Generalizations of the Levi-Civita Bertotti Robinson spacetime

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Abstract

The Levi-Civita Bertotti Robinson (LBR) solution of Einstein's field equations may be interpreted to describe spacetime in a region without matter outside a charged spherical domain wall. In the present paper we investigate the physical properties of some solutions of the Einstein-Maxwell equations, and show that they are generalizations of the LBR solution. One represents a spacetime with charged dust outside a charged shell, and another one a spacetime with Lorentz Invariant Vacuum Energy (LIVE) outside a charged shell. The uniqueness of static, spherically symmetric and conformally flat solutions of the Einstein-Maxwell field equations is also discussed.

1. Introduction

The Levi-Civita Bertotti Robinson (LBR) solution [1-4] of Einstein's field equations has been interpreted [5] to describe spacetime without matter outside a charged spherical domain wall. This solution has also been thoroughly discussed by M. Ortaggio, J. Podolsky and J. B. Griffiths [6-9]. In the present paper we shall investigate the properties of one class of solutions found by A. Banerjee and M. M. Som [10] and another one by S. N. Pandey and R. Tiwari [11]. We show that they are generalizations of the LBR solution. The first one represents a spacetime with charged dust outside a charged shell, and the second one a spacetime with Lorentz Invariant Vacuum Energy (LIVE) outside a charged shell. The physical properties of the shells are found by means of Israel's theory of singular shells [12, 13]. The uniqueness of static, spherically symmetric and conformally flat solutions of the Einstein-Maxwell field equations is also discussed.

2. The LBR spacetime

The LBR solution of Einstein's field equations has recently been thoroughly discussed by the present authors [14]. Due to the many symmetries of the LBR spacetime the line element representing this spacetime has a simple form in several coordinate systems. The solution is static, representing a conformally flat spacetime with a spherically symmetric 3-space.

In this section we shall briefly describe this spacetime by means of four coordinate systems: conformally flat space time (CFS) coordinates, (T,R), isotropic coordinates, (t,r), standard coordinates, (η,χ) and a physical radial coordinate $\hat{r}$.

The CFS coordinates are defined by the condition that in these coordinates the line-element is equal to a conformal factor times the standard Minkowski line-element. In these coordinates the line element of the LBR spacetime (the LBR line element) takes the form,

$$ds^2 = \frac{R_Q^2}{R^2}\left(-dT^2 + dR^2 + R^2 d\Omega^2\right). \tag{1}$$

This line element represents a conformally flat spacetime with a constant electrical field outside a spherical, charged domain wall with radius R_Q, mass M and charge Q given by [5]

$$M = R_Q / G \quad , \quad Q = \sqrt{4\pi\varepsilon_0 / G}\, R_Q. \tag{2}$$

Hence the charge and mass of the wall are not independent parameters, and therefore this spacetime is not a generalization of the Schwarzschild spacetime. If the charge is removed, then also the mass is removed. The LBR spacetime exists for $R > R_Q$. The domain wall is at $R = R_Q$, and there is Minkowski spacetime inside the domain wall, i.e. for $0 \le R < R_Q$.

Note also that the CFS coordinate clocks show the same time as a standard clock at the domain wall independent of their position. The rate of the proper time is slower with increasing distance from the domain wall, meaning that the outwards direction is downwards in the gravitational field outside the wall. Hence there is repulsive gravitation outside the wall due to the stresses in the wall.

Furthermore, the CFS coordinates are comoving with a static reference frame, i.e. a frame that is at rest relative to the domain wall. It may be noted that it is not possible to define a so-called curvature radial coordinate, $\bar{r}$, in the LBR spacetime. This coordinate is defined by the condition that the radius of a spherical surface about $\bar{r}=0$ with radius $\bar{r}$ is $4\pi\bar{r}^2$. But in the LBR spacetime the radius of a spherical surface with radius $\bar{r}$ is $4\pi R_Q^2$ for arbitrary values of $\bar{r}\geq R_Q$.

The isotropic coordinates are defined by the condition that the coordinate velocity of light is independent of the direction in these coordinates. Hence the CFS coordinates are a special type of isotropic coordinates. We shall here need a more general form of the LBR line element in isotropic coordinates,

$$ds^2 = \frac{R_Q^2}{R^2}\left[-\left(A+Br^2\right)^2 dt^2 + dr^2 + r^2 d\Omega^2\right], \tag{3}$$

where $r\neq 0$ and A and B are constants. The constant A is dimensionless, and B has dimension (length)$^{-2}$. The line element (3) reduces to the form (1) with $B=0$ and $T=At$. For $A\neq 0$ this line element can be written in a form that has earlier been used in connection with the Friedmann-Robertson-Walker universe models,

$$ds^2 = A^2 R_Q^2 \frac{1}{r^2}\left(1+\frac{k}{4}\frac{r^2}{r_0^2}\right)^2\left[-dt^2 + \left(1+\frac{k}{4}\frac{r^2}{r_0^2}\right)^{-2}\left(dr^2 + r^2 d\Omega^2\right)\right], \tag{4}$$

where $k=\mathrm{sgn}(B)$ and

$$r_0 = (1/2)\sqrt{kA/B} \tag{5}$$

is the curvature radius of the 3-space defined by the spatial part of the line element inside the bracket parenthesis.

The position dependence of the rate of proper time in the isotropic coordinate system is given by

$$\frac{d\tau}{dt} = AR_Q\left(\frac{1}{r}+\frac{k}{4}\frac{r}{r_0^2}\right), \tag{6}$$

where the rate of coordinate time is position independent. The rate of proper time is decreasing as a function of r for $k=-1$. For $k=1$ the rate has a minimum at $r=r_2=2r_0$, decreases for $r<r_2$ and increases for $r>r_2$. At the position $r=r_2$ the acceleration of gravity vanishes, inside it is directed away from the domain wall and outside towards it.

We shall now show that the line element (3) represents the same spacetime as (1) by finding a coordinate transformation that connects (1) and (3). We start with the line element as expressed in terms of the (dimensionless) standard radial coordinate χ defined by the condition that the spatial part of the line element takes the form [15]

$$dl^2 = F(\chi)^2\left(d\chi^2 + S_k\left(\chi/r_1\right)^2 d\Omega^2\right), \tag{7}$$

where $F(\chi)$ is a function determined by Einstein's field equations, the function $S_k(x)$ is given in equation (A.1), and r_1 is a characteristic length for the actual spacetime. In the case of the LBR spacetime as interpreted physically in [5] we choose $r_1 = R_Q$. With this radial coordinate the line element of the LBR spacetime is [14]

$$ds^2 = \left(1/S_k\left(\chi/R_Q\right)\right)^2\left(-d\eta^2 + d\chi^2\right) + R_Q^2 d\Omega^2, \tag{8}$$

where the so called parametric time η is a rescaled isotropic time

$$\eta = At. \tag{9}$$

Both the isotropic and the standard coordinates of the LBR spacetime reduce to the CFS coordinates for $k = 0$.

