MAXIMAL ISOTROPIC SUBBUNDLES OF ORTHOGONAL BUNDLES OF ODD RANK OVER A CURVE

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Abstract. An orthogonal bundle over a curve has an isotropic Segre invariant determined by the maximal degree of a maximal isotropic subbundle. This invariant and the induced stratifications on moduli spaces of orthogonal bundles were studied for bundles of even rank in [4]. In this paper, we obtain analogous results for bundles of odd rank. We compute the sharp upper bound on the isotropic Segre invariant. Also we show the irreducibility of the induced strata on the moduli spaces of orthogonal bundles of odd rank, and compute their dimensions.

1. Introduction

Let X be a smooth irreducible algebraic curve of genus $g \geq 2$ over $\mathbb{C}$. A vector bundle V is called *orthogonal* if there is a nondegenerate symmetric bilinear form $\omega\colon V \otimes V \to \mathcal{O}_X$. A subbundle E of V is called *isotropic* if E_x is an isotropic subspace of V_x for each $x \in X$. For such E we have $\mathrm{rk}\, E \leq \left\lfloor \frac{\mathrm{rk}\, V}{2} \right\rfloor$. If equality holds then E will be called a *maximal isotropic subbundle*[1].

The aim of this paper is to study maximal isotropic subbundles of orthogonal bundles of odd rank over X. The results in this paper, together with those in [4] on orthogonal bundles of even rank, will provide a complete picture on the Segre-type stratifications corresponding to maximal isotropic subbundles on moduli spaces of orthogonal bundles.

The moduli space $MO_X(r)$ of orthogonal bundles of rank r and trivial determinant over X has two connected components, distinguished by the second Stiefel–Whitney class $w_2(V) \in H^2(X, \mathbb{Z}/2\mathbb{Z})$. We will denote the two components by $MO_X(r)^{\pm}$, where $MO_X(r)^{+}$ is the component containing the trivial orthogonal bundle. The tangent space of $MO_X(r)$ at a stable point V is given by $H^1(X, \wedge^2 V)$. Hence $\dim MO_X(r)^{\pm} = \frac{1}{2} r(r-1)(g-1)$.

For orthogonal V as above, recall that the *isotropic Segre invariant* for maximal isotropic subbundles is defined by

$$t(V) \ := \ -2 \max\{\deg E : E \text{ a maximal isotropic subbundle of } V\}.$$

The invariant $t(V)$ defines a natural stratification on $MO_X(r)$: For each even number t, we define

$$MO_X(r;t) \ := \ \{V \in MO_X(r) : t(V) = t\}.$$

Since the invariant $t(V)$ is semicontinuous, these subloci are constructible sets.

[1]Note that in [4], a "Lagrangian subbundle" indicated a maximal isotropic subbundle of a symplectic bundle, and in the terminology *maximal subbundle* and *maximal Lagrangian subbundle* used in [4], the word "maximal" referred to the degree.

Segre invariants of vector bundles, and the stratifications they define on moduli spaces, were studied in [1], [2], [5] and [16]; and the isotropic Segre invariants of symplectic bundles and orthogonal bundles of even rank were studied in [3, 4]. We refer the reader to [4, §1] for a more detailed introduction to isotropic Segre invariants. In the present article, we obtain parallel results for bundles of odd rank. For $n \geq 1$, we characterize the components $MO_X(2n+1)^{\pm}$ in terms of degrees of maximal isotropic subbundles (Theorem 3.10). We prove the irreducibility of each nonempty stratum $MO_X(2n+1;t)$ and compute the dimension (Theorem 5.6). This yields also a sharp upper bound on $t(V)$ for each component $MO_X(2n+1)^{\pm}$ (Corollary 5.8).

Moreover, for $V \in MO_X(2n+1;t)$, we denote by $M(V)$ the space of maximal isotropic subbundles E of V such that $-2\deg E = t(V)$. In Theorem 6.3 we compute the dimension of $M(V)$ for a general $V \in MO_X(2n+1;t)$.

Central to several proofs is a description of those rank $2n+1$ orthogonal bundles admitting a given rank n bundle as a maximal isotropic subbundle (Proposition 3.6). Another ingredient is a correspondence between maximal isotropic subbundles of a rank $2n+1$ orthogonal bundle and those of a certain rank $2n+2$ orthogonal bundle (Proposition 3.8), which allows one to exploit results on the even rank case from [4].

Notation: Throughout, X denotes a complex projective smooth curve of genus $g \geq 2$. If $0 \to E \to V \to F \to 0$ is an extension of vector bundles over X, we denote the class of V in $H^1(X, \mathrm{Hom}(F,E))$ by $[V]$. **For us, an "orthogonal bundle" will always have trivial determinant.**

2. Orthogonal bundles of even rank

In this section, we quote some results from [4] and [6] on orthogonal bundles of even rank, which are relevant for our later discussion.

Proposition 2.1. ([4, Theorem 1.2]) *Suppose $n \geq 2$, and let V be an orthogonal bundle of rank $2n$.*

(1) *Let F and $\tilde{F}$ be maximal isotropic subbundles of V. Then* $\deg F$ *and* $\deg \tilde{F}$ *have the same parity.*
(2) *The Stiefel–Whitney class $w_2(V)$ is trivial (resp., nontrivial) if and only if the maximal isotropic subbundles of V have even degree (resp., odd degree).*
(3) *Therefore, a semistable V belongs to $MO_X(2n)^+$ (resp., $MO_X(2n)^-$) if and only if its maximal isotropic subbundles have even degree (resp., odd degree).* □

Let F be a maximal isotropic subbundle of an orthogonal bundle V of rank $2n$. Since $V/F \cong (F^{\perp})^* = F^*$, the bundle V fits into an exact sequence $0 \to F \to V \to F^* \to 0$.

Proposition 2.2. ([6, Criterion 2.1]) *Let F be a simple bundle of rank n. An extension class $[V] \in H^1(X, \otimes^2 F)$ is induced by an orthogonal structure with respect to which F is maximal isotropic if and only if $[V]$ belongs to the subspace $H^1(X, \wedge^2 F)$.* □

Now suppose that an orthogonal bundle V of rank $2n$ has two different maximal isotropic subbundles F and $\tilde{F}$. Let H be the locally free part of the intersection.

Both subbundles F and $\tilde{F}$ lie inside $H^\perp$, inducing a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & H^\perp & \longrightarrow & H^\perp/F & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H & \longrightarrow & \tilde{F} & \longrightarrow & \tilde{F}/H & \longrightarrow & 0 \end{array} \tag{2.1}$$

It is easy to check that $H^\perp/H$ is an orthogonal bundle of rank $2(n-r)$ and F/H and $\tilde{F}/H$ are maximal isotropic subbundles, yielding a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F/H & \longrightarrow & (\tilde{F}/H)^* & \longrightarrow & \tau & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F/H & \longrightarrow & H^\perp/H & \longrightarrow & (F/H)^* & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & \tilde{F}/H & = & \tilde{F}/H & & \end{array} \tag{2.2}$$

where τ is torsion. Note that $\deg(\tilde{F}/H) \leq \deg(F/H)^*$. Therefore,

$$\deg H \geq \frac{1}{2}(\deg F + \deg \tilde{F}). \tag{2.3}$$

Let p_H denote the surjection $H^1(X, \wedge^2 F) \to H^1(X, \wedge^2(F/H))$ induced by the quotient map $F \to F/H$. Recall that if $Z \subseteq \mathbb{P}^N$ is a quasi-projective variety, then the kth secant variety $\mathrm{Sec}^k Z$ is the closure of the union of all linear spaces spanned by k general points of Z.

Proposition 2.3. *Let F be a general stable bundle of rank n and degree $-f < 0$. Let $\mathrm{Gr}(2, F)$ be the Grassmannian bundle over X of 2-dimensional subspaces of the fibers of F.*

1. ([4, Lemma 2.2 (2)]) *There is a canonical rational map $\phi_a \colon \mathrm{Gr}(2, F) \dashrightarrow \mathbb{P}H^1(X, \wedge^2 F)$, which is injective on a general fiber.*
2. ([4, Criterion 2.3 (2)]) *Let V be a rank $2n$ orthogonal bundle admitting F as a maximal isotropic subbundle, with $[V] \in H^1(X, \wedge^2 F)$. Then V has another maximal isotropic subbundle $\tilde{F}$ of degree $\geq -\tilde{f}$ inducing a diagram (2.1) for some $H \subset F$ of rank $\leq n-2$ and degree $-h$ if and only if*

$$p_H([V]) \in \mathrm{Sec}^k \mathrm{Gr}(2, F/H),$$

where $k = \frac{1}{2}(f + \tilde{f} - 2h) \geq 0$. In particular, H must satisfy (2.3). □

Remark 2.4. We mention some special cases: When $H = 0$, the statement yields a criterion for isotropic liftings of elementary transformations of F^*. If $\mathrm{rk}\,(F/H) = 2$ then $\mathrm{Gr}(2, F/H) \cong X$.

