

RESOLVENT-TECHNIQUES FOR MULTIPLE EXERCISE PROBLEMS

SÖREN CHRISTENSEN AND JUKKA LEMPA

ABSTRACT. We study optimal multiple stopping of strong Markov processes with random refraction periods. The refraction periods are assumed to be exponentially distributed with a common rate and independent of the underlying dynamics. Our main tool is using the resolvent operator. In the first part, we reduce infinite stopping problems to ordinary ones in a general strong Markov setting. This leads to explicit solutions for wide classes of such problems. Starting from this result, we analyze problems with finitely many exercise rights and explain solution methods for some classes of problems with underlying Lévy and diffusion processes, where the optimal characteristics of the problems can be identified more explicitly. We illustrate the main results with explicit examples.

1. INTRODUCTION

In this paper, we investigate the following optimal multiple stopping problem. Assume that the agent follows the evolution of a continuous-time state variable X and has the possibility to choose N stopping times such that there are exponentially distributed refraction periods between the stopping times. At each stopping time, the agent gets a payoff contingent on the state of X and her objective is to maximize the expected present value of the total payoff.

Optimal multiple stopping problems with deterministic refraction periods have been studied quite extensively over the recent years. In particular, theory of Snell envelopes is understood well in both discrete and continuous time, see [10, 11, 1, 6, 7, 30, 39, 43]. In the recent study [14], the authors develop a general approach for optimal multiple stopping with random refraction periods. In this paper, the refraction periods are assumed to be almost surely finite, nonnegative random variables. Furthermore, the exercise policies are formalized with respect to appropriately extended filtrations, see [14], Section 2, for details. From the mathematical point of view, our study is concerned with a special case of [14]. However, by imposing the additional assumption of the distribution of the refraction periods, we can exploit efficiently the known connection between exponentially distributed random times and the resolvent semigroup in our study. This allows

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us to get a much more detailed picture on the problem. For instance, we can solve the problem explicitly in many cases.

One of the major drawbacks in most of the previously mentioned articles is that it is hard to find examples that allow for an explicit solution, even in the case of simple one-dimensional underlying diffusion processes. Indeed, when studying optimal multiple stopping with deterministic refraction periods, the main obstacle is how to determine the value of the remaining stopping opportunities, the so-called *continuation value*. In our framework with exponentially distributed refraction periods, we can tackle this using the theory of resolvent semigroups. From applications point of view, it is also interesting to study multiple stopping with random refraction periods. As we already mentioned, the majority of the literature is concerned with deterministic refraction periods. If we consider applications in real investment problems, it can very well be that the waiting times between exercises of a multi-strike real option are random. As an example in the real option spirit, consider a firm facing the timing problem of multiple identical investment projects. The firm has the capability to execute only one project at a time and it takes a period of time to complete the project. Here, the refraction period has the interpretation as *time to build*, see, e.g., [29]. Depending on the nature of the project, it can be very difficult to tell in advance how long it takes to complete the project and therefore it can be reasonable to include uncertain times to build into the investment problem. Our study gives a flexible yet tractable model to study problems of this form.

This paper contributes to the theory of optimal multiple stopping in the following ways. We present a thorough analysis of the case of exponential waiting times which does not appear in the literature before. We point out an interesting connection on the multiple stopping problem to a class of stochastic impulse control problems. We give separate account of both the case with infinite and finite number of stopping times: we call these problems *infinite* and *finite* stopping problems, respectively. The infinite stopping problem, which is also interesting in its own right, gives an important point of reference to the finite stopping problem as a natural limiting case. Our main results are presented for general underlying strong Markov dynamics. The general results are complemented with a number of more specific results for underlying Lévy and diffusion dynamics. The results are also illustrated with several explicit examples.

The remainder of the paper is organized as follows. In Section 2 we formalize the optimal multiple stopping problem, whereas in Section 3 we first concentrate on the infinite multiple stopping problem, where the issuer has infinitely many exercise rights. Surprisingly, it turns out

that these problems can be reduced to ordinary optimal stopping problems and therefore can be solved explicitly in many cases of interest. Some of these examples are given in Section 4, including multidimensional underlying processes and processes with jumps. In Section 5 we study some of the general properties of stopping problems with finitely many exercise rights. Section 6 is devoted to the analysis of some specific classes of the problems with underlying diffusion dynamics and Lévy processes. The study is wrapped up with an explicit example of the finite stopping problem in Section 7.

2. THE MULTIPLE STOPPING PROBLEM

We start by laying down the probabilistic foundation for the optimal multiple stopping problem. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ be a complete filtered probability space satisfying the usual conditions, where $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$, see [9], p. 2. We assume that the underlying X is a strong Markov process defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ and taking values in $E \subseteq \mathbf{R}^d$ for some $d \geq 1$ with the initial state $x \in E$. As usual, we augment the state space E with a topologically isolated element Δ if the process X is non-conservative. Then the process X can be made conservative on the augmented state space $E^\Delta := E \cup \{\Delta\}$, see [37]. In what follows, we drop the superscript Δ from the notation. By convention, we augment the definition of functions g on E with $g(\Delta) = 0$.

Denote as \mathbf{P}_x the probability measure \mathbf{P} conditioned on the initial state x and as \mathbf{E}_x the expectation with respect to \mathbf{P}_x . The process X is assumed to evolve under \mathbf{P}_x and the sample paths are assumed to be right-continuous and left-continuous over stopping times meaning the following: if the sequence of stopping times $\tau_n \uparrow \tau$, then $X_{\tau_n} \rightarrow X_\tau$ \mathbf{P}_x -almost surely as $n \rightarrow \infty$. Furthermore, we assume that the underlying probability space is rich enough to carry a time-shift operator θ . for the strong Markov process X . There is a well-established theory of standard optimal stopping for this class of processes, see [35].

For $r > 0$, we denote by L_1^r the class of real valued measurable functions f on E satisfying the integrability condition $\mathbf{E}_x \left\{ \int_0^\infty e^{-rt} |f(X_t)| dt \right\} < \infty$ for all $x \in E$. For a function $f \in L_1^r$, the *resolvent* $R_r f : E \rightarrow \mathbf{R}$ is defined as

$$(R_r f)(x) = \mathbf{E}_x \left\{ \int_0^\infty e^{-rs} f(X_s) ds \right\},$$

for all $x \in E$. It is well known that the family $(R_\lambda)_{\lambda > 0}$ is a strongly continuous contraction resolvent and that it has the following connection to exponentially distributed random times: if

$U \sim \text{Exp}(\lambda)$ and independent of X , then $\lambda(R_{r+\lambda}g)(x) = \mathbf{E}_x \{e^{-rU}g(X_U)\}$ whenever $g \in L_1^r$, see [37].

