the rail must be equal to the velocity of the rail as observed in the rest frame of the trolley.

The relationships given in Eqs. (9), (13), and (14) are the main points of the solution of the trolley paradox with constant-radius wheels. The rest length of the stretched elements of the rim of the rotating wheels is rolled out on the rail, and there is no slipping of the wheels although the angular velocity of the wheels approaches zero as the velocity of the trolley approaches c.

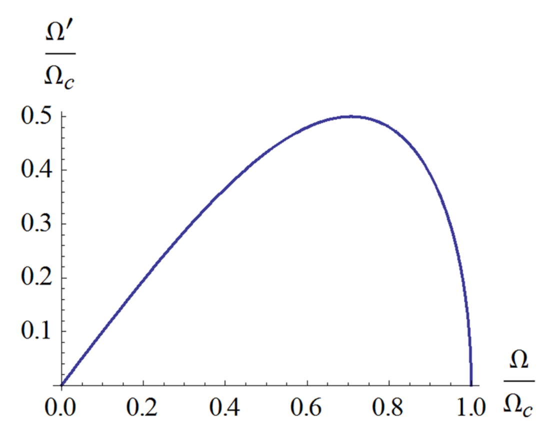

Fig. 2. The angular velocity of the wheels of the trolley as observed in the rest frame of the rails as a function of their angular velocity as observed in the rest frame of the trolley.

IV. SOLUTION OF THE PARADOX WITH CONTRACTING RADIUS OF THE WHEELS

Imagine that a laser signal emitted by a point on the circumference of the wheel is detected at the rail each time the emitter has contact with the rail. The paradox was that as measured in the rail-system the frequency f' of the signal, and hence the angular velocity $2\pi f'$ approaches zero as the velocity of the trolley approaches c. Thus, the wheels appear to be slipping against the rails. But as observed in the rest system of the trolley, there is no slipping, and the phenomenon of slipping is Lorentz invariant.

In Sec. III, this paradox was solved by noting that the rest length of the circumference of the wheel increases towards infinity as the speed of the trolley approaches that of light. Hence, as observed in the rail system, there is no slipping even if the angular velocity of the wheel approaches zero in the velocity of light limit. This solution presupposes that the radius of the wheel is constant. It is the solution of the paradox for a trolley with wheels having constant radii.

But there is an alternative solution of the paradox if one relaxes the condition of a constant radius for the wheels. The increasing rest length of the circumference of the wheels during acceleration of the trolley introduces a tangential tension in the rim of the wheels. Assume now that a wheel is permitted to contract freely in the radial direction so that no tension develops in the tangential direction. Then the rest length of the rim of the wheels must remain constant during the accelerated motion of the trolley. This means that the rim Lorentz contracts, and that the radial extension of the wheels contracts accordingly. The result is that the wheels become infinitely small in the limit that the trolley moves with the velocity of light.

But would not the trolley stop then? The reason that it will not is because the elements of the rim are infinitely Lorentz contracted in the limit $R\Omega \to c$, but it is the rest length of the elements that are rolled out on the rail, not the Lorentz contracted elements of the rim.

Let us first describe the rolling wheels in the rest frame K of the trolley. If v is the velocity on the rim, we have $\Omega = v/R$, where $R = R_0/\gamma$ is the contracted radius of the rotating wheels, and R_0 is their radius when they are at rest. The angular velocity of the rotating wheels is then

$$\Omega = \gamma v/R_0. \tag{15}$$

Hence, in this case the angular velocity Ω must approach an infinitely great value in K when the speed of the rail approaches that of light.

As observed in the rail frame, the distance between the marks on the rail each time a point on a wheel touches it is still given in Eq. (9)

$$l' = \gamma 2\pi R = 2\pi R_0, \tag{16}$$

and this distance is independent of the speed of the trolley, even if the radius of the wheels decreases with increasing velocity. As mentioned above, the reason for this is because the distance between the marks depends upon the rest length of the rim of the wheels and not their Lorentz-contracted length. Also, in this frame the angular velocity of the wheels remains finite even if the wheels have a vanishing radius when the velocity of the trolley approaches that of light

$$\Omega' = \gamma^{-1}\Omega = v/R_0 \tag{17}$$

and hence $\lim_{v\to c}\Omega' = c/R_0$, which is finite.