The position dependence of the rate of proper time in the standard coordinate system is given by

$$\frac{d\tau}{d\eta} = \frac{1}{S_k\left(\chi/R_Q\right)}, \tag{10}$$

where the rate of the coordinate time is position independent. The rate of proper time is decreasing as a function of r for $k = -1$. For $k = 1$ the rate has a minimum at $\chi = \chi_2 = (\pi/2)R_Q$, decreases for $\chi < \chi_2$ and increases for $\chi > \chi_2$. At the position $\chi = \chi_2$ the acceleration of gravity vanishes, inside it is directed away from the domain wall and outside towards it, just as in the isotropic coordinate system.

In passing we note that in terms of the (dimensionless) Robertson-Walker (RW) radial coordinate

$$\bar{r} = S_k\left(\chi/R_Q\right) \tag{11}$$

the line element of the LBR spacetime has the form

$$ds^2 = \frac{R_Q^2}{\bar{r}^2}\left(-d\eta^2 + \frac{d\bar{r}^2}{1 - k\bar{r}^2}\right) + R_Q^2 d\Omega^2. \tag{12}$$

Hence the RW radial coordinate is equal to the CFS radial coordinate for $k = 0$

From equations (3) and (8) it follows that

$$\frac{dr}{r} = \frac{d\chi}{R_Q S_k\left(\chi/R_Q\right)}. \tag{13}$$

Integration by means of the formula (A.9) and demanding agreement between the forms (4) and (8) of the LBR line element gives the transformation equation between the standard and isotropic radial coordinates

$$r = 2R_Q T_k\left(\chi/2R_Q\right). \tag{14}$$

This equation shows that the standard and isotropic coordinates are comoving in the same reference frame, which explains the similarity of the position dependence of the rate of proper time in the isotropic and the standard coordinate system. Also equation (14) gives $r(\pi/2)=2R_Q$ for the position with vanishing acceleration of gravity in these coordinate systems when $k=1$.

In reference [16] it was shown that the transformation between the standard coordinates and the CFS coordinates is

$$gT=\frac{S_k(\eta/R_Q)}{C_k(\eta/R_Q)+C_k(\chi/R_Q)} \quad , \quad gR=\frac{S_k(\chi/R_Q)}{C_k(\eta/R_Q)+C_k(\chi/R_Q)}, \tag{15}$$

where the constant *g* represents the relative acceleration of the reference frames in which (η,χ) and (T,R) are comoving at the moment $T=0$ when these frames are at rest relative to each other. In the case $k=0$ the (η,χ) and (t,r) coordinate systems are comoving in the same reference frame as that of the CFS coordinates, i.e. in the rest frame of the domain wall. For $k=1$ and $k=-1$ the (η,χ) and (t,r) coordinate systems are comoving in a reference frame that accelerates away from or towards the domain wall, respectively. This explains the position dependence of the proper time in the isotropic and standard coordinate systems noted above. For $k=1$ the reference frame accelerates away from the domain wall with a smaller proper acceleration than the acceleration of gravity in the static CFS-frame inside $r=2r_0$, $\chi=\pi/2$, and a larger proper acceleration outside this position. The inverse of the transformation (15) is

$$I_k(\eta/R_Q)=\frac{1-kg^2(T^2-R^2)}{2gT} \quad , \quad I_k(\chi/R_Q)=\frac{1+kg^2(T^2-R^2)}{2gR}. \tag{16}$$

where the function $I_k(x)$ is given in equation (A.4).

Using equations (A.7) and (A.4) we obtain the alternative forms of the transformation equation (14)

$$S_k(\chi/R_Q)=\frac{4r_0r}{4r_0^2+kr^2} \quad , \quad C_k(\chi/R_Q)=\frac{4r_0^2-kr^2}{4r_0^2+kr^2}. \tag{17}$$

Inserting these expressions into equation (15) we find the transformation between the isotropic and CFS coordinates

$$gT=\frac{(4r_0^2+kr^2)S_k(\eta/R_Q)}{(4r_0^2+kr^2)C_k(\eta/R_Q)+4r_0^2-kr^2} \quad , \quad gT=\frac{4r_0r}{(4r_0^2+kr^2)C_k(\eta/R_Q)+4r_0^2-kr^2}, \tag{18}$$

where η is given in terms of t in equation (9).

If $A=0$ the line element (3) takes the form

$$ds^2=\frac{R_Q^2}{r^2}\left(-B^2r^4dt^2+dr^2+r^2d\Omega^2\right). \tag{19}$$

Introducing the coordinates

$$t = T \quad , \quad r = -\frac{1}{BR} \tag{20}$$

in the usual form of the line element of the Minkowski spacetime in spherical coordinates

$$ds_M^2 = -dT^2 + dR^2 + R^2 d\Omega^2 , \tag{21}$$

the line element takes the form

$$ds_M^2 = -dt^2 + \frac{1}{B^2 r^4}\left(dr^2 + r^2 d\Omega^2\right). \tag{22}$$

Hence the line element (19) can be written in the manifestly conformally flat form

$$ds^2 = R_Q^2 B^2 r^2 ds_M^2 . \tag{23}$$

I should be noted that according to equation (20) the isotropic radial coordinate is negative in the case $A = 0,\ B > 0$, but in general equation (18) shows that $r > 0$.

Calculating the Einstein tensor from the line element (3) and using Einstein's field equations we find that the mixed components of the energy-momentum tensor are given by

$$\kappa T_\nu^\mu = \left(1/R_Q\right)^2 \left(-1,-1,1,1\right). \tag{24}$$

This is the energy-momentum tensor of a radial electric field due to a charge given in equation (2). Calculating the Kretschmann curvature scalar for LBR spacetime we find that it is constant and has the value

$$K = 8/R_Q^4 . \tag{25}$$

The physical radial distance, dl, corresponding to a coordinate distance dr is $dl = \sqrt{g_{rr}}\, dr$. The physical radial coordinate is defined by the condition $d\hat{r} = dl$ and hence that $g_{\hat{r}\hat{r}} = 1$. Hence from equation (1) it follows that the coordinate transformation between the CFS and physical radial coordinates is

$$\hat{r} = R_Q\left(1 + \ln\frac{R}{R_Q}\right) \quad , \quad R = R_Q e^{\frac{\hat{r}}{R_Q}-1} \quad , \quad \hat{r}, R \geq R_Q . \tag{26}$$

The line element of the CFS spacetime in terms of the physical radial coordinate is, using $\hat{t} = T$,

$$ds^2 = -e^{2\left(\frac{\hat{r}}{R_Q}-1\right)} d\hat{t}^2 + d\hat{r}^2 + R_Q^2 d\Omega^2 . \tag{27}$$

The LBR spacetime has infinite spatial extension, $\hat{r} \in \left|R_Q, \infty\right\rangle$.

3. A conformally flat spacetime with charged dust outside a charged shell

A generalization of the LBR spacetime containing charged dust has been found by Banerjee and Som [10]. The physical properties of this solution of the Einstein-Maxwell field equations will be further discussed in this section, and we are giving a new interpretation of their solution.