It was overlooked in [4] that the statement (2) is meaningless when $\mathrm{rk}\,(F/H) = 1$. But it is not difficult to show that this case does not arise for a general F. The corresponding case for orthogonal bundles of odd rank will be discussed in the last part of the proof of Proposition 5.5.

3. Orthogonal extensions

3.1. **Orthogonal bundles of odd rank as iterated extensions.** The goal of this section is to to find a criterion similar to that in Proposition 2.2 for constructing orthogonal bundles of odd rank as extensions.

Let E be any subbundle of an orthogonal bundle V. Then we obtain the sequence

$$0 \to E \to V \to (E^{\perp})^* \to 0.$$

The isotropy of E is by definition equivalent to $E \subseteq E^{\perp}$. In this case it is not hard to see that $E^{\perp}/E$ is an orthogonal bundle. In particular, if E is maximal isotropic then $E^{\perp}/E$ is isomorphic to $\mathcal{O}_X$.

Consider an arbitrary extension of vector bundles $0 \to E \xrightarrow{j} F \to \mathcal{O}_X \to 0$. For any extension V of F^* by E, consider the diagram

$$\begin{array}{ccccccccc} & & & & & & \mathcal{O}_X & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & V & \longrightarrow & F^* & \longrightarrow & 0 \\ & & {\scriptstyle j}\downarrow & & & & {\scriptstyle {}^t j}\downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & V^* & \longrightarrow & E^* & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & \mathcal{O}_X & & & & & & \end{array} \tag{3.1}$$

If V has an orthogonal structure with respect to which E is maximal isotropic, then $F \cong E^{\perp}$, and the induced isomorphism $\omega \colon V \to V^*$ fits into the above diagram to yield commutative squares. Furthermore, the induced map $\bar{\omega} \colon \mathcal{O}_X \to \mathcal{O}_X$ described by the Snake Lemma is a (symmetric) isomorphism.

Proposition 3.1. *Let V be a vector bundle fitting into (3.1), with extension class $[V] \in H^1(X, F \otimes E)$. Then there is a symmetric map $\omega \colon V \to V^*$ (not necessarily an isomorphism) extending j and ${}^t j$, if and only if the image of $[V]$ under the map $j_* \colon H^1(X, F \otimes E) \longrightarrow H^1(X, \otimes^2 F)$ lies on the subspace $H^1(X, \wedge^2 F)$.*

Proof. By a standard cohomological argument, there is a map ω extending j and ${}^t j$ if and only if

$$j_*[V] \;=\; {}^t j^*[V^*] \text{ in } H^1(X, \otimes^2 F). \tag{3.2}$$

It is well known that $[V^*] = -\,{}^t[V]$ in $H^1(X, E \otimes F)$. Hence

$${}^t j^*[V^*] \;=\; {}^t j^* \left(-\,{}^t[V]\right) \;=\; -\,{}^t \left(j_*[V]\right).$$

Therefore the condition (3.2) becomes $j_*[V] = -\,{}^t\left(j_*[V]\right)$, that is, $j_*[V] \in H^1(\wedge^2 F)$. A calculation shows that the symmetrization $\frac{1}{2}(\omega + {}^t\omega)$ of ω is a symmetric map with the required properties. □

Lemma 3.2. *Suppose E is stable of negative degree and $0 \to E \to F \to \mathcal{O}_X \to 0$ is a nontrivial extension. Then there is a commutative diagram*

$$\begin{array}{ccccccc} H^0(X, \mathcal{O}_X) & \xrightarrow{\partial} & H^1(X,E) & \longrightarrow & H^1(X,F) & \longrightarrow & \cdots \\ & & \rho\uparrow & & \uparrow & & \\ 0 & \longrightarrow & H^1(X, \wedge^2 F) & \longrightarrow & H^1(X, \otimes^2 F) & \longrightarrow & \cdots \\ & & \uparrow & & j_*\uparrow & & \\ \cdots & \longrightarrow & H^1(X, \wedge^2 E) & \longrightarrow & H^1(X, F \otimes E) & \longrightarrow & \cdots \end{array} \tag{3.3}$$

Moreover, j_* and the two maps from $H^1(X, \wedge^2 E)$ are injective.

Proof. It is easy to check that

$$(\wedge^2 F) \ \cap \ (F \otimes E) \ = \ \wedge^2 E \quad \text{and} \quad \frac{\wedge^2 F}{\wedge^2 E} \ \cong \ E,$$

the latter since $\operatorname{rk} E = \operatorname{rk} F - 1$. The existence of (3.3) follows. Since E is stable of negative degree, $h^0(X, E) = 0$, which shows the injectivity of the map $H^1(X, \wedge^2 E) \to H^1(X, \wedge^2 F)$. Since the composition $H^1(X, \wedge^2 E) \to H^1(X, \wedge^2 F) \to H^1(X, \otimes^2 F)$ is injective, so is the map $H^1(X, \wedge^2 E) \to H^1(X, F \otimes E)$. Finally, the injectivity of j_* would follow from the vanishing of $H^0(X, F)$. But a nonzero section of F would yield a splitting of $0 \to E \to F \to \mathcal{O}_X \to 0$, contrary to hypothesis. □

We write $\mathbb{C}_j$ for the image of $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$ inside $H^1(X, E)$, because it corresponds to the class of the extension $0 \to E \xrightarrow{j} F \to \mathcal{O}_X \to 0$ in $H^1(X, E)$, and the above diagram arises from j.

Proposition 3.3. *Inside the extension space $H^1(X, F \otimes E)$, the locus of extensions $[V]$ admitting a symmetric map $\omega \colon V \to V^*$ extending j and ${}^t j$ in the diagram (3.1) is identified with the preimage $\rho^{-1}(\mathbb{C}_j)$.*

Proof. By Proposition 3.1, this locus is identified with the intersection

$$j_* \left(H^1(X, F \otimes E) \right) \ \cap \ H^1(X, \wedge^2 F).$$

This coincides with the kernel of the composed map $H^1(X, \wedge^2 F) \to H^1(X, \otimes^2 F) \to H^1(X, F)$. By commutativity of (3.3), this kernel is precisely $\rho^{-1}(\mathbb{C}_j)$. □

Henceforth we write $\Pi_j := \rho^{-1}(\mathbb{C}_j)$. We have an exact sequence of vector spaces

$$0 \to H^1(X, \wedge^2 E) \to \Pi_j \to \mathbb{C}_j \to 0. \tag{3.4}$$

Thus we have a criterion for the existence of a symmetric map $\omega \colon V \to V^*$. Now we want to know when such a map is an isomorphism.

Lemma 3.4. *Let E be a stable bundle of negative degree, and consider an extension class $[V] \in \Pi_j$. Then the following statements are equivalent:*

(a) *The extension class $[V]$ belongs to $H^1(X, \wedge^2 E)$.*
(b) *The map $\mathcal{O}_X \to F^*$ lifts to V. In other words, there is a map $\mathcal{O}_X \to V$ whose composition with $V \to F^*$ coincides with the given map $\mathcal{O}_X \to F^*$.*
(c) *The map $\omega \colon V \to V^*$ is not an isomorphism.*

Proof. $(a) \Leftrightarrow (b)$: Firstly, consider the exact commmutative diagram

$$\begin{array}{ccccccc}
H^0(X,F) & \longrightarrow & H^1(X,E) & \longrightarrow & H^1\left(X, \frac{\otimes^2 F}{\otimes^2 E}\right) & \longrightarrow & H^1(X,F) \\
 & & {\scriptstyle\rho}\uparrow & & \uparrow & & \uparrow \\
 & & H^1(X,\wedge^2 F) & \longrightarrow & H^1(X,\otimes^2 F) & \longrightarrow & H^1(X,\mathrm{Sym}^2 F) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & H^1(X,\wedge^2 E) & \longrightarrow & H^1(X,\otimes^2 E) & \longrightarrow & H^1\left(X,\mathrm{Sym}^2 E\right)
\end{array}$$

As was seen in the proof of Lemma 3.2, we have $h^0(X,F) = 0$. It follows that

$$H^1(X, \otimes^2 E) \cap H^1(X, \wedge^2 F) \ = \ H^1(X, \wedge^2 E). \tag{3.5}$$

Now the map $\mathcal{O}_X \to F^*$ lifts to V if and only if $[V]$ belongs to

$$H^1(X, \otimes^2 E) \ = \ \mathrm{Ker}\left(H^1(X, F \otimes E) \to H^1(X,E)\right).$$

Since $[V]$ belongs to $\Pi_j \subset H^1(X, \wedge^2 F)$, this condition is equivalent to $[V] \in H^1(X, \wedge^2 E)$ in view of (3.5).

