

# UNOBSTRUCTEDNESS AND DIMENSION OF FAMILIES OF CODIMENSION 3 ACM ALGEBRAS

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ABSTRACT. The goal of this paper is to study irreducible families of codimension 3, Cohen-Macaulay quotients  $A$  of a polynomial ring  $R = k[x_0, x_1, \dots, x_n]$ ; mainly, we study families of graded Cohen-Macaulay quotients  $A$  of codimension 1 on some codimension 2 Cohen-Macaulay algebra  $B$  defined by a regular section  $\sigma$  of  $(S^2 K_B^\vee)_\lambda$ . We give lower bounds for the dimension of the irreducible components of the Hilbert scheme which contains  $\text{Proj}(A)$ . The components are generically smooth and the bounds are sharp if  $\lambda \gg 0$  and  $n = 4$  and  $5$ .

We also deal with a particular type of codimension 3, Cohen-Macaulay quotients  $A$  of  $R$ ; concretely we restrict our attention to codimension 3 arithmetically Cohen-Macaulay subschemes  $X \subset \mathbb{P}^n$  defined by the submaximal minors of a symmetric homogeneous matrix. We prove that such schemes are glicci and we give lower bounds for the dimension of the corresponding component of the Hilbert scheme.

In the last part of the paper, we collect some questions/problems which naturally arise in our context.

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## 1. INTRODUCTION

The main purpose of this work is to contribute to the classification of codimension  $r$ , Cohen-Macaulay graded quotients of a polynomial ring  $R = k[x_0, \dots, x_n]$  and, in particular, we address the following two problems: (1) to determine the unobstructedness of arithmetically Cohen-Macaulay (briefly ACM) schemes  $X \subset \mathbb{P}^n$  of codimension  $r$ ; and (2) to determine  $\dim_{(X)} \text{Hilb}^{p(t)} \mathbb{P}^n$  being  $X \subset \mathbb{P}^n$  an arithmetically Cohen-Macaulay scheme of codimension  $r$ . In codimension 2, the classification of arithmetically Cohen-Macaulay graded quotients of a polynomial ring  $R = k[x_0, \dots, x_n]$  is well known, it is given by the Hilbert-Burch Theorem; and the unobstructedness of arithmetically Cohen-Macaulay schemes  $X \subset \mathbb{P}^n$  of codimension 2 as well as  $\dim_{(X)} \text{Hilb}^{p(t)} \mathbb{P}^n$  were established

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in 1975 by G. Ellingsrud [8]. There is, in our opinion, little hope of solving the above two problems in full generality and for arbitrary codimension. So, we will restrict our attention to codimension 3, ACM schemes  $X \subset \mathbb{P}^n$  which are divisors on some codimension 2, ACM scheme  $Y \subset \mathbb{P}^n$ . According to [6]; Theorem 3.12, an effective divisor  $D \sim aK_Y + bH$  on  $Y$  is ACM if and only if  $-2 \leq a \leq 1$ . ACM effective divisors  $X \sim bH$  on a codimension 2, ACM scheme  $Y \subset \mathbb{P}^n$  were studied in [19] and [20]; ACM effective divisors  $X \sim K_Y + bH$  on a codimension 2, ACM scheme  $Y \subset \mathbb{P}^n$  were studied in [19]; and ACM effective divisors  $X \sim -K_Y + bH$  on a codimension 2, ACM scheme  $Y \subset \mathbb{P}^n$  were studied in [21]. In this paper we study the remaining case; namely, we study ACM effective divisors  $X \sim -2K_Y + bH$  on a codimension 2 ACM scheme  $Y \subset \mathbb{P}^n$  or, equivalently, graded CM quotients  $A$  given by

$$(1.1) \quad 0 \longrightarrow (S^2K_B)(-\lambda) \xrightarrow{\sigma} B \longrightarrow A \longrightarrow 0$$

where  $B$  is a codimension 2 graded generically complete intersection CM quotient of  $R$ . In this paper, we deal with divisors on codimension 2 ACM schemes and we refer to [13] for general results about the theory of generalized divisors for schemes satisfying the condition  $S_2$  of Serre.

Next we outline the structure of the paper. In section 2, we recall the basic facts on deformation theory needed in the sequel. Sections 3 and 4 are the heart of the paper. In section 3, we study families of graded Cohen-Macaulay quotients  $A$  of codimension 1 on a codimension 2 Cohen-Macaulay algebra  $B$  defined by a regular section  $\sigma$  of  $(S^2K_B^\vee)_\lambda$ ; i.e. graded CM quotients  $A$  given by (1.1). We determine lower bounds for the dimension of any irreducible component of  $\text{Hilb}^{p(x)}(\mathbb{P}^n)$  containing a point  $(X)$ ,  $X = \text{Proj}(A)$ , where  $A$  is given by (1.1) (see Theorem 3.6 and Theorem 3.7), and we show that, for  $n = 5$  and 4, they are sharp for  $\lambda \gg 0$  (see Corollary 3.9 and Remark 3.11). The lower bounds will be computed in terms of

$$\begin{aligned} a &:= {}_0\text{hom}_B(I_B/I_B^2, I_{A/B}) - {}_0\text{ext}_B^1(I_B/I_B^2, I_{A/B}), \\ b &:= {}_0\text{hom}_B(I_{A/B}, B) - {}_0\text{ext}_B^1(I_{A/B}, B), \text{ and} \\ e &:= {}_0\text{ext}_B^2(I_{A/B}, I_{A/B}), \end{aligned}$$

where  $I_{A/B} = S^2K_B(-\lambda)$  and where at least  $a$  and  $b$  are explicitly given as a sum of binomials involving only the degrees of the generators and first syzygies of  $I_B$ .

In section 4, we deal with ideals generated by the submaximal minors of a homogeneous symmetric matrix. A classical scheme that can be constructed in this way is the Veronese surface  $X \subset \mathbb{P}^5$ . Given rational numbers  $a_1, \dots, a_t$  such that  $a_i + a_j \in \mathbb{Z}_+$  for all  $i, j$ , we denote by  $S(\underline{a}) = S(a_1, \dots, a_t)$  the irreducible family of codimension 3, ACM schemes  $X \subset \mathbb{P}^n$  defined by the submaximal minors of a  $t \times t$  symmetric homogeneous matrix  $\mathcal{A} = (f_{ji})_{j=1, \dots, t}^{i=1, \dots, t}$  where  $f_{ji} \in k[x_0, \dots, x_n]$  is a homogeneous polynomial of degree  $a_j + a_i$ . The goal of section 4 is to give lower bounds for the dimension of the irreducible component  $\mathcal{S}(\underline{a})$  of  $\text{Hilb}^p(\mathbb{P}^n)$  containing  $S(\underline{a})$  (cf. Theorem 4.8). We give 2 examples where the first bound turns out to be sharp. Indeed, we guess that, for  $n = 5$ , the first bound may be sharp for any scheme defined by linear forms  $f_{ij}$ . As a byproduct we also prove that any codimension 3, ACM scheme  $X \subset \mathbb{P}^n$  defined by the submaximal minors of a  $t \times t$  symmetric homogeneous matrix is glicci (see Proposition 4.5). This last result has

independently been established by Gorla who has proved that any codimension  $\binom{m-t+2}{2}$ , ACM scheme  $X \subset \mathbb{P}^n$  defined by the  $t \times t$  minors of a  $m \times m$  symmetric homogeneous matrix is glicci ([9]; Corollary 2.7). We end this paper with some questions/problems which naturally arise in our context.

Finally, the authors would like to thank the referee for his/her valuable comments. In particular, we include Remarks 4.1 and 4.2 to answer an imprecision pointed out by the referee.

**Notation:** Throughout this paper  $\mathbb{P}^n$  will be the  $n$ -dimensional projective space over an algebraically closed field  $k$ ,  $R = k[x_0, x_1, \dots, x_n]$  and  $\mathfrak{m} = (x_0, \dots, x_n)$ . The sheafification of a graded  $R$ -module  $M$  will be denoted by  $\widetilde{M}$  and the support of  $M$  by  $Supp(M)$ .

For any closed subscheme  $X$  of  $\mathbb{P}^n$ , we denote by  $\mathcal{I}_X$  the ideal sheaf of  $X$  and  $\mathcal{N}_X$  its normal sheaf. Let  $I(X) = H_*^0(\mathcal{I}_X)$  be the saturated homogeneous ideal of  $X$  unless  $X = \emptyset$ , in which case we let  $I(X) = \mathfrak{m}$ . If  $X$  is equidimensional and Cohen-Macaulay of codimension  $c$ , we set  $\omega_X = \mathcal{E}xt_{\mathcal{O}_{\mathbb{P}^n}}^c(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n})(-n-1)$  to be its canonical sheaf.

For any quotient  $A$  of  $R$ , we let  $I_A = \ker(R \rightarrow A)$  and  $N_A = \text{Hom}_R(I_A, A)$  be the normal module. If  $A$  is Cohen-Macaulay of codimension  $c$ , we let  $K_A = \text{Ext}_R^c(A, R)(-n-1)$  be its canonical module. When we write  $X = \text{Proj}(A)$ , we let  $A = R/I(X)$  and  $K_X = K_A$ . If  $M$  is a finitely generated graded  $A$ -module, let  $\text{depth}_J M$  denote the length of a maximal  $M$ -sequence in a homogeneous ideal  $J$  and let  $\text{depth} M = \text{depth}_{\mathfrak{m}} M$ . Let  $H_J^i(-)$  be the right derived functor of the functor,  $\Gamma_J(-)$ , of sections with support in  $\text{Spec}(A/J)$ .

We denote the Hilbert scheme by  $\text{Hilb}^p(\mathbb{P}^n)$  (cf. [11]). Thus, any point  $p_X \in \text{Hilb}^p(\mathbb{P}^n)$  parameterizes a subscheme  $X \subset \mathbb{P}^n$  with Hilbert polynomial  $p \in \mathbb{Q}[s]$ . By abuse of notation we will write  $(X) \in \text{Hilb}^p(\mathbb{P}^n)$ . By definition a scheme  $X \subset \mathbb{P}^n$  is unobstructed if  $\text{Hilb}^p(\mathbb{P}^n)$  is smooth at  $(X)$ .

## 2. PRELIMINARIES

This section provides the background and basic results on deformation theory needed later on.

Let  $B = R/I_B$  be a graded quotient of the polynomial ring  $R$ , let  $M$  and  $N$  be a finitely generated graded  $B$ -modules and let  $J \subset B$  be an ideal. A Cohen-Macaulay (resp. maximal Cohen-Macaulay)  $B$ -module  $M$  satisfies by definition  $\text{depth} M = \dim M$  (resp.  $\text{depth} M = \dim B$ ), or equivalently,  $H_{\mathfrak{m}}^i(M) = 0$  for  $i < \dim M$  (resp.  $i < \dim B$ ) since  $\text{depth}_J M \geq r$  is equivalent to  $H_J^i(M) = 0$  for  $i < r$ . Note that if  $B$  is Cohen-Macaulay, then the  $v$ -graded piece of  $H_{\mathfrak{m}}^i(M)$  is by Gorenstein duality

$${}_v H_{\mathfrak{m}}^i(M) \simeq {}_{-v} \text{Ext}_B^{\dim B - i}(M, K_B)^\vee.$$

Let  $Z$  be closed in  $Y := \text{Proj}(B)$  and let  $U = Y - Z$ . Then we have an exact sequence

$$0 \rightarrow H_{I(Z)}^0(M) \rightarrow M \rightarrow H_*^0(U, \widetilde{M}) \rightarrow H_{I(Z)}^1(M) \rightarrow 0$$

and isomorphisms  $H_{I(Z)}^i(M) \simeq H_*^{i-1}(U, \widetilde{M})$  for  $i \geq 2$ . More generally if  $\text{depth}_{I(Z)} N \geq i+1$  there is an exact sequence

$$(2.1) \quad {}_0 \text{Ext}_B^i(M, N) \hookrightarrow \text{Ext}_{\mathcal{O}_U}^i(\widetilde{M}|_U, \widetilde{N}|_U) \rightarrow {}_0 \text{Hom}_B(M, H_{I(Z)}^{i+1}(N)) \rightarrow {}_0 \text{Ext}_B^{i+1}(M, N) \rightarrow$$

by [12], exp. VI, where the middle form comes from a spectral sequence also treated in [12]. Moreover, we have (cf. [25], Lemma 5)

**Lemma 2.1.** *Let  $B$  be Cohen-Macaulay, let  $r$  and  $t$  be integers and suppose  $\text{depth}_J B \geq r$ . Then  $\text{depth}_m M \geq \dim B - t$  implies  $\text{depth}_J M \geq r - t$ .*