As was explained in the introduction, we consider an optimal multiple stopping problem with underlying strong Markov process X subject to exponentially distributed refraction periods in between the stopping times. To make a precise statement, let U, U_1, \dots, U_N be IID with $U \sim \text{Exp}(\lambda)$ and independent of X . For brevity, denote the N -tuple (U_1, \dots, U_N) as \bar{U} . Furthermore, denote as $\bar{\tau} = (\tau_1, \dots, \tau_N)$ an N -tuple of random times in the case when N is finite. We first concentrate on the case $N = \infty$, where we consider sequences $\bar{\tau} = (\tau_n)_{n \in \mathbb{N}}$, $\bar{U} = (U_n)_{n \in \mathbb{N}}$, instead of tuples. A priori, the random times U_i are not \mathbb{F} -stopping times, so we need to consider an appropriately augmented filtration to formalize the multiple stopping problem. To this end, we use the approach used in [14] which we review here shortly for the readers' convenience. Define the augmented filtration $\mathbb{F}^{\tau_1+U_1}$ as the smallest right-continuous filtration such that the random time $\tau_1 + U_1$ is a stopping time. Furthermore, define the remaining augmentations $\mathbb{F}^{\tau_1+U_1, \dots, \tau_i+U_i}$, $i = 2, \dots, N$, recursively as

$$\mathbb{F}^{\tau_1+U_1, \dots, \tau_i+U_i} = \left(\mathbb{F}^{\tau_1+U_1, \dots, \tau_{i-1}+U_{i-1}} \right)^{\tau_i+U_i}.$$

This filtration carries the information on the occurrences of the chosen stopping times τ_i and the random times when the refraction period has elapsed after a given stopping time, i.e., the times $\tau_i + U_i$. Now, for each \mathbb{F} -stopping time η and for $n \leq N$, define

$$\mathcal{S}_\eta^n(\hat{U}, \mathbb{F}) = \left\{ \bar{\tau} : \begin{array}{l} \tau_1 \text{ is an } \mathbb{F}\text{-stopping time with } \eta \leq \tau_1, \\ \tau_i \text{ is an } \mathbb{F}^{\tau_1+U_1, \dots, \tau_{i-1}+U_{i-1}}\text{-stopping time} \\ \text{and } \tau_{i-1} + U_{i-1} \leq \tau_i, \quad i = 2, \dots, n \end{array} \right\},$$

where $\hat{U} = (U_1, \dots, U_{n-1})$. Using the set \mathcal{S} , we can formalize the multiple stopping problem under consideration as follows

$$(2.1) \quad V^N(x) = V_\lambda^N(x) = \sup_{\bar{\tau} \in \mathcal{S}_0^N(\hat{U}, \mathbb{F})} \mathbf{E}_x \left\{ \sum_{i=1}^N e^{-r\tau_i} g(X_{\tau_i}) \mathbf{1}_{\{\tau_i < \infty\}} \right\}.$$

Here, $r > 0$ is the constant rate of discounting and the function g satisfies some regularity properties to be specified below.

2.1. Connection to impulse control. Before solving the multiple stopping problem, we want to point out the connection of the multiple and infinite optimal stopping problem to impulse control

problems. We do not use the theory of impulse control in the following, but this point of view sheds light on the nature of the problem. To this end, we introduce a new Markov process \hat{X} on the new state space

$$\hat{E} = E \cup E^\partial,$$

where for each $x \in E$ we introduce a new isolated point ∂_x and write $E^\partial = \{\partial_x : x \in E\}$. Started on E the process \hat{X} behaves as the original process X . Started in ∂_x , the process \hat{X} stays in this point for an exponential time (with parameter λ) and jumps back to E and is restarted with an initial distribution $\mathbf{P}_x(X_S \in \cdot)$, where S is an independent $\text{Exp}(\lambda)$ -distributed random variable.

An admissible impulse control policy for this sequence is a potentially infinite joint sequence $K = (\tau_n, \eta_n)_{n \in \mathbb{N}}$, where $(\tau_n)_{n \in \mathbb{N}}$ is an increasing sequence of stopping times and each η_n is an \mathcal{F}_{τ_n} -measurable random variable with values in the set $\mathcal{A}(X_{\tau_n})$ (in case this set is not empty; for $\mathcal{A}(X_{\tau_n}) = \emptyset$, no control is possible). In our special situation, we consider $\mathcal{A}(x) = \{\partial_x\}$ for all $x \in E$ and $\mathcal{A}(x) = \emptyset$ for all $x \in E^\partial$. For each impulse control strategy K , the controlled process X under the measure \mathbf{P}_x^K behaves as under \mathbf{P}_x until the first control taking place in τ_1 . Then the process is started from η_1 and runs uncontrolled until τ_2 and so on. The construction of the process (on an extension of the original probability space) can be found in [41] to give a reference in English.

We consider the optimal impulse control problem

$$\tilde{V}(x) = \sup_{K=(\tau_n, \eta_n)_{n \in \mathbb{N}}} \mathbf{E}_x^K \left\{ \sum_{n=1}^{\infty} e^{-r\tau_n} g(X_{\tau_n-}) \mathbf{1}_{\{\tau_n < \infty\}} \right\}$$

where $g : E \rightarrow \mathbf{R}$ is as before and we formally set $g(\partial_x) = 0$ for all $x \in E$.

Now, noting that there is an exponential time between each two stopping times of an impulse control strategy, there is a one-to-one correspondence between sequences of stopping times $\bar{\tau} \in \mathcal{S}_0^\infty(\hat{U}, \mathbb{F})$ in the infinite stopping problem and impulse control strategies as given above simply by

$$(\tau_n, \eta_n)_{n \in \mathbb{N}} \mapsto (\tau_n)_{n \in \mathbb{N}}$$

and

$$\mathbf{E}_x^K \left\{ \sum_{n=1}^{\infty} e^{-r\tau_n} g(X_{\tau_n-}) \mathbf{1}_{\{\tau_n < \infty\}} \right\} = \mathbf{E}_x \left\{ \sum_{i=1}^{\infty} e^{-r\tau_i} g(X_{\tau_i}) \mathbf{1}_{\{\tau_i < \infty\}} \right\}.$$

In particular, we see that $V^\infty(x) = \tilde{V}(x)$ and $(\tau_n^*, \eta_n^*)_{n \in \mathbb{N}}$ is optimal for the impulse control problem if and only if $(\tau_n^*)_{n \in \mathbb{N}}$ is optimal for the infinite stopping problem. Therefore, the infinite stopping problem can be seen as an impulse control problem for a strong Markov process from a

special class. The multiple stopping problem for finite N can therefore be seen as the standard approximation of this problem, that is often used for an approximation for general impulse control problems. For connected results in the finite stopping situation, we refer to [12, Subsection 3.6].