$(b) \Leftrightarrow (c)$: By the Snake Lemma, we know $\mathrm{Ker}\,\omega$ coincides with the kernel of the induced map $\bar{\omega} \colon \mathcal{O}_X \to \mathcal{O}_X$ in (3.1). Thus ω fails to be an isomorphism if and only if $\bar{\omega}$ is zero, which is equivalent to the lifting of $\mathcal{O}_X \to F^*$ to V. □

Remark 3.5. The bilinear form on any V satisfying the equivalent conditions of the lemma is degenerate. The class $[V]$ is of the form ${}^t j^* [V_0]$, where $[V_0] \in H^1(X, \wedge^2 E)$ defines an orthogonal extension in the sense of Proposition 2.2. Hence V is an extension $0 \to \mathcal{O}_X \to V \to V_0 \to 0$. However, if V_0 is semistable, then V is S-equivalent as a vector bundle to the orthogonal bundle $V_0 \perp \mathcal{O}_X$, which admits a nondegenerate symmetric form. If V_0 is stable as a vector bundle, then $V_0 \perp \mathcal{O}_X$ is a stable orthogonal bundle by Ramanan [14]. □

From the discussion in this section, we can conclude:

Proposition 3.6. *Let E be a stable bundle of rank n and negative degree. Let F be a nontrivial extension of $\mathcal{O}_X$ by E.*

(1) *An extension $0 \to E \to V \to F^* \to 0$ is induced by an orthogonal structure on V with respect to which E is maximal isotropic, if and only if $[V]$ belongs to the complement of $j_*\left(H^1(X, \wedge^2 E)\right)$ inside Π_j.*
(2) *If $[V']$ in $H^1(X, \wedge^2 E)$ defines a semistable bundle V', then the vector bundle defined by $j_*[V']$ is S-equivalent to a semistable orthogonal bundle.* □

In §4.3 we will discuss stability of such extensions.

3.2. **Interplay between even rank and odd rank.**

Here we exploit and generalize some well-known facts on orthogonal Grassmannians. Let $\mathrm{OG}(n, 2n+1)$ (resp., $\mathrm{OG}(n, 2n)$) be the *odd orthogonal Grassmannian* (resp., *even orthogonal Grassmannian*) parameterizing maximal isotropic subspaces of $\mathbb{C}^{2n+1}$ (resp., $\mathbb{C}^{2n}$).

Lemma 3.7.
(1) *The odd orthogonal Grassmannian $\mathrm{OG}(n, 2n+1)$ is irreducible.*
(2) *The even orthogonal Grassmannian $\mathrm{OG}(n, 2n)$ consists of two disjoint irreducible components $\mathrm{OG}(n, 2n)_1$ and $\mathrm{OG}(n, 2n)_2$ of the same dimension. Two maximal isotropic subspaces F and F' belong to the same component if and only if $\dim(\mathsf{F} \cap \mathsf{F}') \equiv n \mod 2$.*

(3) *There is a canonical isomorphism* $\mathrm{OG}(n, 2n+1) \xrightarrow{\sim} \mathrm{OG}(n+1, 2n+2)_i$ *for* $i = 1, 2$.

(4) *For* $\mathsf{E} \in \mathrm{OG}(n, 2n+1)$*, let* $\mathsf{F}_i \in \mathrm{OG}(n+1, 2n+2)_i$ *be the maximal isotropic subspace obtained via the above isomorphism. Then*

$$T_{\mathsf{E}}\mathrm{OG}(n, 2n+1) \;\cong\; T_{\mathsf{F}_i}\mathrm{OG}(n+1, 2n+2)_i \;\cong\; \wedge^2 \mathsf{F}_i^*.$$

Proof. The proofs of (1) and (2) can be found in Reid [15, §1].

To prove (3), let W be an orthogonal vector space of dimension $2n+2$ and V an orthogonal subspace of dimension $2n+1$. Given $\mathsf{F} \in \mathrm{OG}(n+1, \mathsf{W})_i$, write $\mathsf{E} := \mathsf{F} \cap \mathsf{V}$. Then E is isotropic of dimension n, since the dimension of an isotropic subspace cannot exceed $\frac{1}{2}\dim \mathsf{V}$. It is easy to see that the assignment $\mathsf{F} \mapsto \mathsf{E}$ yields a surjection $\mathrm{OG}(n+1, \mathsf{W})_i \to \mathrm{OG}(n, \mathsf{V})$. To show the injectivity, let F and F' be two maximal isotropic subspaces of W in the same component, both containing E. By (2), we have

$$n \;\leq\; \dim(\mathsf{F} \cap \mathsf{F}') \;\equiv\; n+1 \mod 2,$$

so $\mathsf{F} = \mathsf{F}'$.

The first isomorphism of (4) is immediate from (3). The second isomorphism of (4) is a well-known fact on even orthogonal Grassmannians. □

Now let $0 \to E \to V \to F^* \to 0$ be an orthogonal extension of rank $2n+1$ as in Proposition 3.6. By Proposition 3.1, the class $j_*[V]$ belongs to $H^1(X, \wedge^2 F)$. In view of Proposition 2.2, we obtain an orthogonal extension $0 \to F \to W \to F^* \to 0$ with $[W] = j_*[V]$. Note that the F in this sequence is a new copy of F distinct from $E^\perp \subset V$. Henceforth, we denote it F' for clarity. We obtain an exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & E & \longrightarrow & V & \longrightarrow & F^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & F' & \longrightarrow & W & \longrightarrow & F^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_X & = & \mathcal{O}_X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.6}$$

Proposition 3.8. (1) *The bundle* W *is an orthogonal direct sum* $V \perp \mathcal{O}_X$. *The subbundle* F' *is maximal isotropic in* W *and satisfies* $F' \cap V = E$.

(2) *For every maximal isotropic subbundle* G *of* W*, the intersection* $G \cap V$ *is a maximal isotropic subbundle of* V *with* $\det(G \cap V) = \det G$.

(3) *The association* $G \mapsto G \cap V$ *defines a surjective* $2:1$ *map*

$$\{\textit{maximal isotropic subbundles of } W\} \longleftrightarrow \{\textit{maximal isotropic subbundles of } V\}.$$

Proof. (1) The only statement not clear from the diagram and the discussion before it is that $W = V \perp \mathcal{O}_X$. It is easy to see that the orthogonal complement $V^\perp \subset W$ is isomorphic to $\mathcal{O}_X$. Since the form on V is nondegenerate, $V \cap V^\perp$ is everywhere zero. Hence $V^\perp$ is not isotropic, and is therefore isomorphic to $\mathcal{O}_X$ as an orthogonal bundle. From the fact that $V \cap V^\perp = 0$ it also follows that $W = V \perp \mathcal{O}_X$.

Statement (2) follows from the linear algebra in Lemma 3.7 (3) and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & W & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & G \cap V & \longrightarrow & G & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0. \end{array}$$

As for (3): Suppose E is a maximal isotropic subbundle of V. Consider a trivalization of W over an open subset $U \subseteq X$. By Lemma 3.7 (3), the bundle $E|_U$ can be completed to a maximal isotropic subbundle of $W|_U$ in exactly two ways. Since $\dim X = 1$, each such completion admits a unique extension to the whole of X. Therefore, the map $G \mapsto G \cap V$ is two-to-one. □

Remark 3.9. Given $E \xrightarrow{j} E^\perp = F \subset V$, the two maximal isotropic subbundles $G \subset W$ satisfying $G \cap V = E$ are exchanged by the orthogonal automorphism of $W = V \perp \mathcal{O}_X$ given by $(v, \lambda) \mapsto (v, -\lambda)$.