Their point of departure is a generalization of the line element (3) in isotropic coordinates of the form

$$ds^2 = \frac{BR_Q^2}{A+Br^2}\left[-\left(A+Br^2\right)^2 dt^2 + dr^2 + r^2 d\Omega^2\right]. \tag{28}$$

The line element (23) is recovered by putting $A=0$. Calculating the mixed components of the Einstein tensor from the line element (28) and using Einstein's field equations, we find the following non-vanishing mixed components of the energy-momentum tensor,

$$T_t^t = -\frac{6A+Br^2}{\kappa R_Q^2\left(A+Br^2\right)} \quad , \quad -T_r^r = T_\theta^\theta = T_\phi^\phi = \frac{Br^2}{\kappa R_Q^2\left(A+Br^2\right)}. \tag{29}$$

Hence, this line element represents a spacetime with a mixture of dust and a radial electrical field. From the above equation it follows that the density of the dust and the energy density of the electrical field are

$$\rho_M = \frac{6A}{\kappa R_Q^2\left(A+Br^2\right)} \quad , \quad \rho_E = \frac{Br^2}{\kappa R_Q^2\left(A+Br^2\right)}. \tag{30}$$

Since the densities are positive it follows that $A>0$ and $B>0$ so that $k=1$. This implies that $A+Br^2>0$ for all r.

The expressions (30) for the densities can also be written

$$\rho_M = \frac{24r_0^2}{\kappa R_Q^2\left(4r_0^2+r^2\right)} \quad , \quad \rho_E = \frac{r^2}{\kappa R_Q^2\left(4r_0^2+r^2\right)}, \tag{31}$$

where r_0 is given in equation (5). The density of the dust decreases with increasing r and vanishes in the limit $r\to\infty$. The mass density of the electrical field increases with increasing r and approaches the constant value $1/\left(\kappa R_Q^2\right)$. Note also that $\lim_{r_0\to 0}\rho_M = 0$, but $\lim_{r_0\to 0}\rho_E = 1/\left(\kappa R_Q^2\right)$. This means that when the charged dust is removed, there still remains an electrical field. In this limit the spacetime approaches the LBR spacetime, and the source of the electrical field is a charged domain wall [5]. Hence, this solution of Einstein's field equations is a generalization of the LBR solution. The charged dust modifies the properties of the charged shell, so it is not a domain wall in general. We shall assume that the shell has a physical radius R_Q.

The ratio of the mass density of the electrical field and the density of the dust is

$$\frac{\rho_E}{\rho_M} = \frac{1}{24}\left(\frac{r}{r_0}\right)^2. \tag{32}$$

We see that the ratio of the mass density of the electrical field and the dust increases with *r*. The reason for this will be explained below. Equations (31) imply that

$$\frac{1}{6}\rho_M + \rho_E = \frac{1}{\kappa R_Q^2} . \tag{33}$$

The Kretschmann curvature scalar for the spacetime represented by the line element (29) is found to be

$$K = \frac{8}{R_Q^4} \frac{120 r_0^4 + 12 r_0^2 r^2 + r^4}{\left(4r_0^2 + r^2\right)^2} , \tag{34}$$

which is positive for all r. Note that *K* approaches the value (25) of the LBR spacetime for large *r*.

We get a particularly nice form of the line element (28) by choosing $B = 1/R_Q^2$. Introducing r_0 it then takes the form

$$ds^2 = -e^{\alpha} dt^2 + e^{\beta} dr^2 + e^{\gamma} d\Omega^2 \quad , \quad -e^{\alpha} = e^{-\beta} = r^2 e^{-\gamma} = \frac{4r_0^2 + r^2}{R_Q^2} . \tag{35}$$

Since $k = 1$ the isotropic coordinates are comoving with a reference frame that accelerates away from the charged shell. The proper time τ as measured on standard clocks at rest in the reference frame is given by

$$d\tau = \frac{\sqrt{4r_0^2 + r^2}}{R_Q} dt . \tag{36}$$

The rate of coordinate time is position independent and equal to the rate of proper time at the position

$$r_1 = \sqrt{R_Q^2 - 4r_0^2} . \tag{37}$$

The rate of proper time increases with *r*. This means that there is an acceleration of gravity pointing towards the charged shell. The reason for this inwards directed acceleration of gravity is that the reference frame accelerates in the outwards direction with an acceleration that is larger than the outwards directed acceleration of gravity in the static CFS coordinates system.

We can introduce a radial coordinate, $\hat{r}$, so that $d\hat{r}$ is equal to the physical length corresponding to the coordinate differential , dr ,

$$d\hat{r} = \sqrt{g_{rr}}\, dr = R_Q \left(4r_0^2 + r^2\right)^{-1/2} dr . \tag{38}$$

This shows that the length of the coordinate measuring rods decreases with increasing *r*. Integrating with $r_0 \neq 0$ and the boundary condition $\hat{r}(0) = 0$ gives

$$\hat{r} = R_Q \operatorname{arcsinh}\left(r / 2r_0\right) . \tag{39}$$

This shows that the 3-space of this spacetime is infinitely extended. The inverse transformation is

$$r = 2r_0 \sinh\left(\hat{r} / R_Q\right). \tag{40}$$

In terms of the physical radius $\hat{r}$ the line element (35) takes the form

$$ds^2 = -\left(\frac{2r_0}{R_Q}\right)^2 \cosh^2\left(\frac{\hat{r}}{R_Q}\right)dt^2 + d\hat{r}^2 + R_Q^2 \tanh^2\left(\frac{\hat{r}}{R_Q}\right)d\Omega^2 . \tag{41}$$

The angular part of this line element shows that the geometry of the 3-space in this spacetime is unusual in a similar manner as the geometry of the 3-space of the LBR-spacetime. In the LBR-spacetime a spherical surface with center at $R = 0$ and arbitrary radius has an R-independent area $A_Q = 4\pi R_Q^2$ equal to the area of the domain wall. In the present spacetime, with charged dust outside a charged shell, the area of such a spherical surface is less than A_Q, approaching A_Q infinitely far from the charged shell.

The present spacetime allows a radial curvature coordinate $\overline{r}$ defined by the condition that the area of a spherical surface with center at the spatial origin and radius $\overline{r}$ has area $4\pi\overline{r}^2$, and hence that the angular part of the line element takes the form $\overline{r}^2 d\Omega^2$. Comparison with the line elements (35) and (41) shows that the transformations from the isotropic and physical radii to the curvature radius are, respectively,

$$\overline{r} = \frac{R_Q r}{\sqrt{4r_0^2 + r^2}} \tag{42}$$

and

$$\overline{r} = R_Q \tanh\left(\frac{\hat{r}}{R_Q}\right). \tag{43}$$

Equation (42) shows that the curvature coordinate only exists if $r_0 \neq 0$, i.e. if there is dust outside the charged shell. These equations show that $\overline{r} < R_Q$ and $\lim_{\overline{r}\to\infty} = \lim_{\hat{r}\to\infty} = R_Q$. The inverse transformations are

$$r = \frac{2r_0\overline{r}}{\sqrt{R_0^2 - \overline{r}^2}} \tag{44}$$

and

$$\hat{r} = R_Q \operatorname{artanh}\left(\overline{r} / R_Q\right). \tag{45}$$

The line element of this spacetime in curvature coordinates is

$$ds^2 = \frac{4r_0^2}{R_Q^2 - \overline{r}^2}dt^2 + \frac{R_Q^4}{\left(R_Q^2 - \overline{r}^2\right)^2}d\overline{r}^2 + \overline{r}d\Omega^2 . \tag{46}$$

The reason for the finite upper limit of the curvature radius, $\overline{r} < R_Q$, in spite of the infinity of space, is seen from the relationship

$$d\hat{r} = \left(1 - \frac{\overline{r}^2}{R_Q^2}\right)^{-1} d\overline{r} . \tag{47}$$

The physical distance corresponding to a unit curvature coordinate distance increases with $\overline{r}$. Hence the curvature coordinate measuring rods are longer the farther away they are from the charged shell. So even if there are a finite number of them along a radial line, they cover an infinitely great physical distance.