In dealing with deformations we will need to consider the (co)homology groups of algebras. To define them let

$$(2.2) \quad \dots \rightarrow F_2 := \bigoplus_{j=1}^{r_2} R(-n_{2,j}) \rightarrow F_1 := \bigoplus_{i=1}^{r_1} R(-n_{1,i}) \rightarrow R \rightarrow B \rightarrow 0$$

be a minimal resolution of  $B$  and let  $H_1 = H_1(I_B)$  be the 1. Koszul homology built on a set of minimal generators of  $I_B$ . Then we may take the exact sequence

$$(2.3) \quad 0 \rightarrow H_2(R, B, B) \rightarrow H_1 \rightarrow F_1 \otimes_R B \rightarrow I_B/I_B^2 \rightarrow 0$$

as definition of the 2. algebra homology  $H_2(R, B, B)$  (cf. [28]), and the dual sequence,

$$\rightarrow {}_v\text{Hom}_B(F_1 \otimes B, B) \rightarrow {}_v\text{Hom}_B(H_1, B) \rightarrow {}_vH^2(R, B, B) \rightarrow 0,$$

as a definition of graded 2. algebra cohomology  $H^2(R, B, B)$ . If  $B$  is generically a complete intersection, then it is well known that  $\text{Ext}_B^1(I_B/I_B^2, B) \simeq H^2(R, B, B)$  ([1], Proposition 16.1). We know that  $H^0(Y, \mathcal{N}_Y)$  is isomorphic to the tangent space of  $\text{Hilb}^p(\mathbb{P}^n)$  in general, while  $H^1(Y, \mathcal{N}_Y)$  contains the obstructions of deforming  $Y \subset \mathbb{P}^n$  in the case  $Y$  is locally a complete intersection (l.c.i.) (cf. [11]). If  ${}_0\text{Hom}_R(I_B, H_m^1(B)) = 0$  (e.g.  $\text{depth}_m B \geq 2$ ), we have by (2.1) that  ${}_0\text{Hom}_B(I_B/I_B^2, B) \simeq H^0(Y, \mathcal{N}_Y)$  and  ${}_0H^2(R, B, B) \hookrightarrow H^1(Y, \mathcal{N}_Y)$  is injective in the l.c.i. case, and by [18], Remark 3.7 that  ${}_0H^2(R, B, B)$  contains the obstructions of deforming  $Y \subset \mathbb{P}^n$ . Thus  ${}_0H^2(R, B, B) = 0$  suffices for the unobstructedness of a locally complete intersection arithmetically Cohen-Macaulay (ACM) subscheme  $Y$  of  $\mathbb{P}^n$  of  $\dim B \geq 2$  (for this conclusion we may even entirely skip ‘‘l.c.i.’’ by slightly extending the argument, as done in [18]).

There are cases where we can conclude that some  $X = \text{Proj}(A)$  or  $A$  is unobstructed without assuming  ${}_0H^2(R, A, A) = 0$ . One such case which we need in this paper is treated in [19], Theorem 9.4 and extended in [23]. Following [23] we say ‘‘ $(M, B)$  is unobstructed along any graded deformation of  $B$ ’’ if for every small Artin surjection  $(T, m_T) \rightarrow (S, m_S)$  (i.e. a surjection of local Artinian  $k$ -algebras with residue fields  $k$  whose kernel  $\mathfrak{a}$  satisfies  $\mathfrak{a} \cdot m_T = 0$ ) and every graded deformation  $(M_S, B_S)$  of  $(M, B)$ , there is a graded deformation of  $M_S$  to *any* graded deformation  $B_T$  of  $B_S$ . We need the following special case of [23], Proposition 13.

**Proposition 2.2.** *Let  ${}_0\text{Ext}_B^1(M, M) = 0$ . Then  $(M, B)$  is unobstructed along any graded deformation of  $B$  if for every local Artinian  $k$ -algebra  $T$  with residue field  $k$  and for every graded deformation  $B_T$  of  $B$  to  $T$ , there exists a graded deformation  $M_T$  of  $M$  to  $B_T$ .*

**Example 2.3.** Let  $\text{char}(k) \neq 2$ , let  $B \simeq R/I_B$  be a graded CM quotient of  $R$  of codimension  $c$  and suppose  $\text{depth}_{I(Z)} B \geq 3$  where  $Y - Z$  is locally a complete intersection in  $\mathbb{P}^n$ . Then  $(S^2K_B(v), B)$  is unobstructed along any graded deformation of  $B$  for every integer  $v$ . Indeed the proposition above applies because  $K_{B_T} := \text{Ext}_{B_T}^c(B_T, R_T(-n-1))$  is flat over  $T$  by [16], Proposition A1. Hence  $K_{B_T} \otimes K_{B_T}$ , as well as  $S^2K_{B_T}$ , are  $T$ -flat. It follows that  $S^2K_{B_T}(v)$  is a graded deformation of  $S^2K_B(v)$ . Since  ${}_0\text{Ext}_B^1(S^2K_B, S^2K_B) = 0$

by (2.1) (explained thoroughly later), we conclude easily. In the next section, e.g. in Corollary 3.9, we use the unobstructedness of  $(S^2K_B(v), B)$  to show that the quotient  $A$  of (1.1) is unobstructed provided  $\lambda \gg 0$ .

Finally in the case  $B = R/I_B$  is a generically a complete intersection codimension two CM quotient of  $R$ , we notice some exact sequences frequently used in this paper. Firstly since  $F_2 \hookrightarrow F_1$  is injective in (2.2), we get a minimal resolution

$$(2.4) \quad 0 \rightarrow R \rightarrow F_1^\vee = \bigoplus_{i=1}^{\mu} R(n_{1,i}) \rightarrow F_2^\vee = \bigoplus_{j=1}^{\mu-1} R(n_{2,j}) \rightarrow K_B(n+1) \rightarrow 0$$

by taking  $R$ -duals. If we apply  $\text{Hom}(-, B)$  to (2.4), letting  $K_B^\vee = \text{Hom}_B(K_B, B)$ , we get the exactness to the left in the exact sequence

$$(2.5) \quad 0 \rightarrow K_B(n+1)^\vee \rightarrow \bigoplus_{j=1}^{\mu-1} B(-n_{2,j}) \rightarrow \bigoplus_{i=1}^{\mu} B(-n_{1,i}) \rightarrow I_B/I_B^2 \rightarrow 0$$

which splits into two short exact sequences “via  $\bigoplus B(-n_{2,j}) \twoheadrightarrow H_1 \hookrightarrow \bigoplus B(-n_{1,i})$ ”, one of which is (2.3) with  $H_2(R, B, B) = 0$ . Indeed in this case  $H_1$  is Cohen-Macaulay by [2] and hence  $H_2(R, B, B) = 0$  by (2.3). Moreover since  $\text{Ext}_R^1(I_B, I_B) \simeq N_B$  we showed in [24] that there is an exact sequence of the form

$$(2.6) \quad 0 \rightarrow F_1^\vee \otimes_R F_2 \rightarrow ((F_1^\vee \otimes_R F_1) \oplus (F_2^\vee \otimes_R F_2))/R \rightarrow F_2^\vee \otimes_R F_1 \rightarrow N_B \rightarrow 0.$$

Indeed this sequence is deduced from the exact sequence

$$0 \longrightarrow R \longrightarrow \bigoplus_{i=1}^{\mu} I(n_{1,i}) \longrightarrow \bigoplus_{j=1}^{\mu-1} I(n_{2,j}) \longrightarrow N_B \rightarrow 0$$

which we get by applying  $\text{Hom}_R(-, I_B)$  to (2.2), (cf. [24], (26)). Then it is straightforward to find Castelnuovo-Mumford regularity of  $N_B$ , as well as the formula of  $\dim(N_B)_0$  of [8]. Note also the following frequently used sequences (cf. [7], p. 595 for two of them)

$$(2.7) \quad 0 \rightarrow \wedge^2(F_1^\vee) \rightarrow F_1^\vee \otimes F_2^\vee \rightarrow S^2(F_2^\vee) \rightarrow S^2(K_B)(2n+2) \rightarrow 0$$

$$(2.8) \quad 0 \rightarrow \wedge^3(F_1^\vee) \rightarrow \wedge^2(F_1^\vee) \otimes F_2^\vee \rightarrow F_1^\vee \otimes S^2(F_2^\vee) \rightarrow S^3(F_2^\vee) \rightarrow S^3(K_B)(3n+3) \rightarrow 0$$

$$(2.9) \quad 0 \rightarrow \wedge^2(F_2) \rightarrow F_1 \otimes F_2 \rightarrow S^2(F_1) \rightarrow I_B^2 \rightarrow 0.$$

### 3. CM QUOTIENTS OF CODIMENSION 1 ON A CODIMENSION 2 CM ALGEBRA

In this section we study families of graded CM quotients defined by a regular section  $\sigma$  of  $(S^2K_B^\vee)_\lambda$ ; i.e. CM quotients  $A$  given by

$$(3.1) \quad 0 \longrightarrow (S^2K_B)(-\lambda) \xrightarrow{\sigma} B \longrightarrow A \longrightarrow 0$$

where  $B$  is a codimension 2 graded generically complete intersection CM quotient of  $R$ .

Recall that by [27]; Theorem III.4.2, for any general codimension 2, arithmetically Cohen-Macaulay scheme  $Y \subset \mathbb{P}^n$  with degree matrix  $(u_{ij})_{i=1, \dots, t}^{j=1, \dots, t-1}$ ,  $u_{ij} > 0$ , we have  $\text{Pic}(Y) \cong \mathbb{Z}.H$  or  $\text{Pic}(Y) \cong \mathbb{Z}^2$ , unless  $Y \subset \mathbb{P}^4$  is a Castelnuovo surface and  $\text{Pic}(Y) \cong \mathbb{Z}^9$  or  $Y \subset \mathbb{P}^4$  is a Bordiga surface and  $\text{Pic}(Y) \cong \mathbb{Z}^{11}$ . In addition, if  $n = 4$  then  $\text{Pic}(Y) \cong \mathbb{Z}^2 = \mathbb{Z}.H \oplus \mathbb{Z}.K$  with  $K$  the canonical divisor on  $Y$  and  $H$  the hyperplane divisor.

Moreover, by [6] Theorem 3.12, an effective divisor  $X \sim aK_Y + bH$  is ACM if and only if  $-2 \leq a \leq 1$ . ACM effective divisors  $X \sim bH$  on a codimension 2, ACM  $Y \subset \mathbb{P}^n$  are determinantal schemes of codimension 3 and they have been studied in [19] and [20]; ACM effective divisors  $X \sim -K_Y + bH$  on a codimension 2, ACM  $Y \subset \mathbb{P}^n$  are

arithmetically Gorenstein schemes of codimension 3 and they have been studied in [21]; and ACM effective divisors  $X \sim K_Y + bH$  on a codimension 2, ACM  $Y \subset \mathbb{P}^n$  were studied in [19]. We will devote this section to the remaining case; namely, to study arithmetically Cohen-Macaulay effective divisors  $X \sim -2K_Y + bH$  on a codimension 2, ACM  $Y \subset \mathbb{P}^n$ .

Our goal is to determine good lower bounds for the dimension of any irreducible component of  $\text{Hilb}^{p(x)}(\mathbb{P}^n)$  containing a point  $(X)$ ,  $X = \text{Proj}(A)$ , where  $A$  is given by (3.1), and to show that they are sharp for  $\lambda \gg 0$  (when  $n = 4$  and  $5$ ). The lower bounds will be computed in terms of

$$\begin{aligned} a &:= {}_0\text{hom}_B(I_B/I_B^2, I_{A/B}) - {}_0\text{ext}_B^1(I_B/I_B^2, I_{A/B}), \\ b &:= {}_0\text{hom}_B(I_{A/B}, B) - {}_0\text{ext}_B^1(I_{A/B}, B), \text{ and} \\ e &:= {}_0\text{ext}_B^2(I_{A/B}, I_{A/B}), \end{aligned}$$

where  $I_{A/B} = S^2K_B(-\lambda)$ . Indeed since we will use deformation theory related to the flag of surjections

$$R \longrightarrow B \longrightarrow A \cong B/I_{A/B}$$

it turns out that the groups

$$\begin{aligned} &{}_0\text{Ext}_B^1(I_B/I_B^2, I_{A/B}) \quad (\text{ or } {}_0\text{Ext}_R^1(I_B, I_{A/B}) ) \\ &{}_0\text{Ext}_A^1(I_{A/B}/I_{A/B}^2, A) \quad (\text{ or } {}_0\text{Ext}_B^1(I_{A/B}, A) ) \end{aligned}$$

play a central role.

Let  $Y = \text{Proj}(B)$  and let  $U = \text{Proj}(B) - Z \hookrightarrow \mathbb{P}^n$  be a local complete intersection (l.c.i.). In the following we almost always assume  $\text{depth}_{I(Z)} B \geq 3$  because our proofs often use that  $\widetilde{K_B|U}$  and  $\widetilde{I_B/I_B^2|U}$  are locally free in a large enough open set. Firstly we will make  $a$  more explicit. To do it, we define

$$s^i(\nu) := \dim(S^i K_B)_\nu.$$

**Lemma 3.1.** *If  $\text{depth}_{I(Z)} B \geq 3$ , then  $\text{Ext}_B^2(I_B/I_B^2, S^2(K_B)) = 0$ , and*

$$\begin{aligned} a &= \sum_{i=1}^{\mu} s^2(n_{1,i} - \lambda) - \sum_{i=1}^{\mu-1} s^2(n_{2,i} - \lambda) + s^3(n+1 - \lambda) = \\ &[\dim(\wedge^2 F_1^\vee \otimes F_1^\vee)_{-v} - \dim(\wedge^3 F_1^\vee)_{-v}] - \dim(F_1^\vee \otimes F_1^\vee \otimes F_2^\vee)_{-v} + \\ &\dim(F_1^\vee \otimes F_2^\vee \otimes F_2^\vee)_{-v} - [\dim(S^2 F_2^\vee \otimes F_2^\vee)_{-v} - \dim(S^3 F_2^\vee)_{-v}] \end{aligned}$$

where  $v = 2n + 2 + \lambda$ .