We can now deduce a topological characterization of the two components of $MO_X(2n+1)$, from the analogous one stated in Proposition 2.1 for bundles of even rank:

Theorem 3.10. (1) *Let E_1 and E_2 be maximal isotropic subbundles of an orthogonal bundle $V \in MO_X(2n+1)$. Then $\deg E_1$ and $\deg E_2$ have the same parity.*

(2) *A semistable orthogonal bundle V belongs to $MO_X(2n+1)^+$ (resp., $MO_X(2n+1)^-$) if and only if its maximal isotropic subbundles have even degree (resp., odd degree).*

Proof. This follows from Proposition 2.1 and Proposition 3.8 (2) and (3). □

In a similar way, we obtain an upper bound on $t(V)$:

Proposition 3.11. *For any orthogonal bundle V of rank $2n + 1$, we have $t(V) \leq (n + 1)(g - 1) + 3$.*

Proof. Let $W = V \perp \mathcal{O}_X$. By [4, Theorem 1.3], the bundle W has a maximal isotropic subbundle F with $-2 \deg F \leq (n + 1)(g - 1) + 3$. By Proposition 3.8 (2), the bundle $E = F \cap V$ is a maximal isotropic subbundle of V with $\deg E = \deg F$. Hence $t(V) \leq (n + 1)(g - 1) + 3$. □

In §5, we will show that this upper bound is sharp.

4. Parameter spaces of orthogonal extensions

4.1. **Construction of the parameter space.** Firstly, we construct parameter spaces for certain orthogonal extensions of the form $0 \to E \to V \to F^* \to 0$, where E is a bundle of rank n and degree $-e < 0$ and F is an extension $0 \to E \xrightarrow{j} F \to \mathcal{O}_X \to 0$. The parameter space will be obtained by deforming the space $\mathbb{P}\Pi_j$ obtained in the previous section in a suitable way.

Let $U_X^s(n, -e)$ be the moduli space of stable bundles of rank n and degree $-e$. This is a quasiprojective irreducible variety of dimension $n^2(g - 1) + 1$.

Proposition 4.1. *Let e be a positive integer.*

(1) *There exists a finite étale cover* $\xi_e \colon \tilde{U}_e \to U_X^s(n, -e)$ *and a double fibration*

$$\mathbb{A}_e \xrightarrow{\pi_e} J_e \xrightarrow{\tau_e} \tilde{U}_e$$

such that the fiber $\tau_e^{-1}(\bar{E})$ *with* $\xi_e(\bar{E}) = E$ *is isomorphic to the (projectivized) extension space* $\mathbb{P}H^1(X, E)$*, and the fiber of* π_e *at* $j \in \mathbb{P}H^1(X, E)$ *is isomorphic to* $\mathbb{P}\Pi_j$*.*

(2) *There is a bundle* $\mathcal{V}_e$ *over* $\mathbb{A}_e \times X$ *such that for each* $j \in \mathbb{P}H^1(X, E)$ *and* $[V] \in \mathbb{P}\Pi_j$*, the restriction of* $\mathcal{V}_e$ *to* $\{[V]\} \times X$ *is isomorphic to the orthogonal bundle* V*.*

(3) *The variety* $\mathbb{A}_e$ *has dimension* $\frac{1}{2}n(3n + 1)(g - 1) + ne$*.*

Proof. (1) We follow the construction in [4, §3.1]. By Narasimhan–Ramanan [13, Proposition 2.4], there exists a finite étale cover $\xi_e \colon \tilde{U}_e \to U_X^s(n, -e)$ together with a bundle $\mathcal{E} \to \tilde{U}_e \times X$, such that the restriction of $\mathcal{E}$ to $\{\bar{E}\} \times X$ is isomorphic to the bundle $E = \xi_e(\bar{E})$. By a standard process with universal extensions, we find a fibration $\tau_e \colon J_e \to \tilde{U}_e$ with fiber $\tau_e^{-1}(\bar{E}) = \mathbb{P}H^1(X, E)$, where $E = \xi_e(\bar{E})$. Furthermore, write $p_J \colon J_e \times X \to J_e$ and $p_X \colon J_e \times X \to X$ for the projections. Over $J_e \times X$ there is an exact sequence of bundles

$$0 \to \mathcal{E} \to \mathcal{F} \to p_X^* \mathcal{O}_X \to 0,$$

whose restriction to $\{j\} \times X$ is the extension $0 \to E \xrightarrow{j} F \to \mathcal{O}_X \to 0$.

Globalizing (3.3), we consider the sheaf

$$\mathrm{Ker}\left[R^1(p_J)_*(\wedge^2 \mathcal{F}) \longrightarrow R^1(p_J)_*(\otimes^2 \mathcal{F}) \longrightarrow R^1(p_J)_*(\mathcal{F} \otimes p_X^* \mathcal{O}_X) \right].$$

By Lemma 3.2, the restriction of this kernel to each $j \in J_e$ is of constant dimension. Hence we obtain a vector bundle over J_e whose fiber at j is Π_j. We denote the corresponding projective bundle over J_e by $\mathbb{A}_e$.

(2) The existence of the universal extension $\mathcal{V}_e$ over $\mathbb{A}_e \times X$ follows from Lange [8, Corollary 4.5].

(3) By Lemma 3.2, we have $\dim \mathbb{P}\Pi_j = h^1(X, \wedge^2 E)$. Since E is stable of negative degree, $h^1(X, E) = e + n(g - 1)$ and $h^1(X, \wedge^2 E) = (n - 1)e + \frac{1}{2}n(n - 1)(g - 1)$. Thus $\dim \mathbb{A}_e$ is given by

$$\dim \tilde{U}_e + \dim \mathbb{P}H^1(X, E) + \dim \mathbb{P}\Pi_j = \frac{1}{2}n(3n + 1)(g - 1) + ne.$$

□

4.2. **Rank 3 case.** Consider an orthogonal bundle V of rank 3. In this case, E is a line bundle and $\wedge^2 E = 0$. Therefore, $\Pi_j \cong \mathbb{C}$, corresponding to the extension class $j \in H^1(X, E)$. In other words, for each rank two extension $0 \to E \xrightarrow{j} F \to \mathcal{O}_X \to 0$, there is, up to isomorphism, a unique orthogonal bundle V containing a maximal isotropic subbundle E with $E^\perp \cong F$. The parameter space $\mathbb{A}_e$ coincides with J_e, which admits a fibration $\tau_e \colon J_e \to \mathrm{Pic}^{-e}(X)$ with fiber $\tau_e^{-1}(E) = \mathbb{P}H^1(X, E)$. Note that $\dim \mathbb{A}_e = e + 2g - 2$. From Mumford [12, p. 185] we recall the following statement:

Lemma 4.2. *Every orthogonal bundle* V *of rank 3 is of the form* $L \otimes S^2 F$*, where* L *is a line bundle and* F *is a rank 2 bundle with* $\det F \cong L^*$*.*

Proof. The bundle $V := L \otimes S^2 F$ is orthogonal, due to the symmetric isomorphism

$$V^* \cong L^* \otimes S^2(F^*) \cong L^* \otimes S^2(F \otimes L) \cong L \otimes S^2 F = V.$$

Let V be an orthogonal bundle of rank 3 with a maximal isotropic subbundle E and write $F := E^{\perp}$. Tensoring the exact sequence

$$0 \to E \otimes F \to S^2 F \to \mathcal{O}_X \to 0,$$

by E^*, we obtain $0 \to F \to E^* \otimes S^2 F \to E^* \to 0$. By the uniqueness discussed before the statement of the lemma, V is isomorphic to $E^* \otimes S^2 F$. □

4.3. **Stability of general bundles.** The space $\mathbb{A}_e$ parameterizes those orthogonal bundles of rank $2n+1$ which admit maximal isotropic subbundles of degree $-e$. By the universal property, there is a rational map $\alpha_e \colon \mathbb{A}_e \dashrightarrow MO_X(2n+1)$. Our next step will be to show that a general point of $\mathbb{A}_e$ corresponds to a stable orthogonal bundle. This will imply that the maps α_e are defined on dense subsets.