We shall now describe the physical contents of this spacetime in terms of the curvature radial coordinate and the physical radial coordinate. With the curvature coordinate the energy densities of the dust and of the electrical field are

$$\rho_M = \frac{6}{\kappa R_Q^4}\left(R_Q^2 - \bar{r}^2\right) \quad , \quad \rho_E = \frac{\bar{r}^2}{\kappa R_Q^4} . \tag{48}$$

In order to see how the densities depend upon the physical distance from the origin we express them as functions of the physical radial coordinate,

$$\rho_M = \frac{6}{\kappa R_Q^2}\cosh^{-2}\left(\frac{\hat{r}}{R_Q}\right) \quad , \quad \rho_E = \frac{1}{\kappa R_Q^2}\tanh^2\left(\frac{\hat{r}}{R_Q}\right) . \tag{49}$$

At large distances from the charged shell the density of the dust decreases exponentially and the energy density of the electrical field is nearly constant.

We shall now calculate the charge distribution in the dust, which is most simply performed in curvature coordinates. The charge of the dust inside a radius is $\bar{r}$

$$Q_M\left(\bar{r}\right) = 4\pi\int_0^{\bar{r}} \sigma\left(\tilde{r}\right) e^{\beta(\tilde{r})/2}\tilde{r}^2 d\tilde{r} , \tag{50}$$

where $\sigma\left(\tilde{r}\right)$ is the charge density. Differentiating we get

$$\sigma\left(\bar{r}\right) = \frac{e^{-\beta(\bar{r})/2}}{4\pi\bar{r}^2}\frac{dQ_M}{d\bar{r}} . \tag{51}$$

The electrical field has two contributions, E_S due to the shell and E_M due to the dust, i.e. $E = E_S + E_M$. The energy density of the electrical field is

$$\rho_E = \left(1/2\right)E^2 , \tag{52}$$

where the electrical permittivity has been put equal to one. In the present case with spherical symmetry we have

$$E = \frac{Q\left(\bar{r}\right)}{4\pi\bar{r}^2} . \tag{53}$$

Here $Q = Q_S + Q_M$ where Q_S is the charge of the shell and Q_M is the charge of the dust inside the radius $\bar{r}$. Using equations (52), (53) and the last of equations (48) we obtain

$$\rho_E = \frac{Q\left(\bar{r}^2\right)}{32\pi^2\bar{r}^4} = \frac{\bar{r}^2}{\kappa R_Q^4} , \tag{54}$$

giving

$$Q\left(\bar{r}\right) = \frac{4\pi}{R_Q^2}\sqrt{\frac{2}{\kappa}}\,\bar{r}^3 . \tag{55}$$

From the line element (46) we have

$$e^{-\beta(\bar{r})/2} = 1 - \frac{\bar{r}^2}{R_Q^2} . \tag{56}$$

Inserting these expressions into equation (50) we find the charge distribution as a function of the curvature radius

$$\sigma\left(\bar{r}\right) = \frac{3}{R_Q^4}\sqrt{\frac{2}{\kappa}}\left(R_Q^2 - \bar{r}^2\right). \tag{57}$$

The charge distribution of the dust depends upon the physical radius as

$$\sigma\left(\hat{r}\right) = \frac{3}{R_Q^2}\sqrt{\frac{2}{\kappa}}\cosh^{-2}\left(\frac{\hat{r}}{R_Q}\right). \tag{58}$$

Equation (57) and the transformation (42) give the charge distribution in terms of the isotropic radial coordinate

$$\sigma(\hat{r})=\frac{3}{R_Q^2}\sqrt{\frac{2}{\kappa}}\left[1-\left(\frac{\hat{r}}{2r_0}\right)^2\right]^{-1} \tag{59}$$

in accordance with the expression found by Banerjee and Som [6]. Comparing equations (49) and (58) we see that the charge distribution is identical to the distribution of the dust. Hence the dust is uniformly charged. The densities approach zero at large distances from the shell.

The line element of this spacetime takes a simple form in (η,χ) coordinates with $A=4r_0^2/R_Q^2$ in equation (9). From the transformation (14) with $k=1$ and the line element (35) it then follows that

$$ds^2=\left[1/2\cos(\chi/2R_Q)\right]^2\left(-d\eta^2+d\chi^2+\sin^2\chi d\Omega^2\right). \tag{60}$$

Using the transformation (16) the form of this line element in CFS coordinates is found to be

$$ds^2=\frac{ds_M^2}{g^2R^2+(1/4)\left[1+g^2\left(T^2-R^2\right)\right]^2+\left\{g^2R^2+(1/4)\left[1+g^2\left(T^2-R^2\right)\right]^2\right\}^{1/2}\left[1+g^2\left(T^2-R^2\right)\right]} \tag{61}$$

in agreement with the form of the line element presented by Banerjee and Som [6].

The LBR spacetime corresponds to the case with vanishing dust, $\rho_M=0$, i.e. $r_0=0$. In this case the line element (35) reduces to the LBR line element (1). However in the case $\rho_E=0$ equation (30) implies that $B=0$, and the line element (28) vanishes. Hence, there is no solution of the form (28) corresponding to the special case $\rho_E=0$.

The Israel formalism for singular layers [12, 13] can be used to investigate the properties of the singular shell which replaces the domain wall in the present solution of the Einstein-Maxwell field equations. We take as point of departure the line element (35) and introduce a new radial coordinate $\tilde{r}=-R_Q^2/r$. This transforms the line element to the form

$$ds^2=-\frac{4r_0^2\tilde{r}^2+R_Q^4}{R_Q^2\tilde{r}^2}+\frac{R_Q^6}{\left(4r_0^2\tilde{r}^2+R_Q^4\right)\tilde{r}^2}d\tilde{r}^2+\frac{R_Q^6}{4r_0^2\tilde{r}^2+R_Q^4}d\Omega^2. \tag{62}$$

The mixed components of the energy momentum tensor of the shell are given by

$$\kappa S_t^t=\left[e^{-\beta/2}\gamma,_{\tilde{r}}\right] \quad , \quad \kappa S_\theta^\theta=\kappa S_\phi^\phi=(1/2)\kappa S_t^t+(1/2)\left[e^{-\beta/2}\alpha,_{\tilde{r}}\right], \tag{63}$$

where the brackets denote discontinuity of the function inside at the shell. In the calculations the metric is assumed to be continuous at the shell. Since the mechanical properties of the shell are time independent, we use the position of the shell in the isotropic coordinate system at the point of time $t=\eta=T=0$. According to the transformation (18) this position is $r=r_Q=2gr_0R_Q$. Defining the density (mass per area) of the shell and the pressure if $p_S>0$ or stress if $p_S<0$ by $\rho_S=-S_t^t$, $p_S=S_\theta^\theta$ we find, going back to the isotropic radial coordinate,

$$\rho_S=\frac{2}{\kappa r_Q R_Q}\frac{8r_0^2+r_Q^2}{\sqrt{4r_0^2+r_Q^2}} \quad , \quad p_S=-\frac{2}{\kappa r_Q R_Q}\sqrt{4r_0^2+r_Q^2}. \tag{64}$$

With vanishing dust, $r_0=0$, these expressions reduce to $\rho_S=-p_S=2/\kappa R_Q$. In general the equation of state parameter is

$$w=\frac{p_S}{\rho_S}=-\frac{4r_0^2+r_Q^2}{8r_0^2+r_Q^2}, \tag{65}$$

which reduces to $w=-1$ when $r_0=0$. Hence, with dust outside the shell the stresses are smaller relative to the density than in a domain wall, but when the dust is removed the shell is transformed to a domain wall.

