Unobstructedness and dimension of families of Gorenstein algebras

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Abstract

The goal of this paper is to develop tools to study maximal families of Gorenstein quotients A of a polynomial ring R. We prove a very general Theorem on deformations of the homogeneous coordinate ring of a scheme $\mathrm{Proj}(A)$ which is defined as the degeneracy locus of a regular section of the dual of some sheaf $\widetilde{M}$ of rank r supported on say an arithmetically Cohen-Macaulay subscheme $\mathrm{Proj}(B)$ of $\mathrm{Proj}(R)$. Under certain conditions (notably; M maximally Cohen-Macaulay and $\wedge^r \widetilde{M} \simeq \widetilde{K}_B(t)$ a twist of the canonical sheaf), then A is Gorenstein, and under additional assumptions, we show the unobstructedness of A and we give an explicit formula the dimension of any maximal family of Gorenstein quotients of R with fixed Hilbert function obtained by a regular section as above. The theorem also applies to Artinian quotients A.

The case where M itself is a twist of the canonical module ($r = 1$) was studied in a previous paper, while this paper concentrates on other low rank cases, notably $r = 2$ and 3. In these cases regular sections of the first Koszul homology module and of normal sheaves to licci schemes (of say codimension 2) lead to Gorenstein quotients (of e.g. codimension 4) whose parameter spaces we examine. Our main applications are for Gorenstein quotients of codimension 4 of R since our assumptions are almost always satisfied in this case. Special attention are paid to arithmetically Gorenstein curves in $\mathbb{P}^5$.

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Contents

1 Introduction

In this paper we study deformations of arithmetically Gorenstein (AG) subschemes $X = \mathrm{Proj}(A)$ of a projective space $\mathbb{P}^N = \mathrm{Proj}(R)$. In particular we consider deformations of schemes X defined as the degeneracy locus of a regular section of some maximal Cohen-Macaulay sheaf $\widetilde{M}$ supported on an arithmetically Cohen-Macaulay (ACM) subscheme $Y = \mathrm{Proj}(B)$ of $\mathbb{P}^N$. Our results allow us to study well three natural parameter spaces classifying AG subschemes of $\mathbb{P}^N$ (resp. graded Gorenstein quotients of R) with Hilbert polynomial p (resp. Hilbert function H). The three schemes we have in mind are the open subscheme of Grothendieck's Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^N)$ consisting of AG subschemes of positive dimension, the open part of the postulation Hilbert scheme $\mathrm{Hilb}^H(\mathbb{P}^c)$ consisting of AG zeroschemes with Hilbert function H and Iarrobino-Kanev's determinantal scheme $\mathrm{PGor}(H)$ classifying graded Artinian Gorenstein quotients of R via the Macaulay correspondence.

A common denominator of these parameter spaces is the k-scheme $\mathrm{GradAlg}^H(R)$ of [22] which parametrizes graded quotients $B = R/I_B$ such that $\operatorname{depth} B \geq \min(1, \dim B)$ and $H_B = H$, H_B the Hilbert function of B. Its open subscheme consisting of Gorenstein quotients B of codimension c in R, $\mathrm{Gor}_c^H(R)$, is naturally *isomorphic* to the three schemes in the mentioned order provided the degree of p is greater or equal 1, equal 0 and equal -1 (i.e. $p(t) = 0$) respectively. The isomorphism is even scheme-theoretic except possibly for $\mathrm{PGor}(H)$ where the isomorphism is at least infinitesimal and topological, making questions of dimension and smoothness of $\mathrm{PGor}(H)$ equivalent to the corresponding questions for $\mathrm{GradAlg}(H) := \mathrm{GradAlg}^H(R)$, cf. [23] for details and [16] for a generalization of $\mathrm{GradAlg}(H)$.

If the codimension $c \leq 3$, the smoothness, the irreducibility and the dimension of $\mathrm{Gor}_c^H(R)$ are known ([22], [10], [6], [24]). Hence we focus on Gorenstein quotients of codimension 4 of R which, nowadays, get quite a lot of attention ([9], [19], [23], [18], [3]). Moreover since some of our results require that the degree of p is greater or equal to 1, we mostly consider AG curves in $\mathbb{P}^5$, having also the case $\mathrm{PGor}(H)$ in mind since we have tried very much, and partially succeeded, to generalize our results to the Artinian case.

Let $B \to A$ be a morphism of graded quotients of R. In the background section we recall a basic result of how we can deduce the smoothness and the dimension of $\mathrm{GradAlg}^{H_A}(R)$ from the corresponding properties of $\mathrm{GradAlg}^{H_B}(R)$ (Theorem 15 and Corollary 18). The assumptions of Theorem 15 are satisfied for a complete intersection (c.i.) B and for some other classes of quotients with nice properties (licci is often enough) except for the vanishing "${}_0\mathrm{H}^2(B, A, A) = 0$" of the second algebra cohomology group. Theorem 15 and Corollary 18 are enough to treat the rank $r = 1$ case of M satisfactorily ([23]), while, for higher rank cases, they are insufficient. A main result, Theorem 22, of this paper replaces the assumption "${}_0\mathrm{H}^2(B, A, A) = 0$" with assumptions which are more natural to the application we have in mind; deforming schemes X defined as the degeneracy locus of a regular section. Note, however, that Theorem 15 and Corollary 18 apply also to quotients which not necessarily are given by such a regular section, and we take the opportunity to include a result which generalizes [18], Thm. 4.17 (Corollary 20).

To state the assumptions of Corollary 18 and Theorem 22 and its dimension formulas in a more computable form we consider low rank cases on a licci scheme $Y = \mathrm{Proj}(B)$. Let K_B be the canonical module of B. If M is a graded maximal Cohen-Macaulay B-module of rank $r = 2$ such that $\widetilde{M}|_U$ is locally free and $\wedge^2 \widetilde{M}|_U \simeq \widetilde{K_B}(t)|_U$ in a large enough open set $U = Y - Z$ of Y, then a regular section σ of $\widetilde{M}^*(s)|_U$, $M^* = \mathrm{Hom}_B(M, B)$, defines a graded Gorenstein quotient A given by the exact sequence

$$0 \to K_B(t-2s) \to M(-s) \xrightarrow{\sigma} B \to A \to 0\,, \tag{1}$$

similar to what happens in the usual Hartshorne-Serre correspondence ([14], Thm. 4.1). Moreover $M \simeq \mathrm{Hom}_B(M, K_B(t))$ (Theorem 8). Let $S_2(M)$ be the second symmetric power of M. Using small letters for the k-dimension of the v-graded piece ${}_v\mathrm{Ext}^i_B(-,-)$ of $\mathrm{Ext}^i_B(-,-)$, we define $\gamma(S_2M)_v$ and $\delta(Q)_v$ by

$$\delta(Q)_v = {}_v\hom_B(I_B/I_B^2, Q) - {}_v\mathrm{ext}^1_B(I_B/I_B^2, Q)\ , \quad \text{and}$$

$$\gamma(S_2M)_v = {}_v\hom_B(S_2(M), K_B(t)) - {}_v\mathrm{ext}^1_B(S_2(M), K_B(t))\ .$$

Note that if $\operatorname{depth}_{I(Z)} B \geq 3$ (resp. $\operatorname{depth}_{I(Z)} B \geq 4$) and $char(k) \neq 2$, one may show that $\gamma(S_2M)_0 = {}_0\hom_B(M, M) - {}_0\mathrm{ext}^1_B(M, M) - 1$ (resp. ${}_0\mathrm{Ext}^i_B(S_2(M), K_B(t)) \simeq {}_0\mathrm{Ext}^i_B(M, M)$ for $i = 1, 2$) since $r = 2$, cf. (12). A main result of Section 2 is the following Theorem (Theorem 25) which applies also to an Artinian A.

Theorem 1. *Let $B = R/I_B$ be a graded licci quotient of R, let M be a graded maximal Cohen-Macaulay B-module, and suppose $\widetilde{M}$ is locally free of rank 2 in $U := \mathrm{Proj}(B) - Z$, that $\dim B -$*

$\dim B/I(Z) \geq 2$ *and* $\wedge^2 \widetilde{M}|_U \simeq \widetilde{K_B}(t)|_U$. *Let* A *be defined by a regular section* σ *of* $\widetilde{M}^*(s)$ *on* U, *i.e. given by* (1), *and suppose* ${}_s\mathrm{Ext}^1_B(M,B) = 0$.

A) *If* ${}_t\mathrm{Ext}^2_B(S_2(M), K_B) = 0$ *and* ${}_{-s}\mathrm{Ext}^2_B(I_B/I_B^2, M) = 0$,

then A *is unobstructed as a graded* R*-algebra (indeed* ${}_0\mathrm{H}^2(R,A,A) = 0$*),* A *is Gorenstein, and*

$$\dim_{(A)} \mathrm{GradAlg}(H_A) = \dim(N_B)_0 + \dim(M^*)_s - 1 - \gamma(S_2M)_0 + \dim(K_B)_{t-2s} + \delta(K_B)_{t-2s} - \delta(M)_{-s}.$$

Moreover if $char(k) = 0$ *and* $(B \to A)$ *is general with respect to* ${}_0\mathrm{hom}_R(I_B, I_{A/B})$*, then the codimension of the stratum of quotients given by* (1) *around* (A) *is*

$${}_{-s}\mathrm{ext}^1_B(I_B/I_B^2, M) - \dim(\mathrm{im}\,\beta)$$

where β *is the homomorphism* ${}_{-2s}\mathrm{Ext}^1_B(I_B/I_B^2, K_B(t)) \to {}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, M)$ *induced by* (1).

B) *If* (M,B) *is unobstructed along any graded deformation of* B *and* ${}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, M) = 0$, *then* A *is Gorenstein and* H_B*-generic. Moreover* A *is unobstructed as a graded* R*-algebra and the dimension formula for* $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R)$ *of part* A) *holds.*

Here $N_B = \mathrm{Hom}_B(I_B/I_B^2, B)$, $I_{A/B} := \ker(B \to A)$ and "(M,B) unobstructed along any deformation of B" means that for every deformation (M_S, B_S) of (M,B), S local and Artinian, there is a deformation of M_S to any deformation of B_S (cf. Definition 11). Moreover "$(B \to A)$ is general with respect to $\gamma := {}_0\mathrm{hom}_R(I_B, I_{A/B})$" if and only if ${}_0\mathrm{hom}_R(I_{B'}, I_{A'/B'}) \geq \gamma$ for every $(B' \to A')$ in an *open neighbourhood* of $(B \to A)$ in $\mathrm{GradAlg}(H_B, H_A)$ (see Remark 16(*ii*)). Furthermore "A is H_B-generic" essentially means that the codimension of the stratum mentioned in Theorem 1A) is zero, see the text before Theorem 15 and Theorem 25 for precise definitions.

Remark 2. *If* $\max n_{2,j}$ *is the largest degree of a minimal relation of* I_B *and* a *is an integer such that* $M_v = 0$ *for* $v \leq a$*, then*

$${}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, M) = 0 \quad \text{provided} \quad s \geq \max n_{2,j} - a.$$

In this case $\dim(K_B)_{t-2s} + \delta(K_B)_{t-2s} - \delta(M)_{-s} = 0$. *This significantly simplifies the dimension formula of Theorem 1B) provided we use the theorem for such* s.

On a Cohen-Macaulay (CM) quotient $B = R/I_B$ of codimension 2 in R with minimal resolution

$$0 \to \oplus_{j=1}^{\mu-1} R(-n_{2,j}) \to \oplus_{i=1}^{\mu} R(-n_{1,i}) \to R \to B \to 0\ ,$$

Theorem 1 applies to $M = N_B$, the normal module, and to $M = H_1$, the 1. Koszul homology of I_B. In Section 3 we study these applications in detail. To see that the assumptions of Theorem 1 are satisfied, let $\mathrm{Proj}(B) - Z \hookrightarrow \mathbb{P}^N$ be a local complete intersection (l.c.i.). Then we prove

Proposition 3. *Let* $B = R/I_B$ *be a codimension two CM quotient of* R *and suppose* $\dim B - \dim B/I(Z) \geq 2$. *Then*

(*i*) $\mathrm{Ext}^2_B(I_B/I_B^2, H_1) = \mathrm{Ext}^1_B(H_1, B) = 0$ *and* $S_2(H_1)$ *is a maximal CM* B*-module.*

(*ii*) $\mathrm{Ext}^i_B(I_B/I_B^2, I_B/I_B^2) = 0$ for $1 \leq i \leq \dim B - \dim B/I(Z)$.

Using Proposition 3(*i*) we get the following result, with a sufficiently weak assumption on $\mathrm{depth}_{I(Z)} B = \dim B - \dim B/I(Z)$ so that the result applies also to an Artinian A.

Corollary 4. *Let $B = R/I_B$ be a graded codimension two CM quotient of R, let $U = \operatorname{Proj}(B) - Z \hookrightarrow \mathbb{P}^N$ be an l.c.i. and suppose the number of minimal generators of I_B is $\mu = 4$ and $\operatorname{depth}_{I(Z)} B \geq 2$. If A is defined by a regular section of $\widetilde{H_1}^*(s)$ on U, then A is unobstructed as a graded R-algebra (indeed $\mathrm{H}^2(R, A, A) = 0$), A is Gorenstein of codimension 4 in R, and*

$$\dim_{(A)} \operatorname{GradAlg}(H_A) = \dim(N_B)_0 + \dim(H_1^*)_s - 1 - \gamma(S_2 H_1)_0 + \dim(K_B)_{t-2s} + \delta(K_B)_{t-2s} - \delta(H_1)_{-s}$$

where $t = \dim R - \sum n_{1,i}$. Moreover if $s > \max n_{2,j} - \min n_{2,j}$, then A is H_B-generic, and

$$\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = \dim(N_B)_0 + \dim(H_1^*)_s - 1 - {}_0\hom_B(S_2(H_1), K_B(t)).$$

Furthermore in the situation of Corollary 4 we show how to compute every term of the dimension formulas and we give several examples.

Applying Theorem 1 to the normal module N_B we need the vanishing of the Ext-groups of Proposition 3(ii) to prove that the assumptions of Theorem 1 hold. To do so we have to increase $\operatorname{depth}_{I(Z)} B$. Hence the following result does not automatically apply to Gorenstein quotients A of dimension less or equal to 1 unless we in a given example are able to verify the assumptions of Theorem 1. In Section 3 we give examples of the result.

Corollary 5. *Let $B = R/I_B$ be a graded codimension two CM quotient of R, let $U = \operatorname{Proj}(B) - Z \hookrightarrow \mathbb{P}^N$ be an l.c.i. and suppose $\operatorname{depth}_{I(Z)} B \geq 4$. If A is given by a regular section of $\widetilde{N_B}^*(s)$ on U, and if $s > 2\max n_{2,j} - \min n_{1,i}$ and char$(k) \neq 2$, then A is H_B-generic and unobstructed as a graded R-algebra. Moreover A is Gorenstein of codimension 4 in R, and letting $X = \operatorname{Proj}(A)$ and $\eta(v) := \dim(I_B/I_B^2)_v$, we have*

$$\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = \dim_{(X)} \operatorname{Hilb}^p(\mathbb{P}^N) = \eta(s) + \sum_{j=1}^{\mu-1} \eta(n_{2,j}) - \sum_{i=1}^{\mu} \eta(n_{1,i}) \ .$$

Finally we prove a Theorem, similar to Theorem 1, in the rank $r = 3$ case which also applies to an Artinian A (Theorem 30). It admits the following Corollary,

Corollary 6. *Let $B = R/I$ and $U = \operatorname{Proj}(B) - Z$ be as in Corollary 4 and suppose that $\mu = 5$, char$(k) \neq 2$ and $\operatorname{depth}_{I(Z)} B \geq 3$. If A is defined by a regular section of $\widetilde{H_1}^*(s)$ on U, then A is unobstructed as a graded R-algebra (indeed $\mathrm{H}^2(R, A, A) = 0$), A is Gorenstein of codimension 5 in R, and $\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) =$*

$$\dim(N_B)_0 + \dim(H_1^*)_s + {}_{-s}\hom_B(S_2(H_1), K_B(t)) - {}_0\hom_B(H_1, H_1) - \dim(K_B)_{t-3s} - \delta \ ,$$

where $\delta := \delta(H_1)_{-s} + \delta(K_B)_{t-3s} - \delta(H_2)_{-2s}$ and $t = \dim R - \sum n_{1,i}$. If in addition $s > \max n_{2,j} - \min n_{2,j}$, then A is H_B-generic, $\delta = 0$ and $\dim(K_B)_{t-3s} = 0$.

We thank Chris Peterson and Johannes Kleppe for useful discussions. Indeed we are in this work inspired by the joint works [27] and [28] where we in various ways construct Gorenstein quotients of R. This paper uses deformation theory to vary every object and morphism of these constructions, to see how large the corresponding stratum of its parameter space will be. As we see by the results above, the stratum is often a component. In some cases, however, it is a proper stratum of a codimension which we make explicit. Moreover we thank Rosa M. Miró-Roig for interesting discussions on PGor(H). In particular Corollary 20 was included in the paper after a discussion with her.

1.1 Background

We largely keep the notations of [23], and we recommend the section of Preliminaries of that paper for an overview of cohomology groups and deformation theory. Let B be an n-dimensional graded quotient of a polynomial ring R in $n+c$ variables (of degree 1) over an algebraically closed field k, and let M and N be finitely generated graded B-modules. Let $\operatorname{depth}_J M$ denote the length of a maximal M-sequence in a homogeneous ideal J and let $\operatorname{depth} M = \operatorname{depth}_{\mathfrak{m}} M$ where $\mathfrak{m}$ is the irrelevant maximal ideal. Let $\mathrm{H}^i_J(-)$ be the right derived functor of the functor, $\Gamma_J(-)$, of sections with support in $\operatorname{Spec}(B/J)$. Then

$$\operatorname{depth}_J M \geq r \ \text{ if and only if } \ \text{(iff)} \ \ \mathrm{H}^i_J(M) = 0 \text{ for } i < r. \tag{2}$$

(cf. [13]). Let $Y = \operatorname{Proj}(B)$ and let Z be closed in Y and $U = Y - Z$. Put $\mathrm{H}^0_*(U, \widetilde{M}) := \oplus_v \mathrm{H}^0(U, \widetilde{M}(v))$. Then we have an exact sequence $0 \to \mathrm{H}^0_{I(Z)}(M) \to M \to \mathrm{H}^0_*(U, \widetilde{M}) \to \mathrm{H}^1_{I(Z)}(M) \to 0$ and isomorphisms $\mathrm{H}^i_{I(Z)}(M) \simeq \mathrm{H}^{i-1}_*(U, \widetilde{M})$ for $i \geq 2$. If $\operatorname{depth}_{I(Z)} N \geq i+1$, then the graded group $\operatorname{Ext}^i_B(M, N)$ injects into the corresponding global $\operatorname{Ext}^i_{\mathcal{O}_U}$-group of sheaves. Indeed we have by [12], exp. VI, an exact sequence

$${}_0\mathrm{Ext}^i_B(M,N) \hookrightarrow \mathrm{Ext}^i_{\mathcal{O}_U}(\widetilde{M}|_U, \widetilde{N}|_U) \to {}_0\mathrm{Hom}_B(M, \mathrm{H}^{i+1}_{I(Z)}(N)) \to {}_0\mathrm{Ext}^{i+1}_B(M,N) \to \mathrm{Ext}^{i+1}_{\mathcal{O}_U}(\widetilde{M}|_U, \widetilde{N}|_U) \tag{3}$$

where the form of the middle term comes from a spectral sequence discussed in [12].

A Cohen-Macaulay (resp. maximal Cohen-Macaulay) B-module satisfies $\operatorname{depth} M = \dim M$ (resp. $\operatorname{depth} M = \dim B$) by definition, or equivalently, $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i < \dim M$ (resp. $i < \dim B$). If B is Cohen-Macaulay, then the v-graded piece of $\mathrm{H}^i_{\mathfrak{m}}(M)$ is via Gorenstein duality given by ${}_v\mathrm{H}^i_{\mathfrak{m}}(M) \simeq {}_{-v}\mathrm{Ext}^{n-i}_B(M, K_B)^\vee$ where $K_B = \mathrm{Ext}^c_R(B, R(-n-c))$ is the canonical module of B. In this case $B = R/I_B$ has a minimal R-free resolution of the following form

$$0 \to G_c \to \ldots \to G_1 \to R \to B \to 0 \ , \qquad G_j = \oplus_{i=1}^{r_j} R(-n_{j,i}) \tag{4}$$

and the R-dual sequence $0 \to R \to G_1^* \to \ldots \to G_c^* \to K_B(n+c) \to 0$ is exact. Moreover the Castelnuovo-Mumford regularity of I_B is $\operatorname{reg}(I_B) = \max\{n_{c,i}\} - c + 1$. A coherent non-trivial $\mathcal{O}_Y$-module, $\mathcal{M}$, is called a maximal Cohen-Macaulay sheaf on U if $\mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{M}, \omega_Y)|_U = 0$ for $i > 0$ where $\omega_Y = \widetilde{K}_B$.

Lemma 7. *Let B be Cohen-Macaulay. Let r and t be integers. Let $J \subseteq B$ be an ideal satisfying $\operatorname{depth}_J B \geq r$ and let M be a finitely generated B-module satisfying $\operatorname{depth}_{\mathfrak{m}} M \geq \dim B - t$. Then $\operatorname{depth}_J M \geq r - t$.*

For a proof, see [28], Lem. 5. (What there looks like sheaf-$\mathcal{E}xt^j$ is actually Ext^j as B-modules).

Now we recall a main result of [27] in which B is Cohen-Macaulay. Indeed a main idea of this paper is to use deformation theory to see how general the construction of Gorenstein quotients, given by the Theorem below, is. Note that in this case $\operatorname{depth}_J B = \dim B - \dim B/J$ for any ideal J of B, cf. [7].