To proceed, we need a generalization of Lange–Narasimhan [10, Proposition 1.1]:

Proposition 4.3. *Let E be a stable bundle of negative degree.*

(1) ([4, Lemma 2.2 (2)]) *There is a canonical rational map $\phi \colon \mathbb{P}E \dashrightarrow \mathbb{P}H^1(X, E)$ which is injective on a general fiber.*
(2) (*Adaptation of* [2, Theorem 4.4]) *Let F be an extension of $\mathcal{O}_X$ by E. If there is a subsheaf $\mathcal{O}_X(-D)$ of $\mathcal{O}_X$ lifting to F for some effective divisor D, then the class $[F]$ lies on the kernel of the surjection $H^1(X, E) \to H^1(X, E(D))$. In particular, it lies on $\mathrm{Sec}^d \mathbb{P}E$, where $d = \deg D$.* □

Proposition 4.4. *For $0 < e < \frac{1}{2}(n+1)(g-1)$, a general bundle represented in $\mathbb{A}_e$ is a stable orthogonal bundle.*

Proof. Firstly, we consider the case $n = 1$. Suppose V_0 is an orthogonal bundle of rank 3 and $G \subset V_0$ an isotropic line subbundle of nonnegative degree. From the sequence $0 \to E \to V_0 \to F^* \to 0$ and its dual sequence, we see that G is a subsheaf of both F^* and E^*. From the sequence

$$0 \to \mathcal{O}_X \to F^* \xrightarrow{{}^t j} E^* \to 0,$$

it follows that $G \cong E^*(-D)$ for some D of degree $\leq e$, and that G lifts to a subsheaf of F^*. By Proposition 4.3 (2), we have $[{}^t j] \in \mathrm{Sec}^e X$ for the embedded curve $\mathbb{P}E = X \subset \mathbb{P}H^1(X, E)$. But since $e < g - 1$ by hypothesis,

$$\dim \mathrm{Sec}^e X \ \leq \ 2e - 1 \ < \ e + g - 2 \ = \ \dim \mathbb{P}H^1(X, E).$$

Thus a general extension class $[{}^t j]$ yields a stable orthogonal bundle in Π_j.

Now suppose $n \geq 2$. Let $E \in U_X^s(n, -e)$ be general, and let $0 \to E \xrightarrow{j} F \to \mathcal{O}_X \to 0$ be an extension. Let $0 \to E \to \tilde{V}_0 \to E^* \to 0$ be an extension whose class $[\tilde{V}_0]$ is a general point of $H^1(X, \wedge^2 E)$. By Proposition 2.2, the bundle V_0 admits an orthogonal structure.

Consider now the orthogonal extension $0 \to E \to V_0 \to F^* \to 0$ defined by $[V_0] = {}^t j^* [\tilde{V}_0]$. We obtain a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & \tilde{V}_0 & \longrightarrow & E^* & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow {}^t j & & \\
0 & \longrightarrow & E & \longrightarrow & V_0 & \longrightarrow & F^* & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{O}_X & = & \mathcal{O}_X & &
\end{array}$$

By Lemma 3.4 and Remark 3.5, there is an exact sequence

$$0 \to \mathcal{O}_X \to V_0 \xrightarrow{\omega} V_0^* \to \mathcal{O}_X \to 0$$

where ω defines a degenerate symmetric form on V_0 (the pullback of the form on $\tilde{V}_0$). By Lemma 3.4, however, a generic deformation of V_0 in Π_j admits an orthogonal structure.

Suppose there is an isotropic subbundle G of V_0 of nonnegative degree. Then we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & V_0 & \longrightarrow & \tilde{V}_0 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & G_1 & \longrightarrow & G & \longrightarrow & G_2 & \longrightarrow & 0 \end{array}$$

where G_1 is either zero or $\mathcal{O}_X$. Since $[\tilde{V}_0]$ is general, $\tilde{V}_0$ is a stable orthogonal bundle by [4, Theorem 3.1]. If G_2 is nonzero, then it is isotropic in $\tilde{V}_0$ since G is isotropic in V_0. Hence $\deg G = \deg G_2 < 0$. This shows that the only destabilizing subbundle of V_0 is $\mathcal{O}_X$.

Now we deform V_0 in Π_j to get a family $\{V_\lambda\}$ whose general member is an orthogonal bundle lying on $\mathbb{A}_e$. By semicontinuity, a generic deformation V_λ of V_0 in Π_j can possibly be destabilized only by a line bundle which is a deformation of $\mathcal{O}_X \subset V_0$. Note that this at the same time yields a deformation of $\mathcal{O}_X$ in F^*. Such deformations are parameterized by $H^0(X, \mathrm{Hom}(\mathcal{O}_X, F^*/\mathcal{O}_X)) \cong H^0(X, E^*)$. Since

$$\mu(E^*) \; = \; \frac{e}{n} \; < \; \frac{n+1}{2n}(g-1) \; \leq \; g-1,$$

we have $h^0(X, E^*) = 0$ for a general $E \in U_X^s(n, -e)$. Thus $\mathcal{O}_X$ does not deform nontrivially in F^*. Since by Lemma 3.4 a general deformation V_λ does not have a lifting of $\mathcal{O}_X \subset F^*$, we conclude that it is a stable orthogonal bundle in $\mathbb{A}_e$. □

We remark that a general $V \in \mathbb{A}_e$ is stable also for $\frac{1}{2}(n+1)(g-1) \leq e \leq \frac{1}{2}\left((n+1)(g-1)+3\right)$, as will be shown in next section. We expect the same is true for larger values of e, but we do not require this for the present applications.

5. The Segre stratification

Suppose $1 \leq e \leq \frac{1}{2}\left((n+1)(g-1)+3\right)$, and let $E \in U_X^s(n, -e)$ be general. Let $0 \to E \xrightarrow{j} F \to \mathcal{O}_X \to 0$ be a general extension. The goal of this section is to show that for a general orthogonal extension V represented in Π_j, the maximal isotropic subbundle E lies in $M(V)$. This will confirm that $t(V) = -2 \cdot \deg E$.

Consider an orthogonal bundle V of rank $2n+1$ admitting two different maximal isotropic subbundles E and $\tilde{E}$, with orthogonal complements F and $\tilde{F}$ respectively. Let I and H be the locally free parts of $E \cap \tilde{E}$ and $F \cap \tilde{F}$ respectively.

Lemma 5.1. *The subsheaf $I \subset H$ is a subbundle of corank 1.*

Proof. Since it suffices to give a fiberwise argument, we will regard E and F as vector spaces in the discussion below. We have

$$F \cap \tilde{F} \; = \; E^\perp \cap \tilde{E}^\perp \; = \; (E + \tilde{E})^\perp.$$

Thus if $\dim(E \cap \tilde{E}) = r$, then $\dim(E + \tilde{E}) = 2n - r$ and $\dim(F \cap \tilde{F}) = r + 1$. □

Corollary 5.2. *In the above context, $H/I \cong \mathcal{O}_X(-D)$ for some effective divisor D with $\deg D = \deg I - \deg H$. Furthermore, some extension H of $\mathcal{O}_X(-D)$ by I lifts to F if and only if the class $j \in H^1(X,E)$ belongs to*

$$\mathrm{Ker}\left[H^1(X,E) \to H^1(X,E/I) \to H^1(X,(E/I)(D))\right].$$

Proof. As a consequence of Lemma 5.1, we obtain the following diagram for a torsion sheaf τ_D associated to some effective divisor D:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E/I & \longrightarrow & F/H & \longrightarrow & \tau_D & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & H & \longrightarrow & H/I & \longrightarrow & 0 \end{array} \tag{5.1}$$

From this it is clear that H/I is of the form $\mathcal{O}_X(-D)$ as stated.

For the rest: It is easy to see that some extension $0 \to I \to H \to \mathcal{O}_X(-D) \to 0$ lifts to F if and only if $\mathcal{O}_X(-D) \subseteq \mathcal{O}_X$ lifts to F/I. This is equivalent to the statement that j belongs to $\mathrm{Ker}\left[H^1(X,E) \to H^1(X,E/I) \to H^1(X,(E/I)(D))\right]$. □

Recall now that $H^1(X, \wedge^2 F)$ admits the filtration

$$H^1(X, \wedge^2 E) \ \subset \ \Pi_j \ \subset \ H^1(X, \wedge^2 F).$$

We write $\widetilde{p_H}$ for the restriction to Π_j of the natural surjection $p_H \colon H^1(X, \wedge^2 F) \to H^1(X, \wedge^2 (F/H))$.