It is worth emphasizing that there is no slipping of the wheels, not in the rest frame of the trolley or in the rest frame of the rails.

V. AN ADDITIONAL LENGTH CONTRACTION PARADOX

A referee suggested that we consider the following seeming paradox. The observers in the frame of the rails will see all standard measuring rods used by those in the trolley frame as Lorentz contracted along the line of motion. Likewise they will see the circular wheels of the trolley Lorentz contracted into the shape of an ellipse. In the limit that the speed of the trolley approaches c, the ellipse degenerates to a vertical line of length $2R$, and the perimeter of the ellipse is then equal to $4R$, which is smaller than $2\pi R$. Thus, one would expect one revolution of the wheel on the trolley to move the trolley a distance less than a circle of the same radius, not the distance $\gamma 2\pi R$, which is greater than that of a circle of the same radius. The referee also commented that how one turns a Lorentz contracted rod in motion into a standard size rod at the point of contact and ends up with a rolled distance that is greater than $2\pi R$ is not immediately clear, and suggested we try to explain these matters.

The main question is: How can the rolled out distance of a Lorentz contracted wheel with the shape of an ellipse having a perimeter that is obviously smaller than $2\pi R$, be larger than $2\pi R$? The explanation is found by considering how the velocity, angular velocity, and length of elements on the periphery of the wheels change as the wheels roll in the rail frame K'. When an element touches a rail, it is instantaneously at rest in K'. Therefore, the rest length of the element is "imprinted" on the rail. It is this stretched element that is "imprinted" on the rails as the trolley rolls; the sum of these "prints" is the rolled out length.

Consider a point P on the rim of a wheel. As observed in the rest frame of the rails, it moves along a cycloid like path[24] with varying velocity. Let θ be the angle between the radius vector from the center of the disk to P and the vertical line to the point where the wheel touches the rail. We shall now describe how the velocity component v_x along the rail, the angular velocity of the radius vector ω, and the length l, of an element on the rim of a rolling wheel, vary as functions of θ. In this case Eqs. (19) and (21) of Ref. 24 take the form

$$v_x = \frac{\sqrt{1+\gamma^2\tan^2\theta}\pm 1}{\sqrt{1+\gamma^2\tan^2\theta}\pm v^2/c^2}\,v \tag{18}$$

and

$$\omega = \frac{\left(1+\gamma^2\tan^2\theta\right)^{3/2}\cos^2\theta}{\gamma^2\left(\sqrt{1+\gamma^2\tan^2\theta}\pm v^2/c^2\right)}\,\Omega. \tag{19}$$

Let l_0 be the length of an element on the rim of a wheel when the trolley is at rest. Then the length when the trolley moves is

$$l = \sqrt{1 - v_x^2/c^2}\, l_0. \tag{20}$$

Inserting the expression (18) for v_x gives

$$l = \frac{\sqrt{1+\gamma^2\tan^2\theta - v^2/c^2}}{\sqrt{1+\gamma^2\tan^2\theta}\pm v^2/c^2}\, l_0. \tag{21}$$

In these three formulae, one should use the minus sign for the lower part of the wheels, where $0 \le \theta < \pi/2$ and $3\pi/2 < \theta \le 2\pi$, and the plus sign for the upper part of the wheels, where $\pi/2 < \theta < 3\pi/2$.