3. THE INFINITE STOPPING PROBLEM

Since the time horizon is assumed to be infinite, it is natural to consider the infinite stopping problem $N = \infty$, that is, the problem

$$(3.1) \quad V^\infty(x) = V_\lambda^\infty(x) = \sup_{\bar{\tau} \in \mathcal{S}_0^\infty(\hat{U}, \mathbb{F})} \mathbf{E}_x \left\{ \sum_{i=1}^{\infty} e^{-r\tau_i} g(X_{\tau_i}) \mathbf{1}_{\{\tau_i < \infty\}} \right\}.$$

We assume in the following that g is continuous and nonnegative. The main aim of this section is to show that the solution to this infinite stopping problem can be reduced to the solution of the ordinary optimal stopping problem

$$\hat{V}(x) = \sup_{\tau} \mathbf{E}_x \left\{ e^{-(r+\lambda)\tau} g(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right\}$$

under general assumptions (note that the discounting parameter $r + \lambda$ is used instead of r). To this end, we first give a verification theorem for the value function.

Proposition 3.1. *Assume that v is nonnegative, r -excessive, and*

$$(3.2) \quad v(x) \geq g(x) + \lambda(R_{r+\lambda}v)(x).$$

Then $V^\infty \leq v$.

Furthermore, if τ^* is a stopping time such that for all $x \in E$,

$$(3.3) \quad \mathbf{E}_x \left\{ e^{-r\tau^*} v(X_{\tau^*}) \mathbf{1}_{\{\tau^* < \infty\}} \right\} = v(x),$$

and

$$(3.4) \quad \begin{aligned} & \mathbf{E}_x \left\{ e^{-r\tau^*} v(X_{\tau^*}) \mathbf{1}_{\{\tau^* < \infty\}} \right\} \\ &= \mathbf{E}_x \left\{ e^{-r\tau^*} g(X_{\tau^*}) \mathbf{1}_{\{\tau^* < \infty\}} \right\} + \lambda \mathbf{E}_x \left\{ e^{-r\tau^*} (R_{r+\lambda}v)(X_{\tau^*}) \mathbf{1}_{\{\tau^* < \infty\}} \right\}, \end{aligned}$$

then $V^\infty = v$ and the sequence $\bar{\tau} \in \mathcal{S}_0^\infty(\hat{U}, \mathbb{F})$ associated with τ^* given by $\tau_1 = \tau^*$,

$$\tau_i = \tau^* \circ \theta_{\tau_{i-1} + U_{i-1}} + \tau_{i-1} + U_{i-1} \text{ for all } i \geq 2,$$

is optimal.

Before proving the result, let us note that for each nonnegative, r -excessive function v it holds that $v \in L_1^{r+\lambda}$ since

$$\mathbf{E}_x \left\{ \int_0^\infty e^{-(r+\lambda)t} |v(X_t)| dt \right\} \leq \int_0^\infty e^{-\lambda t} v(x) dt < \infty,$$

i.e. the function $R_{r+\lambda}v$ is well-defined for all x .

Proof. Let $\bar{\tau} \in S_0^\infty(\hat{U}, \mathbb{F})$ be arbitrary. By noting that $\tau_k \rightarrow \infty$, it is clear that $\mathbf{E}_x \{e^{-r\tau_k} \lambda(R_{r+\lambda}v)(X_{\tau_k})\} \rightarrow$

0. Writing $\tau_{i+1} = \tau_i + \sigma_{i+1} \circ \theta_{\tau_i}$ we obtain

$$\begin{aligned} & \mathbf{E}_x \left\{ \sum_{i=1}^\infty e^{-r\tau_i} g(X_{\tau_i}) \mathbf{1}_{\{\tau_i < \infty\}} \right\} \\ & \leq \mathbf{E}_x \left\{ \sum_{i=1}^\infty e^{-r\tau_i} (v(X_{\tau_i}) - \lambda(R_{r+\lambda}v)(X_{\tau_i})) \mathbf{1}_{\{\tau_i < \infty\}} \right\} \\ & = \mathbf{E}_x \{e^{-r\tau_1} v(X_{\tau_1}) \mathbf{1}_{\{\tau_1 < \infty\}}\} - \lim_{k \rightarrow \infty} \mathbf{E}_x \{e^{-r\tau_k} \lambda(R_{r+\lambda}v)(X_{\tau_k})\} \\ & \quad + \lim_{k \rightarrow \infty} \sum_{i=1}^k \mathbf{E}_x \{e^{-r\tau_{i+1}} v(X_{\tau_{i+1}}) \mathbf{1}_{\{\tau_{i+1} < \infty\}} - e^{-r\tau_i} \lambda(R_{r+\lambda}v)(X_{\tau_i}) \mathbf{1}_{\{\tau_i < \infty\}}\} \\ & = \mathbf{E}_x \{e^{-r\tau_1} v(X_{\tau_1}) \mathbf{1}_{\{\tau_1 < \infty\}}\} \\ & \quad + \sum_{i=1}^\infty \mathbf{E}_x \{e^{-r\tau_{i+1}} v(X_{\tau_{i+1}}) \mathbf{1}_{\{\tau_{i+1} < \infty\}} - e^{-r\tau_i} \mathbf{E}_{X_{\tau_i}} \{e^{-rU_i} v(X_{U_i})\} \mathbf{1}_{\{\tau_i < \infty\}}\} \\ & = \mathbf{E}_x \{e^{-r\tau_1} v(X_{\tau_1}) \mathbf{1}_{\{\tau_1 < \infty\}}\} \\ & \quad + \sum_{i=1}^\infty \mathbf{E}_x \{e^{-r\tau_i} (\mathbf{E}_{X_{\tau_i}} \{e^{-r\sigma_{i+1}} v(X_{\sigma_{i+1}})\}) - \mathbf{E}_{X_{\tau_i}} \{e^{-rU_i} v(X_{U_i})\} \mathbf{1}_{\{\tau_i < \infty\}}\} \\ & \leq v(x), \end{aligned}$$

where, in the last step, we used the r -excessivity of v together with the fact that $\sigma_{i+1} \geq U_i$ for all i . This proves the first claim.