4. Conformally flat spacetime with dust, radiation or vacuum energy outside a charged shell

S. N. Pandey and R. Tiwari [11] have found a class of solutions of the Einstein-Maxwell equations representing conformally flat spherically symmetric spacetimes with charged perfect fluids, which have been further investigated by A. Pradhan and P. Pandey [17]. Their solutions were represented by the line element

$$ds^2 = \frac{ds_M^2}{\left[\alpha(R)+\beta(T)\right]^2} . \tag{66}$$

Calculating the components of the Einstein tensor from this line element and utilizing the field equations, we find

$$E_T^T = -\kappa\left(\rho_F + \rho_E\right) \quad , \quad E_R^R = \kappa\left(p_F + p_{E\|}\right) \quad , \quad E_\theta^\theta = E_\phi^\phi = \kappa\left(p_F + p_{E\perp}\right), \tag{67}$$

where ρ_F and ρ_E are the energy densities of a perfect fluid and a radial electric field, p_F is the pressure of the fluid, and $p_{E\|}$ and $p_{E\perp}$ are the stress components of the electrical field along and normal to the field. These quantities are given by

$$\kappa\rho_F = 3\left(\beta_T^2 - \alpha_R^2\right) + 3(\alpha+\beta)\left(\alpha_{RR} + \frac{\alpha_R}{R}\right), \tag{68}$$

$$\kappa p_F = -3\left(\beta_T^2 - \alpha_R^2\right) + (\alpha+\beta)\left(2\beta_{TT} - \alpha_{RR} - \frac{3\alpha_R}{R}\right), \tag{69}$$

$$\kappa\rho_E = -\kappa p_{E\|} = \kappa p_{E\perp} = (\alpha+\beta)\left(\frac{\alpha_R}{R} - \alpha_{RR}\right), \tag{70}$$

where $\beta_T = d\beta/dT$ and so forth. There is no energy transport, so the CSF coordinate system is comoving with the fluid. Note that equation (66) only permits the existence of an electric field if spacetime is inhomogeneous.

The pressure of the perfect fluid is assumed to be isotropic. The anisotropy of the energy momentum tensor (or the Einstein tensor) comes from the electrical field. If

$$R\alpha_{RR} - \alpha_R = 0, \tag{71}$$

the electrical field vanishes. Then

$$\alpha(R) = a + bR^2, \tag{72}$$

where *a* and *b* are integration constants.

In this case the relationship between the density and pressure of the fluid is

$$\kappa\left(\rho_F + p_F\right) = 2(\alpha+\beta)\left(\alpha_{RR} + \beta_{TT}\right) = 2(\alpha+\beta)\left(2b + \beta_{TT}\right). \tag{73}$$

With $\beta(T) = -bT^2$ the equation of state is

$$p_F = -\rho_F, \tag{74}$$

representing Lorentz invariant vacuum energy, LIVE, which can be represented by a cosmological constant Λ. The density is given by

$$\kappa\rho_F = 12ab = \Lambda. \tag{75}$$

In this case the line element takes the form

$$ds^2 = \frac{ds_M^2}{\left[a - b\left(T^2 - R^2\right)\right]^2} . \tag{76}$$

Choosing $a = b = H_\Lambda/2$ with $H_\Lambda = \sqrt{\Lambda/3}$, the line element (76) agrees with the line element (20) in reference [14]. In the special case $a = 0,\ b = 1$ the line element reduces to

$$ds^2 = \frac{ds_M^2}{\left(T^2 - R^2\right)^2}, \tag{77}$$

which represents flat, empty Minkowski spacetime (see equation (61) in ref.[18]).

We now consider the case with both a fluid and an electrical field. Let us assume that the fluid is a LIVE. Then $p_F = -\rho_F$ which requires

$$\alpha_{RR} + \beta_{TT} = 0. \tag{78}$$

The general solution of this equation is

$$\alpha(R)=a_1R^2+a_2R+a_3 \quad , \quad \beta(T)=-a_1T^2+b_2T+b_3 , \tag{79}$$

where a_i, b_i are constants.

As an illustration we shall give the most simple solution with LIVE and an electrical field. The energy density of the electrical field energy is given by

$$\kappa\rho_E=(\alpha+\beta)(a_2/R) . \tag{80}$$

Hence the condition for a non-vanishing electrical field is $a_2 \neq 0$. For simplicity we now assume that $a_1=a_3=b_3=0$. Then $\alpha(R)=a_2R$ and $\beta(T)=b_2T$, and the spacetime is represented by the line element

$$ds^2=\frac{ds_M^2}{(a_2R+b_2T)^2} . \tag{81}$$

The energy density of the electrical field is

$$\rho_E=\rho_{E0}\left(1+\frac{b_2}{a_2}\frac{T}{R}\right) . \tag{82}$$

where $\rho_{E0}=a_2^2/\kappa$ is the energy density of the electrical field at the point of time $T=0$ which is independent of *R.* Furthermore Pandey and Tiwari [11] calculated that the expansion of the fluid is $v^{\mu}_{;\mu}=3b_2$. Hence the constant a_2 represents the energy density of the electrical field, and the constant b_2 represents the expansion of the fluid. The density of the vacuum energy is

$$\rho_F=\rho_{F0}+3(\rho_E-\rho_{E0}) , \tag{83}$$

where $\rho_{F0}=3(b_2/a_2)^2\rho_{E0}$ is the density of the vacuum energy at the point of time $T=0$ which is independent of R.