Theorem 8. *Let R be a polynomial ring, let $B = R/I_B$ be a codimension c graded CM quotient of R and let M be a finitely generated graded maximal CM B-module. Let $Y = \operatorname{Proj}(B)$, let Z be a closed scheme such that $\dim(B) - \dim(B/I(Z)) \geq \max(r, 2)$ and let $U = Y - Z$. Let $M_i = \mathrm{H}^0_*(U, \widetilde{\wedge^i M})$ for $i \geq 0$, and suppose that $\widetilde{M}|_U$ is locally free (of rank r) and that*

$$\wedge^r \widetilde{M}|_U \simeq \widetilde{K}_B(t)|_U \ \text{ for some integer } t \ .$$

Moreover suppose M_i is a maximal CM B-module for $2 \le i \le r/2$. Then any regular section, $\sigma \in \mathrm{H}^0(U, \widetilde{M}^*(s))$ (s an integer), defines a Gorenstein quotient $R \twoheadrightarrow A$ of codimension $r+c$ given by the exact sequence

$$0 \to M_r(-rs) \to M_{r-1}((1-r)s) \to \cdots \to M_2(-2s) \to M(-s) \stackrel{\sigma}{\to} B \to A \to 0. \qquad (5)$$

Indeed all $M_i, 0 \le i \le r$, are maximal CM B-modules, and $M_{r-i} \simeq \mathrm{Hom}_B(M_i, K_B)(t)$ for $0 \le i \le r$.

Remark 9. *By replacing M by M_1 in (5) we see that the maximal CM assumption on M is really superfluous in the rank 2 Artinian case ($r = \dim(B) = 2$). In [28] we succeeded generalizing Theorem 8 by weakening the maximal CM assumptions on M and M_i in several ways.*

In [27] we applied Theorem 8 to different modules M on licci schemes (i.e. schemes whose homogeneous coordinate ring is in the linkage class of a complete intersection, cf. [30] for a survey), notably the normal module, N_B, and the 1. Koszul homology, $\mathrm{H}_1 = \mathrm{H}_1(I_B)$, built on a set of minimal generators of I_B. Note that the module H_1 is essentially given by an exact sequence

$$0 \to \mathrm{H}_2(R,B,B) \to \mathrm{H}_1 \to G_1 \otimes_R B \to I_B/{I_B}^2 \to 0. \qquad (6)$$

in which $\mathrm{H}_2(R,B,B)$ is the 2. algebra homology [31]. An ideal I_B of R is called *syzygetic* if $\mathrm{H}_2(R, R/I_B, R/I_B) = 0$. Using (6) one shows that if $R \to B$ is generically a complete intersection and B is licci, then $\mathrm{H}_2(R,B,B) = 0$ because H_1 is a maximal CM module in the licci case [17]. If B is licci one also knows that N_B is a maximal CM B-module and that $\mathrm{Ext}^1_B(I_B/{I_B}^2, B) = 0$ ([5] and [26]). Note that if $\mu(I_B)$ is the number of minimal generators of I_B, we see from (6) that $\mathrm{rank}\, H_1 = \mu(I_B) - c$ since the rank of $\mathcal{N}_Y = \widetilde{N}_B = \widetilde{I_B/{I_B}^2}^{\,*}$ is the same as the codimension, c, of Y in $\mathrm{Proj}(R)$. We proved

Proposition 10. *Let $B = R/I_B$ be licci of codimension $c \ge 2$, and let $Y = \mathrm{Proj}(B) \hookrightarrow \mathbb{P}^{n+c-1} = \mathrm{Proj}(R)$ be a local complete intersection (l.c.i.) in some open $U = Y - Z$.*

1. *If $\dim B - \dim B/I(Z) \ge c$, then $\mathrm{H}^0_*(U, \wedge^i \mathcal{N}_Y)$ are maximal CM B-modules for every $1 \le i \le c$, and any regular section of $\mathrm{H}^0(U, \mathcal{N}_Y^*(s))$ defines a Gorenstein quotient $R \twoheadrightarrow A$ of codimension $2c$.*
2. *If $\dim B - \dim B/I(Z) \ge \max(2, \mu(I_B) - c)$, then $\mathrm{H}^0_*(U, \wedge^i \widetilde{H}_1)$ are maximal CM B-modules for every $1 \le i \le \mu(I_B) - c$ and any regular section of $\mathrm{H}^0(U, \widetilde{H}_1^*(s))$ defines a Gorenstein quotient $R \twoheadrightarrow A$ of codimension $\mu(I_B)$.*

For the graded group $\mathrm{H}^2(R,B,B)$ we just remark that there is an exact sequence

$$0 \to {}_0\mathrm{Ext}^1_B(I_B/I_B^2, N) \to {}_0\mathrm{H}^2(R,B,N) \to {}_0\mathrm{Hom}_B(\mathrm{H}_2(R,B,B), N) \to \qquad (7)$$

induced from some well known spectral sequence, cf. [1], Prop. 16.1.

Let $p \in \mathbb{Q}[t]$ be a non-trivial polynomial and let $\mathbb{P} = \mathbb{P}^{n+c-1}$. Then the scheme $\mathrm{GradAlg}(H) = \mathrm{GradAlg}^H(R)$ which we study in this paper is the stratum of Grothendieck's Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P})$ (cf. [11]) consisting of points $(Y \subset \mathbb{P})$ with Hilbert function $H_Y = H$ (i.e. its corresponding functor deforms the *homogeneous coordinate ring*, B, of Y flatly as a graded R-algebra), cf. [22]. Note that we define the Hilbert function of Y, or B, by $H_Y(v) = H_B(v) := \dim B_v$. $\mathrm{GradAlg}^H(R)$ has a natural scheme structure whose tangent (resp. "obstruction") space at $(Y \subset \mathbb{P})$ is ${}_0\mathrm{Hom}_B(I_B/I_B^2, B) \simeq {}_0\mathrm{Hom}_R(I_B, B)$ (resp. ${}_0\mathrm{H}^2(R,B,B)$) [21]. Since $H(v)$ does not vanish for large v (i.e. B is non-Artinian), we may look upon $\mathrm{GradAlg}^H(R)$ as parametrizing graded R-quotients, $R \to B$, satisfying

$\operatorname{depth}_{\mathfrak{m}} B \geq 1$ and with Hilbert function $H_B = H$. If B is Artinian, then $\operatorname{GradAlg}^H(R)$ still represents a functor parametrizing graded R-quotients with Hilbert function $H_B = H$. It contains an open subscheme of Gorenstein quotients which, at least topologically and infinitesimally, coincides with $\operatorname{PGor}(H)$, the corresponding scheme of forms with "catalecticant structure" (see [22] or [23] for details). B is called *unobstructed* as a graded R-algebra if $\operatorname{GradAlg}^H(R)$ is smooth at $(R \to B)$. Similarly a closed subscheme Y of $\mathbb{P}$ is called *unobstructed* if $\operatorname{Hilb}^p(\mathbb{P})$ is smooth at $(Y \subset \mathbb{P})$. By [21], Thm. 3.6 and Rem. 3.7,

$$\operatorname{GradAlg}^H(R) \simeq \operatorname{Hilb}^p(\mathbb{P}) \quad \text{at} \quad (Y \subset \mathbb{P}) \tag{8}$$

provided ${}_0\operatorname{Hom}_R(I_B, \operatorname{H}^1_{\mathfrak{m}}(B)) = 0$ (e.g. provided $\operatorname{depth}_{\mathfrak{m}} B \geq 2$, in which case (8) also follows from [8]). In particular if the degree of p is positive, then the open subschemes of $\operatorname{GradAlg}^H(R)$ (resp. of $\operatorname{Hilb}^p(\mathbb{P})$) of Gorenstein quotients (resp. of AG subschemes) are isomorphic as *schemes*.

Similarly we let $\operatorname{GradAlg}(H_B, H_A)$ be the representing object of the functor deforming flags (surjections) $B \to A$ of graded quotients of R of positive depth (if B and/or A are non-Artinian) and with Hilbert functions H_B and H_A of B and A respectively. Let p be the second projection

$$p : \operatorname{GradAlg}(H_B, H_A) \to \operatorname{GradAlg}^{H_A}(R)$$

induced by sending $(B' \to A')$ onto (A'), and let $q : \operatorname{GradAlg}(H_B, H_A) \to \operatorname{GradAlg}^{H_B}(R)$ be the first projection.

Definition 11. *Let $R \to B$ be a graded quotient, let M be a graded B-module and let $\varphi : M \to B$ be a homogeneous B-linear map. Let $(T, m_T) \to (S, m_S)$ be a small Artin surjection (i.e. of local Artinian k-algebras with residue fields k whose kernel $\mathfrak{a}$ satisfies $\mathfrak{a} \cdot m_T = 0$). A graded deformation B_S of B to S is a graded S-flat quotient of $R \otimes_k S$ satisfying $B_S \otimes_S k \simeq B$.*

(i) (M, φ) is said to be unobstructed along any graded deformation of B if for every small Artin $T \twoheadrightarrow S$ and for every *graded deformation $M_S \to B_S$ (of S-flat B_S-modules) of $M \xrightarrow{\varphi} B$ to S, there exists, for* every *graded deformation B_T of B_S to T, a graded deformation $M_T \to B_T$ over $M_S \to B_S$ (i.e. a morphism of T-flat B_T-modules reducing to $M_S \to B_S$ via $(-) \otimes_T S$).*

(ii) One correspondingly defines (M, B) to be unobstructed along any graded deformation of B by forgetting φ, i.e. by considering pairs (M_S, B_S) instead of $M_S \to B_S$. The unobstructedness of (M, B) and $(M(-s), B)$, s an integer, are obviously equivalent. Moreover a surjection of graded quotients of R, $B \to A$, is unobstructed if every graded deformation $B_S \to A_S$ deforms further to T for every small Artin $T \twoheadrightarrow S$. Similarly a quotient A of R is unobstructed if every graded deformation A_S deforms further to T.

Note that we have defined unobstructedness of graded objects by considering graded deformations only. Moreover note that, in deforming quotients B and A of R to S the corresponding deformation of R is always the trivial one (i.e. $R \otimes_k S$) while the deformations B_S and A_S need not be trivial.

Remark 12. *Let $T \to S$ be a small Artin surjection with kernel $\mathfrak{a}$. The obstruction of deforming a graded B_S-module M_S (resp. a graded morphism $M_S \to N_S$) to T sits in ${}_0\operatorname{Ext}^2_B(M, M) \otimes \mathfrak{a}$ (resp. ${}_0\operatorname{Ext}^1_B(M, N) \otimes \mathfrak{a}$) and ${}_0\operatorname{Ext}^1_B(M, M) \otimes \mathfrak{a}$ (resp. ${}_0\operatorname{Hom}_B(M, N) \otimes \mathfrak{a}$) corresponds to their set of graded deformations respectively. For a generically complete intersection, $R \twoheadrightarrow B$, its obstructions sit in ${}_0\operatorname{Ext}^1_B(I_B/I_B^2, B) \otimes \mathfrak{a}$. Note that, by definition, the vanishing of the obstruction is equivalent to the existence of corresponding desired deformation.*

Since the whole deformation theory fits functorially together (by e.g. Laudal's work on deformations of categories in [29]), we get in particular

Proposition 13. *Let M be a finitely generated graded B-module.*

(i) If $\varphi : M \to B$ is a B-module homomorphism and ${}_0\mathrm{Ext}^1_B(M,B) = 0$, then (M,φ) is unobstructed along any graded deformation of B if and only if (M,B) is unobstructed along any graded deformation of B.

(ii) If ${}_0\mathrm{Ext}^2_B(M,M) = 0$, then (M,B) is unobstructed along any graded deformation of B.

(iii) If ${}_0\mathrm{Ext}^1_B(M,M) = 0$ and if for every local Artinian k-algebra T with residue field k and for every *graded deformation B_T of B to T, there exists a graded deformation M_T of M to B_T, then (M,B) is unobstructed along any graded deformation of B.*

Proof. (i) If $T \to S$ is a small Artin surjection (Definition 11), it follows from the exact sequence

$$0 \to B \otimes_k \mathfrak{a} \to B_T \to B_S \to 0$$

that ${}_0\mathrm{Hom}_{B_T}(M_T, B_T) \to {}_0\mathrm{Hom}_{B_S}(M_S, B_S)$ is surjective and hence that ${}_0\mathrm{Hom}_{B_S}(M_S, B_S) \to {}_0\mathrm{Hom}_B(M,B)$ is surjective by induction. This implies that φ lifts to deformations of M and B, and we easily get (i). (Remark 12 may provide a quicker proof.)

Finally (ii) and (iii) follow from Remark 12. Here we leave a few details to the reader, remarking only that for (iii) the assumption ${}_0\mathrm{Ext}^1_B(M,M) = 0$ implies that an isomorphism $M_S \to M'_S$ of deformations of M to B_S lifts further to an isomorphism of given deformations of M_S and M'_S to B_T, i.e. all deformations of M to B_S are isomorphic. □

Example 14. *Let $B = R/I_B$ be a graded n-dimensional licci quotient of R of codimension c and suppose* $\mathrm{depth}_{I(Z)} B \geq 2$ *where $Y - Z$ is an l.c.i. in* $\mathrm{Proj}(R)$.

i) Then (K_B, B) is unobstructed along any graded deformation of B by Proposition 13(ii). Indeed Proposition 13(iii) also applies because $K_{B_T} := \mathrm{Ext}^c_{B_T}(B_T, R_T(-n-c))$ is a graded deformation of K_B to B_T by [20], Prop. A1.

ii) If $c = 2$, then $(I_B/I_B^2, B)$ is unobstructed along any graded deformation of B by Proposition 13(ii) and Proposition 3. Moreover if $\mathrm{depth}_{I(Z)} B \geq 4$ *we will see in Section 3 that (N_B, B) is unobstructed along any graded deformation of B because the assumptions of Proposition 13(iii) holds (here Proposition 13(ii) may not apply, cf. Remark 42).*

We will need the following result on how deformations of $R \to A$ is related to deformations of $R \to B$ provided $A \simeq B/I_{A/B}$ is a graded quotient of B ([23], Thm. 5 and Rem. 6). To state it, let $U \subset \mathrm{GradAlg}^{H_A}(R)$ be a sufficiently small open subset containing (A) and let $p : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}^{H_A}(R)$ be the second projection, i.e. given by $p((B' \to A')) = (A')$. Then the k-points of the subset $p(p^{-1}(U))$ of U correspond to quotients $R \to A'$ with Hilbert function H_A for which there exist some factorization $B' \to A'$ such that B' has Hilbert function H_B. We will call it a *stratum of H_B-factorizations around* (A), and $\dim U - \dim p(p^{-1}(U))$ the *codimension of the H_B-stratum of A*. At least if U is smooth, it is the ordinary codimension of $p(p^{-1}(U))$ in U. A is called *H_B-generic* if there is an *open* subset U_A of $\mathrm{GradAlg}^{H_A}(R)$ such that $(A) \in U_A \subset p(p^{-1}(U))$. The *$H_B$-stratum of A at $(B \to A)$, (resp. its codimension)*, is defined to be $p(U')$ (resp. $\dim U - \dim p(U')$) where $U' \subset p^{-1}(U)$ is an open subset in the union of the irreducible components of $p^{-1}(U)$ which contain $(B \to A)$, and $(B \to A) \in U'$. If $(I_{A/B})_v = 0$ for $v \leq$ the largest degree of the minimal generators of I_B, then $p|_{U'}$ will be unramified and universally injective and the two concepts of codimension above coincide ([23], Lem. 7).

Theorem 15. *Let R be a graded polynomial k-algebra, let $B = R/I_B \twoheadrightarrow A \simeq B/I_{A/B}$ be a graded morphism of quotients of R and suppose ${}_0\mathrm{H}^2(B,A,A) = 0$,* $\mathrm{depth}_{\mathfrak{m}} A \geq \min(1, \dim A)$ *and* $\mathrm{depth}_{\mathfrak{m}} B \geq \min(1, \dim B)$.

A) *If* ${}_0\mathrm{Ext}^1_B(I_B/I_B^2, A) = 0$ *and $(I_B)_\wp$ is syzygetic for any graded $\wp$ of $Ass(A)$,* *then A is unobstructed as a graded R-algebra (indeed ${}_0\mathrm{H}^2(R,A,A)=0$), and*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = {}_0\mathrm{hom}_R(I_B,B) + {}_0\mathrm{hom}_B(I_{A/B},A) - {}_0\mathrm{hom}_R(I_B,I_{A/B}) +$$

$${}_0\mathrm{ext}^1_B(I_B/I_B^2,I_{A/B}) - {}_0\mathrm{ext}^1_B(I_B/I_B^2,B).$$

Moreover let B be unobstructed as a graded R-algebra, let k be of characteristic zero and suppose $(B\to A)$ is general with respect to ${}_0\mathrm{hom}_R(I_B,I_{A/B})$. Then the codimension of the H_B-stratum of A at $(B\to A)$ is

$${}_0\mathrm{ext}^1_B(I_B/I_B^2,I_{A/B}) - {}_0\mathrm{ext}^1_B(I_B/I_B^2,B).$$

B) If ${}_0\mathrm{Ext}^1_B(I_B/I_B^2,I_{A/B})=0$ and $(I_B)_\wp$ is syzygetic for any graded prime $\wp$ of $Ass(I_{A/B})$, then A is H_B-generic. Moreover A is unobstructed as a graded R-algebra if and only if B is unobstructed as a graded R-algebra. Indeed

$${}_0\mathrm{hom}_R(I_A,A) - \dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = {}_0\mathrm{hom}_R(I_B,B) - \dim_{(B)} \mathrm{GradAlg}^{H_B}(R), \quad \textit{and}$$

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + {}_0\mathrm{hom}_B(I_{A/B},A) - {}_0\mathrm{hom}_R(I_B,I_{A/B})\ .$$

Remark 16. *(i) Theorem 15 follows directly from Thm. 5 and Rem. 6 of [23]. Moreover one may replace the assumption " ${}_0\mathrm{Ext}^1_B(I_B/I_B^2,I_{A/B})=0$ and $(I_B)_\wp$ syzygetic for any graded prime $\wp$ of $Ass(I_{A/B})$" of Theorem 15B) by " ${}_0\mathrm{Ext}^1_R(I_B,I_{A/B})=0$" and conclude exactly as in Theorem 15B) because, what's needed to prove part B) is ${}_0\mathrm{H}^2(R,B,I_{A/B})=0$ (cf. the proof of Thm. 5 of [23]). Since we in general have an injection ${}_0\mathrm{H}^2(R,B,I_{A/B}) \hookrightarrow {}_0\mathrm{Ext}^1_R(I_B,I_{A/B})$, we get the claim.*

(ii) To find the codimension of H_B-stratum of A at $(B\to A)$, *the proof of [23], Thm. 5 uses generic smoothness. Indeed it computes the dimension of the image $p(U')$, see the text before Theorem 15 above, in terms of the dimension of* $\mathrm{GradAlg}(H_B,H_A)$ *at* $(B\to A)$, *and the dimension,* ${}_0\mathrm{hom}_R(I_B,I_{A/B})$, *of a* general *fiber. Unfortunately the word "general", i.e. the assumption "$(B\to A)$ is general with respect to ${}_0\mathrm{hom}_R(I_B,I_{A/B})$" is missing in [23], Thm. 5 as well as in Theorem 15 of the published version, [*Collect. Math. *58, 2 (2007), 199-238], of this paper. Indeed it is easily seen from the proof in [23] (cf. [23], Prop 4(i)) that we may suppose U' above is a smooth irreducible scheme such that the restriction of the* first *projection* $q : \mathrm{GradAlg}(H_B,H_A) \to \mathrm{GradAlg}^{H_B}(R)$, *$q((B'\to A'))=(B')$, to U' is a smooth morphism. Hence if we let "$(B\to A)$ is general with respect to ${}_0\mathrm{hom}_R(I_B,I_{A/B})$" mean that, for a given $(B\to A)$, ${}_0\mathrm{hom}_R(I_B,I_{A/B})$ obtains its least possible value in U', or equivalently that ${}_0\mathrm{hom}_R(I_B,I_{A/B}) = {}_0\mathrm{hom}_R(I_{B'},I_{A'/B'})$ for every $(B'\to A')$ in an open neighbourhood of $(B\to A)$ in U', the proof is complete. Note that this assumption of generality is always satisfied if ${}_0\mathrm{hom}_R(I_B,I_{A/B})=0$, in which case we do not need to suppose "$char(k)=0$" either because then $p|_{U'}$ is unramified. In particular in [23]; Theorem 1, Proposition 13, Theorem 16 (and hence in Theorem 23 of [*Collect. Math. *58, 2 (2007), 199-238]), for the* **codimension statement** *in the* A)*-part of the results we need to assume that "(B) is general with respect to ${}_{-s}\mathrm{hom}_R(I_B,K_B)$", i.e. that ${}_{-s}\mathrm{hom}_R(I_B,K_B)$ obtains its least possible value in the open subset $q(U')$ of* $\mathrm{GradAlg}^{H_B}(R)$, *since we in this application use [23], Thm. 5 with $I_{A/B} := K_B(-s)$. Several results of the published version of this paper in* Collect. Math. *58, 2 (2007), 199-238, as well as the first version of this paper on the arXiv, i.e. Thm. 1, Cor. 18, Cor. 20, Thm. 25, Thm. 30, Cor. 37 and Cor. 44, use Theorem 15 and lack the generality assumption on $(B\to A)$ above which is needed for the validity of the* **codimension statements** *of the strata. In this version all these results are corrected, and in the examples where the codimensions of the strata are stated and computed, one may check that ${}_0\mathrm{hom}_R(I_B,I_{A/B})=0$, so no corrections are needed there.*

The following lemma is a slight improvement of [23], Lem. 15 which we will need to transform Theorem 15 into a result on a Gorenstein quotient of a CM algebra.