For the second claim, note that we have equality in each step for the special sequence. \square

Having the previous verification result in mind, we have to find an appropriate candidate function v . However, on the first view it is not clear at all how to find such a candidate and for other refraction time distributions, it seems to be impossible to find such a candidate explicitly. But the exponential distribution allows for such a construction.

3.1. On solving ordinary optimal stopping problems using the resolvent operator. The key idea is to use a representation of the value function of ordinary optimal stopping problems

using the resolvent operator as follows: Consider the ordinary optimal stopping problem

$$\hat{V}(x) = \sup_{\tau} \mathbf{E}_x \{ e^{-\hat{r}\tau} g(X_{\tau}) \mathbf{1}_{\{\tau < \infty\}} \}$$

for some $\hat{r} > 0$. It is well-known that under minimal conditions on X and g , the value function \hat{V} can be characterized as the smallest \hat{r} -excessive majorant of g , see e.g. [40].

A typical class of \hat{r} -excessive functions is given by the \hat{r} -resolvent operator $R_{\hat{r}}$ applied to a nonnegative function σ . On the other hand, the Riesz representation theorem for \hat{r} -excessive functions of a *nice* strong Markov process X with state space E implies that each \hat{r} -excessive function u can be represented uniquely in the form

$$(3.5) \quad u(x) = \int_E G_{\hat{r}}(x, y) \sigma(dy) + h(x),$$

where $G_{\hat{r}}$ denotes the resolvent kernel with respect to some duality measure m , σ is a Radon measure, and h is an \hat{r} -harmonic function. For the exact assumptions on the process X in the framework of Hunt processes, we refer to the discussion in [17, Chapters 13,14]. This integral representation of \hat{r} -excessive functions can be used fruitfully to establish a dual approach to solving ordinary optimal stopping problems by representing \hat{V} in the form (3.5), see [38, 32, 13, 18, 16].

One basic approach to obtain a representation of the form (3.5) is the following. Under weak integrability assumptions on g , it can be seen that \hat{V} is an \hat{r} -potential implying that $h = 0$. Furthermore, if g is smooth enough, it can be seen that – under some further assumptions – the measure σ is absolutely continuous with respect to the duality measure m and the density of σ is given by

$$\sigma(dy) = \begin{cases} 0, & y \in S^c, \\ \sigma(y)m(dy), & y \in S, \end{cases}$$

where S denotes the stopping set, the function σ is defined as $\sigma(y) = (\hat{r} - \mathcal{A})\hat{V}(y)$, and \mathcal{A} denotes the generator/Dynkin operator of X – for local operators \mathcal{A} , it holds that $\sigma(y) = (\hat{r} - \mathcal{A})g(y)$. We write $\sigma(y) = 0$ on S^c . Then, the representation (3.5) reads as

$$\hat{V}(x) = \int_E G_{\hat{r}}(x, y) \sigma(dy) + h(x) = \int_E \sigma(y) G_{\hat{r}}(x, y) m(dy) = R_{\hat{r}}\sigma(x),$$

which yields a representation of the value function as a resolvent of a nonnegative function σ . This representation for value functions of ordinary optimal stopping problems will be the key for solving infinite stopping problems in the rest of this section.

3.2. Reduction of the infinite optimal stopping problem to an ordinary one. Now, we assume that there exists an optimal stopping time τ^* for the problem

$$(3.6) \quad \hat{V}(x) = \sup_{\tau} \mathbf{E}_x \left\{ e^{-(r+\lambda)\tau} g(X_{\tau}) \mathbf{1}_{\{\tau < \infty\}} \right\}$$

and that \hat{V} has the Riesz representation

$$\hat{V}(x) = (R_{r+\lambda}\sigma)(x)$$

for some function $\sigma : E \rightarrow \mathbf{R}_+$, see the previous discussion. Assuming that the integral defining \hat{V} is finite, we define the function v as

$$v(x) = (R_r\sigma)(x) \text{ for all } x \in E;$$

note that we changed the parameter of the resolvent from $\hat{r} = r + \lambda$ to r . Then, v is r -excessive and nonnegative as the r -resolvent of a nonnegative function. Furthermore, since \hat{V} is $(r + \lambda)$ -harmonic on the continuation set S^c , i.e. $\sigma = 0$ on S^c , we see that v is r -harmonic on this set, in particular

$$\mathbf{E}_x \left\{ e^{-r\tau^*} v(X_{\tau^*}) \mathbf{1}_{\{\tau^* < \infty\}} \right\} = v(x).$$

That is, the condition (3.3) holds. Moreover, using the resolvent equation we obtain

$$v(x) - g(x) \geq v(x) - \hat{V}(x) = ((R_r - R_{r+\lambda})\sigma)(x) = \lambda(R_{r+\lambda}(R_r\sigma))(x) = \lambda(R_{r+\lambda}v)(x)$$

for all $x \in E$ with equality for x in the stopping set, and in particular assumption (3.2) holds true. By evaluating these functions at X_{τ^*} , multiplying by $e^{-r\tau^*}$ and taking expectations, we obtain using the optimality of τ^* the condition

$$\begin{aligned} & \mathbf{E}_x \left\{ e^{-r\tau^*} v(X_{\tau^*}) \mathbf{1}_{\{\tau^* < \infty\}} \right\} - \mathbf{E}_x \left\{ e^{-r\tau^*} g(X_{\tau^*}) \mathbf{1}_{\{\tau^* < \infty\}} \right\} \\ &= \mathbf{E}_x \left\{ e^{-r\tau^*} v(X_{\tau^*}) \mathbf{1}_{\{\tau^* < \infty\}} \right\} - \mathbf{E}_x \left\{ e^{-r\tau^*} \hat{V}(X_{\tau^*}) \mathbf{1}_{\{\tau^* < \infty\}} \right\} \\ &= \lambda \mathbf{E}_x \left\{ e^{-r\tau^*} (R_{r+\lambda}v)(X_{\tau^*}) \mathbf{1}_{\{\tau^* < \infty\}} \right\}, \end{aligned}$$

that is, the condition (3.4). This shows that the assumptions of Proposition 3.1 are fulfilled.