The line element (81) with $b_2=a_2$ has been considered by V. V. Kiselev [19]. Then the energy density of the electrical field and the vacuum energy are

$$\rho_E=\frac{a_2^2}{\kappa}\left(1+\frac{T}{R}\right) \quad , \quad \rho_F=3\rho_E \quad , \quad p_{Er}=-\rho_E \quad , \quad p_{E\theta}=p_{E\phi}=\rho_E \quad , \quad p_F=-\rho_F . \tag{84}$$

Defining

$$\rho=\rho_E+\rho_F \quad , \quad p_i=p_{Ei}+p_{Fi} , \tag{85}$$

we obtain

$$\rho=4\rho_E \quad , \quad p_r=-\rho \quad , \quad p_\theta=p_\phi=-\rho/2 . \tag{86}$$

In this form it is tempting to interpret equation (86) as representing a quintessence energy with anisotropic stresses [20, 21]. However, as the present analysis shows, it represents a radial electric field in a space with LIVE, i.e. with a non-vanishing cosmological constant, Λ, given by $\kappa\rho_{F0}=3H_\Lambda^2=\Lambda$. Hence $b_2=H_\Lambda$. It is seen from equation (81) that the line element of this spacetime in CFS coordinates has the form

$$ds^2=\frac{ds_M^2}{H_\Lambda^2(R+T)^2} . \tag{87}$$

Standard coordinates (t,χ) can be introduced by means of the transformation [22, 23]

$$H_\Lambda T=e^{H_\Lambda t}\cosh(\chi/R_Q) \quad , \quad H_\Lambda R=e^{H_\Lambda t}\sinh(\chi/R_Q) . \tag{88}$$

The inverse transformation is

$$e^{2H_\Lambda t}=H_\Lambda^2\left(T^2-R^2\right) \quad , \quad \tanh(\chi/R_Q)=R/T . \tag{89}$$

Hence in this spacetime the reference frame of the standard coordinates, having $\chi=$constant, moves with constant velocity in the positive radial direction in the reference frame of the CFS coordinates, and the simultaneity curves are hyperbolae. In other words the reference frame of this χ-coordinate expands relative to the CFS-reference frame. It is a similar frame in this spacetime to the Milne frame in the Minkowski spacetime. In these coordinates the line element (87) takes the form

$$ds^2 = H_\Lambda^{-2} e^{-2\chi/R_0}\left[-dt^2 + d\chi^2 + \sinh^2\left(\chi/R_Q\right)d\Omega^2\right]. \tag{90}$$

A radial standard coordinate is introduced by the coordinate transformation

$$r = \left(2H_\Lambda\right)^{-1}\left(1 - e^{-2\chi/R_Q}\right) \quad , \quad \chi = -\left(R_Q/2\right)\ln\left(1 - 2H_\Lambda r\right). \tag{91}$$

With this radial coordinate the line element takes the form

$$ds^2 = -\left(1 - 2H_\Lambda r\right)dt^2 + \frac{dr^2}{1 - 2H_\Lambda r} + r^2 d\Omega^2 . \tag{92}$$

The rate of proper time decreases with increasing r. Hence the acceleration of gravity acts in the direction out from the charged shell, i.e. there is repulsive gravitation.

Since for an arbitrary equation of state the coordinates are comoving with the fluid, the CFS coordinates are comoving in an expanding reference frame when $b \neq 0$. If $b = 0$ there is no expansion and for $a_2 = 1/R_Q$ the line element reduces to the form (1) representing the LBR-spacetime outside a charged domain wall. In the static case the energy density of the electrical field due to the domain wall is $\rho_{E0} = 1/\kappa R_Q^2$ and $\rho_{F0} = 0$, i.e. then the vacuum energy vanishes.

On the other hand, if $a_2 = 0$, $b_2 \neq 0$ equation (80) shows that the electrical field vanishes. In this case the line element reduces to

$$ds^2 = \frac{ds_M^2}{H_\Lambda^2 T^2} . \tag{93}$$

This line element represents a universe with LIVE and is in agreement with equation (65) in ref.[22].

4.1. A class of static solutions

We shall now find a class of static solutions of the field equations (68) – (70) for which the fluid obeys an equation of state

$$p_F = w\rho_F , \tag{94}$$

where w is constant. Then $\beta(T) = 0$. Inserting the expressions for p_F and ρ_F from equations (66) and (69) into equation (94) then lead to the differential equation

$$\left(1 + 3w\right)R\alpha\alpha_{RR} + 3\left(1 + w\right)\alpha\alpha_R - 3\left(1 + w\right)R\alpha_R^2 = 0 . \tag{95}$$

Integration gives

$$\alpha_R = b\left(\frac{\alpha}{R}\right)^\gamma \quad , \quad \gamma = \frac{3\left(1 + w\right)}{1 + 3w} \quad , \quad w \neq -1/3 . \tag{96}$$

where b is a constant. For $w = -1/3$, i.e. a texture gas, we have $\alpha\alpha_R = R\alpha_R^2$ giving either $\alpha = a$ where a is a constant, or $\alpha = aR$. New integration for $w \neq -1/3$ gives

$$\alpha = \left(a + bR^{-2/(1+3w)}\right)^{-(1/2)(1+3w)} . \tag{97}$$

This is in agreement with the solution of the Einstein-Maxwell equations found by A. Banerjee and N. O. Santos [24]. Solving equation (84) with respect to α_{RR} and inserting into equation (66) gives

$$\kappa\rho_F = \frac{6}{1 + 3w}\frac{\alpha_R}{R}\left(R\alpha_R - \alpha\right), \tag{98}$$

which shows that the fluid vanishes if $\alpha(R) \propto R$. The solution then reduces to the LBR spacetime. In general the density of the fluid and the electrical field are given by

$$\kappa\rho_F = \frac{6b}{1 + 3w}\left(\frac{\alpha}{R}\right)^{\gamma+1}\left[b\left(\frac{\alpha}{R}\right)^{\gamma-1} - 1\right]. \tag{99}$$

and

$$\kappa\rho_E = b\left(\frac{\alpha}{R}\right)^{\gamma+1} - \frac{1}{2}(1+w)\kappa\rho_F \,. \tag{100}$$

In the case that the fluid is a LIVE, $w=-1$, $\gamma=0$ and

$$\alpha = a + bR \quad , \quad \kappa\rho_F = \frac{3ab}{R} \quad , \quad \kappa\rho_E = \frac{bR}{a+bR} \,. \tag{101}$$

In order to have positive densities this case requires $a>0$, $b>0$. For the case that the fluid is dust, $w=0$, $\gamma=3$ and

$$\alpha = \frac{R}{\left(aR^2+b\right)^{1/2}} \quad , \quad \kappa\rho_F = -\frac{6abR^2}{\left(aR^2+b\right)^3} \quad , \quad \kappa\rho_E = \frac{b\left(4aR^2+b\right)}{\left(aR^2+b\right)^3} \,. \tag{102}$$

Then we must have $a<0$, $b>0$ and $R^2 < -b/4a$. If the fluid is radiation $w=1/3$ and $\gamma=2$. Then

$$\alpha = \frac{r}{aR+b} \quad , \quad \kappa\rho_F = -\frac{3abR}{\left(aR+b\right)^4} \quad , \quad \kappa\rho_E = b\frac{b-aR}{\left(aR+b\right)^4} \,, \tag{103}$$

which requires $a<0$, $b>0$. In all cases, when there is no fluid, the density of the electrical field is constant, and the solution reduces to the LBR spacetime. On the other hand if both the fluid and the electrical field vanishes (requiring positive densities for a non-vanishing fluid and electrical field), and the solution reduces to the Minkowski spacetime.

4.2. A class of homogeneous, time dependent solutions

Homogeneous and time dependent solutions of the field equations (68) and (69) can be considered to be universe models with a conformal scale factor $a(T)=1/\beta(T)$, and will here be briefly presented. From the line element (64) it follows that these universe models are flat. In cosmology it is then usual to call the time T for conformal time and use the symbol η for it. In the present case there is no electric field so these solutions do not represent generalizations of the LBR solution. However, they are included for completeness.