Equation (18) tells how the velocity component along the rails of a point on the rim of a wheel depends upon the angle as the trolley moves, as observed in the rest frame of the rails. There is maximal velocity as the point passes the top of the wheel, and the element is maximally Lorentz contracted. The velocity of the element at the top ($\theta = \pi$) and bottom ($\theta = 0$) of the wheel is

$$v_T = \frac{2v}{1+v^2/c^2}, \quad v_B = 0. \tag{22}$$

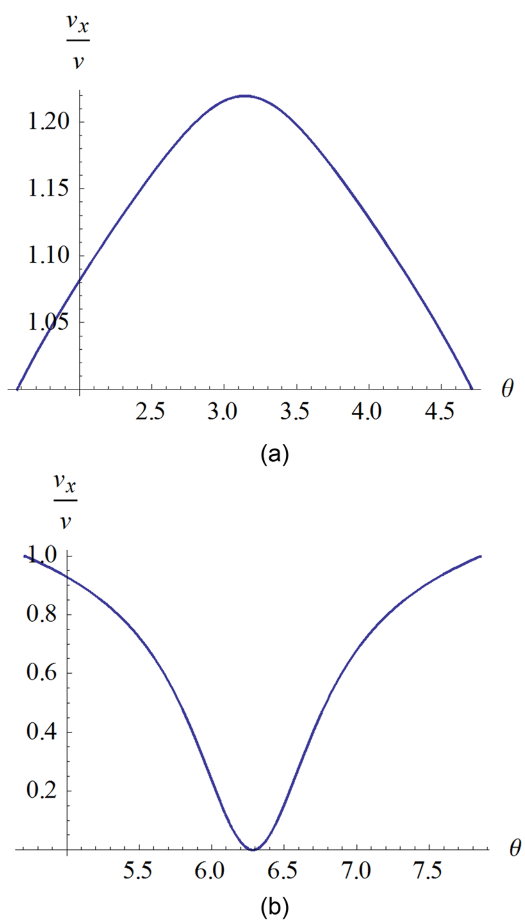

Fig. 3. The velocity component of a point P on the rim of a wheel in the direction of the rail, given in Eq. (18), as a function of its angular position on the wheel as observed in the rest frame of the rail. The upper plot is for the upper half of the wheel and lower plot for the lower half of the wheel.

Meanwhile, the angular velocity at the top and bottom is

$$\omega_T = \frac{c^2 - v^2}{c^2 + v^2}\Omega, \quad \omega_B = \Omega, \tag{23}$$

and the length of an element on the rim at the top and bottom is

$$l_T = \frac{\sqrt{1 - v^2/c^2}}{1 + v^2/c^2} l_0, \quad l_B = \frac{l_0}{\sqrt{1 - v^2/c^2}}. \tag{24}$$

The velocity component of a point P on the rim of a wheel in the direction of the rails, the angular velocity of the radius vector from the center of a wheel to P, and the length of an element on the rim of a wheel as measured in the rest frame of the rails are plotted, from Eqs. (18), (19), and (21), as functions of the angular position on the wheel in Figs. 3–5.

These figures show that a point on the rim is instantaneously at rest as it touches the rail. The angular velocity of a radius vector from the center of the wheel to a point on the rim is smaller when the point is at the upper part of the wheel, and an element on the rim of the wheel is observed to be contracted when it is in the upper part of the wheel, but it is stretched at the lower part, and maximally stretched as it touches the rail. Such stretched elements are rolled out on the rail, and that is the explanation of the strange fact that a Lorentz contracted wheel rolls out a larger distance between successive touches of a point of the wheel on the rail than the length of a circle with the same radius.