Putting pieces together, we have the following:

Proposition 3.2. *Assume that there exists an optimal stopping time τ^* for the problem*

$$\hat{V}(x) = \sup_{\tau} \mathbf{E}_x \left\{ e^{-(r+\lambda)\tau} g(X_{\tau}) \mathbf{1}_{\{\tau < \infty\}} \right\}$$

and that \hat{V} has the Riesz representation

$$(3.7) \quad \hat{V}(x) = (R_{r+\lambda}\sigma)(x)$$

for some function $\sigma : E \rightarrow \mathbf{R}_+$ in L_1^r . Then

$$V^\infty(x) = (R_r\sigma)(x) \text{ for all } x \in E$$

and the sequence $\bar{\tau} \in \mathcal{S}_0^\infty(\hat{U}, \mathbb{F})$ associated with τ^* given by $\tau_1 = \tau^*$,

$$\tau_i = \tau^* \circ \theta_{\tau_{i-1} + U_{i-1}} + \tau_{i-1} + U_{i-1} \text{ for all } i \geq 2,$$

is optimal.

Remark 3.1. We point out that the optimal stopping problem (3.6) has an interpretation as an optimal stopping problem with random time horizon. To this end, let $U \sim \text{Exp}(\lambda)$ independent of X and τ be a stopping time. Then

$$\mathbf{E}_x \{ e^{-r\tau} g(X_\tau) e^{-\lambda\tau} \mathbf{1}_{\{\tau < \infty\}} \} = \mathbf{E}_x \{ e^{-r\tau} g(X_\tau) \mathbf{1}_{\{\tau < U\}} \}.$$

That is, we can interpret the optimal stopping problem (3.6) as a problem with independent and exponentially distributed time horizon. The mean of the random time horizon is the same as the mean waiting time after stopping in the infinite stopping problem.

4. SOME EXAMPLES FOR THE INFINITE STOPPING PROBLEM

4.1. Infinite American call problem for the geometric Brownian motion. As an application of the previous results, we consider the case that $g(x) = (x - K)^+$ and X is a geometric Brownian motion on $(0, \infty)$, i.e., the regular linear diffusion X given as the solution of the Itô equation $dX_t = \mu X_t dt + \sigma X_t dW_t$, where $\mu \in \mathbf{R}$ and $\sigma > 0$. Here, W is a Wiener process. For the problem to be well-defined, we assume that $\mu < r$. The associated ordinary optimal stopping problem

$$(4.1) \quad \hat{V}(x) = \sup_{\tau} \mathbf{E}_x \left\{ e^{-(r+\lambda)\tau} (X_\tau - K)^+ \mathbf{1}_{\{\tau < \infty\}} \right\}$$

has the well-known optimal stopping time

$$\tau^* = \inf \{ t \geq 0 : X_t \geq \hat{x}_\infty \}, \text{ where } \hat{x}_\infty = \frac{\beta}{\beta - 1} K,$$

for

$$\beta = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1,$$

and

$$\hat{V}(x) = \begin{cases} x - K, & x \geq \hat{x}_\infty \\ \frac{\hat{x}_\infty - K}{\hat{x}_\infty} x^\beta, & x \leq \hat{x}_\infty. \end{cases}$$

Writing

$$\sigma(x) = (r + \lambda - \mathcal{A})g(x) = (r + \lambda - \mu)x - K(r + \lambda)$$

for $x \geq \hat{x}_\infty$ and $\sigma(x) = 0$ for $x < \hat{x}_\infty$, we find that \hat{V} can be represented as

$$\hat{V}(x) = (R_{r+\lambda}\sigma)(x),$$

see also [18], Section 5.1. Therefore, by Proposition 3.2 the sequence $\bar{\tau} \in \mathcal{S}_0^\infty(\hat{U}, \mathbb{F})$ associated with τ^* given by

$$\tau_i = \inf\{t \geq \tau_{i-1} + U_{i-1} : X_t \geq \hat{x}_\infty\} \text{ for all } i \geq 2,$$

is optimal and

$$V^\infty(x) = (R_r\sigma)(x).$$

Using the representation of the resolvent for diffusion processes discussed in (6.2) below, a short calculation yields

$$V^\infty(x) = \begin{cases} c_1x + c_2 + c_3x^a, & x \geq \hat{x}_\infty, \\ c_4x^b, & x < \hat{x}_\infty, \end{cases}$$

where

$$b = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \quad a = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0,$$

and

$$\begin{aligned} c_1 &= \frac{r + \lambda - \mu}{r - \mu}, \quad c_2 = -K \frac{r + \lambda}{r}, \\ c_3 &= \frac{2}{\sigma^2} B_r^{-1}(\hat{x}_\infty)^{b+2\mu/\sigma^2-1} \left(-\frac{r + \lambda - \mu}{b + 2\mu/\sigma^2} \hat{x}_\infty - \frac{K(r + \lambda)}{b + 2\mu/\sigma^2 - 1} \right), \\ c_4 &= \frac{2}{\sigma^2} B_r^{-1}(\hat{x}_\infty)^{a+2\mu/\sigma^2-1} \left(-\frac{r + \lambda - \mu}{a + 2\mu/\sigma^2} \hat{x}_\infty - \frac{K(r + \lambda)}{a + 2\mu/\sigma^2 - 1} \right). \end{aligned}$$

Note that in the degenerated case $\lambda = 0$, one obtains $V^\infty = \hat{V}$, as expected.

4.2. Infinite American call problem for geometric Lévy processes. One can generalize the previous result to general geometric Lévy processes, that is, for processes $X = e^Y$, where Y is a Lévy process with $\mathbf{E}e^{Y_1} < e^r$. In this case, it is known that the optimal stopping time for the problem (4.1) is also a threshold time and the optimal threshold is given by

$$\hat{x}_\infty = K\mathbf{E}e^M,$$

where M denotes the running maximum process of Y evaluated at an independent, $\text{Exp}(r + \lambda)$ -distributed time, see [31].

Finding an explicit representation of the form (3.7) for the value function seems to be hard for general Lévy processes. But for spectrally positive Lévy processes [15, Proposition 2.16] is applicable and yields that again

$$\sigma(y) = (r + \lambda - \mathcal{A})(e^y - K) \quad \text{for } y \geq \hat{x}_\infty,$$

where \mathcal{A} denotes the extended infinitesimal generator of Y , which gives the desired representation.