For these solutions $\alpha(R)=0$, and the field equations reduce to

$$\kappa\rho_F = 3\beta_\eta^2 \quad , \quad \kappa p_F = -3\beta_\eta^2 + 2\beta\beta_{\eta\eta} \,. \tag{104}$$

The equation of state $p_F = w\rho_F$ then leads to the differential equation

$$2\beta\beta_{\eta\eta} - 3(1+w)\beta_\eta^2 = 0 \tag{105}$$

with the general solution

$$\beta = \left(K_1 + K_2\eta\right)^{-2/(1+3w)} \quad , \quad w \neq -1/3 \,, \tag{106}$$

where K_1 and K_2 are arbitrary constants.

There are some important special cases. A LIVE dominated universe has $w=-1$. In this case the scale factor with $a(\eta_0)=1$ at the present conformal time η_0, and the density of the vacuum energy are

$$a(\eta) = \left(K_1 + K_2\eta\right)^{-1} \quad , \quad K_2 = \left(1-K_1\right)/\eta_0 \quad , \quad \kappa\rho_F = 3K_2^2 \,. \tag{107}$$

Expansion demands that $K_2 < 0$ and hence that $K_1 > 1$. Even if the universe expands, the density of the LIVE is constant and can be described by a cosmological constant Λ with $K_2 = -H_\Lambda = -\sqrt{\Lambda/3}$ and $K_1 = 1 + H_\Lambda\eta_0$. The line element reduces to the form (93) when $K_1 = 0$. Pradhan and Pandey [17] have considered some similar universe models, but with decreasing density of the vacuum energy. However, by permitting the cosmological "constant" to be a function of time, they have introduced a new unknown function in addition to the pressure and density of a fluid. They have only two field equations. Hence they have not enough field equations to determine the functions, so their models have an ad hoc character.

A dust dominated universe has $w=0$. With $a(0)=0$, $a(\eta_0)=1$ and $\rho_F(\eta_0)=\rho_{F0}$ we obtain

$$a(\eta)=\left(\frac{\eta}{\eta_0}\right)^2 \quad , \quad \rho_F=\rho_{F0}\left(\frac{\eta}{\eta_0}\right)^6 . \tag{108}$$

Similarly for a radiation dominated universe, $w=1/3$, we find

$$a(\eta)=\frac{\eta}{\eta_0} \quad , \quad \rho_F=\rho_{F0}\left(\frac{\eta}{\eta_0}\right)^4 . \tag{109}$$

A texture dominated universe has $w=-1/3$. Then equations (104 give

$$\beta\beta_{\eta\eta}-\beta_\eta^2=0 . \tag{110}$$

The general solution of this equation is

$$\beta=K_1 e^{K_2\eta} , \tag{111}$$

where K_1 and K_2 are constants. Defining the Hubble parameter $H=a_\eta/a$, normalizing the scale factor to $a(\eta_0)=1$, and using that $\rho_F(\eta_0)=\rho_{F0}$, we obtain $\kappa\rho_{F0}=(3/4)H^2$, $K_2=-H/2$, $K_1^2=e^{H\eta_0}$. Then the scale factor of this universe model and the density of the texture gas are [23]

$$a(\eta)=e^{H(\eta-\eta_0)} \quad , \quad \rho_F=\rho_{F0}e^{-H(\eta-\eta_0)} . \tag{112}$$

In terms of the conformal time the universe expands exponentially and the density of the texture gas decreases exponentially. The cosmic time t is defined by $dt=e^{H(\eta-\eta_0)}d\eta$. Integration with the boundary condition $t(\eta_0)=1/H$ gives the transformation

$$Ht=e^{H(\eta-\eta_0)} , \tag{113}$$

and the line element takes the form

$$ds^2=-dt^2+(t/t_0)^2\left(dr^2+r^2d\Omega^2\right) . \tag{114}$$

The scale factor is the same as that of the empty Milne universe, but the spatial geometry is different. The Milne universe has negatively curved spatial geometry, while the texture dominated universe has Euclidean spatial geometry.

5. On the uniqueness of static, conformally flat, spherically symmetric spacetimes

The internal Schwarzschild solution or Schwarzschild interior metric (SIM) is usually defined to be the solution of Einstein's field equations inside a static, spherically symmetric and incompressible fluid matching the metric of the exterior Schwarzschild solution and its derivative continuously at the boundary of the mass distribution. The spacetime of SIM is conformally flat.

In 1973 B. Kuchowicz [24] considered conformally flat spacetimes with spherically symmetric 3-space as described in terms of isotropic coordinates. He then wrote that the internal Schwarzschild solution is the only static and spherically symmetric solution of the Einstein equations that is conformally flat. This conclusion is based upon the requirement that the pressure of the perfect fluid and energy filling the spacetime is isotropic and positive.

The de Sitter spacetime is conformally flat and can be described in isotropic coordinates with a static and spherically symmetric metric. But the pressure of the vacuum energy filling this spacetime is negative. We shall here show how this solution appears as a special case of Kuchowicz's field equations (11) in ref.[24]. Let us assume that all of his four functions F_i are constant. Then his field equations reduce to

$$\kappa\rho=-12\frac{F_2}{F_3} \quad , \quad \kappa p=4\frac{2F_1F_2-F_2^2+(F_1-2F_2)r}{F_3\left(F_1-r^2\right)} . \tag{115}$$

Here the density is constant. In order that the pressure too shall be constant, the functions F_i that are now constants, must fulfill the relationship $F_1=F_2(2F_1-F_2)/(2F_2-F_1)$ or $F_1^2=F_2^2$. This gives $\kappa\rho=4(2F_2-F_1)/F_3$. Hence, putting $2F_2-F_1=3F_2$ or $F_2=-F_1$ leads to $p=-\rho$, which is the equation of state of LIVE. With $F_1 = 3/\Lambda = H_\Lambda^{-2}$, $F_3 = 36/\Lambda^2 = 4H_\Lambda^{-4}$ equation (10) in ref. [24] then leads to the static form of the de Sitter line element in isotropic coordinates,

$$ds^2 = -\left(\frac{1-H_\Lambda^2 r^2}{1+H_\Lambda^2 r^2}\right)dt^2 + \frac{4}{\left(1+H_\Lambda^2 r^2\right)^2}\left(dr^2 + r^2 d\Omega^2\right). \tag{116}$$

On the other hand if $F_2 = F_1$ we get the equation of state $p = -\rho/3$ characterizing a texture gas. It may be noted that this equation of state is obtained by putting $B = 0$ in equation (1.4) of M. Gürses and Y. Gürsey [25]. We must still have $F_2 < 0$ in order that the density of the texture gas shall be positive. Introducing $H^2 = (1/12)\kappa\rho$ and letting $F_2 = F_1 = -H^{-2}$, equation (10) in ref. [24] shows that the line element in isotropic coordinates of this spacetime has the form,

$$ds^2 = -dt^2 + \frac{1}{\left(1+H^2 r^2\right)^2}\left(dr^2 + r^2 d\Omega^2\right). \tag{117}$$

Hence if the condition of a positive and isotropic pressure is relaxed, there exist several solutions of the Einstein's and Maxwell's field equations that are static, spherically symmetric and conformally flat, besides the internal Schwarzschild solution.