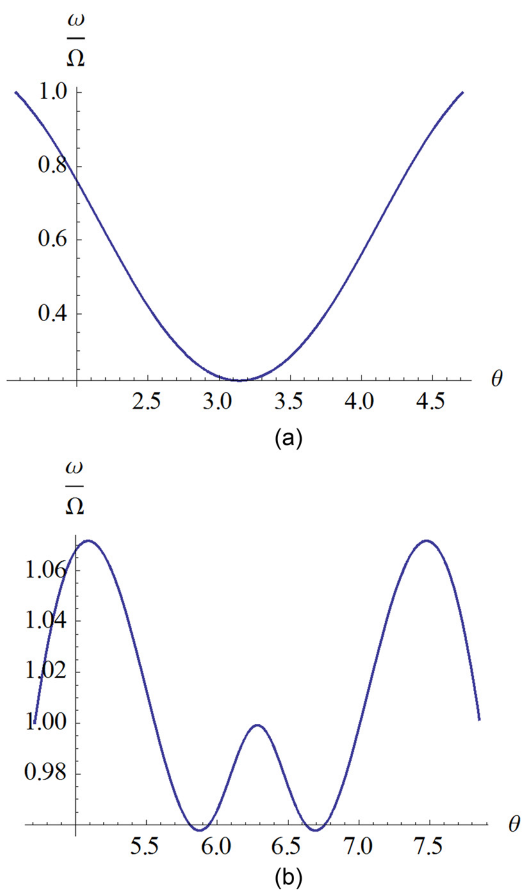

Fig. 4. The angular velocity of a radius vector from the center of the wheel to a point on the rim of the wheel, given in Eq. (19), as a function of it angular position on the wheel as observed in the rest frame of the rail. The left plot is for the upper half of the wheel and the right plot for the lower half of the wheel.

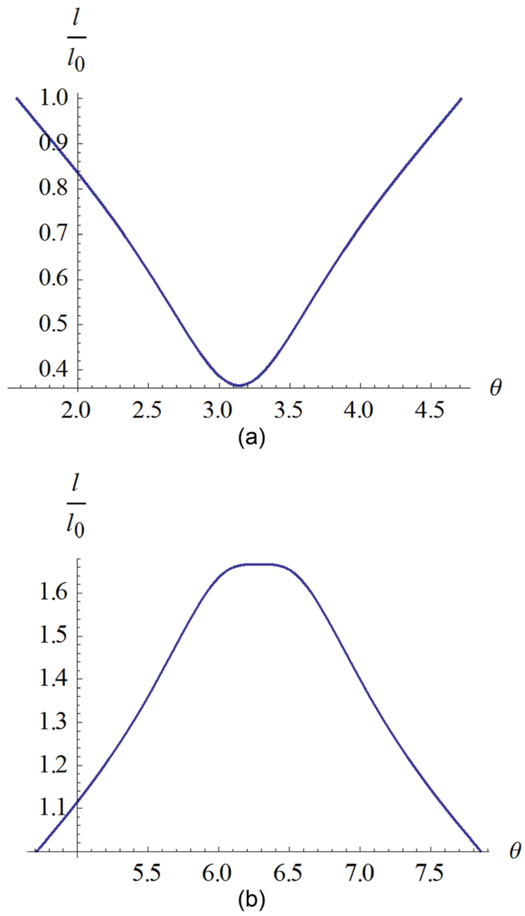

Fig. 5. The length of an element on the rim of the wheel, given in Eq. (21), as a function of it angular position on the wheel as observed in the rest frame of the rail. The upper plot is for the upper half of the wheel and the lower plot for the lower half of the wheel.

VI. THE SHAPE OF A ROLLING WHEEL

In connection with the description of a rolling wheel as given in the first paragraph of Sec. V, a referee has objected. He writes that it is not correct that the wheels have contracted to an elliptical shape and proceeds: The bottom point of the wheel, if it is not slipping on the rail, is momentarily at rest in the rail's frame. Therefore, the top of the wheel has the greatest velocity and points in between are moving at intermediate speeds in that frame. So the rim of the wheel contracts to an ovoid, narrow at the top and broad at the bottom.

This is a tempting and seemingly natural conception. But it is not correct. Due to the naturalness of this misconception, it may be useful to clarify this point. Also, such an explanation contributes in providing a more detailed solution of the trolley paradox.