Lemma 17. *Let B be a graded CM quotient of R and let $A \simeq B/I_{A/B}$ be Gorenstein such that the canonical module $K_A \simeq A(j)$ and $\dim B - \dim A = r$. We have*

(i) If $r > 0$ (resp. $r = 0$), then ${}_0\mathrm{H}^2(B, A, A) = 0$ if and only if ${}_{-j}\operatorname{Ext}^r_B(S_2(I_{A/B}), K_B) = 0$ (resp. iff the natural map ${}_{-j}\operatorname{Hom}_B(I_{A/B}, K_B) \to {}_{-j}\operatorname{Hom}_B(S_2(I_{A/B}), K_B)$ is surjective).

(ii) ${}_0\hom_B(I_{A/B}, A) = {}_{-j}\operatorname{ext}^r_B(B/I^2_{A/B}, K_B) - 1$ for any $r \geq 0$. Moreover if $r \geq 2$, then ${}_0\hom_B(I_{A/B}, A) = {}_{-j}\operatorname{ext}^{r-1}_B(S_2(I_{A/B}), K_B) - 1$.

(iii) Let $\dim A = 0$. If the surjection $S_2(I_{A/B})_j \to (I^2_{A/B})_j$ is an isomorphism, then ${}_0\mathrm{H}^2(B, A, A) = 0$ and ${}_0\hom_B(I_{A/B}, A) = \dim B_j - \dim S_2(I_{A/B})_j - 1$. In particular if $S_2(I_{A/B})_j = 0$, then ${}_0\mathrm{H}^2(B, A, A) = 0$ and ${}_0\hom_B(I_{A/B}, A) = \dim B_j - 1$.

Proof. (i) By [23], Lemma 15 (whose proof is quite close to the proof below), we get the vanishing of ${}_0\mathrm{H}^2(B, A, A)$ from ${}_0\operatorname{Ext}^r_B(S_2(I_{A/B}), K_B(-j)) = 0$, and conversely if $r > 0$. By paying a little extra attention to the case $r = 0$ we get (i).

(ii) We have by $A \simeq K_A(-j)$ and Gorenstein duality (applied to both A and B) that

$$ {}_0\operatorname{Hom}_A(I_{A/B}/I^2_{A/B}, A) \simeq {}_j\mathrm{H}^{\dim A}_{\mathfrak{m}}(I_{A/B}/I^2_{A/B})^\vee \simeq {}_{-j}\operatorname{Ext}^r_B(I_{A/B}/I^2_{A/B}, K_B). $$

Moreover since $\operatorname{Ext}^r_B(A, K_B) \simeq \operatorname{Hom}_A(A, K_A) \simeq A(j)$ and $\operatorname{Ext}^{r-i}_B(D, K_B) = 0$ for $i > 0$ provided D is an A-module, we get ${}_0\hom_B(I_{A/B}, A) = {}_{-j}\operatorname{ext}^r_B(B/I^2_{A/B}, K_B) - 1$ by the long exact sequence of $\operatorname{Hom}(-, K_B)$ applied to $0 \to I_{A/B}/I^2_{A/B} \to B/I^2_{A/B} \to A \to 0$. Moreover applying $\operatorname{Hom}(-, K_B)$ onto $0 \to I^2_{A/B} \to B \to B/I^2_{A/B} \to 0$, we get ${}_{-j}\operatorname{Ext}^r_B(B/I^2_{A/B}, K_B) \simeq {}_{-j}\operatorname{Ext}^{r-1}_B(I^2_{A/B}, K_B)$ provided $r > 1$. Since the kernel of the surjection $S_2(I_{A/B}) \to I^2_{A/B}$ is an A-module (namely $\mathrm{H}_2(B, A, A)$, cf. [31], Section 2.1), we get $\operatorname{Ext}^{r-1}_B(I^2_{A/B}, K_B) \simeq \operatorname{Ext}^{r-1}_B(S_2(I_{A/B}), K_B)$ and we have (ii).

(iii) If $\dim A = 0$ and the surjection $S_2(I_{A/B})_j \to (I^2_{A/B})_j$ is an isomorphism, we get $\dim B = r$ and ${}_j\mathrm{H}_2(B, A, A) = 0$ and hence

$$ {}_{-j}\operatorname{Ext}^r_B(S_2(I_{A/B}), K_B)^\vee \simeq {}_j\mathrm{H}^0_{\mathfrak{m}}(S_2(I_{A/B})) \simeq {}_j\mathrm{H}^0_{\mathfrak{m}}(I^2_{A/B}) $$

by Gorenstein duality. If $r > 0$ (resp. $r = 0$), the last group vanishes (resp. injects into ${}_j\mathrm{H}^0_{\mathfrak{m}}(I_{A/B})$). Hence we get ${}_0\mathrm{H}^2(B, A, A) = 0$ by (i) and since ${}_{-j}\operatorname{Ext}^r_B(B/I^2_{A/B}, K_B)^\vee \simeq {}_j\mathrm{H}^0_{\mathfrak{m}}(B/I^2_{A/B}) = (B/I^2_{A/B})_j$, we conclude by (ii) and by assumption. $\square$

Corollary 18. *Let $B = R/I_B$ be Cohen-Macaulay and let $A \simeq B/I_{A/B}$ be a graded Gorenstein quotient such that $K_A \simeq A(j)$ and $\dim B - \dim A = r$. If*

(i) ${}_0\operatorname{Ext}^1_B(I_B/I^2_B, B) = 0$ and $(I_B)_\wp$ is syzygetic for any graded $\wp$ of $Ass(A) \cup Ass(B)$,

(ii) ${}_{-j}\operatorname{Ext}^r_B(S_2(I_{A/B}), K_B) = 0$, and

(iii) ${}_0\operatorname{Ext}^2_B(I_B/I^2_B, I_{A/B}) = 0$,

then A is unobstructed as a graded R-algebra (indeed ${}_0\mathrm{H}^2(R, A, A) = 0$). Moreover, if $r \geq 2$, then

$$ \dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = {}_0\hom_R(I_B, B) + {}_{-j}\operatorname{ext}^{r-1}_B(S_2(I_{A/B}), K_B) - 1 - \delta(I_{A/B})_0 $$

where $\delta(I_{A/B})_0 := {}_0\hom_R(I_B, I_{A/B}) - {}_0\operatorname{ext}^1_B(I_B/I^2_B, I_{A/B})$. Furthermore if $char(k) = 0$ and $(B \to A)$ is general with respect to ${}_0\hom_R(I_B, I_{A/B})$, then the codimension of the H_B-stratum of A at $(B \to A)$ is ${}_0\operatorname{ext}^1_B(I_B/I^2_B, I_{A/B})$.

Proof. By (i) we have that B is unobstructed. Moreover (i), (iii) and the long exact sequence of $\mathrm{Hom}_B(I_B/I_B^2,-)$ applied to $0 \to I_{A/B} \to B \to A \to 0$ show the first line of assumptions of Theorem 15A). Hence we conclude by Lemma 17. $\square$

Remark 19. *We get* ${}_v\mathrm{H}^2(R,A,A)=0$ *by twisting the three* Ext^i_B*-vanishing assumptions of Corollary 18 by* v *because Theorem 15A) admits such a generalization (cf. [23], Rem. 6(a)).*

Corollary 20. *Let* $B = R/I_B$ *be Cohen-Macaulay and let* $A \simeq B/I_{A/B}$ *be a graded Artinian Gorenstein quotient such that* $K_A \simeq A(j)$ *and* $\dim B - \dim A = r$*. If*

(i) ${}_0\mathrm{Ext}^1_B(I_B/I_B^2, B) = 0$ *and* $(I_B)_\wp$ *is syzygetic for any graded* $\wp$ *of* $Ass(A) \cup Ass(B)$*,*

(ii) $\dim S_2(I_{A/B})_j = \dim(I^2_{A/B})_j$ *, and*

(iii) ${}_0\mathrm{Ext}^2_B(I_B/I_B^2, I_{A/B}) = 0$*,*

then A *is unobstructed as a graded* R*-algebra (indeed* ${}_0\mathrm{H}^2(R,A,A)=0$*), and*

$$\dim_{(A)} \mathrm{PGor}(H_A) = {}_0\hom_R(I_B,B) + \dim B_j - \dim S_2(I_{A/B})_j - 1 - \delta(I_{A/B})_0$$

where $\delta(I_{A/B})_0 := {}_0\hom_R(I_B, I_{A/B}) - {}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B})$*. Furthermore if* $char(k)=0$ *and* $(B \to A)$ *is general with respect to* ${}_0\hom_R(I_B, I_{A/B})$*, then the codimension of the* H_B*-stratum of* A *at* $(B \to A)$ *is* ${}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B})$*.*

The proof is similar to the preceding Corollary. Note that if B is a c.i. and $(I_{A/B})_v = 0$ for $v \le j/2$, then all assumptions of Corollary 20 are satisfied. Hence [18], Thm. 4.17 is generalized by Corollary 20 and we get in addition that $\mathrm{PGor}(H_A)$ is generically smooth along the component of [18], Thm. 4.17.

Remark 21. *If the Gorenstein algebra* A *is given by* (5)*, then condition* (*iii*) *of Corollary 18 and 20 is satisfied provided*

$${}_0\mathrm{Ext}^{i+1}_B(I_B/I_B^2, M_i(-is)) = 0 \quad \text{for } 1 \le i \le r \tag{9}$$

letting $M_1 = M$*. Indeed splitting the exact sequence* (5) *into short exact sequences, we get*

$$0 \to Z_i \to M_i(-is) \to Z_{i-1} \to 0$$

where $Z_i = \ker(M_i(-is) \to M_{i-1}((1-i)s))$ *for* $i \ge 1$ *and* $Z_0 = I_{A/B}$*. Successively applying* ${}_0\mathrm{Ext}^{i+1}_B(I_B/I_B^2,-)$ *onto this sequences and using* (9) *we get* ${}_0\mathrm{Ext}^{i+1}_B(I_B/I_B^2, Z_{i-1}) = 0$ *from* ${}_0\mathrm{Ext}^{i+1}_B(I_B/I_B^2, M_i(-is)) = 0$ *and* ${}_0\mathrm{Ext}^{i+2}_B(I_B/I_B^2, Z_i) = 0$*, i.e. we get* (*iii*) *because* $Z_{r-1} = M_r(-rs)$*.*

It is desirable to weaken the general assumption ${}_0\mathrm{H}^2(B,A,A)=0$ of Theorem 15. In the appendix we prove a nice result, Theorem 47, on deformations of the degeneracy locus of a regular section of a maximal CM sheaf, which leads to a variation of Theorem 15B) (Theorem 22 below) in which the mentioned assumption is replaced by assumptions closer to set-up of Theorem 8. The main idea of the proof of Theorem 47 and hence of Theorem 22 is to use the complete intersection property (in large enough open subset) of a regular section to control all deformations of the degeneracy locus. Since the proofs are probably of most interest only for specialists in deformation theories, we delay them to the appendix. For the case $r=1$ and $\mathrm{depth}_{I(Z)} B = 1$, which requires special attention, we refer to [23].

Theorem 22. *Let R be a finitely generated polynomial k-algebra, let $B = R/I_B$ be a graded Cohen-Macaulay quotient of R of codimension c, and let M be a finitely generated graded maximal Cohen-Macaulay B-module. Let $Y = \operatorname{Proj}(B)$, let Z be a closed scheme such that $\dim B - \dim B/I(Z) \geq \max(r, 2)$ and let $U = Y - Z$. Let A be defined by a regular section σ of $\widetilde{M}^*(s)$ on U, i.e. given by (5). Let $M_i = \mathrm{H}^0_*(U, \wedge^i \widetilde{M})$ for $i \geq 0$, and suppose $\widetilde{M}|_U$ is locally free (of rank $r > 0$) and*

$$\wedge^r \widetilde{M}|_U \simeq \widetilde{K}_B(t)|_U \text{ (for some integer } t).$$

Moreover suppose M_i are maximal Cohen-Macaulay B-modules for $2 \leq i \leq r/2$. If

(i) ${}_0\operatorname{Ext}^i_B(M, M_i(-(i-1)s)) = 0 \quad$ for $\; 2 \leq i \leq r-1$, *and*

(ii) $(M(-s), \sigma)$ *is unobstructed along any graded deformation of B, and*

(iii) either ${}_0\operatorname{Ext}^i_B(I_B/I_B^2, M_i(-is)) = 0$ for $1 \leq i \leq r$ *and $(I_B)_\wp$ is syzygetic for any $\wp$ of $Ass(B)$*, or ${}_0\operatorname{Ext}^i_R(I_B, M_i(-is)) = 0$ for $1 \leq i \leq r$,

then ${}_0\hom_R(I_B, B) - \dim_{(B)} \operatorname{GradAlg}^{H_B}(R) = {}_0\hom_R(I_A, A) - \dim_{(A)} \operatorname{GradAlg}^{H_A}(R)$, *$A$ is H_B-generic and Gorenstein, and*

$$\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = \dim_{(B)} \operatorname{GradAlg}^{H_B}(R) + {}_0\hom_B(I_{A/B}, A) - {}_0\hom_R(I_B, I_{A/B}).$$

Moreover A is unobstructed as a graded R-algebra if and only if B is unobstructed as a graded R-algebra.

Fortunately, by results of the next sections we will see that many of the Ext-groups of Corollary 18 and Theorem 22 vanish, and that the dimension formula turns out to be computable, provided we apply it to a licci quotient B of R of small enough codimension.

2 Families of Gorenstein quotients of low codimension on Cohen-Macaulay algebras

An important issue of this paper is to study families of graded Gorenstein quotients A obtained by taking regular sections of the dual of a maximal Cohen-Macaulay sheaf of rank r, i.e. quotients given by Theorem 8. In the following we will see how Corollary 18 and Theorem 22 enable us to treat the low rank cases $r \leq 3$ on a licci scheme satisfactorially. The different values of r, $1 \leq r \leq 3$, require special attention and lead to the three theorems of this section.

The case $r = 1$ was considered in [23]. There we proved the following result in which $N_B := \operatorname{Hom}_B(I_B/I_B^2, B)$ and $K_B^* := \operatorname{Hom}_B(K_B, B) \simeq \operatorname{Hom}_B(S_2(K_B), K_B)$. The conclusions below about when "the codimension of the stratum is zero" overlap results proved by others ([4], Thm. 3.2, cf. [15], Thm. 3.5). Note that we say "the stratum of quotients given by (10) around or at (A)" for "the H_B-stratum of A at $(B \to A)$" provided any $(B' \to A')$ in a small enough *open* neighbourhood of $(B \to A)$ in $\operatorname{GradAlg}(H_B, H_A)$ is given by (10).

Theorem 23. *Let $B = R/I_B$ be a generically Gorenstein, graded Cohen-Macaulay quotient of a polynomial ring R, and let A be a graded codimension one quotient of B, given by an exact sequence*

$$0 \to K_B(-s) \to B \to A \to 0 \;, \qquad s \;\text{ an }\; \text{integer} \;. \tag{10}$$

A) *If B is licci, then A is unobstructed as a graded R-algebra (indeed $\mathrm{H}^2(R, A, A) = 0$), A is Gorenstein and,*

$$\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = \dim_{(B)} \operatorname{GradAlg}^{H_B}(R) + \dim(K_B^*)_s - 1 - \delta(K_B)_{-s}$$

where $\delta(K_B)_{-s} = {}_{-s}\hom_B(I_B/I_B^2, K_B) - {}_{-s}\operatorname{ext}^1_B(I_B/I_B^2, K_B)$. *Moreover if* $char(k) = 0$ *and* (B) *is general with respect to* ${}_{-s}\hom_B(I_B/I_B^2, K_B)$, *then the codimension of the stratum of quotients given by* (10) *around* (A) *is* ${}_{-s}\operatorname{ext}^1_B(I_B/I_B^2, K_B)$.

B) *If* $s >> 0$ *and* $\operatorname{Proj}(B)$ *is locally licci, then* A *is Gorenstein and the codimension of the stratum of quotients given by* (10) *around* (A) *is zero (so* A *is* H_B*-generic), and*

$$\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = \dim_{(B)} \operatorname{GradAlg}^{H_B}(R) + \dim(K_B^*)_s - 1.$$

Moreover A *is unobstructed as a graded* R*-algebra iff* B *is unobstructed as a graded* R*-algebra.*

Example 24. *(Arithmetically Gorenstein curves* $\operatorname{Proj}(A)$ *in* $\mathbb{P}^5$*, obtained by* (10)*.) Let* R *be a polynomial ring in 6 variables, and let* $Y = \operatorname{Proj}(B)$ *be a generically Gorenstein ACM-surface with resolution*

$$0 \to R(-7)^3 \to R(-6)^3 \oplus R(-5)^3 \to R(-3)^4 \to I_B \to 0$$

and Hilbert function $H_B(v) = 19\binom{v+1}{2} - 26v + 16$, $v \geq 2$. *We may obtain* B *by taking a c.i.* B' *of type* $(2, 2, 2)$ *and linking it to* B *via some c.i. of type* $(3, 3, 3)$. *Since the linkage result, Prop. 33 of [23], connects all invariants of* B*, appearing in Theorem 23, to the corresponding invariants of* B' *and since* $N_{B'} \simeq B'(2)^{\oplus 3}$, $\operatorname{Hom}_{B'}(I_{B'}/I_{B'}^2, K_{B'}) \simeq K_{B'}(2)^{\oplus 3}$ *and* $K_{B'} \simeq B'$*, we get by [23] Prop. 33 that* $\dim(N_B)_0 = \dim(N_{B'})_0 + 3H_B(3) - 3H_{B'}(3) = 96$, *and that*

$$\dim(K_B^*)_v = \dim B_{v-3} + 3\dim(I_{B/D})_v - \delta(K_{B'})_{v-6} = 67 + (19v^2 - 99v)/2$$

because $\delta(K_{B'})_v = {}_v\hom_{B'}(I_{B'}/I_{B'}^2, K_{B'})$. *Similarly,* $\delta(K_B)_{v-6} = \dim B'_{v-3} + 3\dim(I_{B'/D})_v - \dim(K_{B'}^*)_v$. *Hence we get* $\delta(K_B)_v = 0$ *for* $v \leq -7$ *and* $(\delta(K_B)_{-5}, \delta(K_B)_{-6}) = (-6, -1)$. *Let* A *be defined by* (10) *for* $s \geq 5$, *so* $X = \operatorname{Proj}(A)$ *is an AG curve in* $\mathbb{P}^5$ *with Hilbert polynomial* $p_X(v) = (19s - 33)v - s(19s - 33)/2$. *Then* A *is unobstructed by Theorem 23 and (cf.* (8)*),*

$$\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = \dim_{(X)} \operatorname{Hilb}^{p_X}(\mathbb{P}^5) = 162 + (19s^2 - 99s)/2 - \delta(K_B)_{-s} ,$$

where ${}_{-s}\operatorname{ext}^1_B(I_B/I_B^2, K_B) = -\delta(K_B)_{-s}$ *represents the codimension of quotients given by* (10) *at* (A).

In the following we concentrate on the case $r > 1$, notably $r = 2$. If $r = 2$, then Theorem 8 provides us with an exact sequence

$$0 \to K_B(t - 2s) \to M(-s) \xrightarrow{\sigma} B \to A \to 0 , \tag{11}$$

and we have $M \simeq \operatorname{Hom}_B(M, K_B)(t)$. We define $\gamma(S_2M)_v$ by

$$\gamma(S_2M)_v = {}_v\hom_B(S_2(M), K_B(t)) - {}_v\operatorname{ext}^1_B(S_2(M), K_B(t)) .$$

Note that if $char(k) \neq 2$, then we get an exact sequence $0 \to \operatorname{Hom}_B(S_2(M), K_B(t)) \to \operatorname{Hom}_B(M, M) \to B \to 0$ from the split exact sequence $0 \to \wedge^2 M \to M \otimes M \to S_2(M) \to 0$ and (3). The split exact sequence and (3) also lead to isomorphisms

$$\operatorname{Ext}^i_B(M, M) \simeq \operatorname{Ext}^i_B(S_2(M), K_B(t)) \quad \text{for } i = 1 \text{ (resp. } i = 2) \tag{12}$$

provided $\operatorname{depth}_{I(Z)} B \geq 3$ (resp. $\operatorname{depth}_{I(Z)} B \geq 4$). Indeed if $\operatorname{depth}_{I(Z)} B \geq 3$ say, we have

$$\operatorname{Ext}^1_B(M, M) \simeq \operatorname{Ext}^1_{\mathcal{O}_U}(\widetilde{M}|_U, \widetilde{M}|_U) \simeq \operatorname{Ext}^1_{\mathcal{O}_U}(\widetilde{M} \otimes \widetilde{M^*} \otimes \widetilde{K_B}|_U, \widetilde{K_B}|_U) \simeq \operatorname{Ext}^1_B(S_2(M), K_B(t))$$

where we have used $\widetilde{(M^* \otimes K_B)}|_U \simeq \widetilde{M(-t)}|_U$ which follows from $M \simeq \mathrm{Hom}_B(M, K_B)(t)$. Hence if $\mathrm{depth}_{I(Z)} B \geq 3$ and $char(k) \neq 2$, then $\gamma(S_2M)_v$ is given by

$$\gamma(S_2M)_v = \ {}_v\mathrm{hom}_B(M,M) - \ {}_v\mathrm{ext}^1_B(M,M) - \ \dim B_v.$$

If Q is a finitely generated B-module, we define $\delta(Q)_v$ as previously by

$$\delta(Q)_v = \ {}_v\mathrm{hom}_B(I_B/I_B^2, Q) - \ {}_v\mathrm{ext}^1_B(I_B/I_B^2, Q)\ .$$

If A is a quotient of B given by Theorem 8 and hence defined by (11), then a *stratum of quotients given by* (11) *around* (A) is of the form $p(W)$, $p : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}^{H_A}(R)$ the second projection, where W is a maximal closed subset of a small enough open affine neighbourhood of $(B \to A)$ in $\mathrm{GradAlg}(H_B, H_A)$, consisting of quotients $(B' \to A')$ for which there exists an extension $\xi' : 0 \to K_{B'}(t-2s) \to M'(-s) \to I_{A'/B'} \to 0$ which is induced by a regular section of some $\widetilde{M'}|_{U'}$ as in Theorem 8 ($\widetilde{M'}|_{U'}$ locally free and $\mathrm{Proj}(B') - U'$ of codimension at least two). Moreover if $(B_W \to A_W)$ is the pullback to W of the universal element of $\mathrm{GradAlg}(H_B, H_A)$ and $I_{A_W/B_W} = \ker(B_W \to A_W)$, then there is an element $\xi_W \in \mathrm{Ext}^1_{B_W}(I_{A_W/B_W}, K_{B_W}(t-2s))$ whose obvious pullbacks are ξ' above and the extension given by (11). Note that if W is *open* in $\mathrm{GradAlg}(H_B, H_A)$, then the stratum defined above coincides with the H_B-stratum of A at $(B \to A)$.