*Proof.* The exact sequence (2.5) leads to

$$(3.2) \quad 0 \longrightarrow H_1 \longrightarrow \bigoplus_{i=1}^{\mu} B(-n_{1,i}) \longrightarrow I_B/I_B^2 \longrightarrow 0, \text{ and}$$

$$(3.3) \quad 0 \longrightarrow K_B(n+1)^\vee \longrightarrow \bigoplus_{i=1}^{\mu-1} B(-n_{2,i}) \cong F_2 \otimes B \longrightarrow H_1 \longrightarrow 0$$

where  $H_1$  is the 1.Koszul homology. Since  $I_{A/B} = S^2K_B(-\lambda)$  we apply  $\text{Hom}_B(-, S^2K_B(-\lambda))$  to (3.2), and we get

$$(3.4) \quad \begin{aligned} 0 \longrightarrow \text{Hom}_B(I_B/I_B^2, S^2K_B(-\lambda)) \longrightarrow \bigoplus_{i=1}^{\mu} S^2K_B(-\lambda + n_{1,i}) \longrightarrow \\ \text{Hom}_B(H_1, S^2K_B(-\lambda)) \longrightarrow \text{Ext}_B^1(I_B/I_B^2, S^2K_B(-\lambda)) \longrightarrow 0 \end{aligned}$$

and

$$\mathrm{Ext}_B^1(H_1, S^2K_B(-\lambda)) \cong \mathrm{Ext}_B^2(I_B/I_B^2, S^2K_B(-\lambda)).$$

Moreover using the exact sequence (3.3) we obtain

$$(3.5) \quad 0 \longrightarrow \mathrm{Hom}_B(H_1, S^2K_B(-\lambda)) \longrightarrow \bigoplus_{i=1}^{\mu-1} S^2K_B(-\lambda + n_{2,i}) \cong F_2^\vee \otimes S^2K_B(-\lambda) \longrightarrow$$

$$\mathrm{Hom}_B(K_B^\vee(-n-1), S^2K_B(-\lambda)) \cong S^3K_B(n+1-\lambda) \rightarrow \mathrm{Ext}_B^1(H_1, S^2K_B(-\lambda)) \rightarrow 0.$$

Note that we have used (2.8) and Lemma 2.1 to see  $\mathrm{depth}_{I(Z)} S^3K_B \geq 2$  and then (2.1) to get  $\mathrm{Hom}_B(K_B^\vee, S^2K_B) \cong S^3K_B$ . Now since  $F_2^\vee \rightarrow K_B(n+1)$  is surjective, it follows that  $F_2^\vee \otimes S^2K_B \rightarrow S^3K_B(n+1)$  is surjective and we get  $\mathrm{Ext}_B^2(I_B/I_B^2, S^2(K_B)) = 0$ . We deduce that

$$\begin{aligned} a &= {}_0\mathrm{hom}_B(I_B/I_B^2, I_{A/B}) - {}_0\mathrm{ext}_B^1(I_B/I_B^2, I_{A/B}) \\ &= \sum_{i=1}^{\mu} s^2(n_{1,i} - \lambda) - \sum_{i=1}^{\mu-1} s^2(n_{2,i} - \lambda) + s^3(n+1-\lambda) \end{aligned}$$

which proves the first dimension formula. Finally since it is straightforward to write down the minimal resolutions of  $F_i^\vee \otimes S^2K_B(-\lambda)$ ,  $i = 1, 2$  and  $S^3K_B(n+1-\lambda)$  from (2.7) and (2.8), we get the second dimension formula as well and we are done.  $\square$

It is also of interest to see the vanishing of the groups  ${}_0\mathrm{Ext}_B^i(I_B/I_B^2, S^2K_B(-\lambda))$  above.

**Lemma 3.2.** *With notations as above, let  $d_k = \max_i \{n_{k,i}\}$ . Then, for  $i = 0$  and 1,*

$${}_0\mathrm{Ext}_B^i(I_B/I_B^2, S^2K_B(-\lambda)) = {}_0\mathrm{Ext}_R^i(I_B, S^2K_B(-\lambda)) = 0 \text{ provided } \lambda > 3d_2 - 2n - 2$$

and

$${}_0\mathrm{Hom}_R(I_B, S^2K_B(-\lambda)) = 0 \text{ provided } \lambda > d_1 + 2d_2 - 2n - 2.$$

*Proof.* Using (2.2), we deduce that  ${}_0\mathrm{Ext}_R^1(I_B, S^2K_B(-\lambda)) = 0$  provided we can show  $(F_2^\vee \otimes S^2K_B(-\lambda))_0 = 0$ , i.e. provided (cf. (2.7))

$$(F_2^\vee \otimes S^2(F_2^\vee)(-2n-2))_{(-\lambda)} = 0.$$

This follows easily from the first assumption of the lemma since  $F_2^\vee = \bigoplus R(n_{2,i})$ . Using  $F_1^\vee = \bigoplus R(n_{1,i})$  we get the vanishing of the Hom-group from  $(F_1^\vee \otimes S^2(F_2^\vee)(-2n-2-\lambda))_0 = 0$ . Finally since  $d_2 \geq d_1$ , the first assumption also implies the vanishing of the  $\mathrm{Ext}^i$ -group for  $i = 0$  and we are done.  $\square$

Next we will make  $b$  more explicit. Firstly note that, if  $\mathrm{depth}_{I(Z)} B \geq 3$ , we have

$$(3.6) \quad {}_0\mathrm{Ext}_A^1(I_{A/B}/I_{A/B}^2, A) \cong {}_0\mathrm{Ext}_B^1(I_{A/B}, A)$$

and hence  ${}_0H^2(B, A, A) \cong {}_0\mathrm{Ext}_B^1(I_{A/B}, A)$ . Indeed by a well-known spectral sequence it suffices to see  $\mathrm{Hom}_A(\mathrm{Tor}_1^B(I_{A/B}, A), A) = 0$ . Since, however,  $I_{A/B}$  is invertible in  $U = Y - Z$  and the intersection of  $U$  with any irreducible component of  $X$  is non-empty, we get the assertion.

**Lemma 3.3.** *If  $\text{depth}_{I(Z)} B \geq 3$ , then  ${}_0\text{Ext}_B^2(I_{A/B}, B) = {}_0\text{Ext}_B^2(S^3 K_B(-\lambda), K_B) = 0$ , and*

$$b = {}_0\text{hom}_B(S^3 K_B(-\lambda), K_B) - {}_0\text{ext}_B^1(S^3 K_B(-\lambda), K_B) = \\ \dim(S^3 F_2)_v - \dim(F_1 \otimes S^2 F_2)_v + \dim(\wedge^2 F_1 \otimes F_2)_v - \dim(\wedge^3 F_1)_v$$

where  $v = \lambda + 2n + 2$ .

*Proof.* Since  $\text{depth}_{I(Z)} B \geq 3$  and  $\text{depth}_{I(Z)} K_B \geq 3$ , we have by (2.1),

$$\text{Ext}_B^1(I_{A/B}, B) \cong \text{Ext}_{\mathcal{O}_U}^1(\widetilde{S^2 K_B(-\lambda)}, \widetilde{B}) \cong H_*^1(U, \mathcal{H}om(\widetilde{S^2 K_B(-\lambda)}, \widetilde{B})) \\ \cong H_*^1(U, \mathcal{H}om(\widetilde{S^3 K_B(-\lambda)}, \widetilde{K_B})) \cong \text{Ext}_B^1(S^3 K_B(-\lambda), K_B),$$

and correspondingly for the Hom groups. Hence we get the first expression of  $b$  above.

Moreover  $\text{Ext}_B^2(I_{A/B}, B) \cong \text{Ext}_B^2(S^2 K_B(-\lambda) \otimes K_B, K_B)$  by the spectral sequence of [14], Satz 2.1 and  ${}_0\text{Ext}_B^2(S^3 K_B(-\lambda), K_B) = 0$  by (2.8). If  $\Lambda$  is the kernel of the surjective map  $S^2 K_B(-\lambda) \otimes K_B \rightarrow S^3 K_B(-\lambda)$ , then  $\widetilde{\Lambda}|_U = 0$  and since  $\text{Ext}_B^2(\Lambda, K_B) \hookrightarrow \text{Ext}_{\mathcal{O}_U}^2(\widetilde{\Lambda}, \widetilde{K_B})$  is injective by (2.1), we get  $\text{Ext}_B^2(\Lambda, K_B) = 0$  from  $\text{Ext}_{\mathcal{O}_U}^2(\widetilde{\Lambda}, \widetilde{K_B}) \cong H_*^2(U, \mathcal{H}om(\widetilde{\Lambda}, \widetilde{K_B})) = 0$ . Then the exact sequence

$$\rightarrow \text{Ext}_B^2(S^3 K_B(-\lambda), K_B) \rightarrow \text{Ext}_B^2(S^2 K_B(-\lambda) \otimes K_B, K_B) \rightarrow \text{Ext}_B^2(\Lambda, K_B) \rightarrow$$

shows that  $\text{Ext}_B^2(S^2 K_B(-\lambda) \otimes K_B, K_B) = 0$  and we get the vanishing of both  $\text{Ext}^2$ -groups of the lemma.

Finally using Gorenstein duality twice (i.e. over  $B$  and over  $R$ ) we get

$${}_0\text{Ext}_R^i(S^3 K_B(-\lambda), R(-n-1)) \cong {}_0\text{Ext}_B^{i-2}(S^3 K_B(-\lambda), K_B)$$

for  $i = 2$  and  $3$  and we have  ${}_0\text{Ext}_R^i(S^3 K_B(-\lambda), R(-n-1)) = 0$  for  $i = 0$  and  $1$ . Hence if we apply the contravariant functor  ${}_0\text{Hom}(-, R(\lambda + 2n + 2))$  to the exact sequence (2.8), we obtain a complex (where  $v = \lambda + 2n + 2$ );

$$(3.7) \quad 0 \rightarrow (S^3 F_2)_v \rightarrow (F_1 \otimes S^2 F_2)_v \rightarrow (\wedge^2 F_1 \otimes F_2)_v \rightarrow (\wedge^3 F_1)_v \rightarrow 0,$$

which is exact except at the spots which correspond to  ${}_0\text{Ext}_R^i(S^3 K_B(-\lambda), R(-n-1)) \neq 0$ . Therefore, the alternating sum of the dimension of this complex must be  $b$  and we are done.  $\square$

By using (2.1) as above we get the following information about the number  $e := {}_0\text{ext}_B^2(I_{A/B}, I_{A/B})$ ;

**Lemma 3.4.** *If  $\text{depth}_{I(Z)} B \geq 3$ , then  ${}_0\text{hom}_B(I_{A/B}, I_{A/B}) = 1$ ,  ${}_0\text{Ext}_B^1(I_{A/B}, I_{A/B}) = 0$  and there is an exact sequence*

$$(3.8) \quad 0 \rightarrow \text{Ext}_B^2(I_{A/B}, I_{A/B}) \rightarrow H_*^2(U, \widetilde{B}) \rightarrow \text{Hom}_B(I_{A/B}, H_{I(Z)}^3(I_{A/B})) \rightarrow \text{Ext}_B^3(I_{A/B}, I_{A/B}).$$

*In particular*

- (i) if  $\text{depth}_{I(Z)} B \geq 4$  then  $\text{Ext}_B^2(I_{A/B}, I_{A/B}) = 0$ , i.e.  $e = 0$ ,
- (ii) if  $\text{depth}_{I(Z)} B = 3$ ,  $I(Z) = \mathfrak{m}$  and  $B$  has a semi-linear resolution, i.e.

$$(3.9) \quad 0 \rightarrow R(-s-2)^{\beta_2} \oplus R(-s-1)^{\alpha_2} \rightarrow R(-s-1)^{\beta_1} \oplus R(-s)^{\alpha_1} \rightarrow R \rightarrow B \rightarrow 0$$



then

$$e = \beta_2 \cdot \binom{s+1}{4} + (\alpha_2 - \beta_1) \cdot \binom{s}{4} - \alpha_1 \cdot \binom{s-1}{4}.$$

**Remark 3.5.** (a) Suppose  $\text{depth}_{I(Z)} B \geq 3$ . Since  $\text{Ext}_B^2(I_{A/B}, B) = 0$  and  $\text{Ext}_B^1(I_{A/B}, I_{A/B}) = 0$  by the lemmas above, the sequence

$$0 \rightarrow \text{Ext}_B^1(I_{A/B}, B) \rightarrow \text{Ext}_B^1(I_{A/B}, A) \rightarrow \text{Ext}_B^2(I_{A/B}, I_{A/B}) \rightarrow 0,$$

deduced from  $0 \rightarrow I_{A/B} \rightarrow B \rightarrow A \rightarrow 0$ , is exact. The following sequence related to (3.8) is also useful in computing  $e$ ;

$$(3.10) \quad 0 \rightarrow \text{Ext}_B^1(I_{A/B}, A) \rightarrow H_*^1(U, \mathcal{H}om(\widetilde{I_{A/B}}, \widetilde{A})) \rightarrow \text{Hom}_B(I_{A/B}, H_{I(Z)}^2(A)) \rightarrow$$