Lemma 5.3. *The map $\widetilde{p_H} \colon \Pi_j \to H^1(X, \wedge^2(F/H))$ is surjective.*

Proof. It suffices to show that the restriction of $\widetilde{p_H}$ to the subspace $H^1(X, \wedge^2 E)$ of Π_j is surjective. By (5.1), we have a commutative diagram

$$\begin{array}{ccc} H^1(X, \wedge^2 E) & \longrightarrow & H^1(X, \wedge^2 F) \\ \downarrow & & \downarrow \\ H^1(X, \wedge^2 (E/I)) & \longrightarrow & H^1(X, \wedge^2(F/H)). \end{array}$$

where the composition $H^1(X, \wedge^2 E) \to H^1(X, \wedge^2(E/I)) \to H^1(X, \wedge^2(F/H))$ is surjective. The statement follows by commutativity of the diagram. □

Now we will obtain a modification of Proposition 2.3 (2). Let E be a general stable bundle of rank n and degree $-e < 0$. Let V be an orthogonal bundle of rank $2n+1$, admitting E as a maximal isotropic subbundle, so that $E^\perp$ is an extension $0 \to E \xrightarrow{j} F \to \mathcal{O}_X \to 0$.

Lemma 5.4. *Suppose that V has another maximal isotropic subbundle $\tilde{E}$ of degree $-\tilde{e}$ with $\tilde{E}^\perp = \tilde{F}$. As before, write I and H for the locally free parts of $E \cap \tilde{E}$ and $F \cap \tilde{F}$ respectively, and write $\mathrm{rk}\, I = r$ and $\deg H = -h$.*

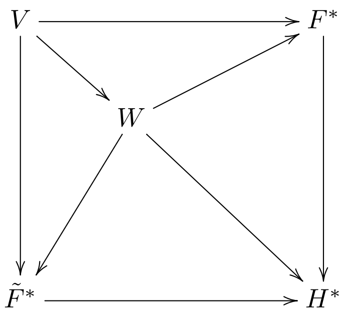
1. *The image of the class j under the surjection $H^1(X,E) \to H^1(X,E/I)$ lies inside the affine cone of $\mathrm{Sec}^d \mathbb{P}(E/I)$, where $d = \deg I + h \geq 0$.*
2. *Write $k := \frac{1}{2}(e + \tilde{e} - 2h)$. (Note that $e + \tilde{e} \equiv 0 \mod 2$, by Theorem 3.10 (1).) Then $k \geq 0$.*

(3) *If* $n \geq 2$ *and* $r \leq n-2$*, the image of the class* $[V]$ *under* $\widetilde{p_H}\colon \Pi_j \to H^1(X, \wedge^2(F/H))$ *lies inside the affine cone of* $\mathrm{Sec}^k \mathrm{Gr}(2, F/H)$*.*

In particular, when E *and* $\tilde{E}$ *intersect generically in zero, the above statements hold with* $\deg I = 0$ *and* $r = 0$*. When* $n = 1$*, parts (1) and (2) hold with* $I = 0$*.*

Proof. Statement (1) follows from a geometric interpretation of Corollary 5.2 by using Proposition 4.3.

For the rest: By Proposition 3.8 (1), the orthogonal bundle $W = V \perp \mathcal{O}_X$ contains two maximal isotropic subbundles F' and $\tilde{F}'$ isomorphic to F and $\tilde{F}$ respectively. We claim that $F' \cap \tilde{F}'$ has locally free part isomorphic to H in W. To see this, note that by Lemma 5.1 and the diagram (3.6) there exists a commutative diagram



Dualizing, we see that F' and $\tilde{F}'$ in $W \cong W^*$ intersect in a copy of H. Now (2) and (3) follow from Proposition 2.3 (2). □

Now we can prove the following statement.

Proposition 5.5. *Assume that* E *is general in* $U_X(n, -e)$ *and that* F *is a general extension of* $\mathcal{O}_X$ *by* E*. For* $0 < e < \frac{1}{2}(n+1)(g-1)$*, a general orthogonal extension* $0 \to E \to V \to F^* \to 0$ *has no maximal isotropic subbundle of degree* $\geq -e$ *other than* E*. Therefore,* $t(V) = 2e$ *and* $M(V) = \{E\}$*.*

Proof. We will prove the proposition in the following way: Recall the double fibration $\mathbb{A}_e \xrightarrow{\pi_e} J_e \xrightarrow{\tau} \tilde{U}_e$ described in Proposition 4.1. By Proposition 3.6, for fixed $\bar{E} \in \tilde{U}_e$ mapping to $E \in U_X(n, -e)$, the fiber $(\tau_e \circ \pi_e)^{-1}(\bar{E})$ parameterizes all orthogonal bundles V containing E as a maximal isotropic subbundle. We will show that for general E, the locus of such V containing a maximal isotropic subbundle of degree $\geq -e$ apart from the original E has positive codimension in $(\tau_e \circ \pi_e)^{-1}(\bar{E})$.

Assume, then, that an orthogonal extension $0 \to E \to V \to F^* \to 0$ has another maximal isotropic subbundle $\tilde{E}$ of degree $-\tilde{e} \geq -e$. Let $\tilde{F}, I$ and H be as in Lemma 5.4, with $\mathrm{rk}\, I = r$ and $\deg H = -h$.

Firstly, suppose $n = 1$, so $r = 0$. By (5.1), we see that $H = \mathcal{O}_X(-D)$ and $d = h$. Here also $\mathbb{P}E \cong X$. By Lemma 5.4 (1), the class $j \in \mathbb{P}H^1(X, E)$ lies inside $\mathrm{Sec}^h X$, where $h \leq \frac{1}{2}(e + \tilde{e}) \leq e$. As $e < g - 1$, we have

$$\dim \mathrm{Sec}^h X \;\leq\; 2e - 1 \;<\; e + g - 2 \;=\; \dim \mathbb{P}H^1(X, E).$$

Thus a general j lies outside $\mathrm{Sec}^h X$ in $H^1(X, E)$, meaning that $M(V) = \{E\}$.

Now suppose $n \geq 2$ and $0 \leq r \leq n-2$. To bound the dimension of those V containing such an $\tilde{E}$, we need to compute four dimensions. Firstly, by Lemma 5.4

(1), the class j varies inside an algebraic subset of dimension bounded by

$$D_1 := \dim \mathrm{Sec}^d \mathbb{P}(E/I) + 1 + \dim \mathrm{Ker}\left[H^1(X,E) \to H^1(X, E/I) \right].$$

We have $\dim \mathrm{Sec}^d \mathbb{P}(E/I) \leq d(n-r+1) - 1 = (\deg I + h)(n-r+1) - 1$. Moreover, the dimension of the fiber of $H^1(X,E) \to H^1(X,E/I)$ is bounded by $h^1(X,I) = -\deg I + r(g-1)$, since $h^0(X,I) \leq h^0(X,E) = 0$. Thus we have

$$D_1 \leq (\deg I + h)(n-r+1) - \deg I + r(g-1).$$

Secondly, $\dim \mathrm{Gr}(2, F/H) = 2(n-r-2)+1$ and

$$D_2 := \dim \mathrm{Sec}^{\frac{1}{2}(e+\tilde{e}-2h)} \mathrm{Gr}(2, F/H) \leq (e+\tilde{e}-2h)(n-r-1) - 1.$$

Thirdly, since $\mathrm{rk}(\wedge^2(F/H)) = \frac{1}{2}(n-r)(n-r-1)$ and $\deg(\wedge^2(F/H)) = -(n-r-1)(e-h)$, we have

$$\begin{aligned} D_3 &:= \dim \Pi_j - h^1(\wedge^2(F/H)) \\ &\leq \dim \Pi_j - (n-r-1)(e-h) - \frac{(n-r)(n-r-1)}{2}(g-1). \end{aligned}$$

By Lemma 5.4 (3), for fixed j, the classes $[V]$ vary in a locus of dimension $(D_2 + 1) + D_3$.