Introducing a radial curvature coordinate

$$\overline{r} = \frac{r}{1+H^2 r^2} \tag{118}$$

with inverse transformation

$$r = \frac{2\overline{r}}{1+\sqrt{1-H^2\overline{r}^2}}, \tag{119}$$

the line element (117) takes the form

$$ds^2 = -dt^2 + \frac{d\overline{r}^2}{1-H^2\overline{r}^2} + \overline{r}^2 d\Omega^2. \tag{120}$$

Introducing a physical radial coordinate $\hat{r}$ so that $d\hat{r} = \left(1-H^2\overline{r}^2\right)^{-1/2} d\overline{r}$ we find the coordinate transformation

$$H\hat{r} = \arcsin\left(H\overline{r}\right) \quad , \quad H\overline{r} = \sin\left(h\hat{r}\right), \tag{121}$$

and the line element (120) takes the form

$$ds^2 = -dt^2 + d\hat{r}^2 + H^{-2}\sin^2\left(H\hat{r}\right)d\Omega^2. \tag{122}$$

The transformation to CFS coordinates is

$$HT = \frac{\sin\left(Ht\right)}{\cos\left(Ht\right)+\cos\left(H\hat{r}\right)} \quad , \quad HR = \frac{\sin\left(H\hat{r}\right)}{\cos\left(Ht\right)+\cos\left(H\hat{r}\right)}. \tag{123}$$

In these coordinates the line element of the texture dominated spacetime takes the form

$$ds^2 = \frac{4}{4H^2T^2 + \left[1-H^2\left(T^2-R^2\right)\right]^2} ds_M^2. \tag{124}$$

The line element (120) is in accordance with equations (1.1) – (1.3) of Gürses and Gürsey [25] for the case $B = 0$. It seems to represent a counter example to their proof that the internal Schwarzschild solution is the only static, spherically symmetric and conformally flat solution of Einstein's field equations. However, they define the "Schwarzschild interior metric" (SIM) in a wider way than usual. They write the line element of SIM in curvature coordinates in the following form

$$ds^2 = -\left[A - B\left(1-H^2\overline{r}^2\right)^{1/2}\right]^2 d\overline{t}^2 + \left(1-H^2\overline{r}^2\right)^{-1} d\overline{r}^2 + \overline{r}^2 d\Omega^2, \tag{125}$$

where A, B and H are arbitrary constants. If there is an exterior solution, the constants may be determined by matching the metric and its radial derivative to the external solution at the boundary. With external Schwarzschild geometry one obtains $A = (3/2)\sqrt{1-R_S/R}$, $B = 1/2$, $H = \sqrt{R_S/R^3}$

where R_S is the Schwarzschild radius of the mass in the interior region, and R is the radius of the mass distribution. Usually one defines the interior Schwarzschild solution with the constants determined in this way. With the definition of Gürses and Gürsey [25] one may for example put $A=0$ or $B=0$ and obtain particular cases of the SIM. This is not possible with the conventional definition of SIM.

The interior Schwarzschild spacetime contains an ideal fluid with constant density and isotropic pressure. Gürses and Gürsey have given the pressure distribution in the form

$$p=-\frac{B\left(1-H^2\overline{r}^2\right)^{1/2}-A/3}{B\left(1-H^2\overline{r}^2\right)^{1/2}-A}\rho \,. \tag{126}$$

Hence, for $A=0$ the fluid has the equation of state $p=-\rho$ of LIVE, and the line element takes the form

$$ds^2=-\left(1-H_\Lambda^2\overline{r}^2\right)d\overline{t}^2+\left(1-H_\Lambda^2\overline{r}^2\right)^{-1}d\overline{r}^2+\overline{r}^2d\Omega^2 \,, \tag{127}$$

which is the static form of the line element in curvature coordinates representing the de Sitter spacetime. In this case $B=1$ and $R_0=1/H_\Lambda$. For $B=0$ the fluid has the equation of state $p=-(1/3)\rho$ of a texture gas, and with $A=1$ the line element (125) takes the form (120). So with the definition of Gürses and Gürsey the De Sitter spacetime and the texture gas dominated spacetime are both special cases of SIM. This explains their uniqueness theorem concerning static, spherically symmetric and conformally flat solutions of Einstein's field equations. With the conventional definition SIM is not the only static, spherically symmetric and conformally flat solution of Einstein's field equations.

6. Conclusion

The LBR-solution of Einstein's equations is static, spherically symmetric and conformally flat. In this solution the pressure is anisotropic due to the presence of a radial electric field in this spacetime. This spacetime represents the region outside a charged, spherical domain wall.

In the present paper we have discussed the physical significance of two classes of solutions found by A. Banerjee and M. M. Som [10] and another one by S. N. Pandey and R. Tiwari [11]. We have shown that they represent static, spherically symmetric and conformally flat spacetimes that are generalizations of the LBR spacetime. Two such generalizations have been discussed, one with charged dust outside a charged spherical shell, and another with LIVE outside a charged spherical shell.

Finally we have discussed the uniqueness of static, spherically symmetric and conformally flat solutions of the Einstein-Maxwell field equations and pointed out that the internal Schwarzschild solution is the unique such solution only in spacetimes with a perfect fluid having positive and isotropic pressure. Allowing negative pressure there is no such uniqueness unless the definition of the term "internal Schwarzschild solution" is extended. In the presence of an electrical field the status of the internal Schwarzschild solution as the only static, spherically symmetric and conformally flat solution of the Einstein-Maxwell field equations is lost.

Appendix. Elements of calculus of k-functions

In reference [10] we presented a new k-calculus. We here collect the formulae from this calculus used in the present article. The following functions are used,

$$S_k(x)=\begin{cases}\sin x & for \quad k=1\\ x & for \quad k=0\\ \sinh x & for \quad k=-1\end{cases} \,, \tag{A.1}$$

$$C_k(x)=\begin{cases}\cos x & for \quad k=1\\ 1 & for \quad k=0\\ \cosh x & for \quad k=-1\end{cases}\quad , \tag{A.2}$$

$$T_k(x)=\begin{cases}\tan x & for \quad k=1\\ x & for \quad k=0\\ \tanh x & for \quad k=-1\end{cases}\quad , \tag{A.3}$$

$$I_k(x)=\begin{cases}\cot x & for \quad k=1\\ 1/x & for \quad k=0\\ \coth x & for \quad k=-1\end{cases}\quad , \tag{A.4}$$

From the usual trigonometric and hyperbolic identities and the definitions (A.1) and (A.2) it follows that

$$C_k(x)^2+kS_k(x)^2=1\,, \tag{A.5}$$

$$S_k(2x)=2S_k(x)\,C_k(x) \tag{A.6}$$

and

$$C_k(2x)=C_k(x)^2-kS_k(x)^2\,. \tag{A.7}$$

From equations (A.4) and (A.6) follows

$$C_k(x)=\frac{C_k(x/2)^2-kS_k(x/2)^2}{C_k(x/2)^2+kS_k(x/2)^2}=\frac{1-kT_k(x/2)^2}{1+kT_k(x/2)^2}\,. \tag{A.8}$$

We shall also need the formula

$$\int\frac{1}{S_k(x)}dx=\ln\left[n_kT_k(x/2)+K_k\right]\quad , \quad n_k=\begin{cases}1 & , \quad k\neq 0\\ 2 & , \quad k=0\end{cases}, \tag{A.9}$$

where K_k is a k-dependent integration constant.

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