Note that the corresponding sequence for  ${}_0\text{Ext}_B^1(I_{A/B}, B)$  implies

$${}_0\text{Ext}_B^1(I_{A/B}, B) \cong H^1(U, \mathcal{H}om(\widetilde{I_{A/B}}, \widetilde{B})) \cong H^1(U, \widetilde{I_{A/B}}^\vee).$$

(b) Let  $T \rightarrow S$  be a small Artin surjection with kernel  $\mathfrak{a}$  and let  $\psi_S : B_S \rightarrow A_S$  (resp.  $B_T$ ) be a graded deformation of  $B \rightarrow A$  to  $S$  (resp. of  $B_S$  to  $T$ ), see Proposition 2.2 and Example 2.3. By (3.6) and [11] (see [21], Sect. 1.1 for a general introduction) the group  ${}_0\text{Ext}_B^1(I_{A/B}, A) \otimes_k \mathfrak{a}$  contains the obstruction  $o(\psi_S; B_T)_0$  of deforming  $\psi_S$  further to  $B_T$ . Since  $(I_{A/B}, B)$  is unobstructed along any graded deformation of  $B$  and since  $o(\psi_S; B_T)_0$  maps onto the obstruction in  ${}_0\text{Ext}_B^2(I_{A/B}, I_{A/B}) \otimes_k \mathfrak{a}$  of deforming  $\ker \psi_S$  to  $B_T$ , it maps to zero!! It follows that  $o(\psi_S; B_T)_0$  sits in  ${}_0\text{Ext}_B^1(I_{A/B}, B) \otimes_k \mathfrak{a}$ , i.e. we say that  ${}_0\text{Ext}_B^1(I_{A/B}, B)$  contains all graded obstructions of deforming  $B \rightarrow A$  to any given deformation of  $B$ .

*Proof.* By (2.7)  $I_{A/B} = S^2 K_B(-\lambda)$  is a maximal CM module and we have  $\text{depth}_{I(Z)} I_{A/B} \geq 3$ . Hence, by (2.1),

$$\begin{aligned} \text{Ext}^1(I_{A/B}, I_{A/B}) &\cong \text{Ext}_{\mathcal{O}_U}^1(\widetilde{I_{A/B}}, \widetilde{I_{A/B}}) \\ &\cong H_*^1(U, \mathcal{H}om(\widetilde{I_{A/B}}, \widetilde{I_{A/B}})) \cong H_*^1(U, \widetilde{B}) \cong H_{I(Z)}^2(B) = 0 \end{aligned}$$

and

$${}_0\text{Hom}(I_{A/B}, I_{A/B}) \cong H_*^0(U, \widetilde{B})_0 \cong B_0 \cong k.$$

In the same way, by using (2.1), we get (3.8).

Now we will use (3.8) to show (i) and (ii). (i) is clear. To see (ii) it suffices to show  ${}_0\text{Hom}_B(I_{A/B}, H_{\mathfrak{m}}^3(I_{A/B})) = 0$  because, by (2.4) and duality,

$$h^2(Y, \widetilde{B}) = \dim(K_B)_0 = \sum_{1 \leq i \leq \mu-1} \binom{n_{2,i}-1}{n} - \sum_{1 \leq i \leq \mu} \binom{n_{1,i}-1}{n}$$

and  $n = 4$  and we get the expression of  $e$  in the lemma. We have  $I_{A/B} = S^2 K_B(-\lambda)$ . To see  ${}_0\text{Hom}_B(S^2 K_B, H_{\mathfrak{m}}^3(S^2 K_B)) = 0$  it suffices by (2.7) to show

$${}_0\text{Hom}_B(S^2(F_2^\vee), H_{\mathfrak{m}}^5(\wedge^2(F_1^\vee))) = 0,$$

i.e. to prove  ${}_0H_{\mathfrak{m}}^5(S^2(F_2) \otimes \wedge^2(F_1^\vee)) = 0$ . Since, however,  ${}_0H_{\mathfrak{m}}^5(R(t)) = 0$  for  $t \geq -4$ , we are done.  $\square$

Now we come to the two main theorems of this section. Recall that  $B = R/I_B$  has normal module  $N_B := \text{Hom}_B(I_B, B)$  and minimal resolution

$$0 \rightarrow F_2 := \bigoplus_{j=1}^{\mu-1} R(-n_{2,j}) \rightarrow F_1 := \bigoplus_{i=1}^{\mu} R(-n_{1,i}) \rightarrow R \rightarrow B \rightarrow 0$$

and that  $A$  is defined by (3.1), i.e.  $A \cong B/(S^2 K_B(-\lambda))$ . We include a criterion of unobstructedness which we may apply if we are able to compute  $h^0(\mathcal{N}_X)$  ( $\mathcal{N}_X$  the normal sheaf of  $X = \text{Proj}(A) \subset \mathbb{P}^n$ ).

**Theorem 3.6.** *Let  $B$  be a codimension 2 CM quotient of  $R = k[x_0, \dots, x_n]$ , and let  $U = \text{Proj}(B) - Z \hookrightarrow \mathbb{P}^n$  be a local complete intersection such that  $\text{depth}_{I(Z)} B \geq 3$ . Let  $X = \text{Proj}(A) \subset \mathbb{P}^n$  be the codimension 3 ACM scheme defined by (3.1). Let  $h_A^i := \dim {}_0H^i(R, A, A)$  and let  $v = \lambda + 2n + 2$ . Then*

$$b - 1 + \dim(N_B)_0 - a - e = h^0(\mathcal{N}_X) - h_A^2 \leq \dim_{(X)} \text{Hilb } \mathbb{P}^n,$$

where  $e := {}_0\text{ext}_B^2(I_{A/B}, I_{A/B})$  and

$$\begin{aligned} b-1+\dim(N_B)_0-a = & - \sum_{1 \leq i < j < k \leq \mu} \binom{-n_{1,i} - n_{1,j} - n_{1,k} + n + v}{n} + \sum_{\substack{1 \leq i < j \leq \mu \\ 1 \leq k \leq \mu-1}} \binom{-n_{1,i} - n_{1,j} - n_{2,k} + n + v}{n} \\ & - \sum_{\substack{1 \leq i \leq \mu \\ 1 \leq j \leq k \leq \mu-1}} \binom{-n_{1,i} - n_{2,j} - n_{2,k} + n + v}{n} + \sum_{1 \leq i \leq j \leq k \leq \mu-1} \binom{-n_{2,i} - n_{2,j} - n_{2,k} + n + v}{n} + \sum_{\substack{1 \leq i \leq \mu \\ 1 \leq j \leq \mu-1}} \binom{n_{2,j} - n_{1,i} + n}{n} \\ & + \sum_{\substack{1 \leq i \leq \mu \\ 1 \leq j \leq \mu-1}} \binom{n_{1,i} - n_{2,j} + n}{n} - \sum_{1 \leq i, j \leq \mu} \binom{n_{1,j} - n_{1,i} + n}{n} - \sum_{1 \leq i, j \leq \mu-1} \binom{n_{2,i} - n_{2,j} + n}{n} \\ & - \sum_{\substack{1 \leq i < j \leq \mu \\ k \leq j}} \binom{n_{1,i} + n_{1,j} + n_{1,k} + n - v}{n} + \sum_{\substack{1 \leq i, j \leq \mu \\ 1 \leq k \leq \mu-1}} \binom{n_{1,i} + n_{1,j} + n_{2,k} + n - v}{n} \\ & - \sum_{\substack{1 \leq i \leq \mu \\ 1 \leq j, k \leq \mu-1}} \binom{n_{1,i} + n_{2,j} + n_{2,k} + n - v}{n} + \sum_{\substack{1 \leq i, j, k \leq \mu-1 \\ i \leq j > k}} \binom{n_{2,i} + n_{2,j} + n_{2,k} + n - v}{n} \end{aligned}$$

Moreover if  $h^0(\mathcal{N}_X) + e = b - 1 + \dim(N_B)_0 - a$ , then  $X$  is unobstructed.

*Proof.* A general theorem of Laudal, which establishes a lower bound for dimension of the hull of any deformation functor, implies, thanks to Theorem 1.3 and Theorem 2.2 of [18], that  $h_A^1 - h_A^2$  is a lower bound for the dimension of the hull of the graded deformation functor of  $R \rightarrow A$ . Since Theorem 3.6 and Remark 3.7 of [18] (or one may use [8], Proposition 1 since  $A$  is CM) implies that the graded deformation functor of  $R \rightarrow A$  is isomorphic to the local Hilbert functor of  $(X = \text{Proj}(A) \subset \mathbb{P}^n)$ , we get

$$\dim_{(X)} \text{Hilb } \mathbb{P}^n \geq h_A^1 - h_A^2.$$

Now to see  $b - 1 - e + \dim(N_B)_0 - a = h_A^1 - h_A^2$ , we consider the diagram

$$(3.11) \quad \begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & {}_0\mathrm{Hom}(I_B, I_{A/B}) & & & \\ & & & \downarrow & & & \\ & & & H^0(\widetilde{N_B}) & & & \\ & & & \downarrow & & & \\ {}_0\mathrm{Hom}(I_{A/B}, A) & \hookrightarrow & {}_0H^1 & \rightarrow & {}_0\mathrm{Hom}_R(I_B, A) & \rightarrow & {}_0H^2(B, A, A) \rightarrow {}_0H^2 \rightarrow 0 \\ & & & & \downarrow & & \\ & & & & {}_0\mathrm{Ext}^1(I_B/I_B^2, I_{A/B}) & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

where  $H^i = H^i(R, A, A)$  (cf. [21], Section 1.1). Note that we have used Lemma 3.1 to see that

$${}_0H^2(R, B, A) \cong {}_0\mathrm{Ext}^1(I_B/I_B^2, A) \cong {}_0\mathrm{Ext}^2(I_B/I_B^2, I_{A/B}) = 0$$

and the fact that  $B$  is licci and  $\widetilde{K_B|U}$  locally free to see

$${}_0\mathrm{Ext}^i(I_B/I_B^2, B) \cong {}_0\mathrm{Ext}^i(I_B/I_B^2 \otimes K_B, K_B) = 0 \text{ for } 1 \leq i \leq 2.$$

Using the vertical sequence in the diagram, we get  ${}_0\mathrm{hom}_R(I_B, A) = \dim(N_B)_0 - a$  by the definition of  $a$ . Hence it suffices to show that

$${}_0\mathrm{hom}_B(I_{A/B}, A) - {}_0h^2(B, A, A) = b - 1 - e.$$

Since one knows that  ${}_0H^2(B, A, A) \cong {}_0\mathrm{Ext}_B^1(I_{A/B}, A)$  by (3.6) and we have  ${}_0h^2(B, A, A) = e + {}_0\mathrm{ext}_B^1(I_{A/B}, B)$  by Remark 3.5(a), we conclude by Lemma 3.4.