In this case, \mathcal{A} acts as

$$\begin{aligned} \mathcal{A}f(y) &= \frac{c^2}{2} \frac{d^2}{dy^2} f(y) + b \frac{d}{dy} f(y) \\ &\quad + \int_{(0, \infty)} \left(f(y+z) - f(y) - z \frac{d}{dz} f(z) 1_{\{z < 1\}} \right) \pi(dz), \end{aligned}$$

where (b, c, π) is the Lévy triple of X . Therefore, σ can be identified to have the following easy form:

$$(4.2) \quad \sigma(y) = (r + \lambda)(e^y - K) + e^y \left(\frac{c^2}{2} + b + \int_{(0, \infty)} (e^z - 1 - y 1_{\{z < 1\}}) \pi(dz) \right)$$

$$(4.3) \quad = -(r + \lambda)K + (r + \lambda + \psi(1))e^y,$$

for $y \geq \hat{x}_\infty$ and $\sigma(y) = 0$ for $y < \hat{x}_\infty$, where $\psi(1) = \log \mathbf{E}(e^{Y_1})$. This yields the optimal strategies and the semi-explicit representation for the value function

$$V^\infty(x) = (R_r \sigma)(x).$$

A more explicit representation of the resolvent operator for spectrally one-sided Lévy processes is discussed in Subsection 6.2 below.

4.3. Infinite investment problem. To illustrate that the theory is not restricted to one-dimensional problems, we now consider one of the most studied multidimensional ordinary optimal stopping problems, namely the problem

$$v^1(x) = \sup_{\tau} \mathbf{E}_x \left\{ e^{-\beta\tau} (X_{\tau}^{(0)} - X_{\tau}^{(1)} - \dots - X_{\tau}^{(d)}) \right\}, \quad x \in (0, \infty)^{d+1}, \beta > 0,$$

where $(X^{(0)}, \dots, X^{(d)})$ is a $d+1$ -dimensional (correlated) geometric Brownian motion. This problem is motivated by an investment situation: An investor is able to choose a time point to make an investment. She has to pay the sum of the cost factors $X^{(1)}, \dots, X^{(d)}$ and gets out $X^{(0)}$. By a change of measure argument, it can immediately be seen that the problem can be reduced to the following

$$(4.4) \quad \hat{V}(x) = \sup_{\tau} \mathbf{E}_x \left\{ e^{-\beta\tau} \left(1 - \sum_{i=1}^d X_{\tau}^{(i)} \right)^+ \right\}, \quad x \in (0, \infty)^d.$$

The problem was studied in [20, 34, 22, 33, 13, 16] from different points of view. Recently, the problem was solved [16] by characterizing the boundary of the optimal stopping set S as the unique solution to an integral equation using the Riesz representation approach as described in Subsection 3.1. Then, the value function can be represented as

$$\hat{V}(x) = R_{\beta}\sigma(x)$$

with σ given by $\sigma(y) = (\beta - \mathcal{A})g(y)$ on S and $\sigma(y) = 0$ on S^c .

Now, we consider the corresponding infinite stopping problem: Over the time, new investment opportunities arise, but after making an investment, the investor has to wait for the next investment opportunity; as before, we assume this refraction period to be exponentially distributed. Due to the memoryless property of the exponential distribution, this assumption seems to be reasonable in this situation. We are faced with the problem

$$V^{\infty}(x) = \sup_{\bar{\tau} \in \mathcal{S}_0^{\infty}(\hat{U}, \mathbb{F})} \mathbf{E}_x \left\{ \sum_{i=1}^{\infty} e^{-r\tau_i} g(X_{\tau_i}) \mathbf{1}_{\{\tau_i < \infty\}} \right\},$$

where $X = (X^{(1)}, \dots, X^{(d)})$ is the geometric Brownian motion and $g(x) = (1 - x_1 - \dots - x_d)^+$. Again, this problem can immediately be solved by applying Proposition 3.2: We consider the ordinary optimal investment problem discussed above with parameter $\beta = r + \lambda$ and obtain S and the function σ . Then

$$V^{\infty}(x) = R_r\sigma(x)$$

and the sequence of optimal stopping times is given recursively by

$$\tau_i^* = \inf\{t \geq \tau_{i-1}^* + U_{i-1} : X_t \in S\}.$$

This result can be extended to multidimensional geometric Lévy processes with only negative jumps by using the results from [16].

5. THE FINITE STOPPING PROBLEM

In the previous sections we studied the limiting case of the optimal multiple stopping problem (2.1) where the number of exercise rights was infinite, i.e. $N = \infty$. In this section, we focus on the finite case $N < \infty$. The following theorem gives the solution of the optimal multiple stopping problem (2.1) for finite N via a strip of N optimal single stopping problems with modified payoff functions. For brevity, denote

$$(5.1) \quad H^i(x) = g(x) + \lambda(R_{r+\lambda}V_\lambda^{i-1})(x),$$

for all $i = 1, \dots, N$, where $V_\lambda^0 = 0$. We make the following assumptions on the payoff structure:

- (A1) the payoff $g : E \rightarrow \mathbf{R}_+$ is lower semicontinuous and in L_r^1 ,
- (A2) there exist an r -harmonic function $h : E \rightarrow \mathbf{R}_+$ such that the function $x \mapsto \frac{g(x)}{h(x)}$ is bounded.

We have the following result.

Theorem 5.1. *Assume that (A1) and (A2) hold. Then, for all $i = 1, \dots, N$, the value function V_λ^i exists and can be identified recursively as the least r -excessive majorant of the function H^i . Furthermore, if, in addition, the function g is continuous for all $i = 1, \dots, N$, then the optimal stopping time τ_i^* exists and can be expressed as*

$$\tau_i^* = \inf\{t \geq \tau_{i-1}^* + U_{i-1} : H^{N-i+1}(X_t) = V_\lambda^{N-i+1}(X_t)\}.$$

Proof. For notational convenience, we concentrate on the case $N = 2$; the general case then holds by a straightforward induction.