Theorem 25. *Let $B = R/I_B$ be a graded licci quotient of R, let M be a graded maximal Cohen-Macaulay B-module, and suppose $\widetilde{M}$ is locally free of rank 2 in $U := \mathrm{Proj}(B) - Z$, that $\dim B - \dim B/I(Z) \geq 2$ and $\wedge^2 \widetilde{M}|_U \simeq \widetilde{K_B}(t)|_U$. Let A be defined by a regular section σ of $\widetilde{M}^*(s)$ on U, i.e. given by* (11) *and suppose ${}_s\mathrm{Ext}^1_B(M, B) = 0$.*

A) *If* ${}_t\mathrm{Ext}^2_B(S_2(M), K_B) = 0$ *and* ${}_{-s}\mathrm{Ext}^2_B(I_B/I_B^2, M) = 0$,

then A is unobstructed as a graded R-algebra (indeed ${}_0\mathrm{H}^2(R, A, A) = 0$), A is Gorenstein, and

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) \ = \ \dim(N_B)_0 + \ \dim(M^*)_s - 1 - \gamma(S_2M)_0 + \ \dim(K_B)_{t-2s} + \delta(K_B)_{t-2s} - \delta(M)_{-s}.$$

Moreover if $char(k) = 0$ and $(B \to A)$ is general with respect to ${}_0\mathrm{hom}_R(I_B, I_{A/B})$, then the codimension of the H_B-stratum of A at $(B \to A)$ is

$${}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B}) = \ {}_{-s}\mathrm{ext}^1_B(I_B/I_B^2, M) - \dim(\mathrm{im}\,\beta)$$

where β is the homomorphism ${}_{-2s}\mathrm{Ext}^1_B(I_B/I_B^2, K_B(t)) \to {}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, M)$ induced by (11). *This codimension also equals the codimension of the stratum of quotients given by* (11) *around* (A).

B) *If (M, B) is unobstructed along any graded deformation of B and ${}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, M) = 0$, then A is Gorenstein and the stratum of quotients given by* (11) *around* (A) *is open in* $\mathrm{GradAlg}^{H_A}(R)$ *(so A is H_B-generic). Moreover A is unobstructed as a graded R-algebra and the dimension formula for $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R)$ of part* A) *holds.*

Remark 26. *i) Looking to the proof below we can weaken the assumption "B is a licci quotient" of Theorem 25B) to "B is a generically syzygetic unobstructed CM quotient satisfying ${}_0\mathrm{Ext}^2_B(I_B/I_B^2, K_B(t-2s)) = 0$", and conclude as in Theorem 25B).*

ii) The most natural vanishing condition for ${}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, M)$ in Theorem 25B) seems to be $s \geq \max n_{2,j} - a$ where $\max n_{2,j}$ is the largest degree of a minimal relation of I_B (cf. (4)*) and a is an integer which satisfies $M_v = 0$ for $v \leq a$. In this case we get ${}_{-s}\mathrm{Hom}_R(I_B, M) = {}_{-s}\mathrm{Ext}^1_R(I_B, M) = 0$ and hence $\delta(M)_{-s} = 0$. Since $K_B(t)_v = 0$ for $v \leq a - s$ by* (11) *we get ${}_{-2s}\mathrm{Ext}^i_R(I_B, K_B(t)) = 0$ for $i \leq 1$ and $\delta(K_B)_{t-2s} = 0$, as well as $\dim(K_B)_{t-2s} = 0$. It follows that*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \ \dim(N_B)_0 + \ \dim(M^*)_s - 1 - \gamma(S_2M)_0.$$

iii) Arguing as in ii) one shows ${}_{-v}\mathrm{Ext}^i_B(I_B/I_B^2, K_B) = 0$ *for* $i \le 1$ *and hence* $\delta(K_B)_{-v} = 0$ *provided* $v > \max n_{c,i} + \max n_{2,i} - n - c$, *e.g. provided* $v > 2\,\mathrm{reg}(I_B) - n$.

Remark 27. *If* $\dim B - \dim B/I(Z) \ge 4$ *we can replace* $\mathrm{GradAlg}^{H_A}(R)$ *by* $\mathrm{Hilb}^p(\mathbb{P})$ *in all conclusions, obviously modified (A by* $\mathrm{Proj}(A)$ *etc.), of Theorem 25 by* (8), *cf. Remark 49.*

Proof. A) We need to verify the assumptions of Corollary 18. Since B is licci and generically Gorenstein ($\widetilde{K_B}(t)|_U$ is locally free), one knows that $\mathrm{Ext}^1_B(I_B/I_B^2, B) = 0$ ([5]) and $\mathrm{Ext}^i_B(I_B/I_B^2, K_B) = 0$ for $i \ge 2$ ([31], Thm.4.2.6), and that I_B is syzygetic (see [23], proof of Thm. 16). It follows from Lemma 28 below and Remark 21 that all assumptions of Corollary 18 hold. To see that the dimension formulas of $\mathrm{GradAlg}^{H_A}(R)$ in Theorem 25 and Corollary 18 coincide, we use (11) and we get that $\delta(I_{A/B})_0 = \delta(M)_{-s} - \delta(K_B)_{t-2s}$ as well as

$$ {}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B}) = {}_{-s}\mathrm{ext}^1_B(I_B/I_B^2, M) - \dim(\mathrm{im}\,\beta). $$

Then we conclude by Lemma 28. For the final codimension statement, we refer to Lemma 29.

B) A generically Gorenstein licci quotient is generically syzygetic (cf. [23], proof of Thm. 16) and satisfies ${}_0\mathrm{Ext}^2_B(I_B/I_B^2, K_B(t-2s)) = 0$. Hence we get Theorem 25B) from Proposition 13(i) and Theorem 22, since the dimension formulas of Theorem 25 coincide with that in Theorem 22 by Lemma 28 and Lemma 29 (see part A) of the proof). □

Lemma 28. *Let B be Cohen-Macaulay, let M and Z be as in (the three first lines of) Theorem 25 and let* $A \simeq B/I_{A/B}$ *be given by* (11) *(so* $r = \dim B - \dim A = 2$ *and* $K_A \simeq A(j)$ *where* $j = 2s - t$*). If* ${}_s\mathrm{Ext}^1_B(M, B) = 0$, *then*

$$ {}_0\hom_B(I_{A/B}, A) = \dim(M^*)_s - 1 - \gamma(S_2M)_0 + \dim(K_B)_{t-2s}. $$

If in addition ${}_t\mathrm{Ext}^2_B(S_2(M), K_B) = 0$, *then* ${}_{-j}\mathrm{Ext}^2_B(S_2(I_{A/B}), K_B) = 0$, *i.e.* ${}_0\mathrm{H}^2(B, A, A) = 0$.

Proof. To show the dimension formula we remark that ${}_0\hom_B(I_{A/B}, A) = {}_{-j}\mathrm{ext}^1_B(S_2(I_{A/B}), K_B) - 1$ by Lemma 17. Moreover note that there exists an exact sequence

$$ M \otimes K_B(t-3s) \to S_2(M(-s)) \to S_2(I_{A/B}) \to 0 \tag{13} $$

whose leftmost map becomes injective if we sheafify and restrict to U. (Indeed $\wedge^2 \widetilde{K_B}|_U = 0$, or if we prefer, the morphism $U \cap \mathrm{Proj}(A) \hookrightarrow \mathrm{Proj}(B)$ is an l.c.i.). Let $MK := \ker(S_2(M(-s)) \to S_2(I_{A/B}))$ and $\wedge := \ker(M \otimes K_B(t-3s) \to MK)$. Since $\wedge$ is supported at Z, we get $\mathrm{Hom}_B(\wedge, K_B) = 0$. It follows that $\mathrm{Hom}_B(MK, K_B) \simeq \mathrm{Hom}_B(M \otimes K_B(t-3s), K_B)$ and that

$$ {}_{t-2s}\mathrm{Ext}^1_B(MK, K_B) \subseteq {}_{t-2s}\mathrm{Ext}^1_B(M \otimes K_B(t-3s), K_B) \simeq {}_0\mathrm{Ext}^1_B(M(-s), B) = 0. $$

Hence (13) induces a long exact sequence (i.e. exact in degree $t - 2s$)

$$ \ldots \to \mathrm{Hom}_B(M(t-3s), B) \to \mathrm{Ext}^1_B(S_2(I_{A/B}), K_B) \to \mathrm{Ext}^1_B(S_2(M(-s)), K_B) \to 0. \tag{14} $$

Since it is easy to see that the socle degree j of A is $j = 2s - t$ from (11), we get the conclusion by counting dimensions in (14) in degree $t - 2s$ and by observing that $\mathrm{Hom}_B(S_2(I_{A/B}), K_B) \simeq \mathrm{Hom}_B(I_{A/B}^2, K_B) \simeq \mathrm{Hom}_B(I_{A/B}, K_B) \simeq K_B$. Indeed these isomorphisms follow from the fact that $\ker(S_2(I_{A/B}) \to I_{A/B}^2)$ and $\mathrm{coker}(I_{A/B}^2 \hookrightarrow I_{A/B})$ are A-modules (see the proof of Lemma 17).

By Lemma 17, we get ${}_0\mathrm{H}^2(B, A, A) \simeq {}_{-j}\mathrm{Ext}^2_B(S_2(I_{A/B}), K_B)$. Now continuing the exact sequence (14) we get an injection ${}_{-j}\mathrm{Ext}^2_B(S_2(I_{A/B}), K_B) \hookrightarrow {}_{t-2s}\mathrm{Ext}^2_B(S_2(M(-s)), K_B)$ and we conclude easily. □

Lemma 29. *Let B be Cohen-Macaulay, let M and Z be as in (the three first lines of) Theorem 25 and let $A \simeq B/I_{A/B}$ be given by* (11) *(so $r = \dim B - \dim A = 2$ and $K_A \simeq A(2s-t)$). Then the H_B-stratum of A at $(B \to A)$ and the stratum of quotients given by* (11) *around (A) coincide. Moreover if ${}_0\mathrm{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0$, then the mentioned strata are open in* $\mathrm{GradAlg}^{H_A}(R)$.

Proof. To see that the H_B-stratum and the stratum of quotients given by (11) coincide, it suffices to show that any $(B' \to A')$ in a small enough open neighbourhood of $t := (B \to A)$ in $G := \mathrm{GradAlg}(H_B, H_A)$ is given by (11). Let (S, m_S) be the local ring of G at t and let $S_i = S/m_S^i$. Since G is a scheme of finite type which represents the corresponding functor of graded flat quotients, there exists a universal quotient whose pullback to $\mathrm{Spec}(S)$ is denoted by $B_S \to A_S$ (with kernel I_{A_S/B_S}, flat over S). Since $K_{B_S} := \mathrm{Ext}^c_{R_S}(B_S, R_S(-n-c))$ where $R_S := R \otimes_k S$, is S-flat (cf. [20], Prop. A1), it suffices to show that the natural map ${}_t\mathrm{Ext}^1_{B_S}(I_{A_S/B_S}, K_{B_S}(-2s)) \to {}_t\mathrm{Ext}^1_B(I_{A/B}, K_B(-2s))$ is surjective because then, there is an extension of I_{A_S/B_S} by $K_{B_S}(t-2s)$ over S, which reduces to the given extension of $I_{A/B}$ by $K_B(t-2s)$, and which extends to extensions in an open neighbourhood of t in G (the assumptions of Theorem 8 needed to get (11), i.e. $\widetilde{M'}|_{U'}$ is locally free and $\mathrm{Proj}(B') - U'$ is of codimension at least two is quite easy to get. Note that the maximal Cohen-Macaulayness of M' may be deduced from the extension). To show the surjectivity, let F_S be a graded B_S-free module which surjects into I_{A_S/B_S}, and let $Q_S := \ker(F_S \to I_{A_S/B_S})$. Let $Q_{S_i} := Q_S \otimes_S S_i$ and $Q := Q_S \otimes_S k$. A simple diagram chasing shows that it suffices to prove the surjectivity of ${}_t\mathrm{Hom}(Q_S, K_{B_S}(-2s)) \to {}_t\mathrm{Hom}(Q, K_B(-2s))$. As in the proof of [23], Prop. 13, it suffices to show that

$$\eta_i : {}_t\mathrm{Hom}(Q_{S_i}, K_{B_{S_i}}(-2s)) \to {}_t\mathrm{Hom}(Q_{S_{i-1}}, K_{B_{S_{i-1}}}(-2s))$$

is surjective. Since $\mathrm{Ext}^1_B(Q, K_B) \simeq \mathrm{Ext}^2_B(I_{A/B}, K_B) = 0$, this follows by applying ${}_{t-2s}\mathrm{Hom}(Q_{S_i}, -)$ onto $0 \to B \otimes_k \mathfrak{a} \to B_{S_i} \to B_{S_{i-1}} \to 0$ where $\mathfrak{a} := m_S^{i-1}/m_S^i$, i.e. we get the surjectivity of η_i from ${}_t\mathrm{Ext}^1_B(Q, K_B(-2s)) = 0$ and we get what we want. Finally note that by Prop. 4(ii) and (5) of [23] the assumption ${}_0\mathrm{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0$ implies that the second projection $\mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}^{H_A}(R)$ maps small enough open sets of $(B \to A)$ to open sets of (A) and we are done. □

Since we need some vanishing results on Ext-groups to use Theorem 25 effectively, we will delay specific examples to the next section. Instead we consider the rank 3 case which will allow us to treat certain Gorenstein families of codimension 5 or more. Here there are even more Ext-groups involved. Fortunately they all vanish for "good" modules on licci schemes. Now if $r = 3$, then Theorem 8 provides us with an exact sequence

$$0 \to K_B(t-3s) \to M^\vee(t-2s) \to M(-s) \to B \to A \to 0 \ , \tag{15}$$

where $M^\vee = \mathrm{Hom}_B(M, K_B)$. Let

$$\gamma(S_2M)_v = {}_v\mathrm{hom}_B(S_2(M), K_B(t)) - {}_v\mathrm{ext}^1_B(S_2(M), K_B(t)) + {}_v\mathrm{ext}^2_B(S_2(M), K_B(t))$$

(one more term than in the rank 2 case !), let $\gamma(M,M)_v = {}_v\mathrm{hom}_B(M,M) - {}_v\mathrm{ext}^1_B(M,M)$ while let $\delta(Q)_v = {}_v\mathrm{hom}_B(I_B/I_B^2, Q) - {}_v\mathrm{ext}^1_B(I_B/I_B^2, Q)$ be as previously.

Theorem 30. *Let $B = R/I_B$ be a graded licci quotient of R, let M be a graded maximal Cohen-Macaulay B-module, and suppose $\widetilde{M}$ is locally free of rank 3 in $U := \mathrm{Proj}(B) - Z$, that $\dim B - \dim B/I(Z) \geq 3$ and $\wedge^3 \widetilde{M}|_U \simeq \widetilde{K_B}(t)|_U$. Let A be defined by a regular section σ of $\widetilde{M}^*(s)$ on U, i.e. given by* (15) *and suppose ${}_s\mathrm{Ext}^1_B(M, B) = 0$.*

A) *If ${}_0\mathrm{Ext}^2_B(M^\vee \otimes M, K_B) = {}_{-s}\mathrm{Ext}^3_B(S_2(M), K_B(t)) = 0$ and ${}_{-s}\mathrm{Ext}^2_B(I_B/I_B^2, M) =$*

${}_{-2s}\mathrm{Ext}^3_B(I_B/I_B^2, M^\vee(t)) = 0$, *then A is unobstructed as a graded R-algebra (indeed ${}_0\mathrm{H}^2(R,A,A) = 0$), A is Gorenstein, and*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim(N_B)_0 + \dim(M^*)_s + \gamma(S_2M)_{-s} - \gamma(M,M)_0 - \dim(K_B)_{t-3s} - \delta \ ,$$

where

$$\delta := \delta(M)_{-s} + \delta(K_B)_{t-3s} - \delta(M^\vee)_{t-2s} - {}_{-2s}\,\mathrm{ext}^2_B(I_B/I_B^2, M^\vee(t)).$$

Moreover if char$(k) = 0$ and $(B \to A)$ is general with respect to ${}_0\hom_R(I_B, I_{A/B})$, then the codimension of the H_B-stratum of A at $(B \to A)$ is

$${}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B}) = {}_{-s}\mathrm{ext}^1_B(I_B/I_B^2, M) + {}_{-2s}\,\mathrm{ext}^2_B(I_B/I_B^2, M^\vee(t)) - \dim(\mathrm{im}\,\beta)$$

where β is the homomorphism ${}_{-2s}\mathrm{Ext}^1_B(I_B/I_B^2, M^\vee(t)) \to {}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, M)$ induced by (15).

B) *If (M,B) is unobstructed along any graded deformation of B and ${}_{-s}\mathrm{Ext}^2_B(M, M^\vee(t)) = {}_{-s}\mathrm{Ext}^2_B(S_2(M), K_B(t)) = 0$ and ${}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, M) = {}_{-2s}\mathrm{Ext}^2_B(I_B/I_B^2, M^\vee(t)) = 0$, then A is Gorenstein and H_B-generic. Moreover A is unobstructed as a graded R-algebra and the dimension formula for $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R)$ of part* A) *holds (this formula simplifies a little due to the assumed vanishing of the* Ext*-groups).*

Remark 31. *i) Using* (3) *one may see that ${}_{-s}\mathrm{Ext}^2_B(M, M^\vee(t)) \simeq {}_{-s}\mathrm{Ext}^2_B(S_2(M), K_B(t))$ provided* $\mathrm{depth}_{I(Z)} B \geq 4$ *and char$(k) \neq 2$, cf. the assumptions of Theorem 30B).*

ii) Moreover one may replace "${}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, M) = {}_{-2s}\mathrm{Ext}^2_B(I_B/I_B^2, M^\vee(t)) = 0$" in Theorem 30B) by

$${}_{-s}\mathrm{Ext}^1_R(I_B, M) = {}_{-2s}\mathrm{Ext}^2_R(I_B, M^\vee(t)) = 0$$

and still conclude as in Theorem 30B). This follows from Theorem 22. This variation is particularly useful if the codimension of B in R is 2, in which case $\mathrm{Ext}^2_R(I_B, M^\vee(t))$ vanishes. A natural vanishing condition for ${}_{-s}\mathrm{Ext}^1_R(I_B, M)$ is again $s \geq \max n_{2,j} - a$ where $\max n_{2,j}$ is the largest degree of a minimal relation of I_B and a is an integer which satisfies $M_v = 0$ for $v \leq a$. In this case we get ${}_{-s}\mathrm{Ext}^i_R(I_B, M) = 0$ for $i = 0,1$ and hence $\delta(M)_{-s} = 0$.

Lemma 32. *Let B be Cohen-Macaulay, let M and Z be as in (the three first lines of) Theorem 30 and let $A \simeq B/I_{A/B}$ be given by* (15) *(so $r = \dim B - \dim A = 3$ and $K_A \simeq A(j)$ where $j = 3s - t$) and suppose ${}_s\mathrm{Ext}^1_B(M, B) = 0$.*

A) *If ${}_0\mathrm{Ext}^2_B(M^\vee \otimes M, K_B) = 0$, then*

$${}_0\hom_B(I_{A/B}, A) = \dim(M^*)_s + \gamma(S_2M)_{-s} - \gamma(M,M)_0 - \dim(K_B)_{t-3s}$$

If in addition ${}_{-s}\mathrm{Ext}^3_B(S_2(M), K_B(t)) = 0$, then ${}_{-j}\mathrm{Ext}^3_B(S_2(I_{A/B}), K_B) \simeq {}_0\mathrm{H}^2(B,A,A) = 0$.

B) *If ${}_{-s}\mathrm{Ext}^2_B(S_2(M), K_B(t)) = 0$ then ${}_0\hom_B(I_{A/B}, A)$ is given as in part* A).