It remains to compute  $b$ ,  $a$ , and  $\dim(N_B)_0$  in terms of  $n_{j,i}$ . Thanks to Lemma 3.1, we get

$$(3.12) \quad a = \sum_{\substack{1 \leq i < j \leq \mu \\ k \leq j}} \binom{n_{1,i} + n_{1,j} + n_{1,k} + n - v}{n} - \sum_{\substack{1 \leq i, j \leq \mu \\ 1 \leq k \leq \mu-1}} \binom{n_{1,i} + n_{1,j} + n_{2,k} + n - v}{n} \\ + \sum_{\substack{1 \leq i \leq \mu \\ 1 \leq j, k \leq \mu-1}} \binom{n_{1,i} + n_{2,j} + n_{2,k} + n - v}{n} - \sum_{\substack{1 \leq i, j, k \leq \mu-1 \\ i > j > k}} \binom{n_{2,i} + n_{2,j} + n_{2,k} + n - v}{n}.$$

Using Lemma 3.3, we get

$$(3.13) \quad b = - \sum_{1 \leq i < j < k \leq \mu} \binom{-n_{1,i} - n_{1,j} - n_{1,k} + n + v}{n} + \sum_{\substack{1 \leq i < j \leq \mu \\ 1 \leq k \leq \mu-1}} \binom{-n_{1,i} - n_{1,j} - n_{2,k} + n + v}{n} \\ - \sum_{\substack{1 \leq i \leq \mu \\ 1 \leq j \leq k \leq \mu-1}} \binom{-n_{1,i} - n_{2,j} - n_{2,k} + n + v}{n} + \sum_{1 \leq i \leq j \leq k \leq \mu-1} \binom{-n_{2,i} - n_{2,j} - n_{2,k} + n + v}{n}.$$

Moreover one knows by [8] that

$$(3.14) \quad \dim(N_B)_0 = \sum_{\substack{1 \leq i \leq \mu \\ 1 \leq j \leq \mu-1}} \binom{n_{2,j} - n_{1,i} + n}{n} + \sum_{\substack{1 \leq i \leq \mu \\ 1 \leq j \leq \mu-1}} \binom{n_{1,i} - n_{2,j} + n}{n} -$$

$$\sum_{1 \leq i, j \leq \mu} \binom{n_{1,i} - n_{1,j} + n}{n} - \sum_{1 \leq i, j \leq \mu-1} \binom{n_{2,i} - n_{2,j} + n}{n} + 1.$$

Putting (3.12), (3.13) and (3.14) together we get what we want.  $\square$

Looking to the very first part of the proof above, we see that the lower bound is actually a lower bound for the dimension of the moduli space parameterizing graded quotients  $A$  of  $R$  with fixed Hilbert function. We get the bound because the local ring of this space at  $(A)$  is isomorphic to the local ring of  $\text{Hilb } \mathbb{P}^n$  at  $(X)$ . There is also another very natural moduli space to consider, namely the Hilbert-flag scheme  $D$  parameterizing graded surjections  $B \rightarrow A$  of  $R$  with fixed Hilbert functions. If we write  $a$  as  $a = a^0 - a^1$  where

$$(3.15) \quad a^i := {}_0\text{ext}_B^i(I_B/I_B^2, I_{A/B}),$$

one knows that the local ring of  $D$  at  $(B \rightarrow A)$  is isomorphic to (resp. a subring of) the local ring of  $\text{Hilb } \mathbb{P}^n$  at  $(X)$  provided  $a^0 = a^1 = 0$  (resp.  $a^0 = 0$ ). Note that Lemma 3.2 gives a simple criterion for the vanishing of  $a^0$  and  $a^i$ ,  $i = 0, 1$ . The main ingredient of our next result is that we can prove that  $b - 1 + \dim(N_B)_0$  is a lower bound for  $\dim \mathcal{O}_{D, (B \rightarrow A)}$ .

To this end, let us fix some more notation. We denote by

$$\alpha : H^0(\widetilde{N}_B) \rightarrow {}_0\text{Hom}(I_B, A) \rightarrow {}_0H^2(B, A, A) \cong {}_0\text{Ext}_B^1(I_{A/B}, A)$$

the composition of maps appearing in (3.11); see (3.6) for the last isomorphism. If  $\text{depth}_{I(Z)} B \geq 3$ , we see by Remark 3.5 that  $\alpha$  factors through  $\text{Ext}_B^1(I_{A/B}, B)$  giving rise to a map

$$\alpha' : H^0(\widetilde{N}_B) \longrightarrow {}_0\text{Ext}_B^1(I_{A/B}, B).$$

We have

**Theorem 3.7.** *Let  $B$  be a codimension 2 CM quotient of  $R = k[x_0, \dots, x_n]$ ,  $\text{char}(k) \neq 2$ , and let  $U = Y - Z \hookrightarrow \mathbb{P}^n$ ,  $Y = \text{Proj}(B)$ , be a local complete intersection such that  $\text{depth}_{I(Z)} B \geq 3$ . Let  $X = \text{Proj}(A) \subset \mathbb{P}^n$  be the codimension 3 ACM scheme defined by (3.1). Then*

$$b - 1 + \dim(N_B)_0 - a^0 \leq \dim_{(X)} \text{Hilb } \mathbb{P}^n,$$

where  $\dim(N_B)_0$  and  $a^0$  are given by (3.14) and (3.15) with  $v = \lambda + 2n + 2$ , and  $b$  is given by (3.13), or by  $b = h^0(U, \omega_Y^{-2}(\lambda)) - h^1(U, \omega_Y^{-2}(\lambda))$ . Moreover suppose that  $\alpha'$  is surjective (e.g.  ${}_0\text{Ext}_B^1(I_{A/B}, B) = 0$ ). Then,

$$\dim_{(X)} \text{Hilb } \mathbb{P}^n \leq b - 1 + \dim(N_B)_0 - a,$$

and  $X$  is unobstructed if this upper bound is sharp. In particular, if  $a^0 = a^1 = 0$  and  $\alpha'$  is surjective, then  $X$  is unobstructed and

$$\dim_{(X)} \text{Hilb } \mathbb{P}^n = b - 1 + \dim(N_B)_0.$$

*Proof.* Let  $k[\epsilon] := k[x]/(x^2)$  be the ring of dual numbers. We claim that  $b - 1 + \dim(N_B)_0$  is a lower bound for  $\dim \mathcal{O}_{D, (B \rightarrow A)}$ . Indeed if  $\gamma \in H^0(\widetilde{N}_B)$  corresponds to a deformation  $B_\epsilon$  of  $R \rightarrow B$  to  $k[\epsilon]$ , then  $\alpha(\gamma) = o(B \rightarrow A; B_\epsilon)_0$  by [11], see also the proof of Theorem

9.4 in [19] and see Remark 3.5 for notations. Let  ${}_0A_{B \rightarrow A}^1$  be defined by the cartesian diagram

$$\begin{array}{ccc} {}_0A_{B \rightarrow A}^1 & \longrightarrow & H^0(\widetilde{N}_B) \\ \downarrow & \square & \downarrow \\ {}_0H^1 & \longrightarrow & {}_0\mathrm{Hom}_R(I_B, A) \end{array}$$

of morphisms appearing in (3.11). Then  ${}_0A_{B \rightarrow A}^1$  is the tangent space of the Hilbert-flag scheme  $D$  at  $(B \rightarrow A)$  ([21], Sect. 1.1). Moreover, if  $B$  is unobstructed as a graded algebra, then  $\mathrm{coker} \alpha$  contains all obstructions of deforming  $B \rightarrow A$  as graded algebras by a result of the unpublished thesis of the first author (cf. [22], (1.9) for a closely related result). By Remark 3.5,  $\mathrm{coker} \alpha' \subset \mathrm{coker} \alpha$  contains all obstructions of deforming  $B \rightarrow A$  in the case  $(I_{A/B}, B)$  is unobstructed along any graded deformation of  $B$ . Then we can again conclude by Laudal's theorem ([26], Theorem 4.2.4) that

$$(3.16) \quad \dim {}_0A_{B \rightarrow A}^1 - \dim \mathrm{coker} \alpha' \leq \dim \mathcal{O}_{D, (B \rightarrow A)}.$$

Now looking once more to (3.11) and the proof of Theorem 3.6, we easily get

$$(3.17) \quad \dim {}_0A_{B \rightarrow A}^1 - \dim \mathrm{coker} \alpha' = b - 1 + \dim(N_B)_0$$

and the claim is proved.

Let  $p : D \rightarrow \mathrm{Hilb} \mathbb{P}^n$  be the second projection morphism, i.e. induced by  $p((B' \rightarrow A')) = (\mathrm{Proj}(A'))$ . Since the tangent space of the fiber  $p^{-1}((\mathrm{Proj}(A)))$  is  ${}_0\mathrm{Hom}_B(I_B/I_B^2, I_{A/B})$  at  $(B \rightarrow A)$  ([18], Theorem 1.6 or [21], Proposition 4) i.e. an  $a^0$ -dimensional vector space, we get  $\dim \mathcal{O}_{D, (B \rightarrow A)} - a^0 \leq \dim_{(X)} \mathrm{Hilb} \mathbb{P}^n$  and hence the lower bound of the theorem.

Finally suppose  $\mathrm{coker}(\alpha') = 0$ . Looking at (3.11) and using the definition of  $a^i$  and  ${}_0A_{B \rightarrow A}^1$ , we get

$$h^0(\mathcal{N}_X) = \dim {}_0H^1 \leq \dim {}_0A_{B \rightarrow A}^1 - a^0 + a^1.$$

Since  $\dim_{(X)} \mathrm{Hilb} \mathbb{P}^n \leq h^0(\mathcal{N}_X)$  we get the upper bound from (3.17). Moreover if the upper bound is sharp, we deduce  $\dim_{(X)} \mathrm{Hilb} \mathbb{P}^n = h^0(\mathcal{N}_X)$ , and it follows that  $X$  is unobstructed. Since we get the final conclusions by observing that the upper and lower bounds coincide under the assumption  $a^0 = a^1 = 0$ , we are done.  $\square$

**Remark 3.8.** (a) Comparing the lower bounds of Theorem 3.6 and Theorem 3.7, we remark that  $a = a^0 - a^1$ . Hence if  $e > a^1$  then the lower bound of Theorem 3.7 is larger and conversely if  $e < a^1$ .

(b) Moreover note that if  $\alpha'$  is surjective, we see from the proof above that  $e = \dim {}_0H^2(R, A, A)$ . In particular if  $X$  is licci, then  $e = 0$  ([19], Proposition 6.17).

**Corollary 3.9.** *Let  $B$  be a codimension 2 CM quotient of  $R = k[x_0, \dots, x_n]$ ,  $\mathrm{char}(k) \neq 2$  and let  $Y := \mathrm{Proj}(B) \hookrightarrow \mathbb{P}^n$  be a local complete intersection such that  $\dim Y \geq 2$ . Let  $X = \mathrm{Proj}(A) \subset \mathbb{P}^n$  be the codimension 3 ACM scheme defined by (3.1) with  $\lambda \gg 0$ . Then  $X$  is unobstructed and*

$$\dim_{(X)} \mathrm{Hilb} \mathbb{P}^n = b - 1 + \dim(N_B)_0.$$

*For the numbers  $a$  and  $e$  above, we have  $a = 0$ . Moreover if  $\dim Y \geq 3$  then  $e = 0$ .*

*Proof.* By Theorem 3.7 it suffices to show  $a^0 = a^1 = 0$  and  ${}_0\mathrm{Ext}_B^1(S^2K_B(-\lambda), B) = 0$ . The first assumption is taken care of by Lemma 3.2. Moreover using the l.c.i. assumption on  $Y$  we get that  $\widetilde{K}_B$  is locally free, and hence that

$$(3.18) \quad {}_0\mathrm{Ext}_B^1(S^2K_B(-\lambda), B) \cong H^1(Y, S^2\widetilde{K}_B^\vee(\lambda)) \cong H^{\dim Y - 1}(Y, S^3\widetilde{K}_B(-\lambda))^\vee,$$

which vanishes for  $\lambda \gg 0$ . Finally if  $\dim Y = 3$ , it follows from Lemma 3.4(i) that  $e = 0$ , and we are done.  $\square$

One may prove the following result directly from Theorem 9.4 of [19] by using the fact that  $(S^2K_B(-\lambda), B)$  is unobstructed along any graded deformation of  $B$  (Example 2.3). We, however, get the corollary by combining Theorems 3.6 and 3.7.

**Corollary 3.10.** *Let  $Y = \mathrm{Proj}(B)$ ,  $U = Y - Z \hookrightarrow \mathbb{P}^n$  and  $X = \mathrm{Proj}(A)$  be as in Theorem 3.7 and suppose  $\mathrm{depth}_{I(Z)} B \geq 3$  and  ${}_0\mathrm{Ext}_B^1(I_{A/B}, A) = 0$ . Then  $X$  is unobstructed and*

$$\dim_{(X)} \mathrm{Hilb} \mathbb{P}^n = b - 1 + \dim(N_B)_0 - a.$$

Moreover the number  $b$  of (3.13) is also equal to  $b = {}_0\mathrm{hom}(I_{A/B}, B) = h^0(U, \omega_Y^{-2}(\lambda))$ .

*Proof.* Using Remark 3.5 (a), we get  $e = 0$  and that  $\alpha'$  is surjective. It follows that the upper bound of Theorem 3.7 and the lower bound of Theorem 3.6 coincide and we conclude by Theorem 3.7.  $\square$

**Remark 3.11.** In this remark we will use (3.18) to make the assumption  $\lambda \gg 0$  of Corollary 3.9 into an explicit bound provided the number of minimal generators of  $I_B$  is  $\mu = 3$  (and  $I(Z) = \mathfrak{m}$ ). Firstly we consider the exact sequence of locally free sheaves on  $Y$ ,

$$(3.19) \quad 0 \longrightarrow \widetilde{K}_B(m) \longrightarrow \bigoplus_{i=1}^3 \widetilde{B}(-n_{1,i}) \longrightarrow \widetilde{I_B/I_B^2} \longrightarrow 0$$

where  $m := n + 1 - \sum_{i=1}^3 n_{1,i}$ , associated to the exact sequences (3.2) and (3.3). Dualizing it and building the exact sequence of  $S^2(\widetilde{K}_B^\vee)(-2m)$  in the usual way, we get the sequence

$$0 \rightarrow \wedge^2(\widetilde{I_B/I_B^2})^\vee \rightarrow (\widetilde{I_B/I_B^2})^\vee \otimes (\bigoplus_i \widetilde{B}(n_{1,i})) \rightarrow S^2(\bigoplus_i \widetilde{B}(n_{1,i})) \rightarrow S^2(\widetilde{K}_B^\vee)(-2m) \rightarrow 0$$

which simplifies to the following exact sequence of locally free  $\mathcal{O}_Y$ -sheaves

$$(3.20) \quad 0 \rightarrow \widetilde{K}_B(n+1) \rightarrow \bigoplus_i \widetilde{N}_B(n_{1,i}) \rightarrow \bigoplus_{i \leq j} \widetilde{B}(n_{1,i} + n_{1,j}) \rightarrow S^2(\widetilde{K}_B^\vee)(-2m) \rightarrow 0.$$

If  $\dim Y = 3$  and hence  $n = 5$ , we have by (3.20) an injection  $H^1(Y, S^2(\widetilde{K}_B^\vee)(\lambda)) \hookrightarrow H^3(Y, \widetilde{K}_B(n+1+2m+\lambda))$ , which vanishes if  $6 + 2m + \lambda > 0$ , i.e.  $\lambda > 2 \cdot \sum_{i=1}^3 n_{1,i} - 18$ . Combining with Lemma 3.4(i) and Remark 3.5 (a) we get  ${}_0\mathrm{Ext}_B^1(I_{A/B}, A) = 0$  and Corollary 3.10 applies. Hence  $X$  is unobstructed (indeed  ${}_0H^2(R, A, A) = 0$ ) and

$$(3.21) \quad \dim_{(X)} \mathrm{Hilb} \mathbb{P}^n = b - 1 + \dim(N_B)_0 - a$$

provided

$$\lambda > 2 \sum_{i=1}^3 n_{1,i} - 18.$$

We have (3.21) with  $a = 0$  if  $\lambda > \max\{3d_2 - 12, 2 \sum_{i=1}^3 n_{1,i} - 18\}$  by Lemma 3.2.