Finally, the subbundle I of E varies in a Quot scheme whose dimension is $h^0(X, \mathrm{Hom}(I, E/I))$. By Laumon [9, Proposition 3.5], since E is general, it is very stable. Thus by Lange–Newstead [11, Lemma 3.3] we have $h^1(X, \mathrm{Hom}(I, E/I)) = 0$ for all subbundles $I \subset E$. Thus we compute

$$D_4 := h^0(X, \mathrm{Hom}(I, E/I)) = -n \cdot \deg I - re - r(n-r)(g-1).$$

We now compare the sum $S_1 = D_1 + (D_2+1) + D_3 + D_4$ with the dimension S_2 of the fiber $(\tau_e \circ \pi_e)^{-1}(\bar{E})$. We have

$$S_2 := \dim \Pi_j + \dim H^1(X,E) = \dim \Pi_j + e + n(g-1).$$

Computing, we find that $S_1 < S_2$ if

$$2h - (r+1)e + \tilde{e}(n-r-1) - r \cdot \deg I < \frac{(n-r)(n+r+1)}{2}(g-1). \tag{5.2}$$

From Corollary 5.2 it follows that $-\deg I \leq h$. Furthermore, $\tilde{e} \leq e$ by hypothesis. Thus by Lemma 5.4 (2), we have $h \leq \frac{1}{2}(e+\tilde{e}) \leq e$. Applying these inequalities, we see that the expression on the left is bounded above by

$$2e - (r+1)e + e(n-r-1) + re = e(n-r).$$

Thus the required inequality (5.2) would follow from

$$e < \frac{(n+r+1)(g-1)}{2}. \tag{5.3}$$

This is satisfied for $r \geq 0$, since $e < \frac{1}{2}(n+1)(g-1)$ by hypothesis.

Now we need only to deal with the case when $r = n-1 \geq 1$. Since E is general,

$$-(n-1)e - n \cdot \deg I \geq (n-1)(g-1).$$

Combining with the inequality $-\deg I \leq h \leq e$, we get $(n-1)(g-1) \leq e$. From the hypothesis $e < \frac{1}{2}(n+1)(g-1)$, we have $n < 3$. Thus it remains to consider the case when $n = 2$ and $r = \mathrm{rk}\, I = 1$. In this case, we claim that given a general E, the extensions $j \in H^1(X,E)$ admitting a diagram of the form (5.1) are special. Indeed, from the previous computations, the dimension of the locus of such extensions is bounded by $D_1 + D_4$, which is computed in this case as:

$$D_1 + D_4 \ \leq \ [-\deg I + (g-1)] \ + \ [-2\deg I - e - (g-1)] \ = \ e - 3\deg I \ \leq \ 2e.$$

On the other hand, $h^1(X, E) = e + 2(g-1)$. From the assumption $e < \frac{3}{2}(g-1)$, we get

$$D_1 + D_4 \ \leq \ 2e \ < \ e + \frac{3}{2}(g-1) \ < \ h^1(X, E).$$

This confirms the claim. □

Now we consider the consequences of Proposition 5.5. By Proposition 4.4, for $e < \frac{1}{2}(n+1)(g-1)$, a general point $V \in \mathbb{A}_e$ represents a stable orthogonal bundle. As discussed in §4.3, there is a rational moduli map $\alpha_e \colon \mathbb{A}_e \dashrightarrow MO_X(2n+1)$.

Theorem 5.6. (1) *For each even number t with $0 < t < (n+1)(g-1)$, the stratum $MO_X(2n+1; t)$ is nonempty and irreducible of dimension equal to $\frac{1}{2}n(3n+1)(g-1) + \frac{1}{2}nt$.*

(2) *For $(n+1)(g-1) \leq t \leq (n+1)(g-1) + 3$, the stratum $MO_X(2n+1; t)$ is dense in a component $MO_X(2n+1)^{\pm}$.*

Proof. (1) For $t = 2e$ in the range $0 < t < (n+1)(g-1)$, the stratum $MO_X(2n+1; t)$ is nonempty, by Proposition 5.5. It contains the image $\alpha_e(\mathbb{A}_e)$, which is irreducible. To show the irreducibility of $MO_X(2n+1; t)$, we need to show that $\alpha_e(\mathbb{A}_e)$ is dense in $MO_X(2n+1; t)$. Any bundle $V_0 \in MO_X(2n+1; t)$ contains a maximal isotropic subbundle E_0 of degree $-e$, which might be unstable. Considering a versal deformation of E, we can find a one-parameter family $\{E_\lambda\}$ containing E_0 with general E_λ being stable. Along this family, we can build a family $\{V_\lambda\}$ of orthogonal bundles such that $E_\lambda \in M(V_\lambda)$, using the parameter space in Proposition 4.1. Since a general orthogonal bundle in this family is represented in $\mathbb{A}_e$, the bundle $V_0 \in MO_X(2n+1)$ lies on the closure of $\alpha_e(\mathbb{A}_e)$, as required.

For $t = 2e < (n+1)(g-1)$, the map α_e is generically finite, by Proposition 5.5. Therefore, $MO_X(2n+1; t)$ has the same dimension as $\mathbb{A}_e$. By the computation in Proposition 4.1 (3), this is $\frac{1}{2}n(3n+1)(g-1) + \frac{1}{2}nt$.

(2) For $t < (n+1)(g-1)$ we have $\dim MO_X(2n+1; t) < \dim MO_X(2n+1)$, while we know that $t(V) \leq (n+1)(g-1)+3$ by Proposition 3.11. Therefore, the two strata corresponding to even numbers in the range $(n+1)(g-1) \leq t \leq (n+1)(g-1)+3$ must be nonempty and dense in the components $MO_X(2n+1)^{\pm}$. □

Remark 5.7. In particular, the last statement shows a general bundle represented in either of the corresponding parameter spaces $\mathbb{A}_e$ must be stable. □

Together with Proposition 3.11, we get the following sharp upper bound on $t(V)$.

Corollary 5.8. *For any orthogonal bundle V of rank $2n+1$, we have $t(V) \leq (n+1)(g-1)+3$. This bound is sharp in the sense that the two even numbers t with $(n+1)(g-1) \leq t \leq (n+1)(g-1)+3$ correspond to the values of $t(V)$ for a general $V \in MO_X(2n+1)^{\pm}$.* □

Remark 5.9. Holla and Narasimhan [7] proved a bound on generalized Segre invariants for arbitrary principal bundles. For our case, their upper bound reads:

$$t(V) \ \leq \ \frac{n(n+1)}{n-1} g.$$

From the computation in Proposition 5.5 we now deduce a statement for generic orthogonal bundles of odd rank, analogous to Lange–Newstead [11, Theorem 2.3] for vector bundles and [4, Theorem 4.1 (3)] for symplectic bundles:

Corollary 5.10. *As before, write* $\deg E = -e$. *Suppose that* $g \geq 5$ *and*

$$\frac{(n+1)(g-1)}{2} \;\leq\; e \;\leq\; \frac{(n+1)(g-1)+3}{2}.$$

Then a general orthogonal extension $0 \to E \to V \to F^* \to 0$ *admits no maximal isotropic subbundle of degree* $\geq -e$ *intersecting* E *in generically positive rank.*

Proof. We must exclude the lifting of an $\tilde{E}$ of degree $\geq -e$ for which $I = E \cap \tilde{E}$ has rank $r \geq 1$. By (5.3), this would follow from

$$\frac{(n+1)(g-1)+3}{2} \;<\; \frac{(n+2)(g-1)}{2},$$

which is satisfied for all $g \geq 5$. □

Remark 5.11. According to Hirschowitz [5] (see also [2]), every vector bundle V of rank $2n+1$ and degree 0 has a subbundle of rank n and degree $-f$, where

$$f \;\leq\; \left\lceil \frac{n(n+1)(g-1)}{2n+1} \right\rceil. \tag{5.4}$$

By direct computation, one can check that for $n \geq 1$ and $g \geq 2$, the bound in (5.4) is strictly smaller than $\frac{1}{2}t(V)$ for a general $V \in MO_X(2n+1)$, except in the following cases:

(i) $g = 2$, n is odd and $t(V) = n+1$
(ii) $g = 2$, n is even and $t(V) = n+2$
(iii) $g = 3$, $t(V) = 2(n+1)$
(iv) $g = 4$, n is odd and $t(V) = 3(n+1)$

This implies that apart from these cases, a general orthogonal bundle of rank $2n+1$ has the property that no rank n subbundle of maximal degree is isotropic. For a similar property in the case of even rank, see [4, Remark 5.5]. □

We conclude this section with a result on the topology of the strata, which can be deduced from the analogous statement in [4] for orthogonal bundles of even rank:

Theorem 5.12. *For each* $t < (g-1)(n+1)$, *the stratum* $MO_X(2n+1;t)$ *is contained in the closure of* $MO_X(2n+1;t+4)$.

Proof. We define a map $\Psi \colon MO_X(2n+1) \to MO_X(2n+2)$ by $\Psi(V) = V \perp \mathcal{O}_X$. This is clearly an injective morphism, and $\Psi(V)$ is a stable orthogonal bundle if V is stable.