Proof. A) Thanks to Lemma 17 we have ${}_0\hom_B(I_{A/B}, A) = {}_{-j}\mathrm{ext}^2_B(S_2(I_{A/B}), K_B) - 1$. To compute ${}_{-j}\mathrm{ext}^2_B(S_2(I_{A/B}), K_B)$, we look to the exact sequence

$$M^\vee \otimes M(t-3s) \to S_2(M(-s)) \to S_2(I_{A/B}) \to 0 \qquad (16)$$

where one knows that the kernel of $\widetilde{M^\vee \otimes M}(t-3s)|_U \to \widetilde{S_2(M)}(-2s)|_U$ is $(\wedge^2\widetilde{M^\vee}(2t-4s) \oplus \widetilde{K_B}(t-3s))|_U$ $(U \cap \mathrm{Proj}(A) \hookrightarrow \mathrm{Proj}(B)$ is an l.c.i.). Let $M^\vee M := \ker(S_2(M(-s)) \to S_2(I_{A/B}))$. Since $\wedge^2\widetilde{M^\vee}|_U \simeq \widetilde{M} \otimes \widetilde{K_B}(-t)|_U$, we have an exact sequence

$$0 \to (\widetilde{K_B}(t-3s) \oplus (\widetilde{M} \otimes \widetilde{K_B})(t-4s))|_U \to \widetilde{M^\vee \otimes M}(t-3s)|_U \to \widetilde{M^\vee M}|_U \to 0$$

to which we apply $\mathrm{Hom}(-,\widetilde{K_B}|_U)$. Since $\mathrm{Ext}^i_B(-,N) \simeq \mathrm{Ext}^i_{\mathcal{O}_U}(\widetilde{-}|_U,\widetilde{N}|_U)$ for $i \le 1$ for every maximal CM module N by (3), we get a long exact sequence starting as

$$0 \to \mathrm{Hom}_B(M^\vee M, K_B(-j)) \to \mathrm{Hom}_B(M^\vee \otimes M(t-3s), K_B(-j)) \to \mathrm{Hom}_B(K_B,K_B) \oplus H \to ...$$

where $H := \mathrm{Hom}_B(M \otimes K_B(t-4s), K_B(-j)) \simeq \mathrm{Hom}_B(M(-s),B)$ and stopping at ${}_0\mathrm{Ext}^1_B(M \otimes K_B(t-4s),K_B(-j)) \simeq {}_s\mathrm{Ext}^1_B(M,B) = 0$. Noting that

$${}_0\mathrm{Ext}^i_B(M^\vee \otimes M, K_B) \simeq \mathrm{Ext}^i_{\mathcal{O}_U}(\widetilde{M}^* \otimes \widetilde{K_B} \otimes \widetilde{M}|_U, \widetilde{K_B}|_U) \simeq {}_0\mathrm{Ext}^i_B(M,M) \quad \text{for} \quad i \le 1,$$

we get

$$\sum_{i=0}^{1}(-1)^i{}_{-j}\mathrm{ext}^i_B(M^\vee M, K_B) = \sum_{i=0}^{1}(-1)^i{}_0\mathrm{ext}^i_B(M,M) - 1 - {}_s\mathrm{hom}_B(M,B)\ .$$

Note that the long exact sequence above and the assumption ${}_0\mathrm{Ext}^2_B(M^\vee \otimes M, K_B) = 0$ also show ${}_0\mathrm{Ext}^2_B(M^\vee M, K_B) = 0$. Now combining with the long exact sequence which we get by applying ${}_0\mathrm{Hom}(-,K_B(-j)) = 0$ to $0 \to M^\vee M \to S_2(M)(-2s) \to S_2(I_{A/B}) \to 0$;

$$... \to {}_0\mathrm{Ext}^1_B(M^\vee M, K_B(-j)) \to {}_0\mathrm{Ext}^2_B(S_2(I_{A/B}),K_B(-j)) \to {}_0\mathrm{Ext}^2_B(S_2(M)(-2s),K_B(-j)) \to 0 \quad (17)$$

which implies

$$\sum_{i=0}^{2}(-1)^i{}_{-j}\mathrm{ext}^i_B(S_2(I_{A/B}),K_B) = \gamma(S_2M)_{-s} - \sum_{i=0}^{1}(-1)^i{}_{-j}\mathrm{ext}^i_B(M^\vee M, K_B)\ ,$$

we get the dimension formula because

$$\mathrm{Hom}_B(S_2(I_{A/B}),K_B) \ \simeq\ \mathrm{Hom}_B(I^2_{A/B},K_B) \ \simeq\ \mathrm{Hom}_B(I_{A/B},K_B) \simeq K_B$$

and $\mathrm{Ext}^1_B(S_2(I_{A/B}),K_B) \simeq \mathrm{Ext}^1_B(I_{A/B},K_B) = 0$ ($\ker(S_2(I_{A/B}) \to I^2_{A/B})$ and $\mathrm{coker}(I^2_{A/B} \hookrightarrow I_{A/B})$ are A-modules). Part B) is proven in exactly the same way (the sequence (17) stops by "one module earlier"). Moreover continuing (17), we get the "if in addition" statement of part A) from the assumption ${}_{-s}\mathrm{Ext}^3_B(S_2(M),K_B(t)) = 0$. ☐

Proof of Theorem 30. A) follows from Corollary 18, Remark 21 and Lemma 32 because B is licci and generically Gorenstein (cf. the proof of Theorem 25 for details). Note that we need to split (15) into two short exact sequences, and apply ${}_0\mathrm{Hom}_B(I_B/I_B^2,-)$ to them, to see the formula of δ in Theorem 30 (δ obviously equals $\delta(I_{A/B})_0$ by definition of the latter). The same splitting, together with ${}_{-s}\mathrm{Ext}^2_B(I_B/I_B^2,M) = 0$ and ${}_{-3s}\mathrm{Ext}^i_B(I_B/I_B^2,K_B(t)) = 0$ for $i > 1$, shows that

$${}_0\mathrm{ext}^1_B(I_B/I_B^2,I_{A/B}) = {}_{-s}\mathrm{ext}^1_B(I_B/I_B^2,M) + {}_{-2s}\mathrm{ext}^2_B(I_B/I_B^2,M^\vee(t)) - \dim(\mathrm{im}\,\beta),$$

and we get the codimension statement.

B) We get Theorem 30B) from Proposition 13(i), Theorem 22 and Lemma 32 (see part A) of the proof for the dimension formula). ☐

Remark 33. *If we in Theorem 25A) and Theorem 30A) replace the vanishing of all* ${}_v\mathrm{Ext}^i_B$*-groups by the vanishing of the corresponding* Ext^i_B*-groups (skipping the index v), we get that A is strongly unobstructed in the sense* $\mathrm{H}^2(R,A,A) = 0$ *(cf. Remark 19).*

3 Rank two and three sheaves on codimension two quotients

In this section we will see that Theorem 25 applies to the normal module, $M = N_B$ and Theorem 25 and 30 to the 1. Koszul homology module, $M = H_1$. Of course Theorem 25 also applies to modules of the form $M = S_2(K_B) \oplus K_B^*(t)$, where t is an integer. For such decomposable modules Theorem 25 does not really lead to new results since they may be treated by applying Theorem 23.

Before giving our applications we need two propositions to handle the vanishing of the Ext-groups involved. In what follows, $B = R/I$ is an n-dimensional codimension two CM quotient of R,

$$0 \to G_2 := \oplus_{j=1}^{\mu-1} R(-n_{2,j}) \to G_1 := \oplus_{i=1}^{\mu} R(-n_{1,i}) \to I \to 0 \tag{18}$$

is a minimal resolution and $Y = \operatorname{Proj}(B)$ is an l.c.i. in an open set $U = Y - Z$ where $\operatorname{depth}_{I(Z)} B \geq 1$. Taking R-duals, we get a minimal resolution

$$0 \to R \to \oplus R(n_{1,i}) \to \oplus R(n_{2,j}) \to K_B(n+2) \to 0 \tag{19}$$

to whom we apply $\operatorname{Hom}(-, B)$ to see the exactness to the left in the exact sequence

$$0 \to K_B(n+2)^* \to \oplus B(-n_{2,j}) \to \oplus B(-n_{1,i}) \to I/I^2 \to 0 \tag{20}$$

Note that (20) splits into two short exact sequences "via $\oplus B(-n_{2,j}) \twoheadrightarrow H_1 \hookrightarrow \oplus B(-n_{1,i})$", one of which is (6) with $\mathrm{H}_2(R, B, B) = 0$.

Proposition 34. *Let l be a natural number, let $B = R/I$ be a codimension two CM quotient of R, and suppose* $\operatorname{depth}_{I(Z)} B \geq l$. *Then*

$$\operatorname{Ext}_B^i(I/I^2, I/I^2) = 0 \quad \text{for } 1 \leq i \leq l.$$

Proof. Since $pd_R I = 1$, we have $\operatorname{Ext}_R^i(I, I/I^2) = 0$ and $\operatorname{Tor}_i^R(I, B) = 0$ for $i \geq 2$. Moreover, using the spectral sequence $\operatorname{Ext}_B^p(\operatorname{Tor}_q^R(I, B), I/I^2)$ which converges to $\operatorname{Ext}_R^{p+q}(I, I/I^2)$, we get isomorphisms $\operatorname{Ext}_B^{i-2}(\operatorname{Tor}_1^R(I, B), I/I^2) \simeq \operatorname{Ext}_B^i(I/I^2, I/I^2)$ for $i > 2$ and an exact sequence

$$0 \to \operatorname{Ext}_B^1(I/I^2, I/I^2) \to \operatorname{Ext}_R^1(I, I/I^2) \to \operatorname{Hom}_B(\operatorname{Tor}_1^R(I, B), I/I^2) \to \operatorname{Ext}_B^2(I/I^2, I/I^2) \to 0.$$

Note that $\operatorname{Ext}_R^1(I, -)$ is right exact by $pd_R I = 1$. In particular $\operatorname{Ext}_R^1(I, R) \simeq \operatorname{Ext}_R^1(I, A)$ and it follows that $N_B \simeq \operatorname{Ext}_R^1(I, I) \simeq \operatorname{Ext}_R^1(I, R) \otimes I \simeq K_B(n+2) \otimes I/I^2$. Similarly we get

$$\operatorname{Ext}_R^1(I, I/I^2) \simeq \operatorname{Ext}_R^1(I, R) \otimes I/I^2 \simeq N_B\ .$$

Hence there is an injection $\mathrm{H}_{I(Z)}^0(\operatorname{Ext}_B^1(I/I^2, I/I^2)) \hookrightarrow \mathrm{H}_{I(Z)}^0(N_B)$. Since $\widetilde{I/I^2}|_U$ is locally free, we get $\mathrm{H}_{I(Z)}^0(\operatorname{Ext}_B^1(I/I^2, I/I^2)) \simeq \operatorname{Ext}_B^1(I/I^2, I/I^2)$. Moreover one knows that N_B is a maximal CM module (cf. the text after (6)), and we conclude that $\operatorname{Ext}_B^1(I/I^2, I/I^2) = 0$.

Let $p \leq l$ be a natural number and suppose we have proved $\operatorname{Ext}_B^i(I/I^2, I/I^2) = 0$ for $1 \leq i < p$. By the spectral sequence (resp. the two displayed formulas) above, it suffices to show $\operatorname{Ext}_B^{p-2}(\operatorname{Tor}_1^R(I, B), I/I^2) = 0$ for $p > 2$ (resp. $\operatorname{Ext}_B^{p-2}(\operatorname{Tor}_1^R(I, B), I/I^2) \hookrightarrow N_B$ a graded injection for $p = 2$). We have $\mathrm{H}_{I(Z)}^i(I/I^2) = 0$ for $i \leq p-2$ by (6). Hence we get an injective graded map

$$\operatorname{Ext}_B^{p-2}(\operatorname{Tor}_1^R(I, B), I/I^2) \hookrightarrow \operatorname{Ext}_{\mathcal{O}_U}^{p-2}(\widetilde{\operatorname{Tor}_1^R(I, B)}|_U, \widetilde{I/I^2}|_U) \simeq \operatorname{Ext}_{\mathcal{O}_U}^{p-2}(\widetilde{\operatorname{Tor}_1^R(I, K_B(n+2))}|_U, \widetilde{N_B}|_U), \tag{21}$$

(cf. (3)), noting that $\widetilde{\operatorname{Tor}_1^R(I, K_B)} \simeq \widetilde{\operatorname{Tor}_1^R(I, B)} \otimes K_B$ and $\widetilde{I/I^2 \otimes K_B} \simeq \widetilde{N_B}$ are isomorphic on U. Since $\mathrm{H}_{I(Z)}^i(N_B) = 0$ for $i \leq p-1$, the rightmost Ext-group in (21) is, by (3) mainly, further isomorphic to $\mathrm{Ext}_B^{p-2}(\mathrm{Tor}_1^R(I, K_B(n+2)), N_B)$. We *claim* that $\mathrm{Tor}_1^R(I, K_B(n+2)) \simeq R/I$. Indeed we have by (19) that $\mathrm{Tor}_1^R(I, K_B(n+2))$ is the homology group in the middle of the complex

$$0 \to I \to \oplus I(n_{1,i}) \to \oplus I(n_{2,j}) \to 0.$$

Applying $\mathrm{Hom}_R(-, I)$ to (18), we see that $\ker[\oplus I(n_{1,i}) \to \oplus I(n_{2,j})] \simeq \mathrm{Hom}(I, I) \simeq R$ and putting things together we get the claim. Now using the claim we get $\mathrm{Ext}_B^{p-2}(\mathrm{Tor}_1^R(I, B), I/I^2) \simeq \mathrm{Ext}_B^{p-2}(R/I, N_B)$ which vanishes for $p > 2$ and equals N_B for $p = 2$, and we are done. □

Remark 35. *Applying* $\mathrm{Hom}_R(-, I/I^2)$ *to* (18)*, noting that* $\mathrm{Hom}_R(I, I/I^2) \simeq \mathrm{Hom}_B(I/I^2, I/I^2)$ *and* $\mathrm{Ext}_R^1(I, I/I^2) \simeq N_B$ *(see the proof above) we get the following exact sequence*

$$0 \to \mathrm{Hom}_B(I/I^2, I/I^2) \to \oplus I/I^2(n_{1,i}) \to \oplus I/I^2(n_{2,j}) \to N_B \to 0.$$

This sequence of graded B-modules can be used to find ${}_v\mathrm{hom}_B(I/I^2, I/I^2)$ *because* $\dim(I^2)_v$ *and* $\dim(N_B)_v$ *can be computed from*

$$0 \to \wedge^2(\oplus R(-n_{2,j})) \to (\oplus R(-n_{1,i})) \otimes (\oplus R(-n_{2,j})) \to S_2(\oplus R(-n_{1,i})) \to I^2 \to 0\ , \tag{22}$$

$$0 \to G_1^* \otimes_R G_2 \to ((G_1^* \otimes_R G_1) \oplus (G_2^* \otimes_R G_2))/R \to G_2^* \otimes_R G_1 \to N_B \to 0 \tag{23}$$

Indeed the latter sequence is deduced from the exact sequence $0 \to R \to \oplus I(n_{1,i}) \to \oplus I(n_{2,j}) \to N_B \to 0$ *which we get by applying* $\mathrm{Hom}_R(-, I)$ *to* (18)*, (cf. [23], (26)). Note that if* $s > \max n_{2,j} - 2\min n_{1,i}$*, then* $I^2(n_{i,j})_{-s} = 0$ *for all* i, j *and we get* ${}_{-s}\mathrm{Hom}_B(I/I^2, I/I^2) \simeq R_{-s}$*.*

Now we consider the 1.Koszul homology module H_1. In this case all assumptions of Theorem 25A) are satisfied, due to

Proposition 36. *Let $B = R/I$ be a codimension two CM quotient of R, and suppose* $\mathrm{depth}_{I(Z)} B \geq 2$*. Then $S_2(H_1)$ is a maximal CM B-module. Moreover*

$$\mathrm{Ext}_B^1(H_1, H_1) \simeq \mathrm{Ext}_B^2(I/I^2, H_1) = 0 \quad \text{and} \quad \mathrm{Ext}_B^1(H_1, B) = 0.$$

Proof. In the sequence (6), $\mathrm{H}_2(R, B, B) = 0$. Applying $\mathrm{Hom}(I/I^2, -)$ to (6) we get isomorphisms $\mathrm{Ext}_B^1(I/I^2, I/I^2) \simeq \mathrm{Ext}_B^2(I/I^2, H_1)$ because $\mathrm{Ext}_B^i(I/I^2, B) = 0$ for $1 \leq i \leq 2$ ([27], (7)). Hence $\mathrm{Ext}_B^2(I/I^2, H_1) = 0$ by Proposition 34. Moreover applying $\mathrm{Hom}(-, B)$ (resp. $\mathrm{Hom}(-, H_1)$) to (6) we get $\mathrm{Ext}_B^1(H_1, B) \simeq \mathrm{Ext}_B^2(I/I^2, B) = 0$ (resp. $\mathrm{Ext}_B^1(H_1, H_1) \simeq \mathrm{Ext}_B^2(I/I^2, H_1)$). Finally to see that $S_2(H_1)$ is a maximal CM module, we consider the short exact sequence deduced from (20) "to the left". It induces an exact sequence

$$0 \to K_B(n+2)^* \otimes (\oplus B(-n_{2,j})) \xrightarrow{\psi} S_2(\oplus B(-n_{2,j})) \to S_2(H_1) \to 0 \tag{24}$$

because $\ker \psi$, which is supported at Z and contained in a maximal CM module, must vanish. It follows from (24) that $S_2(H_1)$ has codepth at most one. Moreover dualizing (24) we get

$$0 \to \mathrm{Hom}_B(S_2(H_1), K_B) \to K_B \otimes S_2(\oplus B(n_{2,j})) \to \oplus S_2(K_B)(n_{2,j} + n + 2) \tag{25}$$

because $\mathrm{Hom}_B(S_2(\oplus B(-n_{2,j})), K_B) \simeq K_B \otimes S_2(\oplus B(n_{2,j}))$ and

$$\mathrm{Hom}_B(K_B(n+2)^* \otimes (\oplus B(-n_{2,j})), K_B) \simeq \oplus S_2(K_B)(n_{2,j} + n + 2).$$

Note that the cokernel of $K_B \otimes S_2(\oplus B(n_{2,j})) \to \oplus S_2(K_B)(n_{2,j} + n + 2)$ is $\mathrm{Ext}_B^1(S_2(H_1), K_B)$. One may, however, easily see that this map is surjective because it is, via symmetrization and tensorization, obtained from the surjective map $\oplus B(n_{2,j}) \to K_B(n+2)$. Hence $\mathrm{Ext}_B^1(S_2(H_1), K_B) = 0$, and by Gorenstein duality and the fact that the codepth of $S_2(H_1)$ is at most one, we get that $S_2(H_1)$ is maximally Cohen-Macaulay. □

If the number of minimal generators of I is $\mu(I) = 4$, then the rank of H_1 is $r = 2$ by (6) and Theorem 25A) applies to B-module $M = H_1$. Since all assumptions of Theorem 8 are satisfied ([27] and Proposition 10) we have an exact sequence

$$0 \to K_B(t-2s) \to H_1(-s) \to I_{A/B} \to 0 \tag{26}$$

with $t = n + 2 - \sum n_{1,i}$. Hence

Corollary 37. *Let $B = R/I$ be a graded codimension two CM quotient of R, let $U = \operatorname{Proj}(B) - Z \hookrightarrow \mathbb{P}^{n+1}$ be an l.c.i. and suppose $\mu(I) = 4$ and $\operatorname{depth}_{I(Z)} B \geq 2$. If A is defined by a regular section of $\widetilde{H_1}^*(s)$ on U, i.e. given by (26), then A is unobstructed as a graded R-algebra (indeed ${}_0\mathrm{H}^2(R, A, A) = 0$), A is Gorenstein of codimension 4 in R, and*

$$\dim_{(A)} \operatorname{GradAlg}(H_A) = \dim(N_B)_0 + \dim(H_1^*)_s - 1 - \gamma(S_2H_1)_0 + \dim(K_B)_{t-2s} + \delta(K_B)_{t-2s} - \delta(H_1)_{-s}.$$

Moreover if $char(k) = 0$ and $(B \to A)$ is general with respect to ${}_0\hom_R(I_B, I_{A/B})$, then the stratum of quotients given by (26) around (A) is irreducible and its codimension is ${}_{-s}\mathrm{ext}^1_B(I/I^2, H_1) - \dim(\operatorname{im}\beta)$ where β is the homomorphism ${}_{-2s}\mathrm{Ext}^1_B(I/I^2, K_B(t)) \to {}_{-s}\mathrm{Ext}^1_B(I/I^2, H_1)$ induced by (26). Furthermore if $(B') \in \operatorname{GradAlg}^{H_B}(R)$ satisfies the same assumptions as B above and defines A' as B defined A, then the closures in $\operatorname{GradAlg}^{H_A}(R)$ of the stratum of quotients given by (26) around (A) and the corresponding stratum around (A') coincide. If in addition $s > \max n_{2,j} - \min n_{2,j}$, then the stratum of quotients given by (26) around (A) is open (so A is H_B-generic), and

$$\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = \dim(N_B)_0 + \dim(H_1^*)_s - 1 - {}_0\hom_B(S_2(H_1), K_B(t)) .$$

Note that $\dim(H_1^*)_s$ is easily computed from the exact sequence

$$0 \to N_B \to \oplus_{i=1}^{\mu} B(n_{1,i}) \to H_1^* \to 0 \tag{27}$$

(the B-dual sequence of (6) is short exact due to $\mathrm{Ext}^1_B(I/I^2, B) = 0$). Moreover $\gamma(S_2H_1)_0 = {}_0\hom_B(S_2(H_1), K_B(t))$ is computed from (25) and the minimal resolution ([7], p. 595),

$$0 \to \wedge^2(\oplus R(n_{1,i})) \to (\oplus R(n_{1,i})) \otimes (\oplus R(n_{2,j})) \to S_2(\oplus R(n_{2,j})) \to S_2(K_B)(2n+4) \to 0 \tag{28}$$

Proof. Since all assumptions of Theorem 25A) are taken care of by Proposition 36, we get Corollary 37 from Theorem 25A) except the irreducibility of the stratum, its uniqueness (i.e. that the strata above coincide, up to closure) and the final statement of Corollary 37. The irreducibility is trivial because the stratum is the image of an irreducible set. Indeed $\operatorname{GradAlg}(H_B, H_A)$ is smooth at $(B \to A)$ because B is unobstructed and ${}_0\mathrm{H}^2(B, A, A) = 0$ by Lemma 28. The proof of the uniqueness is essentially the same as for [23], Prop. 23(i) and Thm. 24 (see the two first lines of the proof of [23], Thm. 24) because the open subscheme of $\operatorname{GradAlg}^{H_B}(R)$ consisting of CM quotients is irreducible. Moreover, since $\oplus B(-n_{2,i}) \twoheadrightarrow H_1$ is surjective (cf. (20)) we have $(H_1)_v = 0$ for $v < \min n_{2,j}$. By Remark 26(ii) and the assumption $s > \max n_{2,j} - \min n_{2,j}$, it follows that ${}_{-s}\mathrm{Ext}^1_B(I/I^2, H_1) = 0$. Hence A is H_B-generic and the formula for $\dim_{(A)} \operatorname{GradAlg}^{H_A}(R)$ simplifies as in Corollary 37 according to Remark 26(ii). □

We illustrate Corollary 37 by a example in which we compute all numbers in the dimension formula of $\dim_{(A)} \operatorname{GradAlg}^{H_A}(R)$ as well as the number leading to the codimension of the H_B-stratum. Even though the example is relatively simple, our methods of computation may be used quite generally (to treat sections of $H_1^*(s)$ on a codimension 2 CM quotient of R generated by $\mu(I) = 4$ generators).