If  $\dim Y = 2$  and  $n = 4$ , we can use (3.20) to see that  $H^1(Y, S^2(\widetilde{K}_B^\vee)(\lambda)) = 0$  provided  $H^2(Y, \widetilde{N}_B(n_{1,i} + 2m + \lambda)) = 0$  for any  $i$ . Using (2.2) and (2.6) we see that  $H^2(Y, \widetilde{N}_B(v))$  vanishes if  $H^4(\mathbb{P}^4, \widetilde{F}_1^\vee \otimes \widetilde{F}_2(v)) = 0$ , i.e. for  $v + s > d_2 - 5$  where  $s = \min\{n_{1,i}\}$ . Hence  $H^1(Y, S^2(\widetilde{K}_B^\vee)(\lambda)) = 0$  provided  $2s + 2m + \lambda > d_2 - 5$ , i.e.  $\lambda > 2 \cdot \sum_{i=1}^3 n_{1,i} - 2s + d_2 - 15$ . Combining with Lemma 3.2 and Theorem 3.7, we get that  $X$  is unobstructed and that (3.21) holds with  $a = 0$  provided

$$\lambda > \max\{3d_2 - 10, 2 \sum_{i=1}^3 n_{1,i} - 2s + d_2 - 15\}.$$

Sometimes the Kodaira vanishing theorem, or related arguments, show  $H^1(Y, \omega_Y^{-2}(\lambda)) = 0$  more effectively than the arguments of Remark 3.11.

**Example 3.12.** Let  $Y = \text{Proj}(R/I_B)$  be a smooth Castelnuovo surface in  $\mathbb{P}^4$  and let  $X \sim -2K_Y + \lambda H$ ,  $\lambda \geq 0$ , be an effective divisor on  $Y$ . In the usual basis of  $\text{Pic}(Y) \cong \mathbb{Z}^9$ , then  $X_0 := -2K_Y$  corresponds to  $(6; 2^8)$  and  $H$  to  $(4; 2, 1^7)$ . By Kodaira vanishing theorem, or simply by  $K_Y H = -3 < 0$ , we get

$$H^1(\mathcal{N}_{X/Y}^\vee) = H^0(Y, \mathcal{O}_Y(-X + K_Y)) = 0.$$

Hence,  $H^1(Y, \mathcal{O}_Y(X)) = 0$  for any  $\lambda \geq 0$  while Remark 3.11 implies the vanishing of the same group for  $\lambda > 2 \sum n_{1i} - 2s + d_2 - 15 = 1$ , because of

$$0 \longrightarrow R(-4)^2 \longrightarrow R(-3)^2 \oplus R(-2) \longrightarrow I_B \longrightarrow 0.$$

In this case we have  ${}_0\text{Ext}^1(I_{A/B}, A) = 0$  by Remark 3.5(a) and Corollary 3.10 applies for  $\lambda \geq 0$ . We get that  $X$  is unobstructed (indeed  ${}_0H^2(R, A, A) = 0$  by Remark 3.9(b) and Lemma 3.4) and

$$\dim_{(X)} \text{Hilb } \mathbb{P}^n = h^0(\mathcal{O}_Y(X)) - 1 + h^0(\mathcal{N}_Y) - a =$$

$$X(X - K_Y)/2 + 32 - a = 35 + 5\lambda(\lambda + 3)/2 - a.$$

By Lemma 3.2,  $a = 0$  provided  $\lambda \geq 3$  and we have  $a = 6, -2, -2$  for  $\lambda = 0, 1, 2$  respectively by (3.12). Hence  $\dim_{(X)} \text{Hilb } \mathbb{P}^n = 29, 47, 62$  for  $\lambda = 0, 1, 2$  respectively and  $35 + 5\lambda(\lambda + 3)/2$  for  $\lambda \geq 3$ . Note that if  $d$  and  $g$  is the degree and genus of the curve  $X$ , we have  $h^0(\mathcal{N}_X) - h^1(\mathcal{N}_X) = 5d + 1 - g$  which is equal to 29, 47 and 60 for  $\lambda = 0, 1$  and 2 and more generally equal to  $29 + (41\lambda - 5\lambda^2)/2$  for  $\lambda \geq 0$ . Thus

$$h^1(\mathcal{N}_X) = 5\lambda^2 - 13\lambda + 6 - a \text{ for } \lambda \geq 0.$$

#### 4. IDEALS GENERATED BY SUBMAXIMAL MINORS OF SYMMETRIC MATRICES

The goal of this section is to write down lower bounds for the dimension of  $\text{Hilb}_{(X)}^{p(t)}(\mathbb{P}^n)$  being  $X \subset \mathbb{P}^n$  a codimension 3, arithmetically Cohen-Macaulay scheme defined by the submaximal minors of a  $t \times t$  homogeneous symmetric matrix. We will also analyze when the mentioned bounds are sharp. A classical scheme that can be constructed in this way

is the Veronese surface  $X \subset \mathbb{P}^5$ . Indeed, the Veronese surface  $X \subset \mathbb{P}^5$  can be defined by the  $2 \times 2$  minors of the symmetric matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}.$$

Let us first fix the notation we will use throughout this section. From now on,  $X \subset \mathbb{P}^n$  will be a codimension 3, arithmetically Cohen-Macaulay scheme defined by the submaximal minors of a  $t \times t$  homogeneous symmetric matrix  $\mathcal{A} = (f_{ji})_{i,j=1,\dots,t}$  where  $f_{ji} \in k[x_0, \dots, x_n]$  are homogeneous polynomials of degree  $a_i + a_j$  and let  $A = R/I(X)$  be the homogeneous coordinate ring of  $X$ . We denote by

$$\mathcal{U} = \begin{pmatrix} 2a_1 & a_1 + a_2 & \cdots & a_1 + a_t \\ a_1 + a_2 & 2a_2 & \cdots & a_2 + a_t \\ \vdots & \vdots & \ddots & \vdots \\ a_1 + a_t & a_2 + a_t & \cdots & 2a_t \end{pmatrix}$$

the degree matrix of  $\mathcal{A}$ . The determinant of  $\mathcal{A}$  is a homogeneous polynomial of degree  $\ell = 2(a_1 + a_2 + \cdots + a_t)$ . Note that  $a_i + a_j$  is a positive integer for all  $1 \leq i \leq j \leq t$  while  $a_i$  does not necessarily need to be an integer.

Let  $\mathcal{B}$  be the matrix obtained by deleting the last row, let  $I_B = I_{t-1}(\mathcal{B})$  be the ideal defined by the maximal minors of  $\mathcal{B}$  and let  $I_A = I_{t-1}(\mathcal{A})$  be the ideal generated by the submaximal minors of  $\mathcal{A}$ . Set  $A = R/I_A = R/I(X)$  and  $B = R/I_B$ .

**Remark 4.1.** Assume  $\text{char}(k) \neq 2$ . After a basis change that preserves the symmetry of  $\mathcal{A}$ , if necessary, we have that the assumption  $\text{codim}_R A = 3$  implies that  $\text{codim}_R B = 2$  and  $I_B$  is Cohen-Macaulay. In fact, we well know that  $\text{codim}_R B \leq 2$ . Following the approach of the proof of Theorem 2 in [5], and strongly using the fact that  $\mathcal{A}$  is symmetric and  $\text{char}(k) \neq 2$ , we get (see also [9], Theorem 1.22)

$$ht(I_A/I_B) = ht(I_{t-1}(\mathcal{A})/I_{t-1}(\mathcal{B})) \leq 1.$$

Therefore, we obtain  $ht(I_B) = ht(I_{t-1}(\mathcal{B})) \geq ht(I_A) - 1 = ht(I_{t-1}(\mathcal{A})) - 1 = 2$  and we are done.

**Remark 4.2.** With the above notations, let  $Y \subset \mathbb{P}^n$  be the codimension 2, ACM scheme defined by the maximal minors of  $\mathcal{B}$ . We *claim* that  $Y$  is a generically complete intersection. In fact, consider

$$0 \longrightarrow F \xrightarrow{t\mathcal{B}} G \longrightarrow I_B = I(Y) \longrightarrow 0$$

the resolution of  $I(Y)$  given by Hilbert-Burch theorem. Let  $P$  be a minimal associated prime of  $I(Y) = I_{t-1}(\mathcal{B})$ . We have to see that  $I(Y)_P$  is a complete intersection. We have  $ht(P) = ht(I_{t-1}(\mathcal{B})) = 2 < 3 = ht(I(X)) = ht(I_{t-1}(\mathcal{A})) \leq ht(I_{t-2}(\mathcal{B}))$ . So,  $P \not\subseteq I_{t-2}(\mathcal{B})$ . Denote by  $\mu(I(Y)_P)$  the number of minimal generators of  $I(Y)_P$ . By [4]; Proposition 16.3,  $\mu(I(Y)_P) \leq 2$  and we are done.

**Remark 4.3.** If the entries of  $\mathcal{A}$  and  $\mathcal{B}$  are sufficiently general polynomials of degree  $a_i + a_j$  with  $a_i + a_j$  a positive integer for all  $1 \leq i \leq j \leq t$ , then  $\text{codim}_R B = 2$  and



$\text{codim}_R A = 3$ . In fact, given rational numbers  $a_i \in \mathbb{Q}$ ,  $1 \leq i \leq t$ , with  $a_i + a_j$  a positive integer for all  $1 \leq i \leq j \leq t$ , we can consider the  $t \times t$  symmetric homogeneous matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & x_1^{a_1+a_{t-1}} & x_0^{a_1+a_t} \\ 0 & 0 & \cdots & 0 & 0 & x_1^{a_2+a_{t-2}} & x_0^{a_2+a_{t-1}} & x_2^{a_2+a_t} \\ 0 & 0 & \cdots & 0 & x_1^{a_3+a_{t-3}} & x_0^{a_3+a_{t-2}} & x_2^{a_3+a_{t-1}} & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_0^{a_1+a_t} & x_2^{a_2+a_t} & \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and let  $\mathcal{B}$  be the matrix obtained by deleting the last row of  $\mathcal{A}$ . We clearly have  $\text{codim}_R B = 2$  and  $\text{codim}_R A = 3$  which proves what we want.

**Proposition 4.4.** *With the above notation and assumptions  $\text{codim}_R A = 3$  and  $\text{codim}_R B = 2$ , we have an exact sequence*

$$0 \longrightarrow (S^2 K_B)(2n + 2 + p) \longrightarrow B \longrightarrow A \longrightarrow 0$$

where  $p = 2a_t - 3\ell$  and  $K_B = \text{Ext}_B^2(B, R(-n - 1))$  is the canonical module of  $B$ .

*Proof.* First of all we observe that the degrees of the minors of  $\mathcal{B}$  are  $\ell - a_i - a_t$ , i.e.  $I_B$  has the following minimal free  $R$ -resolution

$$(4.1) \quad 0 \longrightarrow \bigoplus_{i=1}^{t-1} R(-a_i + a_t - \ell) \xrightarrow{^t \mathcal{B}} \bigoplus_{i=1}^t R(a_i + a_t - \ell) \longrightarrow I_B \longrightarrow 0.$$

By [17]; Theorem 3.1,  $I_A$  has a minimal free  $R$ -resolution of the following type:

$$(4.2) \quad 0 \longrightarrow \bigoplus_{1 \leq i < j \leq t} R(-a_i - a_j - \ell) \longrightarrow \bigoplus_{1 \leq i, j \leq t} R(-\ell - a_i + a_j) / R(-\ell) \xrightarrow{^A} \bigoplus_{1 \leq i \leq j \leq t} R(a_i + a_j - \ell) \longrightarrow I_A \longrightarrow 0.$$

The natural injection  $I_B \hookrightarrow I_A$  induces a map from the complex (4.1) to the complex (4.2) in the most obvious way, i.e. factors in the complex (4.2) which are not present in the complex (4.1) are mapped to zero's, otherwise it is mapped by the identity.