By Proposition 3.8 (3) we see that any $V \in MO_X(2n+1)$ admits a maximal isotropic subbundle of degree $-e$ if and only if $V \perp \mathcal{O}_X$ admits a maximal isotropic subbundle of degree $-e$. Therefore,

$$MO_X(2n+1;t) \;\cong\; \Psi\left(MO_X(2n+1)\right) \;\cap\; MO(2n+2;t)$$

for each t. Since all the spaces under consideration are constructible sets, we deduce from [4, Theorem 1.3 (2)]that the stratum $MO_X(2n+1;t)$ is contained in the closure of $MO_X(2n+1;t+4)$ for each $t < (n+1)(g-1)$, as required. □

6. Maximal isotropic subbundles of maximal degree

By Proposition 5.5, the points of a general fiber $\alpha_e^{-1}(V)$ in $\mathbb{A}_e$ correspond to the elements $[E \subset V] \in M(V)$ which are stable as vector bundles. For $t = 2e < (n+1)(g-1)$, we already know by Proposition 5.5 (1) that a general $V \in MO_X(2n+1;t)$ has a unique maximal isotropic subbundle of degree $-e$. In this section we will compute the dimension of the space $M(V)$ for a general $V \in MO_X(2n+1)$ with $t(V) \geq (n+1)(g-1)$. We first observe:

Proposition 6.1. *Let $E \subset V$ be a maximal isotropic subbundle. Then the tangent space of $M(V)$ at E is isomorphic to $H^0(X, \wedge^2(E^\perp)^*)$.*

Proof. To give a maximal isotropic subbundle $E \subset V$ is equivalent to giving a global section $\sigma\colon X \to \mathrm{OG}(n,V)$ of the orthogonal Grassmannian bundle $\mathrm{OG}(n,V)$ over X. A tangent vector to $M(V)$ at E, corresponding to an infinitesimal deformation of E in $M(V)$, is equivalent to a global section of the pullback by σ of the tangent bundle $T_{\mathrm{OG}(n,V)}$. This is the bundle $T \to X$ with $T_x \cong T_{E_x}\mathrm{OG}(n,V_x)$. By Lemma 3.7 (4), we have

$$T_{E_x}\mathrm{OG}(n,V_x) \ \cong\ T_{F'_x}\mathrm{OG}(n+1,W_x) \ \cong\ \wedge^2 \left(F'_x\right)^*,$$

where $W \cong V \perp \mathcal{O}_X$ and $F' \subset W$ is a maximal isotropic subbundle isomorphic as a vector bundle to $E^\perp \subset V$. Therefore, the bundle $T \to X$ can be identified with $\wedge^2(F')^*$. □

By the proposition, the dimension of a general fiber $\alpha_e^{-1}(V)$ is given by $h^0(X, \wedge^2 F^*)$, where F is a general extension $0 \to E \to F \to \mathcal{O}_X \to 0$ for a general $E \in U_X(n,-e)$.

Lemma 6.2. *Let F be a general extension $0 \to E \to F \to \mathcal{O}_X \to 0$ for a general $E \in U_X(n,-e)$. Then*

$$h^0(\wedge^2 F^*) = \begin{cases} 0 & \text{if } e \leq \frac{1}{2}(n+1)(g-1), \\ ne - \frac{1}{2}n(n+1)(g-1) & \text{if } e > \frac{1}{2}(n+1)(g-1). \end{cases}$$

Proof. Firstly, suppose that $e \leq \frac{1}{2}(n+1)(g-1)$, so that $\mu(\wedge^2 F^*) \leq g-1$. If F were general in $U_X(n+1,-e)$, then we would have $h^0(X, \wedge^2 F^*) = 0$ by the variant [3, Lemma A.1] of Hirschowitz' lemma; but F is clearly not general. Note however that the locus of such F is birationally characterized by the property $h^0(X, F^*) > 0$, and that by Sundaram [18] the Brill–Noether locus in $U_X(n+1,-e)$ defined by the condition $h^0(X,F^*) > 0$ is irreducible. Thus it suffices to find one bundle F satisfying both $h^0(X,F^*) > 0$ and $h^0(X, \wedge^2 F^*) = 0$. We will do this for $e = \frac{1}{2}(n+1)(g-1)$, since the lower degree cases are easier.

For $n = 1$, we need to find a rank 2 bundle F^* of degree $g-1$, so that $h^0(X, \det F^*) = 0$ and $h^0(X, F^*) > 0$. Let L be a general line bundle of degree $g-1$ with $h^0(X,L) = 0$. Then a general extension $0 \to \mathcal{O}_X \to F^* \to L \to 0$ satisfies the requirements.

For $n \geq 2$, consider a polystable bundle F^* of the form $G \oplus H$, where $G \in U_X(2, g-1)$ and $H \in U_X\left(n-1, \frac{1}{2}(n-1)(g-1)\right)$. Note that

$$\wedge^2 F^* \ \cong\ (\det G) \oplus (\wedge^2 H) \oplus (G \otimes H).$$

By Lange–Narasimhan [10], the bundle G has finitely many maximal line subbundles of degree zero. Choose a general such G with a maximal line subbundle M. Then by the Hirschowitz lemma [5, §4.6] we have $h^0(X, G \otimes H) = 0$ for general H.

Now put $\tilde{G} = G \otimes M^{-1}$ and $\tilde{H} = H \otimes M$. Then $h^0(X, \tilde{G}) > 0$ and $h^0(X, \tilde{G} \otimes \tilde{H}) = 0$. The vanishing of $H^0(X, \wedge^2 \tilde{H})$ follows again from [3, Lemma A.1].

Now let $\tilde{F}^* = \tilde{G} \oplus \tilde{H}$. Since $h^0(X, \tilde{G}) > 0$, we have $h^0(X, \tilde{F}^*) > 0$. To obtain the vanishing of $h^0(X, \wedge^2 \tilde{F}^*)$, we must show that $h^0(X, \det \tilde{G}) = 0$. Since G has rank two, $\det G \otimes M^{-2} \cong \mathrm{Hom}(M, G/M)$. By generality of M and G, there are no deformations of M in G, so $h^0(X, \mathrm{Hom}(M, G/M)) = 0$.

The statement for $e > \frac{1}{2}(n+1)(g-1)$ is equivalent to the vanishing of $h^1(X, \wedge^2 F^*)$. By Serre duality, this is in turn equivalent to the vanishing of $h^0(X, \wedge^2(F \otimes \kappa))$ for a theta characteristic κ. But if $e > \frac{1}{2}(n+1)(g-1)$, then $\deg(F \otimes \kappa) < \frac{1}{2}(n+1)(g-1)$ and we reduce to the case proven above. □

Theorem 6.3. *Let V be a general orthogonal bundle in $MO_X(2n+1; t)$, for t even. If $t < (n+1)(g-1)$, then $M(V)$ is a single point. If $t = (n+1)(g-1)$, then $\dim M(V) = 0$. If $(n+1)(g-1) < t \leq (n+1)(g-1)+3$, then $\dim M(V) > 0$ as computed below.* □

As in the even rank case [4, §5.4], for a general $V \in MO_X(2n+1)^{\pm}$ the invariants $t(V)$ and $\dim M(V)$ depend on the congruence class of $N := (n+1)(g-1)$ modulo 4. By Lemma 6.2, we have

$$\dim M(V) \;=\; ne - \frac{1}{2}n(n+1)(g-1) \;=\; \frac{1}{2}nt - \frac{1}{2}n(n+1)(g-1) \;=\; \frac{n}{2}(t(V) - N).$$

We deduce the following table, by analogy with that in [4, §5.4]:

$N \equiv 0 \mod 4$:

$t(V)$	Component	$\dim M(V)$
N	$MO_X(2n+1)^+$	0
$N+2$	$MO_X(2n+1)^-$	n

$N \equiv 1 \mod 4$:

$t(V)$	Component	$\dim M(V)$
$N+1$	$MO_X(2n+1)^-$	$n/2$
$N+3$	$MO_X(2n+1)^+$	$3n/2$

$N \equiv 2 \mod 4$:

$t(V)$	Component	$\dim M(V)$
N	$MO_X(2n+1)^-$	0
$N+2$	$MO_X(2n+1)^+$	n

$N \equiv 3 \mod 4$:

$t(V)$	Component	$\dim M(V)$
$N+1$	$MO_X(2n+1)^+$	$n/2$
$N+3$	$MO_X(2n+1)^-$	$3n/2$

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