Example 38. *It is well known how to find a minimal resolution of H_1. Indeed by [2],*

$$0 \to \wedge^2 G_2 \to \wedge^2 G_1 \to G_2 \to H_1 \to 0 \tag{29}$$

where the free modules G_i belong to (18). *Applying the mapping cone construction to* (26) *we get the following resolution of the Gorenstein algebra $A = R/I_A$ of Corollary 37,*

$$0 \to R(b) \to G_1^*(b) \oplus \wedge^2 G_2(-s) \to G_2^*(b) \oplus \wedge^2 G_1(-s) \oplus G_2 \to G_2(-s) \oplus G_1 \to I_A \to 0 \tag{30}$$

where $b := -\sum n_{1,i} - 2s$ and $\mu(I) = 4$. As a special case we suppose the resolution (18) *is linear (i.e. all $n_{1,i} = 3$). We get the following resolution (obviously minimal for $s > 0$) of A,*

$$\begin{aligned} &0 \to R(-12-2s) \to R(-9-2s)^4 \oplus R(-8-s)^8 \\ &\to R(-8-2s)^3 \oplus R(-6-s)^6 \oplus R(-4)^3 \to R(-4-s)^3 \oplus R(-3)^4 \to R \to A \to 0 \ . \end{aligned} \tag{31}$$

Let R be a polynomial ring in four variables. Then $Y = \mathrm{Proj}(B)$ is a curve of degree $d = 6$ and with Hilbert polynomial $p_Y(v) = 6v - 2$. Moreover A is Artinian of socle degree $2s+8$ and with h-vector $(1, 4, 10, .., 6s+16, 6s+19, 6s+16, .., 10, 4, 1)$. We suppose $s \geq -1$ to avoid discussing very special cases. If Y is an l.c.i., then A is unobstructed by Corollary 37 of a dimension which we now calculate. Indeed $\dim(N_B)_0 = h^0(\widetilde{N_B}) = 4d = 24$ while, by (27),

$$\dim(H_1^*)_s = 4 \cdot p_Y(s+3) - h^0(\widetilde{N_B(s)}) = 24(s+3) - 8 - 4d - 2ds = 12s + 40$$

for $s \geq -1$ (one may see that $(N_B)_s = 12 = 4d + 2ds$ also for $s = -1$ by (23)*). Moreover*

$${}_0\mathrm{hom}_B(S_2(H_1), K_B(-8)) = 6\dim(K_B)_0 - 3\dim(S_2(K_B)_0) = 0$$

by (25), (19) *and* (28). *Hence by Corollary 37, if $s > 0$, then $\dim_{(A)} \mathrm{PGor}(H_A) =$*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim(N_B)_0 + \dim(H_1^*)_s - 1 - {}_0\mathrm{hom}_B(S_2(H_1), K_B(-8)) = 12s + 63$$

and A is H_B-generic, i.e. the closure of the family given by (26) *forms a $(12s+63)$-dimensional generically smooth, irreducible component of $\mathrm{GradAlg}^{H_A}(R)$. It is interesting to observe that we can also use Lemma 17(iii) to compute ${}_0\mathrm{hom}_B(I_{A/B}, A)$ and hence $\dim(H_1^*)_s - 1 - {}_0\mathrm{hom}_B(S_2(H_1), K_B(-8))$ (cf. Lemma 28). Indeed since we get $\dim S_2(I_{A/B})_{2s+8} = 6$ from* (31), *it follows that*

$${}_0\mathrm{hom}_B(I_{A/B}, A) = \dim B_{2s+8} - \dim S_2(I_{A/B})_{2s+8} = 12s + 40$$

which again leads to $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = 12s + 63$.

Finally suppose $s \leq 0$. We still have $\delta(K_B)_{-8-2s} = 0$ and ${}_{-2s}\mathrm{Ext}^1_B(I/I^2, K_B(-8)) = 0$ for $s > -2$ by Remark 26(iii) while $(K_B)_{-8-2s} = 0$ by (19). *It remains to compute $\delta(H_1)_{-s}$ and ${}_{-s}\mathrm{ext}^1_B(I/I^2, H_1)$. To do so we apply $\mathrm{Hom}(I/I^2, -)$ to* (6), *and we get*

$$0 \to \mathrm{Hom}(I/I^2, H_1) \to \mathrm{Hom}(I/I^2, \oplus B(-n_{1,i})) \to \mathrm{Hom}(I/I^2, I/I^2) \to \mathrm{Ext}^1_B(I/I^2, H_1) \to 0 \tag{32}$$

Note that $(N_B)_v = 0$ for $v < -1$ by (23). *Hence*

$${}_{-s}\mathrm{ext}^1_B(I/I^2, H_1) = {}_{-s}\mathrm{hom}(I/I^2, I/I^2) = \dim R_{-s}$$

and $\delta(H_1)_{-s} = -{}_{-s}\mathrm{ext}^1_B(I/I^2, H_1)$ for $-1 \leq s \leq 0$ by Remark 35. By Corollary 37,

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = 12s + 63 + {}_{-s}\mathrm{ext}^1_B(I/I^2, H_1) \quad \text{for} \quad -1 \leq s \leq 0,$$

and the codimension, ${}_{-s}\mathrm{ext}^1_B(I/I^2, H_1)$, of the H_B-stratum of A is $1, 4$ for $s = 0, -1$ respectively.

Remark 39. *In view of Remark 35, we can always use* (32) *to find* $\delta(H_1)_{-s}$. *Moreover* $\delta(K_B)_{t-2s}$ *is always computable from the exact sequence we get by applying* $\operatorname{Hom}(-,K_B(t))$ *to* (6), *cf. [23], Remark 14(d). Since* $\dim(K_B)_{t-2s}$ *is given by* (19), *we see that all members of the dimension formula for* $\operatorname{GradAlg}^{H_A}(R)$ *in Corollary 37 are easily computed, even by hand. On the other side, the formula for the codimension is not always straightforward. However, in the range* $\max n_{2,j} - 2\min n_{1,i} < s \le \max n_{2,j} - \min n_{2,j}$, *one may show*

$$ {}_{-s}\mathrm{ext}^1_B(I/I^2, H_1) = -\delta(H_1)_{-s} = \dim R_{-s}. $$

Indeed by (32) *it suffices to show* $N_B(-n_{1,i})_{-s} = 0$ *for every* i. *Since* $\oplus R(n_{2,j} - n_{1,i'}) \to N_B$ *is surjective and* $\oplus R(n_{2,j} - n_{1,i'} - n_{1,i})_{-s} = 0$ *we conclude by Remark 35. Finally note that in the range* $\max n_{2,j} - 2\min n_{1,i} < s \le \max n_{2,j} - \min n_{2,j}$, ${}_{-s}\mathrm{ext}^1_B(I/I^2, H_1)$ *is the codimension of the* H_B*-stratum of Corollary 37 because we have* ${}_{-2s}\mathrm{Ext}^1_B(I/I^2, K_B(t)) = 0$. *Indeed by Remark 26(iii) we get the last mentioned vanishing provided* $2s-t > 2\max n_{2,j} - n - 2$, *i.e. provided* $s > \max n_{2,j} - \sum n_{1,i}/2$ *(by* $t = n+2-\sum_{i=1}^{i=4} n_{1,i}$*) which holds since* $\sum_{i=1}^{i=4} n_{1,i}/2 \ge 2\min n_{1,i}$.

Example 40. *(Arithmetically Gorenstein curves* $\operatorname{Proj}(A)$ *in* $\mathbb{P}^5$, *obtained by* (26).*) Here we reconsider the preceding example, in dimension two higher. Let* R *be a polynomial ring in 6 variables, and let* $Y = \operatorname{Proj}(B)$ *be a threefold satisfying* $\operatorname{depth}_{I(Z)} B \ge 2$ *whose resolution* (18) *is linear with* $\mu = 4$. *Let* A *be defined by a regular section of* $\widetilde{H_1}^*(s)|_{Y-Z}$, *so* $X = \operatorname{Proj}(A)$ *is an AG curve in* $\mathbb{P}^5$. *The Hilbert polynomial of* Y *is given by "integrating"* $6v-2$ *two times, or more precisely* $p_Y(v) = 6\binom{v+2}{3} - 2\binom{v+2}{2} + 3\binom{v+1}{1}$. *The h-vector of* A *is of course still* $(1,4,10,..,6s+16,6s+19,6s+16,..,10,4,1)$ *and the Hilbert polynomial of* X *is* $p_X(v) = (6s^2+44s+81)(v-s-3)$. *This time we suppose* $s \ge -2$, *only avoiding the degenerate case.* A *is unobstructed by Corollary 37. To find the dimension of* $\operatorname{GradAlg}^{H_A}(R)$, *let* $\eta(v) := \dim(I/I^2)_v$ *and note* $\eta(v) = \dim(I)_v$ *for* $v < 6$. *Then* $\dim(N_B)_0 = 3\eta(4) - 4\eta(3) = 48$ *by Remark 35. Moreover, by* (27), $\dim(H_1^*)_s = 4\cdot\dim B_{(3+s)} - \dim(N_B)_s$ *where* $\dim(N_B)_s = 3\dim I_{(4+s)} - 4\dim I_{(3+s)} + \dim R_s$ *by Remark 35. Hence*

$$ \dim(H_1^*)_s = 4\cdot\dim R_{(3+s)} - 3\dim I_{(4+s)} - \dim R_s = 4\binom{s+8}{5} - 12\binom{s+6}{5} + 8\binom{s+5}{5} = 2(s+4)^2(s+5). $$

Moreover ${}_0\mathrm{hom}_B(S_2(H_1), K_B(-6)) = 6\dim(K_B)_2 - 3\dim S_2(K_B)_4 = 0$ *by* (25), (19) *and* (28). *Hence by Corollary 37 and* (8), *if* $s > 0$, *then*

$$ \dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = \dim_{(X)} \operatorname{Hilb}^{p_X}(\mathbb{P}^5) = 2(s+4)^2(s+5) + 47. $$

and the quotients given by (26) *generate a generically smooth, irreducible component of* $\operatorname{GradAlg}^{H_A}(R)$. *Now* $\delta(K_B)_{-6-2s} = 0$ *and* ${}_{-2s}\mathrm{Ext}^1_B(I/I^2, K_B(-6)) = 0$ *for* $s > -2$ *by Remark 26(iii) while* $(K_B)_{-6-2s} = 0$ *for* $s \ge -2$ *by* (19). *To compute* $\delta(H_1)_{-s}$ *and* ${}_{-s}\mathrm{ext}^1_B(I/I^2, H_1)$ *we use* (32) *and we get* ${}_{-s}\mathrm{ext}^1_B(I/I^2, H_1) = \dim R_{-s}$ *and* $\delta(H_1)_{-s} = -{}_{-s}\mathrm{ext}^1_B(I/I^2, H_1)$ *for* $-1 \le s \le 0$ *by Remark 39. By Corollary 37,*

$$ \dim_{(A)} \operatorname{GradAlg}(H_A) = 2(s+4)^2(s+5) + 47 + {}_{-s}\mathrm{ext}^1_B(I/I^2, H_1) \quad \text{for} \quad -1 \le s \le 0, $$

and the codimension, ${}_{-s}\mathrm{ext}^1_B(I/I^2, H_1)$, *of the* H_B*-stratum of* A *is* $1, 6$ *for* $s = 0, -1$ *respectively. Furthermore, for* $s = -2$, *we compute* $\delta(H_1)_2$ *and* $\delta(K_B)_{-2}$ *exactly as described in Remark 39 above and we get its values to be* -3 *and* 3 *respectively. Hence* $\dim_{(A)} \operatorname{GradAlg}^{H_A}(R) = 2(s+4)^2(s+5) + 47 = 71$ *in this case.*

Finally we pay some extra attention to the case $s = 0$ *of AG curves of degree* $d = 81$ *and genus* $g = 244$. *In this case the stratum given by* (26) *forms a 207 dimensional irreducible family contained in an irreducible component of* $\operatorname{GradAlg}^{H_A}(R)$ *or of* $\operatorname{Hilb}^{p_X}(\mathbb{P}^5)$ *of dimension 208. It is* *interesting to observe that we have exactly the same degree and genus as for the AG curve with $s = 6$ of Example 24 where a 207 dimensional stratum in a 208 dimensional component was constructed by means of Theorem 23! Their resolutions, however, differ (see* (31) *for the AG curve of the first family where $R(-4)^3$ appears as a repeated factor, and use linkage to see that $R(-5)^3$ appears as a repeated factor in the resolution of the second family). We would like to pose the following two questions. Are these two strata subschemes of codimension one of the same irreducible component of* $\mathrm{GradAlg}^{H_A}(R)$*? Is the minimal resolution of the generic Gorenstein algebra of this component obtained from the resolution of* (31) *by deleting all repeated factors?*

Now we apply Theorem 25 to $M = N_B$. By Theorem 8 and Proposition 10 we have the exact sequence

$$0 \to K_B(t-2s) \to N_B(-s) \to I_{A/B} \to 0 \tag{33}$$

where $t = n+2$. Since we have not been able to verify the assumption ${}_t\mathrm{Ext}^2_B(S_2(N_B), K_B) = 0$ of Theorem 25A), even by increasing $\mathrm{depth}_{I(Z)} B$ to its largest possible value, we must use the B)-part of the Theorem. Here we get advantage of developing the concept "unobstructed along any graded deformation of B", and we must suppose s "large enough". Indeed we get

Corollary 41. *Let $B = R/I$ be a graded codimension two CM quotient of R, let $U = \mathrm{Proj}(B) - Z \hookrightarrow \mathbb{P}^{n+1}$ be an l.c.i. and suppose $\mathrm{depth}_{I(Z)} B \geq 4$. If A is given by a regular section of $\widetilde{N_B}^*(s)$ on U and if $s > 2\max n_{2,j} - \min n_{1,i}$ and char$(k) \neq 2$, then A is unobstructed as a graded R-algebra, and the stratum of quotients given by* (33) *around (A) is open and irreducible (so A is H_B-generic). Moreover A is Gorenstein of codimension 4 in R, and*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(X)} \mathrm{Hilb}^p(\mathbb{P}^{n+1}) = \dim(N_B)_0 + \dim(I/I^2)_s - {}_0\mathrm{hom}(I/I^2, I/I^2)\ .$$

Letting $\eta(v) := \dim(I/I^2)_v$, we also have

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(X)} \mathrm{Hilb}^p(\mathbb{P}^{n+1}) = \eta(s) + \sum_{j=1}^{\mu-1} \eta(n_{2,j}) - \sum_{i=1}^{\mu} \eta(n_{1,i})\ .$$

Furthermore if $(B') \in \mathrm{GradAlg}^{H_B}(R)$ satisfies the same assumptions as B above and defines A' as B defined A, then the closures in $\mathrm{GradAlg}^{H_A}(R)$ of the stratum of quotients given by (33) *around (A) and the corresponding stratum around (A') coincide, i.e. they form the same irreducible component of* $\mathrm{GradAlg}^{H_A}(R)$.

Proof. Firstly to see that (N_B, B) is unobstructed along any graded deformation of B, it suffices by Proposition 13(iii) to show that ${}_0\mathrm{Ext}^1_B(N_B, N_B) = 0$ since $N_{B_T} := \mathrm{Hom}(I_T/I_T^2, B_T)$ ($B_T := R_T/I_T$ a graded deformation of B to an Artinian T) is a graded deformation of N_B to B_T by [20], Prop. A1 and ${}_0\mathrm{Ext}^1_B(I/I^2, B) = 0$. Since $\mathrm{depth}_{I(Z)} B \geq 4$ implies $\mathrm{depth}_{I(Z)} N_B \geq 4$ and $\mathrm{depth}_{I(Z)} I/I^2 \geq 3$ we get by (3) and Proposition 34,

$${}_0\mathrm{Ext}^1_B(N_B, N_B) \simeq \mathrm{H}^1_*(U, \mathcal{H}om_{\mathcal{O}_U}(\widetilde{I/I^2}^*|_U, \widetilde{I/I^2}^*|_U)) \simeq \mathrm{Ext}^1_B(I/I^2, I/I^2) = 0$$

and we get what we want. Similarly $\mathrm{Ext}^1_B(N_B, B) \simeq \mathrm{H}^1_*(U, \mathcal{H}om_{\mathcal{O}_U}(\widetilde{I/I^2}^*, \widetilde{B})) \simeq \mathrm{H}^1_*(U, \widetilde{I/I^2}) = 0$.

Secondly to show the remaining assumption of Theorem 25B), we use Remark 26(ii). Since $\oplus R(n_{2,j} - n_{1,i}) \to N_B$ is surjective by Remark 35, it follows that $(N_B)_v = 0$ for $v < \min n_{1,i} - \max n_{2,j}$, and we conclude by Remark 26(ii). Note that also the dimension formula above follows from Remark 26 and (12) (see the text before (12) for ${}_0\mathrm{hom}(I/I^2, I/I^2) = {}_0\mathrm{hom}(N_B, N_B) = {}_0\mathrm{hom}(S_2(N_B), K_B(t)) + 1$). Finally the proof of the irreducibility is trivial because the stratum is the image of an irreducible set. Indeed $\mathrm{GradAlg}(H_B, H_A)$ is smooth at $(B \to A)$ by the first conclusion of Theorem 47, cf. proofs of Theorem 25B) Theorem 22. The proof of the uniqueness (i.e. that the strata above coincide, up to closure) is essentially the same as for [23], Prop. 23(i) and Thm. 24 (see the two first lines of the proof of [23], Thm. 24) because the open subscheme of $\mathrm{GradAlg}^{H_B}(R)$ consisting of CM quotients is irreducible. For the connection with the Hilbert scheme, see (8). □

Remark 42. *Let* $\mathrm{depth}_{I(Z)} B \geq 4$ *and* $char(k) \neq 2$. *Using* (3) *(see the proof above), we get*

$${}_0\mathrm{Ext}^2_B(N_B, N_B) \simeq \mathrm{Ext}^2_{\mathcal{O}_U}(\widetilde{I/I^2}|_U, \widetilde{I/I^2}|_U) \simeq {}_0\mathrm{Hom}_B(I/I^2, \mathrm{H}^3_{I(Z)}(I/I^2)) ,$$

and $\mathrm{H}^3_{I(Z)}(I/I^2) \simeq \mathrm{H}^4_{I(Z)}(I^2)$. *Sometimes we can use this connection to prove the vanishing of* ${}_0\mathrm{Ext}^2_B(N_B, N_B)$ *in which case Theorem 25A) applies to* $M = N_B$ *provided we can show* ${}_{-s}\mathrm{Ext}^2_B(I/I^2, N_B) = 0$. *To show this vanishing, we play on* $N_B \simeq I/I^2 \otimes K_B(n+2)$ *and we get* $\mathrm{Ext}^2_B(I/I^2, N_B) \simeq \mathrm{Ext}^2_B(\mathrm{Hom}(I/I^2, I/I^2), K_B(n+2)) = 0$ *from the fact that the codepth of* $\mathrm{Hom}(I/I^2, I/I^2)$ *is at most one by Remark 35. The benefit of using Theorem 25A) (instead of Theorem 25B) which requires* ${}_{-s}\mathrm{Ext}^1_B(I/I^2, N_B) = 0$*) is that we don't need to assume* $s > 2\max n_{2,j} - \min n_{1,i}$. *Hence if we in the case* $s \leq 2\max n_{2,j} - \min n_{1,i}$ *suppose* ${}_0\mathrm{Hom}_B(I/I^2, \mathrm{H}^4_{I(Z)}(I^2)) = 0$, *we get that* A *is an unobstructed graded Gorenstein quotient of codimension 4 in* R, *and*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \epsilon + \delta + \dim(K_B)_{t-2s}$$

where $\delta := \delta(K_B)_{t-2s} - \delta(N_B)_{-s}$ *and* $\epsilon := \dim(N_B)_0 + \dim(I/I^2)_s - {}_0\mathrm{hom}(I/I^2, I/I^2)$, *i.e.* ϵ *is equal to the expression of* η'*s in Corollary 41. Continuing this argument (for this final statement we omit the details of the proof) we may even show that we can skip* $s > 2\max n_{2,j} - \min n_{1,i}$ *in Corollary 41 and at least get inequalities*

$$\epsilon + \delta + \dim(K_B)_{t-2s} - {}_0\mathrm{ext}^2_B(N_B, N_B) \leq \dim_{(A)} \mathrm{GradAlg}^{H_A}(R) \leq \epsilon + \delta + \dim(K_B)_{t-2s} .$$

Moreover, if the inequality to the right is an equality, then A *is unobstructed.*