Fix $\lambda > 0$ and $x \in E$. Let τ be an arbitrary \mathbb{F} -stopping time. First, we write

$$\mathbf{E}_x \{e^{-r\tau} g(X_\tau) \mathbf{1}_{\{\tau < \infty\}}\} = \mathbf{E}_x^h \left\{ \frac{g(X_\tau)}{h(X_\tau)} \mathbf{1}_{\{\tau < \infty\}} \right\} h(x),$$

where \mathbf{E}_x^h is the expectation with respect to the probability \mathbf{P}_x^h associated with the Doob's h -transform X^h , see [19]. Since the function $x \mapsto \frac{g(x)}{h(x)}$ is bounded and lower semicontinuous and the resolvent of zero is zero, we conclude that the function

$$V_\lambda^1(x) = \sup_{\tau_1} \mathbf{E}_x \left\{ e^{-r\tau_1} g(X_{\tau_1}) \mathbf{1}_{\{\tau_1 < \infty\}} \right\}$$

is finite and is the least r -excessive majorant of $x \mapsto g(x)$. Since the underlying X is strong Markov, we find that

$$\begin{aligned} \mathbf{E}_x \left\{ e^{-r(\tau+t)} V_\lambda^1(X_{\tau+t}) \mathbf{1}_{\{\tau < \infty\}} \right\} &= \mathbf{E}_x \left\{ \mathbf{E}_x \left\{ e^{-r(\tau+t)} V_\lambda^1(X_{\tau+t}) \middle| \mathcal{F}_\tau \right\} \mathbf{1}_{\{\tau < \infty\}} \right\} \\ &= \mathbf{E}_x \left\{ e^{-r\tau} \mathbf{E}_{X_\tau} \left\{ e^{-rt} V_\lambda^1(X_t) \right\} \mathbf{1}_{\{\tau < \infty\}} \right\}, \end{aligned}$$

holds for all $t \geq 0$. By integrating this expression over the positive reals with respect to t with weight $\lambda e^{-\lambda t}$ we obtain

$$\begin{aligned} \mathbf{E}_x \left\{ e^{-r(\tau+U)} V_\lambda^1(X_{\tau+U}) \mathbf{1}_{\{\tau < \infty\}} \right\} &= \mathbf{E}_x \left\{ e^{-r\tau} \mathbf{E}_{X_\tau} \left\{ \lambda \int_0^\infty e^{-(r+\lambda)t} V_\lambda^1(X_t) dt \right\} \mathbf{1}_{\{\tau < \infty\}} \right\} \\ &= \mathbf{E}_x \left\{ e^{-r\tau} \lambda (R_{r+\lambda} V_\lambda^1)(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right\}. \end{aligned}$$

Now, let $(\tau_1, \tau_2) \in \mathcal{S}_0^2(\hat{U}, \mathbb{F})$ be arbitrary. Then we find that

$$\begin{aligned} &\mathbf{E}_x \left\{ e^{-r\tau_1} g(X_{\tau_1}) \mathbf{1}_{\{\tau_1 < \infty\}} + e^{-r\tau_2} g(X_{\tau_2}) \mathbf{1}_{\{\tau_2 < \infty\}} \right\} \\ &\leq \mathbf{E}_x \left\{ \left(e^{-r\tau_1} g(X_{\tau_1}) + e^{-r(\tau_1+U_1)} V_\lambda^1(X_{\tau_1+U_1}) \right) \mathbf{1}_{\{\tau_1 < \infty\}} \right\} \\ &= \mathbf{E}_x \left\{ e^{-r\tau_1} (g(X_{\tau_1}) + \lambda (R_{r+\lambda} V_\lambda^1)(X_{\tau_1})) \mathbf{1}_{\{\tau_1 < \infty\}} \right\}, \end{aligned}$$

which, in turn, yields the inequality

$$\begin{aligned} &\sup_{(\tau_1, \tau_2) \in \mathcal{S}_0^2(\hat{U}, \mathbb{F})} \mathbf{E}_x \left\{ e^{-r\tau_1} g(X_{\tau_1}) \mathbf{1}_{\{\tau_1 < \infty\}} + e^{-r\tau_2} g(X_{\tau_2}) \mathbf{1}_{\{\tau_2 < \infty\}} \right\} \\ &\leq \sup_{\tau_1} \mathbf{E}_x \left\{ e^{-r\tau_1} (g(X_{\tau_1}) + \lambda (R_{r+\lambda} V_\lambda^1)(X_{\tau_1})) \mathbf{1}_{\{\tau_1 < \infty\}} \right\}. \end{aligned}$$

To prove the opposite inequality, let τ_1 be arbitrary. Find an optimal stopping sequence such that

$$\mathbf{E}_x \left\{ e^{-r\tau_n^*} g(X_{\tau_n^*}) \mathbf{1}_{\{\tau_n^* < \infty\}} \right\} \uparrow \sup_{\tau} \mathbf{E}_x \left\{ e^{-r\tau} g(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right\} = V_\lambda^1(x)$$

and write

$$\tau_{2,n} := (\tau_1 + U_1) + \tau_n^* \circ \theta_{\tau_1 + U_1}.$$

By Fubini's theorem and the strong Markov property

$$\begin{aligned}
& \mathbf{E}_x \left\{ e^{-r\tau_{2,n}} g(X_{\tau_{2,n}}) \mathbf{1}_{\{\tau_{2,n} < \infty\}} \right\} \\
&= \mathbf{E}_x \left\{ e^{-r((\tau_1+U_1)+\tau_n^* \circ \theta_{\tau_1+U_1})} g(X_{(\tau_1+U_1)+\tau_n^* \circ \theta_{\tau_1+U_1}}) \mathbf{1}_{\{\tau_{2,n} < \infty\}} \right\} \\
&= \mathbf{E}_x \left\{ e^{-r\tau_1} \mathbf{E}_x \left\{ e^{-rU_1} \mathbf{E}_x \left\{ e^{-r\tau_n^* \circ \theta_{\tau_1+U_1}} g(X_{(\tau_1+U_1)+\tau_n^* \circ \theta_{\tau_1+U_1}}) \middle| \mathcal{F}_{\tau_1+U_1} \right\} \middle| \mathcal{F}_{\tau_1} \right\} \mathbf{1}_{\{\tau_{2,n} < \infty\}} \right\} \\
&= \int_0^\infty \mathbf{E}_x \left\{ e^{-r\tau_1} \mathbf{E}_x \left\{ e^{-rt} \mathbf{E}_x \left\{ e^{-r\tau_n^* \circ \theta_{\tau_1+t}} g(X_{\eta_{2,n}(t)}) \middle| \mathcal{F}_{\tau_1+t} \right\} \middle| \mathcal{F}_{\tau_1} \right\} \mathbf{1}_{\{\eta_{2,n}(t) < \infty\}} \right\} \lambda e^{-\lambda t} dt \\
&= \int_0^\infty \mathbf{E}_x \left\{ e^{-r\tau_1} \mathbf{E}_{X_{\tau_1}} \left\{ e^{-rt} \mathbf{E}_{X_t} \left\{ e^{-r\tau_n^*} g(X_{\tau_n^*}) \mathbf{1}_{\{\tau_n^* < \infty\}} \right\} \right\} \mathbf{1}_{\{\tau_1 < \infty\}} \right\} \lambda e^{-\lambda t} dt \\
&= \mathbf{E}_x \left\{ e^{-r\tau_1} \mathbf{E}_{X_{\tau_1}} \left\{ e^{-rU_1} \mathbf{E}_{X_{U_1}} \left\{ e^{-r\tau_n^*} g(X_{\tau_n^*}) \mathbf{1}_{\{\tau_n^* < \infty\}} \right\} \right\} \mathbf{1}_{\{\tau_1 < \infty\}} \right\},
\end{aligned}$$