Dualizing the exact sequence (4.1) we get

$$0 \longrightarrow R \longrightarrow \bigoplus_{i=1}^t R(-a_i - a_t + \ell) \xrightarrow{^B} \bigoplus_{i=1}^{t-1} R(a_i - a_t + \ell) \longrightarrow K_B(n + 1) \longrightarrow 0$$

leading to the exact sequence (see [7]; Theorem A2.10)

$$(4.3) \quad 0 \longrightarrow \bigoplus_{1 \leq i < j \leq t} R(-\ell - a_i - a_j) \longrightarrow \bigoplus_{\substack{1 \leq i \leq t-1 \\ 1 \leq j \leq t}} R(-\ell + a_i - a_j) \xrightarrow{^B} \bigoplus_{1 \leq i \leq j \leq t-1} R(-\ell + a_i + a_j) \longrightarrow S^2 K_B(2n + 2 + p) \longrightarrow 0.$$

One easily checks that the natural maps from the complex of  $I_A$  onto the complex of  $S^2 K_B(2n + 2 + p)$  are a morphism of complexes, and hence we get the exact sequence

$$0 \longrightarrow I_B \longrightarrow I_A \longrightarrow (S^2 K_B)(2n + 2 + p) \longrightarrow 0$$

and thus there is an exact sequence

$$0 \longrightarrow (S^2 K_B)(2n + 2 + p) \longrightarrow B \longrightarrow A \longrightarrow 0$$

which proves what we want.  $\square$

As an interesting consequence of the above result we have

**Proposition 4.5.** *Assume  $\text{char}(k) \neq 2$ . Let  $X \subset \mathbb{P}^n$  be a codimension 3, ACM scheme defined by the submaximal minors of a  $t \times t$  homogeneous symmetric matrix  $\mathcal{A}$ . Then  $X$  is glicci.*

*Proof.* Let  $Y$  be the codimension 2, ACM subscheme of  $\mathbb{P}^n$  defined by the maximal minors of the  $t \times (t-1)$  matrix  $\mathcal{B}$  obtained deleting the last row of  $\mathcal{A}$ , after a basis change that preserves the symmetry of  $\mathcal{A}$ , if necessary. By Remark 4.2,  $Y$  is a generically complete intersection. Hence  $Y$  satisfies the property  $G_0$  and we have the concept of generalized divisors on  $Y$ . By Proposition 4.4,  $X \in |-2K_Y - (2n+2+p)H|$  where  $K_Y$  is the canonical divisor on  $Y$  and  $H$  is a hyperplane divisor. By [19]; Corollary 5.5, for  $m \gg 0$ ,  $G \sim mH - K_Y$  is arithmetically Gorenstein and G-links  $X$  to an element of the linear system  $|K_Y + \beta H|$  for a suitable  $\beta \in \mathbb{Z}$ . So, it suffices to check that an element of the linear system  $|K_Y + \beta H|$  is glicci. To this end, we consider a codimension 3 standard determinantal scheme  $D \subset Y \subset \mathbb{P}^n$  defined by the maximal minors of the matrix  $[\mathcal{B}, \mathcal{L}]$  obtained adding to  $\mathcal{B}$  a sufficiently general column  $\mathcal{L}$  such that  $[\mathcal{B}, \mathcal{L}]$  is again homogeneous. By [19]; Theorem 3.6,  $D \in |K_Y + tH|$  for some  $t \in \mathbb{Z}$  and  $D$  is glicci. Moreover, by [19]; Corollary 5.13,  $K_Y + tH$  and  $D$  are G-bilinked, so any effective divisor of type  $K_Y + dH$  is ACM and glicci and we are done.  $\square$

**Remark 4.6.** The above proposition has also been proved by Gorla. Our proofs are independent except possibly for Remark 4.1, which for some time has been known to experts in the field. Note, however, that Remark 4.3 implies Remark 4.1 in the generic case. In [9]; Corollary 2.7, she has proved that any codimension  $\binom{m-t+2}{2}$ , ACM scheme  $X \subset \mathbb{P}^n$  defined by the  $t \times t$  minors of a  $m \times m$  symmetric homogeneous matrix is glicci.

**Lemma 4.7.** *With the above notation and assumptions  $\text{codim}_R A = 3$  and  $\text{codim}_R B = 2$ , suppose in addition  $a_1 \leq a_2 \leq \dots \leq a_t$ . Then, for  $i = 1$  and 0,*

${}_0\text{Ext}_B^i(I_B/I_B^2, S^2K_B)(2n+2+p) = {}_0\text{Ext}_R^i(I_B, (S^2K_B)(2n+2+p)) = 0$  provided  $a_t > 3a_{t-1}$  and

$${}_0\text{Hom}_R(I_B, S^2K_B)(2n+2+p) = 0 \text{ provided } a_t > 2a_{t-1} - a_1.$$

*Proof.* It follows from Lemma 3.2.  $\square$

Given rational numbers  $a_1, \dots, a_t$  such that  $a_i + a_j \in \mathbb{Z}_+$  for all  $i, j$  we denote by  $S(\underline{a}) = S(a_1, \dots, a_t)$  the irreducible family of codimension 3, ACM schemes  $X \subset \mathbb{P}^n$  defined by the submaximal minors of a  $t \times t$  symmetric homogeneous matrix  $\mathcal{A} = (f_{ji})_{i,j=1,\dots,t}$  where  $f_{ji} \in k[x_0, \dots, x_n]$  is a homogeneous polynomial of degree  $a_j + a_i$ . Our next goal is to determine a lower bound for the dimension of the irreducible component  $\mathcal{S}(\underline{a})$  of  $\text{Hilb}^p(\mathbb{P}^n)$  containing  $S(\underline{a})$ .

Set  $h_A^i := \dim {}_0H^i(R, A, A)$  where  $A = R/I(X)$ . Applying Theorem 3.6, we get

$$b - 1 - e + \dim(N_B)_0 - a = h_A^1 - h_A^2 = h^0(\mathcal{N}_X) - h_A^2 \leq \dim_{(X)} \text{Hilb}^p \mathbb{P}^n.$$

Using this and Theorem 3.7, we are ready to write down bounds for  $\dim_{(X)} \text{Hilb}^{p(t)} \mathbb{P}^n$  in terms of  $a_1, \dots, a_t$ , and  $e := {}_0\text{ext}_B^2(I_{A/B}, I_{A/B})$ , resp.  $a^0 := {}_0\text{hom}_B(I_B/I_B^2, I_{A/B})$ . Note that  $a^0$  (resp.  $a$ ) vanishes if  $a_t > 2a_{t-1} - a_1$  (resp.  $a_t > 3a_{t-1}$ ) by the previous lemma.

**Theorem 4.8.** *Let  $X \subset \mathbb{P}^n$  be a codimension 3, ACM scheme defined by the submaximal minors of a  $t \times t$  symmetric matrix  $\mathcal{A} = (f_{ji})_{i,j=1,\dots,t}$  where  $f_{ji} \in k[x_0, \dots, x_n]$  is a homogeneous polynomial of degree  $a_j + a_i$ . Let  $\mathcal{B}$  be the matrix obtained by deleting the last row, let  $Y = \text{Proj}(\mathcal{B}) \subset \mathbb{P}^n$  be the codimension 2 ACM scheme defined by the maximal minors of  $\mathcal{B}$ , and suppose  $U = Y - Z \hookrightarrow \mathbb{P}^n$  is a local complete intersection for some closed subset  $Z$  such that  $\text{depth}_{I(Z)} B \geq 3$ . Then,*

$$\dim_{(X)} \text{Hilb}^{p(t)}(\mathbb{P}^n) \geq b - 1 + \dim(N_B)_0 - a - e$$

where  $e := {}_0\text{ext}_B^2(S^2K_B, S^2K_B)$ ,

$$\begin{aligned} b - 1 + \dim(N_B)_0 = & \sum_{1 \leq i \leq j \leq k \leq t-1} \binom{a_t - a_i - a_j - a_k + n}{n} - \sum_{\substack{1 \leq i \leq t \\ 1 \leq j \leq k \leq t-1}} \binom{a_t + a_i - a_k - a_j + n}{n} + \\ & \sum_{\substack{1 \leq i \leq j \leq t \\ 1 \leq k \leq t-1}} \binom{a_t + a_j + a_i - a_k + n}{n} - \sum_{1 \leq i < j < k \leq t} \binom{a_t + a_j + a_i + a_k + n}{n} + \sum_{\substack{1 \leq i \leq t \\ 1 \leq j \leq t-1}} \binom{a_i + a_j + n}{n} - \\ & \sum_{1 \leq i, j \leq t} \binom{a_i - a_j + n}{n} - \sum_{1 \leq i, j \leq t-1} \binom{a_j - a_i + n}{n} \end{aligned}$$

and

$$a = - \sum_{\substack{1 \leq i, j, k \leq t-1 \\ i \leq j > k}} \binom{a_i + a_j + a_k - a_t + n}{n} - \sum_{\substack{1 \leq k, j \leq t \\ 1 \leq i \leq t-1}} \binom{a_i - a_j - a_k - a_t + n}{n} + \sum_{\substack{1 \leq k, i \leq t-1 \\ 1 \leq j \leq t}} \binom{a_i - a_j + a_k - a_t + n}{n}.$$

Moreover if  $\text{char}(k) \neq 2$ , then

$$\dim_{(X)} \text{Hilb} \mathbb{P}^n \geq b - 1 + \dim(N_B)_0 - a^0.$$

*Proof.* It follows from Theorem 3.6 and Theorem 3.7 taking into account that, by (4.1)  $n_{1i} = \ell - a_i - a_t$  for  $1 \leq i \leq t$  and  $n_{2j} = \ell + a_j - a_t$  for  $1 \leq j \leq t - 1$ .  $\square$

We are led to pose the following question.

**Question 4.9.** Under which extra hypothesis the bounds given in Theorem 4.8 are sharp?

We will now give an example.

**Example 4.10.** Set  $R = k[x_0, x_1, \dots, x_5]$ . Let  $X \subset \mathbb{P}^5 = \text{Proj}(R)$  be the Veronese surface defined by the  $2 \times 2$  minors of the symmetric matrix

$$\mathcal{A} = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}.$$

Let  $I_A$  be the ideal generated by the  $2 \times 2$  minors of  $\mathcal{A}$  and  $I_B$  the ideal generated by the  $2 \times 2$  minors of  $\mathcal{B} = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \end{pmatrix}$ . Set  $A = R/I_A$  and  $B = R/I_B$ . It is well known that  $\dim_{(X)} \text{Hilb}(\mathbb{P}^5) = 27$ . If we apply Theorem 4.8 we obtain  $\dim_{(X)} \text{Hilb}(\mathbb{P}^5) \geq 29 - e$  where  $e = {}_0\text{ext}_B^2(I_{A/B}, I_{A/B})$ . Using Macaulay program [3] we have computed the dimension of  ${}_0\text{Ext}_B^2(I_{A/B}, I_{A/B})$  and we have got  $e = 2$ . Thus,  $\dim_{(X)} \text{Hilb}(\mathbb{P}^5) \geq 27$  and hence the first bound given in Theorem 4.8 is sharp.

**Example 4.11.** Set  $R = k[x_0, x_1, \dots, x_5]$ . Let  $X \subset \mathbb{P}^5 = \text{Proj}(R)$  be the surface defined by the  $3 \times 3$  minors of the symmetric matrix

$$\mathcal{A} = \begin{pmatrix} x_0 & x_1 & x_2 & L_1 \\ x_1 & x_3 & x_4 & L_2 \\ x_2 & x_4 & x_5 & L_3 \\ L_1 & L_2 & L_3 & L_4 \end{pmatrix}$$

where  $L_i$  are general linear forms. Let  $I_A$  be the ideal generated by the  $3 \times 3$  minors of  $\mathcal{A}$

and  $I_B$  the ideal generated by the  $3 \times 3$  minors of  $\mathcal{B} = \begin{pmatrix} x_0 & x_1 & x_2 & L_1 \\ x_1 & x_3 & x_4 & L_2 \\ x_2 & x_4 & x_5 & L_3 \end{pmatrix}$ . Set  $A = R/I_A$

and  $B = R/I_B$ . If we apply Theorem 4.8 we obtain  $\dim_{(X)} \text{Hilb}(\mathbb{P}^5) \geq 59 - e$  where  $e = {}_0\text{ext}_B^2(I_{A/B}, I_{A/B})$ . Using the Macaulay program [3] we have computed the dimension of  ${}_0\text{Ext}_B^2(I_{A/B}, I_{A/B})$  and we have got  $e = 14$ . Thus,  $\dim_{(X)} \text{Hilb}(\mathbb{P}^5) \geq 45$ . Using again the Macaulay program we have computed  $h^0(\mathcal{N}_X)$  and we have got  $h^0(\mathcal{N}_X) = 45$ . Hence the first bound given in Theorem 4.8 is sharp.