Imagine a plate behind a wheel where a circle is drawn on the plate around the periphery of the wheel. The wheel must have the same shape as the drawn circle because its periphery just covers it. Moreover, the shape of the circle will obviously be elliptical because the circle has only a translational motion and no rotation. Hence, *a rolling wheel has an elliptical shape*. But the upper part of the rim moves faster and should have a greater Lorentz contraction, while

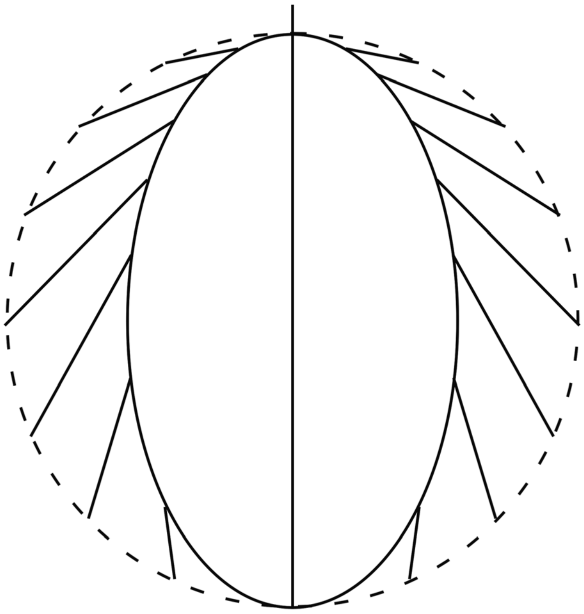

Fig. 6. The dashed circle marks the rim of a non-rotating wheel where 16 (equidistant) points have been marked. Consider a wheel rolling so that the axis of the wheel moves with 80% of the speed of light. The positions of the points at the moment when the axis passes that of the wheel at rest, connected by lines to the corresponding points of the non-rotating wheel, have been calculated (see Ref. 24) and plotted. The points are all positioned on an ellipse, but the distances between the points are smallest at the top, where the rim moves fast, and largest at the bottom, where the velocity of the rim is small. At the bottom, the distances between the points on the rolling wheel are even larger than the corresponding distances for the non-rotating wheel, illustrating that the rim of a rotating wheel is stretched as observed in the rest frame of an element of the rim.

the elements of the rim near the rail move slowly and should not be Lorentz contracted. This seems like a new paradox.

The solution is hidden in the relativity of simultaneity. What we mean by "the wheel as observed in the rail system" is that all parts of the wheel are observed at a fixed moment in the rail frame. But the simultaneity of the rail frame is different from the simultaneities in the instantaneous rest frames of the elements of the rim of a rolling wheel.

In Ref. 24 the rim was divided into 16 elements with 16 dividing points. First, the points were drawn on a non-rotating wheel as shown in Fig. 6. Next, a rolling wheel was considered. The positions of these points were calculated at a fixed moment in the rest frame of the rail, at the moment that the axes of the non-rotating and rotating wheels have the same position. The result of the calculation is shown in Fig. 6.

All the points on the rim of the rolling wheel are positioned on an ellipse—the same ellipse as that of the Lorentz contracted drawing on the plane behind a wheel. But the points are not equidistant on the ellipse. Indeed, there is a greater Lorentz contraction of the upper part of the wheel than that of the lower part. Hence, an axially symmetric mass distribution of a wheel in the rest frame of the trolley leads to a dipole structure in the rail frame, which has been termed the relativistic Hall effect.[27]

The shapes of the spokes of the wheels in the trolley frame and the rail frame are shown in Fig. 7. We see from the shapes of the spokes of the rolling wheel on the right in Fig. 7 that the bottom part of the wheel is stretched.

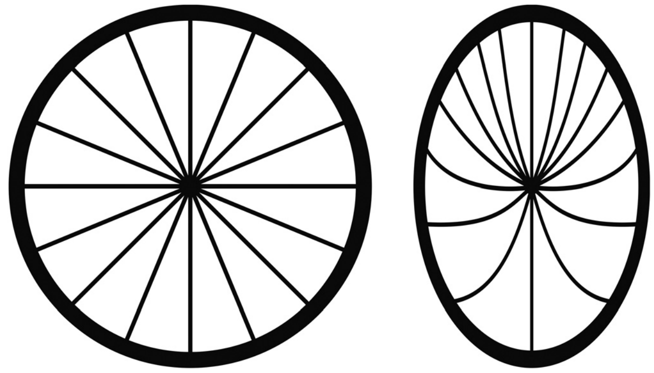

Fig. 7. To the left is shown a non-rotating wheel with straight spokes. To the right is the same wheel, but now rolling with speed $v = 0.7c$. The spokes point to the corresponding points of Fig. 6.