Example 43. *(Arithmetically Gorenstein curves* $\mathrm{Proj}(A)$ *in* $\mathbb{P}^5$*, obtained by* (33)*.) Let* R *be a polynomial* k*-algebra in 6 variables, let* $B = R/I$ *be a codimension two quotient with minimal resolution*

$$0 \to R(-3)^2 \to R(-2)^3 \to R \to B \to 0 . \tag{34}$$

and suppose $Y = \mathrm{Proj}(B)$ *is an l.c.i in* $\mathbb{P}^5$. *Let* A *be given by a regular section of* $\widetilde{I/I^2}(s)$. *The Hilbert polynomial/function of* Y *is* $p_Y(v) = H_B(v) = 3\binom{v+2}{3} + \binom{v+2}{2}$ *for* $v \geq 0$ *by* (34) *(e.g. by "integrating"* $3v+1$ *two times). Thanks to* (23) *and* (19) *and the mapping cone construction applied to* (33) *we get the following resolution (obviously minimal for* $s > 4$*) of the Gorenstein algebra* $A = R/I_A$ *of Corollary 41,*

$$\begin{aligned} &0 \to R(-2s) \to R(2-2s)^3 \oplus R(-1-s)^6 \\ &\to R(3-2s)^2 \oplus R(-s)^{12} \oplus R(-3)^2 \to R(1-s)^6 \oplus R(-2)^3 \to R \to A \to 0 \quad . \end{aligned} \tag{35}$$

Let $h^i(\mathcal{O}_X(v)) = \dim \mathrm{H}^i(X, \mathcal{O}_X(v))$. *Since* $K_A \simeq A(2s-6)$, *the Hilbert polynomial of* X *is of the form* $p_X(v) = h^0(\mathcal{O}_X(v)) - h^1(\mathcal{O}_X(v)) = d(v-s+3)$ *because* $h^1(\mathcal{O}_X(s-3)) = h^0(\mathcal{O}_X(s-3))$. *Moreover looking to the resolution of* I_A *we see that*

$$p_X(s-2) = h^0(\mathcal{O}_X(s-2)) - h^0(\mathcal{O}_X(s-4)) = h^0(\mathcal{O}_Y(s-2)) - h^0(\mathcal{O}_Y(s-4)) = 3s^2 - 10s + 9.$$

So $X = \mathrm{Proj}(A)$ is an AG curve of degree $d = 3s^2 - 10s + 9$ and arithmetic genus $g = 1 + d(s-3)$ in $\mathbb{P}^5$. If $s > 2\max n_{2,j} - \min n_{1,i} = 4$, then Corollary 41 applies. Letting $\eta(v) := \dim(I/I^2)_v$, we get $\sum \eta(n_{2,j}) - \sum \eta(n_{1,i}) = 2\dim I_3 - 3\dim I_2 = 23$. Calculating $\eta(s)$ and using (22) which implies

$$0 \to R(-6) \to R(-5)^6 \to R(-4)^6 \to I^2 \to 0 \tag{36}$$

we get $\eta(s) = (s+1)(s-1)^2$. Hence, if $s \geq 5$, then A is unobstructed and

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(X)} \mathrm{Hilb}^{p_X}(\mathbb{P}^5) = \eta(s) + 2\eta(3) - 3\eta(2) = (s+1)(s-1)^2 + 23 \ .$$

Finally we discuss the cases $3 \leq s \leq 4$. By (36), we get an injection $\mathrm{H}^4_{\mathfrak{m}}(I^2) \to \mathrm{H}^6_{\mathfrak{m}}(R(-6))$. By Remark 42, ${}_0\mathrm{Ext}^2_B(N_B, N_B) = 0$ and it follows that A is unobstructed and $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = (s+1)(s-1)^2 + 23 + \delta$ where $\delta := \dim(K_B)_{6-2s} + \delta(K_B)_{6-2s} - \delta(N_B)_{-s}$. We have $(K_B)_{6-2s} = 0$ by (19). Moreover applying $\mathrm{Hom}(-, K_B(6))$ to (6), we get $-\delta(K_B)_{6-2s} = {}_{-2s}\mathrm{hom}(H_1, K_B(6))$ for $s \geq 3$. Since H_1 has rank one by (6) we get $H_1 \simeq \mathrm{Hom}(H_0, K_B(6 - \sum n_{1,i})) \simeq K_B$ by the last conclusion of Theorem 8. Hence $\delta(K_B)_{6-2s} = -\dim B_{(6-2s)}$, i.e. $\delta(K_B)_{6-2s} = -1, 0$ for $s = 3, 4$ respectively. It remains to compute $\delta(N_B)_{-s}$ for which we use $H_1 \simeq K_B$ and the following exact sequence

$$0 \to \mathrm{Hom}(I/I^2, N_B) \to \mathrm{Hom}(I/I^2, \oplus B(n_{1,i})) \to \mathrm{Hom}(I/I^2, K_B^*) \to \mathrm{Ext}^1_B(I/I^2, N_B) \to 0 \tag{37}$$

which we get by applying $\mathrm{Hom}(I/I^2, -)$ to (27). Since $\mathrm{Hom}(I/I^2, K_B^*) \simeq \mathrm{Hom}(I/I^2 \otimes K_B(6), B(6)) \simeq I/I^2(6)$, we get $\delta(N_B)_{-s} = 2, -3$ for $s = 3, 4$ respectively. By Remark 42, $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = 36, 71$ for $s = 3, 4$ respectively. Note that this result confirms the well known formula

$$6d + 2(1-g) \leq \dim_{(A)} \mathrm{GradAlg}^{H_A}(R) \leq 6d + 2(1-g) + h^1(\mathcal{N}_X)$$

because for $s = 3$ (resp. $s = 4$) the curve has degree $d = 6$ and genus $g = 0$ (resp. $d = 17$ and genus $g = 18$, and one may verify $h^1(\mathcal{N}_X) = 3$ by other methods).

We will finish the section by looking to the rank $r = 3$ case, i.e. we apply Theorem 30A) to the B-module $M = H_1$ provided the number of minimal generators of I is $\mu(I) = 5$. Note that (15) translates to $t = n + 2 - \sum n_{1,i}$, $H_2 = H_1^\vee(t) \simeq \mathrm{Hom}(H_1, K_B(t))$ and

$$0 \to K_B(t - 3s) \to H_2(-2s) \to H_1(-s) \to I_{A/B} \to 0 \ . \tag{38}$$

Corollary 44. *Let $B = R/I$ be a graded codimension two CM quotient of R, let $U = \mathrm{Proj}(B) - Z \hookrightarrow \mathbb{P}^{n+1}$ be an l.c.i. and suppose $\mu(I) = 5$, $char(k) \neq 2$ and $\mathrm{depth}_{I(Z)} B \geq 3$. If A is defined by a regular section of $\widetilde{H_1}^*(s)$ on U, i.e. given by (38), then A is unobstructed as a graded R-algebra (indeed ${}_0\mathrm{H}^2(R, A, A) = 0$), A is Gorenstein of codimension 5 in R, and $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) =$*

$$\dim(N_B)_0 + \dim(H_1^*)_s + {}_{-s}\mathrm{hom}_B(S_2(H_1), K_B(t)) - {}_0\mathrm{hom}_B(H_1, H_1) - \dim(K_B)_{t-3s} - \delta \ ,$$

where $\delta := \delta(H_1)_{-s} + \delta(K_B)_{t-3s} - \delta(H_2)_{-2s}$. Moreover if $char(k) = 0$ and $(B \to A)$ is general with respect to ${}_0\mathrm{hom}_R(I_B, I_{A/B})$, then the codimension of the H_B-stratum of A at $(B \to A)$ is

$${}_{-s}\mathrm{ext}^1_B(I_B/I_B^2, H_1) - \dim(\mathrm{im}\,\beta)$$

where β is the homomorphism ${}_{-2s}\mathrm{Ext}^1_B(I/I^2, H_2) \to {}_{-s}\mathrm{Ext}^1_B(I/I^2, H_1)$ induced by (38). If in addition $s > \max n_{2,j} - \min n_{2,j}$, then A is H_B-generic, $\delta = 0$ and $\dim(K_B)_{t-3s} = 0$.

Proof. This is a corollary to Theorem 30A). Indeed, thanks to Proposition 36 we only need to show ${}_0\mathrm{Ext}^2_B(H_2(-t)\otimes H_1, K_B)=0$ and ${}_{-2s}\mathrm{Ext}^i_B(I/I^2,H_2)=0$ for $2\le i\le 3$ and the "if in addition"-statement. Using (6) we get ${}_{-2s}\mathrm{Ext}^i_B(I/I^2,H_2)\simeq {}_0\mathrm{Ext}^{i-1}_B(H_1,H_2(-2s))$. Since $\operatorname{depth}_{I(Z)} B\ge 3$, we have

$$ {}_0\mathrm{Ext}^1_B(H_1,H_2(-2s))\simeq \mathrm{Ext}^1_{\mathcal{O}_U}(\widetilde{H_1}|_U,\widetilde{H_1^*\otimes K_B}(t-2s)|_U)\simeq \mathrm{Ext}^1_{\mathcal{O}_U}(\widetilde{H_1\otimes H_1}|_U,\widetilde{K_B}(t-2s)|_U) $$

by (3) which vanishes because $0\to \widetilde{H}_2|_U\simeq \wedge^2\widetilde{H}_1|_U\to \widetilde{H_1\otimes H_1}|_U\to S_2(H_1)|_U\to 0$ is exact and H_2 and $S_2(H_1)$ are maximal CM modules (this argument also shows ${}_0\mathrm{Ext}^2_B(H_1,H_2(-2s))=0$ provided $\operatorname{depth}_{I(Z)} B\ge 4$).

To show ${}_0\mathrm{Ext}^2_B(H_1,H_2(-2s))=0$ under the assumption $\operatorname{depth}_{I(Z)} B\ge 3$, we use (20). Hence it suffices to show the ${}_v\mathrm{Ext}^1_B(K_B^*,H_2)=0$ for every v. By (3) this Ext-group is isomorphic to

$$ \mathrm{Ext}^1_{\mathcal{O}_U}(\widetilde{K_B}^*|_U,\widetilde{H_1^*\otimes K_B}(t+v)|_U)\simeq \mathrm{Ext}^1_{\mathcal{O}_U}(\widetilde{K_B^*\otimes H_1}|_U,\widetilde{K_B}(t+v)|_U)\simeq {}_v\mathrm{Ext}^1_B(\mathrm{Hom}_B(K_B,H_1),K_B(t)) $$

which vanishes since $\mathrm{Hom}_B(K_B,H_1)$ is a maximal CM module. Indeed by using (20), we showed in [25] the exactness of

$$ 0\to \mathrm{Hom}_B(K_B(n+2),H_1)\to \wedge^2(\oplus B(-n_{2,j}))\to H_2\to 0 \tag{39} $$

(for that sequence it suffices to have $\operatorname{depth}_{I(Z)} B\ge 2$), from which we get that $\mathrm{Hom}_B(K_B,H_1)$ is a maximal CM B-module.

To prove ${}_t\mathrm{Ext}^2_B(H_2\otimes H_1,K_B)=0$, we use again (20) to get $0\to \widetilde{K_B(n+2)}^*\otimes\widetilde{H}_2|_U\to (\oplus\widetilde{B}(-n_{2,j}))\otimes\widetilde{H}_2|_U\to \widetilde{H}_2\otimes\widetilde{H}_1|_U\to 0$. Applying $\mathrm{H}^0_*(U,-)$ to it, we get the exact sequence

$$ 0\to \mathrm{Hom}_B(K_B(n+2),H_2)\to (\oplus B(-n_{2,j}))\otimes H_2\to H_2\otimes H_1/\tau\to 0 $$

where $\tau:=\mathrm{H}^0_{I(Z)}(H_2\otimes H_1)$. Invoking also (3) it follows that ${}_{v-n-2}\mathrm{Ext}^2_B(H_2\otimes H_1/\tau,K_B)\simeq {}_v\mathrm{Ext}^1_B(\mathrm{Hom}_B(K_B,H_2),K_B)\simeq \mathrm{Ext}^1_{\mathcal{O}_U}(\widetilde{K_B}^*\otimes\widetilde{H_1^*\otimes K_B(t)}|_U,\widetilde{K_B}(v)|_U)\simeq {}_v\mathrm{Ext}^1_B(H_1^*(t),K_B)$, and that the latter vanishes since H_1^* is a maximal CM B-module. Indeed due to (27), H_1^* has codepth at most one, and applying $\mathrm{Hom}(-,K_B)$ to (27), we will see $\mathrm{Ext}^1_B(H_1^*,K_B)=0$ as well because $\mathrm{Hom}(N_B,K_B)\simeq N_B(-n-2)\simeq I/I^2\otimes K_B$. Then we conclude by the exact sequence

$$ \mathrm{Ext}^2_B(H_2\otimes H_1/\tau,K_B)\to \mathrm{Ext}^2_B(H_2\otimes H_1,K_B)\to \mathrm{Ext}^2_B(\tau,K_B) $$

because $\mathrm{Ext}^2_B(\tau,K_B)\hookrightarrow \mathrm{Ext}^2_{\mathcal{O}_U}(\widetilde{\tau}|_U,\widetilde{K_B}|_U)$ is injective by (3) and $\widetilde{\tau}|_U=0$.

Finally if $s>\max n_{2,j}-\min n_{2,j}$ we get ${}_{-s}\mathrm{Hom}_R(I,H_1)={}_{-s}\mathrm{Ext}^1_R(I,H_1)=0$ and hence $\delta(H_1)_{-s}=0$ by Remark 31(*ii*) and the "left" short exact sequence deduced from (20). Using (39) instead of (20) we similarly get $\delta(H_2)_{-2s}=0$ as well as $\delta(K_B)_{t-3s}=0$ and we conclude easily. □

Remark 45. *Due to Remark 33 the quotients A of Corollary 37 and Corollary 44 are strongly unobstructed in the sense* $\mathrm{H}^2(R,A,A)=0$. *This follows from the proofs of the corollaries because the proofs of the vanishing of the ${}_v\mathrm{Ext}^i_B$-groups involved are easily extended to the vanishing of the corresponding Ext^i_B-groups. As a consequence of this, look to the scheme $ZGor(H)$ parametrizing not necessarily graded Gorenstein quotients $R\to A$ with Hilbert function H ([18], p.126). Since ${}_v\mathrm{H}^2(R,A,A)=0$ for $v\ge 0$, we can use [22], Thm. 1.10 to prove the smoothness of $ZGor(H)$ at (A), and one may also find the dimension of the scheme $ZGor(H)$ at (A) by a formula analogous to the dimension formulas of Corollary 37 and Corollary 44, cf. [23], Prop. 29.*

Example 46. *Let R be a polynomial ring in five variables, and let $B = R/I$ be a CM-quotient with $\mu(I) = 5$ and with linear resolution (i.e. all $n_{1,i}$ of* (18) *are $n_{1,i} = 4$). Then $Y = \mathrm{Proj}(B)$ is a surface with Hilbert polynomial $p_Y(v) = 5v^2 - 5v + 5$. By [2],*

$$0 \to \wedge^3 G_2 \to \wedge^3 G_1 \to \wedge^2 G_2 \to H_2 \to 0 \tag{40}$$

is exact, and the mapping cone construction applied to (38) *lead to the following resolution of the Gorenstein algebra $A = R/I_A$ of Corollary 44A),*

$$\begin{aligned} &0 \to R(-20-3s) \to R(-16-3s)^5 \oplus R(-15-2s)^8 \to R(-15-3s)^4 \oplus R(-12-2s)^{10} \oplus \\ &R(-10-s)^6 \to R(-10-2s)^6 \oplus R(-8-s)^6 \oplus R(-5)^4 \to R(-5-s)^4 \oplus R(-4)^5 \to I_A \to 0. \end{aligned} \tag{41}$$

By (38) *or* (41) *A is Artinian of socle degree $3s+15$ and with symmetric h-vector $(1, ..., 1)$ given by $H_A(v) = p_Y(v)$ for $1 \le v \le s+4$, $H_A(v) = p_Y(v) - 4$ for $v = s+5$ and*

$$H_A(v) = p_Y(v) - 15(v-s-4)^2 + 35(v-s-4) - 30 \quad \text{for} \quad s+5 < v < 2(s+5).$$

So if $s = -3, -2, ...$, then the h-vector of A is $(1,5,11,15,11,5,1)$, $(1,5,15,31,45,45,31,15,5,1), ...$ respectively. Suppose $s \ge -3$ and Y an l.c.i.. Then A is unobstructed by Corollary 44. By Remark 35, $\dim(N_B)_s = 4\dim I_{s+5} - 5\dim I_{s+4} + \dim R_s$, so $\dim(N_B)_0 = 60$. Moreover

$$\dim(H_1^*)_s = 5\dim B_{s+4} - \dim(N_B)_s = 5\binom{s+8}{4} - 4\dim I_{s+5} - \binom{s+4}{4} = 15s^2 + 125s + 265$$

by (27). *Applying* $\mathrm{Hom}(-, H_1)$ *to* (6), *and we get*

$$0 \to \mathrm{Hom}(I/I^2, H_1) \to \mathrm{Hom}(\oplus B(-n_{1,i}), H_1) \to \mathrm{Hom}(H_1, H_1) \to \mathrm{Ext}^1_B(I/I^2, H_1) \to 0 \tag{42}$$

Hence ${}_0\mathrm{hom}(H_1, H_1) = 1$ by Remark 39 and ${}_0\mathrm{hom}_B(S_2(H_1), K_B(-15-s)) = 0$ by (25) *and* (19). *Finally $\dim(K_B)_{-15-3s} = 0$ and the sum $\dim(H_1^*)_s + {}_{-s}\mathrm{hom}_B(S_2(H_1), K_B(t)) - {}_0\mathrm{hom}_B(H_1, H_1) - \dim(K_B)_{t-3s}$ of Corollary 44 is equal to $15s^2 + 125s + 264$. We get*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(A)} \mathrm{PGor}(H_A) = 15s^2 + 125s + 324 - \delta.$$

Moreover if $s > 0$, then $\delta = 0$ and A is H_B-generic. Looking to $s \le 0$, we note that if $2s > \max n_{2,j} - 2\min n_{2,j}$, then ${}_{-2s}\mathrm{Hom}_R(I, H_2) = {}_{-2s}\mathrm{Ext}^1_R(I, H_2) = 0$ and hence $\delta(H_2)_{-2s} = 0$ by Remark 31 and (39). *Combining with Remark 39, we get the codimension of the H_B-stratum of Corollary 44 to be $-\delta = -\delta(H_1)_{-s} = {}_{-s}\mathrm{ext}^1_B(I/I^2, H_1) = \dim R_{-s}$ for $-2 \le s \le 0$. In particular*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = 15s^2 + 125s + 324 + \binom{-s+4}{4} \quad \text{for} \quad -2 \le s \le 0.$$

In the final case $s = -3$ (where we skip a few details), one may see $\delta(H_2)_{-2s} = 5\dim H_2(4)_6 - {}_6\mathrm{hom}_B(S_2(H_1), K_B(-15)) = 0$ (from the sequence we get by applying $\mathrm{Hom}(-, H_2)$ *to* (6)), *and*

$$\delta(H_1)_{-s} = 5\dim N_B(-4)_3 - {}_3\mathrm{hom}_B(I/I^2, I/I^2) = 5 \cdot 20 - (4\dim I_8^2 + \dim R_3) = 5$$

by Remark 39, and we get $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = 15s^2 + 125s + 324 - 5 = 79$.

4 Appendix: Deformations of quotients of zerosections

Let $X = \mathrm{Proj}(A)$ be a subscheme of $Y = \mathrm{Proj}(B) \subset \mathbb{P} = \mathrm{Proj}(R)$ defined as the degeneracy locus of a regular section of some sheaf $\mathcal{M}$ supported on Y. In this appendix we prove a quite general result, Theorem 47, concerning the unobstructedness and the "family-dimension" of a quotient A obtained from B as the homogeneous coordinate ring of a zerosection as above, in which we neither assume B to be Cohen-Macaulay, nor A to be Gorenstein. This leads to a main result of this paper (Theorem 22). We have included a version of Theorem 47 for the Hilbert scheme (Corollary 48). This result essentially generalizes Thm. 9.4 of [26], which treats the case where $\mathcal{M}$ is locally free of rank $r = 1$, to higher ranks. Unfortunately our methods often lead to assumptions on $\mathrm{depth}_{I(Z)} B$ which imply that A is non-Artinian. We have, however, succeeded in proving Theorem 22 also for an Artinian Gorenstein algebra A. Below $B = R/I_B \to A = R/I_A$ is a graded surjection with kernel $I_{A/B}$ and $M^* = \mathrm{Hom}_B(M, B)$.