**Example 4.12.** Let  $R = k[x_0, x_1, \dots, x_5]$  and let  $I_A$  be the ideal generated by the  $2 \times 2$  minors of the symmetric matrix

$$\mathcal{A} = \begin{pmatrix} x_0 & x_1 & x_2^2 \\ x_1 & x_3 & x_4^2 \\ x_2^2 & x_4^2 & x_5^3 \end{pmatrix}.$$

Let  $I_B$  the ideal generated by the  $2 \times 2$  minors of  $\mathcal{B} = \begin{pmatrix} x_0 & x_1 & x_2^2 \\ x_1 & x_3 & x_4^2 \end{pmatrix}$ . Set  $X = \text{Proj}(R/I_A)$

and  $I_{A/B} = I_A/I_B$ . If we apply Theorem 4.8 with  $a_1 = a_2 = 1/2$  and  $a_3 = 3/2$  we obtain  $\dim_{(X)} \text{Hilb}(\mathbb{P}^5) \geq 92 - e$  where  $e = {}_0\text{ext}_B^2(I_{A/B}, I_{A/B})$ . Indeed  $b - 1 + h^0(\tilde{N}_B) = 90$ ,  $a = -2$  and we find  $\dim_{(X)} \text{Hilb}(\mathbb{P}^5) \geq b - 1 + h^0(\tilde{N}_B) - a - e = 92 - e = 84$  since we got  $e = 8$  by Macaulay 2. In this case, however,  ${}_0\text{hom}_B(I_B/I_B^2, I_{A/B}) = 0$  by Lemma 4.5 and the second bound of Theorem 4.6 applies. We get  $\dim_{(X)} \text{Hilb}(\mathbb{P}^5) \geq b - 1 + h^0(\tilde{N}_B) = 90$  which is sharp or 1 from being sharp because we have computed  $h^0(\mathcal{N}_X)$  by Macaulay 2 and got 91.

**Remark 4.13.** We want to point out that  $b - 1 + h^0(\mathcal{N}_Y) - a - e$  is a correct formula for  $h^0(\mathcal{N}_X) - h^1(\mathcal{N}_X)$  if  $X$  is a l.c.i. and  $\dim(X) \geq 2$ . Moreover, examples computed using Macaulay 2 show that the dimension of the corresponding component might be significantly different from our lower bounds if  $h^1(\mathcal{N}_X)$  is large. If  $\dim(X) = 1$  then  $b - 1 + h^0(\mathcal{N}_Y) - a - e$  is a correct formula for  $h^0(\mathcal{N}_X) - {}_0h^2(R, A, A)$ .

Let  $CM_c^H(R)$  (resp.  $Gor_c^H(R)$ ) be the open subscheme of  $GradAlg^H(R)$  parameterizing Cohen-Macaulay (resp. Gorenstein) quotients of codimension  $c$  in  $R$ . If  $c \leq \dim R - 2$ , one may alternatively define  $CM_c^H(R)$  (resp.  $Gor_c^H(R)$ ) as the open subscheme of  $\text{Hilb}^p(\mathbb{P}^n)$  parameterizing arithmetically Cohen-Macaulay (resp. arithmetically Gorenstein) subschemes of codimension  $c$  in  $\mathbb{P}^n$ . In [21], the first author established a strong connection (an incidence correspondence) between  $Gor_{c+1}^{H'}(R)$  and  $CM_c^H(R)$  for some  $H'$ , in which

Gorenstein quotients  $(G) \in \text{Gor}_{c+1}^{H'}(R)$  were defined by some exact sequence

$$(4.4) \quad 0 \longrightarrow K_A(-s) \longrightarrow A \longrightarrow G \longrightarrow 0$$

with  $(A) \in \text{CM}_c^H(R)$  and  $s$  a fixed integer. In particular, he proved that if  $s$  is large enough (see [21] for explicit bounds) then this connection essentially determines a well defined injective map from the set of irreducible components of  $\text{CM}_c^H(R)$  to the set of irreducible components of  $\text{Gor}_{c+1}^{H'}(R)$ . Note that if  $\text{Proj}(G)$  is obtained as the intersection of two geometrically G-linked schemes, then  $G$  is given by (4.4) for some  $s$ . In the next example, we will prove the existence of at least two irreducible components  $\mathcal{C}_1$  and  $\mathcal{C}_2$  of  $\text{CM}_3^H(R)$  mapping to the same irreducible component of  $\text{Gor}_4^{H'}(R)$  in the correspondence above, showing that we can not skip the assumption that  $s$  is sufficiently large. Moreover, we will see that the generic graded Cohen-Macaulay quotients  $A_1 \in \mathcal{C}_1$  and  $A_2 \in \mathcal{C}_2$  have both a linear resolution with the same graded Betti numbers.

**Example 4.14.** We keep the notation introduced in Example 4.10.  $B = R/I_B$  is a codimension 2 Cohen-Macaulay graded quotient and the minimal  $R$ -free resolution of  $B$  is

$$0 \longrightarrow R(-3)^2 \longrightarrow R(-2)^3 \longrightarrow R \longrightarrow B \longrightarrow 0.$$

So  $Y = \text{Proj}(B) \subset \mathbb{P}^5$  is a codimension 2, arithmetically Cohen-Macaulay scheme.  $A = R/I_A$  is the codimension 3 Veronese surface and the minimal free  $R$ -resolution of  $A$  is

$$0 \longrightarrow R(-4)^3 \longrightarrow R(-3)^8 \longrightarrow R(-2)^6 \longrightarrow R \longrightarrow A \longrightarrow 0.$$

So,  $X = \text{Proj}(A) \subset \mathbb{P}^5$  is a codimension 3, arithmetically Cohen-Macaulay scheme.

Any effective divisor of the linear system  $| -K_Y |$  is an arithmetically Gorenstein, codimension 3 subscheme  $X_0 \subset \mathbb{P}^5$  and its homogeneous coordinate ring  $A_0$  has a minimal  $R$ -free resolution ( $X_0 = \text{Proj}(A_0)$ ,  $A_0 = R/I_{A_0}$ ):

$$0 \longrightarrow R(-6) \longrightarrow R(-4)^3 \oplus R(-3)^2 \longrightarrow R(-3)^2 \oplus R(-2)^3 \longrightarrow R \longrightarrow A_0 \longrightarrow 0.$$

We G-link  $A$  to  $A'$  using some codimension 3, graded Gorenstein quotient  $A_0$ .  $A'$  is a codimension 3, graded Cohen-Macaulay quotient with a  $R$ -free resolution:

$$0 \longrightarrow R(-4)^3 \longrightarrow R(-3)^{10} \longrightarrow R(-2)^6 \oplus R(-3)^2 \longrightarrow R \longrightarrow A' \longrightarrow 0.$$

Let us check that  $X' = \text{Proj}(A')$  and  $X = \text{Proj}(A)$  belong to different components of  $\text{Hilb}^{p(t)}(\mathbb{P}^5)$ .

It is well known that  $h^0 \mathcal{N}_X = 27$  and  $H^i \mathcal{N}_X = 0$  for  $i > 0$ . Therefore,  $X$  belongs to a generically smooth irreducible component  $\mathcal{C}_1$  of  $\text{Hilb}^{p(t)} \mathbb{P}^5$  of dimension 27. On the other hand,  $X' \in |K_Y + 4H|$  and a general element  $X_2 = \text{Proj}(A_2)$  of  $|K_Y + 4H|$  is a codimension 3 determinantal scheme defined by the maximal minors of a  $2 \times 4$  matrix obtained by adding to  $\mathcal{B}$  a column of linear forms. Hence  $A_2$  has a linear resolution. Moreover, by [19]; §10,  $X_2$  belongs to a generically smooth irreducible component of  $\text{Hilb}^{p(t)} \mathbb{P}^5$  of dimension 29.

So, we conclude that the Veronese surface  $X \subset \mathbb{P}^5$  and a determinantal surface defined by the maximal minors of a  $2 \times 4$  matrix with linear entries have a linear resolution with the same graded Betti numbers but they belong to different components of  $\text{Hilb}^{p(t)} \mathbb{P}^5$ .

Finally, since  $X \in |-2K_Y - 4H|$  and  $X' \in |K_Y + 4H|$  are G-linked by  $X_0 \in |-K_Y|$  we have that  $Z = X \cap X' = \text{Proj}(R/I_A + I_{A'}) \subset \mathbb{P}^5$  is an arithmetically Gorenstein scheme of codimension 4 and  $I_C := I_A + I_{A'}$  has a minimal free  $R$ -resolution

$$0 \longrightarrow R(-6) \longrightarrow R(-4)^9 \longrightarrow R(-3)^{16} \longrightarrow R(-2)^9 \longrightarrow I_C \longrightarrow 0.$$

Thus, we conclude that there are two generically smooth irreducible components of  $CM_3^{HA}(R)$  mapping to the same irreducible component of  $Gor_4^{HC}(R)$  in the correspondence described in (4.4).

## 5. FINAL REMARKS AND OPEN PROBLEMS

The results proved in this paper give rise to a number of quite interesting questions and possible generalizations that we gather together in this last section. In fact, Examples 4.10 and 4.11 suggest - and prove for  $t \leq 4$  and  $n = 5$  - the following question:

**Question 5.1.** Let  $\mathcal{A} = (f_{ji})_{i,j=1,\dots,t}$  be a  $t \times t$  symmetric matrix where  $f_{ji} \in k[x_0, \dots, x_5]$  are general linear forms. Let  $X \subset \mathbb{P}^5$  be a codimension 3, ACM scheme defined by the submaximal minors of  $\mathcal{A}$ . Is it true that

$$\begin{aligned} \dim_{(X)} \text{Hilb}^{p(t)} \mathbb{P}^5 = 6t[(t-1)^2 + (t-1)] - t^2 - (t-1)(t^2-1) - t \binom{t}{2} \\ - \binom{7}{2} \binom{t}{3} - 6 \binom{t+1}{3} - e \quad ? \end{aligned}$$

More generally, we would like to know if under certain numerical conditions on  $a_i$  the bounds given in Theorem 4.8 are sharp. So, we are led to pose the following problem:

**Problem 5.2.** Find numerical conditions on the rational numbers  $a_i$  which allow us to assure that the bounds given in Theorem 4.8 are sharp.

We consider a  $t \times t$  homogeneous symmetric matrix  $\mathcal{A} = (f_{ji})_{i,j=1,\dots,t}$  where  $f_{ji} \in k[x_0, \dots, x_n]$  are homogeneous polynomials of degree  $a_i + a_j \in \mathbb{Z}_+$  and let  $X \subset \mathbb{P}^n$  be the codimension 3, arithmetically Cohen-Macaulay scheme defined by the submaximal minors of  $\mathcal{A}$ . Let  $\mathcal{B}$  be the matrix obtained by deleting the last row of  $\mathcal{A}$  and let  $Y \subset \mathbb{P}^n$  be the codimension 2, arithmetically Cohen-Macaulay scheme defined by the maximal minors of  $\mathcal{B}$ .

If  $\dim(X) \geq 2$  then Theorem 4.8 only gives a lower bound for  $\dim_{(X)} \text{Hilb}^{p(t)} \mathbb{P}^n \geq h^0(\mathcal{N}_X) - h^1(\mathcal{N}_X)$  and, using Macaulay 2, we have checked that  $h^1(\mathcal{N}_X)$  may be large in many examples. So, we are led to pose the following problems

**Problem 5.3.** Find an explicit formula for  $\dim_{(X)} \text{Hilb}^{p(t)} \mathbb{P}^n$  and for  $h^0(\mathcal{N}_X)$ .

Keeping the notation introduced in §4, we denote by  $S(a_1, \dots, a_t)$  the irreducible family of codimension 3, ACM schemes  $X \subset \mathbb{P}^n$  defined by the submaximal minors of a  $t \times t$  homogeneous symmetric matrix  $\mathcal{A} = (f_{ji})_{i,j=1,\dots,t}$  where  $f_{ji} \in k[x_0, \dots, x_n]$  are homogeneous polynomials of degree  $a_i + a_j \in \mathbb{Z}_+$  and let  $S(\mathcal{B})$  be the part of  $\text{Hilb} \mathbb{P}^n$  consisting of those  $Y'$  obtained by deforming  $\mathcal{B}$  as a matrix with symmetric "left"  $(t-1) \times (t-1)$  matrix.

**Problem 5.4.** Find an explicit formula for  $\dim S(a_1, \dots, a_t)$  and  $\dim S(\mathcal{B})$ .

**Problem 5.5.** Is  $\dim_{(X)} \text{Hilb}^{p^{(t)}} \mathbb{P}^n = \dim_{(X)} S(a_1, \dots, a_t)$ ?

The computations we have made, using Macaulay 2 [10], suggest the following conjecture

**Conjecture 5.6.** *If  $t = 3$  and  $2a_i = p$  for all  $i$ , then*

$$\dim_{(X)} \text{Hilb}^{p^{(t)}} \mathbb{P}^n = \dim_{(X)} S(a_1, \dots, a_t) = h^0(\mathcal{N}_X) = 6 \binom{p+n}{n} - 9.$$

We have checked that the right hand equality of the conjecture is true for  $n = 5$  and  $1 \leq p \leq 20$ , for some choices of  $X$ .

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