VII. CONCLUSION

We have introduced a new special relativistic "paradox," which has been called the "trolley paradox," in order to illustrate the physical reality of the Lorentz contraction. It concerns the kinematics of a trolley with a speed approaching that of light, and in particular, the relativistic description of the wheels in the rest frame K of the trolley and the rest frame K' of the rail. It is assumed that as observed in K there is no slipping of the wheels on the rail. Then a seeming contradiction appears. In K' the frequency of a signal emitted from a point on the rim of a wheel each time it passes a fixed point on the trolley approaches zero as the velocity of the trolley approaches the velocity of light.

In order that the trolley can move with nearly the speed of light it then seems necessary that the wheels must slide along the rail as observed in K'. However, the phenomenon of slipping is Lorentz invariant, and since there is not any slipping in K there can be no slipping in K' either. Hence, we have arrived at a contradiction. This is the trolley paradox.

We have presented two solutions to this paradox. The first one assumes that the radius of the wheels remains constant during the motion. We have shown that the distance in K between the emission points of the signal is not $2\pi R$, but $\gamma 2\pi R$ which approaches infinity as the speed of the trolley approaches that of light. Then no slipping is needed for the motion of the trolley even if the angular velocity of the wheels approaches zero as the velocity of the trolley approaches that of light.

The point of departure of the second solution is the observation that since the rim of the wheels tend to Lorentz contract there will be increasing tangential stresses in the rim material when the velocity of the trolley increases. Due to the curvature of the circular rim material, these stresses have an inward component at every point, producing a contraction in the radial direction of the wheels. If the material of the wheel is such that it can contract freely, it will get an induced contraction in the radial direction of the same magnitude as the Lorentz contraction.

Hence, in this solution, the wheels get infinitesimally small as the speed of the trolley approaches that of light. One might then think that the angular velocity of the wheels in the rail frame has to approach infinity in order that the trolley shall be able to move with nearly the speed of light. However, that is not the case. The wheels can be considered as clocks. Due to the relativistic slowing down of a clock with increasing speed, the angular velocity of the wheels remains finite in the rail frame, even in the limit that the trolley moves nearly as fast as light. The reason that the trolley can move very fast in spite of its small wheels with a finite angular velocity is that the rest length of the elements of the rim of the wheels is rolled out on the rail, and not the Lorentz contracted elements.

Again there is no slipping. This resolves the paradox, but is represents a different physical situation than that of the first solution.

Also we have explained the strange fact that even for a wheel that is Lorentz contracted in the direction of motion of the wheel and therefore has an elliptical shape, the distance between two successive touches of a point upon the rail is larger than the length of a circle with the same radius as the wheels. This is due to the following circumstance. The acceleration program during the period with angular acceleration of the wheels corresponds to simultaneous blows on all point of the wheels as observed in the rest frame of the trolley. Hence, due to the relativity of simultaneity, each element of the rim will receive blows a little earlier at their front end than at their rear end. The elements will therefore be stretched. When the wheels roll, these stretched elements will be rolled out on the rails.

In the rail frame, the elements at the upper part of the wheels move faster and those at the lower part more slowly. This implies that the upper elements are more Lorentz contracted than the element on the lower part of the wheels. In spite of this a rolling wheel is not narrower at the top than at the bottom. This is due to the relativity of simultaneity. When transforming a set of points on the rim considered simultaneously in the rest frame of the trolley to a set of points on the rim considered simultaneously in the rail frame, the points on the upper end of the wheel come closer to each other, while the distances between those point closer to the rails increase.

In this way, one may understand that the wheels move without slipping in the rail system in spite of the Lorentz contraction of the wheels and the time dilation of the "wheel clocks."

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