Theorem 47. *Let $r \geq 1$ and s be integers. Let B be a graded quotient of a finitely generated polynomial k-algebra R. Let M be a finitely generated graded B-module, let $Y := \mathrm{Proj}(B)$ and $U = Y - Z$ be an open subset of $\mathrm{Proj}(B)$ such that $\mathrm{depth}_{I(Z)} B \geq 2$ and such that $\widetilde{M}|_U$ is locally free of rank r. Let $M_i = \mathrm{H}^0_*(U, \wedge^i \widetilde{M})$ for $0 \leq i \leq r$, and let $\sigma \in \mathrm{H}^0(U, \widetilde{M}^*(s))$ be a regular section on U. Let $X = \mathrm{Proj}(A)$ be the zero locus of σ defined by $A := \mathrm{coker}(\mathrm{H}^0_*(U, \widetilde{M}(-s)) \xrightarrow{\sigma} B)$. Let $K_1 = \ker \sigma(s)$ and suppose*

(i) $\mathrm{H}^i_*(U, \widetilde{M}_{i+1}) = 0$ for $1 \leq i \leq r - 1$

(ii) ${}_0\mathrm{Ext}^2_B(M_1, K_1) = 0$, *and*

(iii) $(M_1(-s), \sigma)$ *is unobstructed along any graded deformation of B.*

Then the first projection $\mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}^{H_B}(R)$ is smooth at $(B \to A)$ and,

$$\dim_{(B \to A)} \mathrm{GradAlg}(H_B, H_A) = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + {}_0hom_B(I_{A/B}, A).$$

Moreover, $\mathrm{depth}_{\mathfrak{m}} A \geq 1$. If, in addition,

(iv) either ${}_0\mathrm{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0$ *and $(I_B)_\wp$ is syzygetic for any graded prime $\wp$ of $Ass(I_{A/B})$,* or ${}_0\mathrm{Ext}^1_R(I_B, I_{A/B}) = 0$,

then ${}_0\hom_R(I_B, B) - \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) = {}_0\hom_R(I_A, A) - \dim_{(A)} \mathrm{GradAlg}^{H_A}(R)$, *$A$ is H_B-generic, and*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + {}_0\hom_B(I_{A/B}, A) - {}_0\hom_R(I_B, I_{A/B}).$$

Moreover A is unobstructed as a graded R-algebra if and only if B is unobstructed as a graded R-algebra.

Using the arguments appearing in (46) below, we will see that the condition (i) of Theorem 47 implies $\mathrm{H}^0_{I(Z)}(A) = 0$. In particular A is the homogeneous coordinate ring of X. If we in addition suppose

$$(v) \qquad \mathrm{H}^i_*(U, \widetilde{M}_i) = 0 \text{ for } 1 \leq i \leq r \quad ,$$

we can mainly argue as in (46) (or as in the proof of [28], Lemma 12) to see that $\mathrm{H}^1_{I(Z)}(A) = 0$, i.e. that $\mathrm{depth}_{I(Z)} A \geq 2$. Hence $\mathrm{GradAlg}^H(R) \simeq \mathrm{Hilb}^p(\mathbb{P})$ at $(X \subset \mathbb{P})$ by (8).

Let $X \subset Y$ be closed subschemes of $\mathbb{P}$ of Hilbert polynomials p_X and p_Y respectively, let $\mathrm{D}(p_X, p_Y)$ be the Hilbert-flag scheme parametrizing all such "pairs" of closed subschemes and let $p_1 : \mathrm{D}(p_X, p_Y) \to \mathrm{Hilb}^{p_X}(\mathbb{P})$ be the projection induced by $p_1((X' \subset Y')) = (X')$. X is called *p_Y-generic* if there is an *open* subset U_X of $\mathrm{Hilb}^{p_X}(\mathbb{P})$ such that $(X) \in U_X \subset p_1(\mathrm{D}(p_X, p_Y))$.

Corollary 48. *In addition to the notations and assumptions of Theorem 47 (i)-(iv), suppose (v). Then* X *is* p_Y*-generic, and*

$$\dim_{(X)} \operatorname{Hilb}^{p_X}(\mathbb{P}) = \dim_{(Y)} \operatorname{Hilb}^{p_Y}(\mathbb{P}) + \hom_{\mathcal{O}_Y}(\mathcal{I}_{X/Y}, \mathcal{O}_X) - \hom_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_Y, \mathcal{I}_{X/Y}).$$

Moreover X *is unobstructed if and only if* Y *is unobstructed.*

Remark 49. *Since* $\operatorname{depth}_{I(Z)} B \geq 2$*, the modules* M_i *of Theorem 47 satisfy* $M_i = (\wedge^i M)^{**}$ *(cf. [28], Remark 8). In particular* $\widetilde{M}_i = (\wedge^i \widetilde{M})^{**}$ *and clearly* $\widetilde{M}_i|_U = (\wedge^i \widetilde{M})|_U$*. Moreover note that Theorem 47(i) holds if*

$$\operatorname{depth}_{I(Z)} M_i \geq i+1 \quad \text{for} \quad 2 \leq i \leq r.$$

This holds in particular if each M_i *is a maximal CM* B*-module and* $\operatorname{depth}_{I(Z)} B \geq r+1$*. In this case it follows from [28], Prop. 6 that* A *is equidimensional and satisfies Serre's condition* S_1 *provided* B *is Cohen-Macaulay. Indeed since a regular section by definition leads to an exact Koszul resolution of* $\widetilde{A}$ *on* U*, we get that* $U \cap X$ *is equidimensional and without embedded components, and [28], Prop. 6 applies. In the same way (v) above holds if each* M_i *is maximally Cohen-Macaulay and* $\operatorname{depth}_{I(Z)} B \geq r+2$*, in which case* A *satisfies Serre's condition* S_2 *if* B *is Cohen-Macaulay.*

Proof. Once we have proved the smoothness of the first projection

$$q : \operatorname{GradAlg}(H_B, H_A) \to \operatorname{GradAlg}^{H_B}(R) \quad \text{at} \quad (B \to A),$$

we shall see that we get all the conclusions of the Theorem rather quickly. To prove the smoothness of q, let $(T, m_T) \to (S, m_S)$ be a small Artin surjection with kernel $\mathfrak{a}$. Let $B_S \to A_S$ be a (flat and graded) deformation of $B \to A$ to S and let B_T be a deformation of B_S to T. It suffices to find a deformation A_T of A_S to T and a map $B_T \to A_T$ over $B_S \to A_S$. Let $I_{A_S/B_S} = \ker(B_S \to A_S)$.

Firstly we *claim* that there exists a graded deformation $\phi_S : M_{1S}(-s) \to I_{A_S/B_S}$ (of S-flat B_S-modules) of $\phi : M_1(-s) \to I_{A/B}$ to S where ϕ composed with $I_{A/B} \hookrightarrow B$ is the map σ in the definition of A. Indeed we can by induction suppose there is a graded deformation $\phi_{S_1} : M_{1S_1}(-s) \to I_{A_{S_1}/B_{S_1}}$ of ϕ to S_1 where $S \to S_1$ is small Artin surjection (with kernel $\mathfrak{a}_1$). Composing ϕ_{S_1} with $I_{A_{S_1}/B_{S_1}} \to B_{S_1}$ we deduce by the assumption of *(iii)* the existence of a deformation M'_{1S} of M_{1S_1} to S.

By Remark 12 the existence of a homogeneous map $\phi'_S : M'_{1S}(-s) \to I_{A_S/B_S}$ such that $\phi'_S \otimes_S id_{S_1} = \phi_{S_1}$ is equivalent to the vanishing of a well defined element (obstruction) $o_0(M'_{1S}, I_{A_S/B_S}) \in {}_0\operatorname{Ext}^1_B(M_1, I_{A/B}(s)) \otimes_k \mathfrak{a}_1$. Since

$${}_0\operatorname{Ext}^1_B(M_1, M_1) \otimes_k \mathfrak{a}_1 \stackrel{\psi}{\to} {}_0\operatorname{Ext}^1_B(M_1, I_{A/B}(s)) \otimes_k \mathfrak{a}_1 \to {}_0\operatorname{Ext}^2_B(M_1, K_1) \otimes_k \mathfrak{a}_1$$

is exact, there is by *(ii)* an element $\lambda \in {}_0\operatorname{Ext}^1_B(M_1, M_1) \otimes \mathfrak{a}_1$ such that $\psi(\lambda) = o_0(M'_{1S}, I_{A_S/B_S})$. On the other hand, by deformation theory, one knows more generally that $M'_{1S} - \lambda$ defines a graded deformation M_{1S} of M_{1S_1} such that $\psi(\lambda) = o_0(M'_{1S}, I_{A_S/B_S}) - o_0(M_{1S}, I_{A_S/B_S})$ (analogous to [23], last part of Remark 3). Hence $o_0(M_{1S}, I_{A_S/B_S}) = 0$ for some deformation M_{1S} and the claim is proved.

The composition of $\phi_S : M_{1S}(-s) \to I_{A_S/B_S}$ with $I_{A_S/B_S} \to B_S$ yields a homogeneous map σ_S such that $\sigma_S \otimes_S id_k = \sigma$ where $id_k : k \to k$ is the identity. By *(iii)* there is a deformation $\sigma_T : M_{1T}(-s) \to B_T$ over σ_S.

Secondly we *claim* that $A_T := \operatorname{coker} \sigma_T$ is a graded deformation of A_S to T, i.e. that the surjection $I_{A_T/B_T} \otimes_T k \twoheadrightarrow I_{A/B}$ is an isomorphism (where $I_{A_T/B_T} := \operatorname{im} \sigma_T$). Indeed since $A_T \otimes_T S \simeq A_S$, it follows that A_T is a deformation of A_S if it is T-flat, i.e. if $\operatorname{Tor}_1^T(A_T, k) = 0$ or equivalently, if

$I_{A_T/B_T} \otimes_T k \simeq I_{A/B}$. The rough idea for proving this isomorphism of ideals is just to see that the following part of the Koszul complex

$$\wedge^2 M_1(-2s) \longrightarrow M_1(-s) \longrightarrow B \longrightarrow A \tag{43}$$

(induced by the section σ), which obviously commutes with the corresponding complex, $\wedge^2 M_{1T}(-2s) \to M_{1T}(-s) \to B_T \to A_T$ over T (because σ_T commutes with σ), is exact (or "exact enough", cf. below for a precise formulation).

More precisely since σ is a regular section on U, the Koszul complex induced by σ is *exact on* U ([28], Thm. 7(4)). Applying $\mathrm{H}^0_*(U,-)$ to this Koszul complex (which is really what we did in the proof of Theorem 8 in [28]), we get in particular the following part of a complex

$$M_2(-2s) \longrightarrow M_1(-s) \longrightarrow B \longrightarrow A \longrightarrow 0 \tag{44}$$

where $M_1(-s) \longrightarrow B \longrightarrow A \longrightarrow 0$ is exact. Using correspondingly σ_T we get a complex $M_{2T}(-2s) \to M_{1T}(-s) \to B_T \to A_T$ over T which commutes with (44), where $M_{2T} := \mathrm{H}^0_*(U, \wedge^2 \widetilde{M}_{1T})$ (slightly abusing the notation of U by letting U be the set in $\mathrm{Proj}(B_T)$ which corresponds to $U \subseteq \mathrm{Proj}(B)$). Moreover note that $M_{1T} \xrightarrow{\simeq} \mathrm{H}^0_*(U, \widetilde{M}_{1T})$ follows from $\mathrm{H}^0_*(U, \widetilde{M}_1) \simeq M_1$ and the fact that M_{1T} is a deformation of M_1. Indeed since we by induction may suppose $\mathrm{H}^0_*(U, \widetilde{M}_{1S}) \simeq M_{1S}$, we conclude easily by applying $\mathrm{H}^0_*(U,-)$ to the exact sequence

$$0 \to \widetilde{M}_1|_U \otimes_k \mathfrak{a} \to \widetilde{M}_{1T}|_U \to \widetilde{M}_{1S}|_U \to 0, \tag{45}$$

and by comparing with $0 \to M_1 \otimes_k \mathfrak{a} \to M_{1T} \to M_{1S} \to 0$. Let $Z_i = \ker(M_i(-is) \to M_{i-1}((1-i)s))$ for $i \geq 1$ and let $Z_{1T} := \ker(M_{1T}(-s) \to I_{A_T/B_T})$. Applying $(-) \otimes_T k$ to

$$Z_{1T} \longrightarrow M_{1T}(-s) \longrightarrow I_{A_T/B_T} \longrightarrow 0,$$

we see that the exactness of $Z_1 \to M_1(-s) \to I_{A/B} \to 0$ and the isomorphism $M_{1T} \otimes_T k \simeq M_1$ imply the claim provided we can prove that $Z_{1T} \to Z_1$ is surjective.

Now we will show that $\mathrm{H}^1_*(U, \widetilde{Z}_1) = 0$ implies the surjectivity of $Z_{1T} \to Z_1$. To prove this we remark that the T-flatness of $\widetilde{Z}_{1T}|_U$ (which is true because the claim is true locally in U) yields an exact sequence (45) in which we have replaced every $\widetilde{M}_1$ by $\widetilde{Z}_1$. Applying $\mathrm{H}^0_*(U,-)$ to such an exact sequence, we get the exact sequence

$$Z_{1T} \to Z_{1S} \to \mathrm{H}^1_*(U, \widetilde{Z}_1) \otimes_k \mathfrak{a}.$$

because we have $\mathrm{H}^0_*(U, \widetilde{Z}_{1T}) \simeq Z_{1T}$ from $\mathrm{H}^0_*(U, \widetilde{M}_{1T}) \simeq M_{1T}$. If $\mathrm{H}^1_*(U, \widetilde{Z}_1) = 0$, we get the surjectivity of $Z_{1T} \to Z_{1S}$ and hence the surjectivity of $Z_{1T} \to Z_1$ by induction.

Hence it suffices to prove $\mathrm{H}^1_*(U, \widetilde{Z}_1) = 0$. Taking cohomology of the sequence $0 \to Z_i \to M_i(-is) \to Z_{i-1} \to 0$, the assumption (i) leads to injections

$$\mathrm{H}^1_*(U, \widetilde{Z}_1) \hookrightarrow \mathrm{H}^2_*(U, \widetilde{Z}_2) \hookrightarrow \ldots \hookrightarrow \mathrm{H}^{r-1}_*(U, \widetilde{Z}_{r-1}) \tag{46}$$

and we get the vanishing $\mathrm{H}^1_*(U, \widetilde{Z}_1) = 0$ because $\widetilde{Z}_{r-1}|_U \simeq \widetilde{M}_r(-rs)|_U$ and $\mathrm{H}^{r-1}_*(U, \widetilde{M}_r) = 0$ and the second *claim* is proved.

Combining the two claims we get the smoothness of the projection $q : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}^{H_B}(R)$ at $(B \to A)$.

From (46) we have $\mathrm{H}^2_{I(Z)}(Z_1) \simeq \mathrm{H}^1_*(U, \widetilde{Z}_1) = 0$ and using the exact sequence

$$0 \to Z_1 \to M_1(-s) \to B \to A \to 0 \tag{47}$$

we get $\mathrm{H}^1_{I(Z)}(I_{A/B}) = 0$ and $\mathrm{H}^0_{I(Z)}(A) = 0$. It follows that $\operatorname{depth}_{\mathfrak{m}} A \geq 1$.

Once we have the smoothness of q and the assumption (iv), all remaining conclusions of Theorem 47 follow by exactly the same proof as in the proof of Thm. 5B) in [23]. Indeed (iv) implies that the *second* projection $p : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}^{H_A}(R)$ is smooth at $(B \to A)$ (cf. [23], Prop. 4(ii) and Remark 16(i) to conclude that the assumption ${}_0\mathrm{Ext}^1_R(I_B, I_{A/B}) = 0$ also implies the smoothness of p). Since the tangent space $T_{B\to A}$ of $\mathrm{GradAlg}(H_B, H_A)$ at $(B \to A)$ is given by the *cartesian* square in the following diagram of exact sequences

$$\begin{array}{ccccccc} & & & & & & {}_0\mathrm{Hom}_R(I_B, I_{A/B}) \\ & & & & & & \downarrow \\ & & & & T_{B\to A} & \stackrel{T_q}{\to} & {}_0\mathrm{Hom}_R(I_B, B) \\ & & & & \downarrow & \square & \downarrow \\ 0 & \to & {}_0\mathrm{Hom}_R(I_{A/B}, A) & \to & {}_0\mathrm{Hom}_R(I_A, A) & \to & {}_0\mathrm{Hom}_R(I_B, A) \end{array} \tag{48}$$

([22], (10)) where the top vertical map is injective and the tangent map T_q is surjective by the smoothness of q, we get

$$\dim_{(B\to A)} \mathrm{GradAlg}(H_B, H_A) = \dim T_{B\to A} = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + {}_0\hom_R(I_{A/B}, A)$$

as well as the other conclusions (by e.g. using that the tangent map $T_p : T_{B\to A} \to {}_0\mathrm{Hom}_R(I_A, A)$ of p is surjective by (iv) and (48)). ☐

Remark 50. *If we, instead of* (i) $\mathrm{H}^i_*(U, \widetilde{M_{i+1}}) = 0$ for $1 \leq i \leq r-1$, *assume*

$$(i') \quad \mathrm{H}^1_*(U, \widetilde{M_2}) = 0 \quad \text{and} \quad \mathrm{H}^i_*(U, \widetilde{M_{i+2}}) = 0 \quad \text{for} \quad 1 \leq i \leq r-2,$$

and keep the other assumptions of Theorem 47, we still get all conclusions of Theorem 47. Indeed, looking to the proof of (in particular the second claim of) Theorem 47, it suffices to prove that $M_{2T} \to M_2$ *is surjective and that* (44) *is exact. Now applying* $\mathrm{H}^0_*(U, \wedge^2 -)$ *to the exact sequence* (45)*, remarking that* $\widetilde{M_2}|_U \simeq \wedge^2 \widetilde{M_1}|_U$*, we get the exact sequence* $M_{2T} \to M_{2S} \to \mathrm{H}^1_*(U, \widetilde{M_2}) \otimes_k \mathfrak{a}$*. Since we have* $\mathrm{H}^1_*(U, \widetilde{M_2}) = 0$ *by the assumption* (i')*, we get the surjectivity of* $M_{2T} \to M_{2S}$ *and hence the surjectivity of* $M_{2T} \to M_2$ *by induction. Finally to show that* (44) *is exact, it suffices to show the surjectivity of* $\mathrm{H}^0_*(U, \widetilde{M_2}(-2s)) \to \mathrm{H}^0_*(U, \widetilde{Z}_1)$ *or the vanishing of* $\mathrm{H}^1_*(U, \widetilde{Z}_2)$*. Since the conditions of* (i') *lead to inclusions as in* (46) *provided we have replaced* Z_i *by* Z_{i+1}*, we get precisely* $\mathrm{H}^1_*(U, \widetilde{Z}_2) = 0$ *and we are done.*
Finally note that the assumption $\mathrm{H}^1_*(U, \widetilde{M_2}) = 0$ *in* (i') *is superfluous if we in proving the second claim above can show that there is a deformation* M'_{2T} *of* M_2 *which locally on* U *is* M_{2T} *for any* T*. Indeed the argument in the proof of Theorem 47 where we showed* $\mathrm{H}^0_*(U, \widetilde{M_{1T}}) \simeq M_{1T}$ *as consequence of* $\mathrm{H}^0_*(U, \widetilde{M_1}) \simeq M_1$ *and the fact that* M_{1T} *is a deformation of* M_1*, apply to get* $M'_{2T} \simeq M_{2T}$ *and hence the surjectivity of* $M_{2T} \to M_2$*.*

If we apply Theorem 47 under the assumptions of Theorem 8, we get Theorem 22 stated in the background section. Here we include a proof of Theorem 22.

Proof. It suffices to verify (i), (ii) and (iv) of Theorem 47. Firstly we show (ii). Note that (5) is given by applying $\mathrm{H}^0_*(U, -)$ onto the Koszul resolution induced by the regular section σ. Now splitting the exact sequence (5) into short exact sequences, we get $0 \to Z_i \to M_i(-is) \to Z_{i-1} \to 0$ where $Z_i = \ker(M_i(-is) \to M_{i-1}((1-i)s))$ for $i \geq 2$ and $Z_1(s) = K_1$. Hence ${}_0\mathrm{Ext}^2_B(M, K_1) = 0$ provided ${}_0\mathrm{Ext}^2_B(M, M_2(-s)) = 0$ and ${}_0\mathrm{Ext}^3_B(M, Z_2(s)) = 0$, while ${}_0\mathrm{Ext}^3_B(M, Z_2(s)) = 0$ provided

${}_0\mathrm{Ext}^3_B(M, M_3(-2s)) = 0$ and ${}_0\mathrm{Ext}^4_B(M, Z_3(s)) = 0$ etc. By the assumption (i) of Theorem 22, it suffices to show ${}_0\mathrm{Ext}^r_B(M, Z_{r-1}(s)) = 0$. Since $Z_{r-1} = M_r(-rs) = K_B(t-rs)$, cf. Theorem 8, we get this vanishing by Gorenstein duality. Hence we conclude by the assumption (i) of Theorem 22. Moreover (iv) of Theorem 47 is proven by the same argument, using (iii) of Theorem 22 (cf. Remark 21), and remarking that $Ass(I_{A/B}) \subset Ass(B)$. Note that the argument shows that ${}_0\mathrm{Ext}^i_R(I_B, M_i(-is)) = 0$ for $1 \le i \le r$ implies ${}_0\mathrm{Ext}^1_R(I_B, I_{A/B}) = 0$ and we conclude as required.

It remains to show (i) of Theorem 47 or (i') of Remark 50. If $\mathrm{depth}_{I(Z)} B = \dim B - \dim B/I(Z) \ge r+1$, we have

$$\mathrm{H}^i_*(U, \widetilde{M}_{i+1}) \simeq \mathrm{H}^{i+1}_{I(Z)}(M_{i+1}) = 0 \quad \text{for} \quad 1 \le i \le r-1$$

since all M_i are maximal CM B-modules by Theorem 8. If $\mathrm{depth}_{I(Z)} B = r$ and $r \ge 3$ we similarly verify (i') of Remark 50. In the final case $\mathrm{depth}_{I(Z)} B = r = 2$ we can use the two final sentences in Remark 50 to conclude. Indeed in this case $M_2 = K_B(t)$ and since $\mathrm{Ext}^j_R(B, R(-n-c)) = 0$ for $j \ne c$ and $K_B = \mathrm{Ext}^c_R(B, R(-n-c))$ we may take $M'_{2T} = K_{B_T} := \mathrm{Ext}^c_{R_T}(B_T, R_T(-n-c))$ and prove that K_{B_T} is a deformation of K_B to B_T by e.g. [20], Prop. (A1). This concludes the proof. $\square$

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