where the stopping time $\eta_{2,n}(t) = (\tau_1 + t) + \tau_n^* \circ \theta_{\tau_1+t}$. Since

$$\mathbf{E}_{X_{U_1}} \left\{ e^{-r\tau_n^*} g(X_{\tau_n^*}) \mathbf{1}_{\{\tau_n^* < \infty\}} \right\} \uparrow V_\lambda^1(X_{U_1}),$$

we obtain by monotone convergence that

$$\begin{aligned}
\mathbf{E}_x \left\{ e^{-r\tau_{2,n}} g(X_{\tau_{2,n}}) \mathbf{1}_{\{\tau_{2,n} < \infty\}} \right\} &\rightarrow \mathbf{E}_x \left\{ e^{-r\tau_1} \mathbf{E}_{X_{\tau_1}} \left\{ e^{-rU_1} V_\lambda^1(X_{U_1}) \right\} \mathbf{1}_{\{\tau_1 < \infty\}} \right\} \\
&= \mathbf{E}_x \left\{ e^{-r\tau_1} \lambda (R_{r+\lambda} V_\lambda^1)(X_{\tau_1}) \mathbf{1}_{\{\tau_1 < \infty\}} \right\}.
\end{aligned}$$

We have proved

$$\begin{aligned}
& \sup_{(\tau_1, \tau_2) \in \mathcal{S}_0^2(\hat{U}, \mathbb{F})} \mathbf{E}_x \left\{ e^{-r\tau_1} g(X_{\tau_1}) \mathbf{1}_{\{\tau_1 < \infty\}} + e^{-r\tau_2} g(X_{\tau_2}) \mathbf{1}_{\{\tau_2 < \infty\}} \right\} \\
&\geq \sup_{\tau_1} \mathbf{E}_x \left\{ e^{-r\tau_1} (g(X_{\tau_1}) + \lambda (R_{r+\lambda} V_\lambda^1)(X_{\tau_1})) \mathbf{1}_{\{\tau_1 < \infty\}} \right\}.
\end{aligned}$$

The form of the optimal stopping times holds by the general theory of optimal stopping under the stated assumptions. \square

Remark 5.1. *The previous result can also be obtained by using the general theory developed in [14]. Furthermore, one sees that the optimal stopping times are first-entrance-times into stopping sets S^n , where S^1 is the set for one stopping opportunity, S^2 for two etc. By the general theory of multiple optimal stopping with general random refraction times, see [14, Lemma 4.3 ff.], it is clear that $S^1 \subseteq S^2 \subseteq \dots \subseteq S^\infty$, where S^∞ is the stopping set for the infinite stopping problem discussed in the previous sections. This fact turns out to be important for solving multiple stopping problems (N finite) in the following.*

6. CLOSED-FORM SOLUTIONS FOR THE FINITE STOPPING PROBLEM USING THE RESOLVENT OPERATOR

To obtain closed-form solutions of the previous problems using Theorem 5.1, it is crucial to have explicit representations for the resolvent of the underlying process. Here, we consider the particularly interesting cases of diffusion processes and (spectrally one-sided) Lévy processes.

Most solvable ordinary optimal stopping problems have a one-sided solution, that is, the optimal stopping time is of threshold-type. Therefore, it is our aim in the section to find sufficient conditions that guarantee that the optimal stopping times in the multiple stopping problem are also of this type. We give conditions that can be checked a priori, i.e. without solving the sequence of stopping problems using Theorem 5.1 explicitly.

6.1. Diffusion dynamics. We assume that the state process X evolves on \mathbf{R}_+ and follows the regular linear diffusion given as the weakly unique solution of the Itô equation

$$(6.1) \quad dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x.$$

Here, W is a Wiener process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ and the real valued functions μ and $\sigma > 0$ are assumed to be continuous. Using the terminology of [9], the boundaries 0 and ∞ are natural, see [9], pp. 18–20, for the boundary classification of diffusions. As usual, we denote as $\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$ the second order linear differential operator associated with X . Furthermore, we denote as, respectively, $\psi_r > 0$ and $\varphi_r > 0$ the increasing and decreasing solution of the ODE $\mathcal{A}u = ru$, where $r > 0$, defined on the domain of the characteristic operator of X . By posing appropriate boundary conditions depending on the boundary classification of the diffusion X , the functions ψ_r and φ_r are defined uniquely up to a multiplicative constant and can be identified as the minimal r -harmonic functions – for the boundary conditions and further properties of ψ_r and φ_r , see [9], pp. 18–20. Finally, we define the speed measure m and the scale function S of X via the formulæ $m'(x) = \frac{2}{\sigma^2(x)}e^{B(x)}$ and $S'(x) = e^{-B(x)}$ for all $x \in \mathbf{R}_+$, where $B(x) := \int^x \frac{2\mu(y)}{\sigma^2(y)}dy$, see [9], pp. 17.

We know from the literature that for a given $f \in L^1_1$ the resolvent $R_r f$ can be expressed as

$$(6.2) \quad (R_r f)(x) = B_r^{-1}\varphi_r(x) \int_0^x \psi_r(y)f(y)m'(y)dy + B_r^{-1}\psi_r(x) \int_x^\infty \varphi_r(y)f(y)m'(y)dy,$$

for all $x \in \mathbf{R}_+$, where $B_r = \frac{\psi_r'(x)}{S'(x)}\varphi_r(x) - \frac{\varphi_r'(x)}{S'(x)}\psi_r(x)$ denotes the Wronskian determinant, see [9], pp. 19. Finally, we remark that the value of B_r does not depend on the state variable x but depends on the rate r .

Given the underlying X , we consider the multiple optimal stopping problem

$$(6.3) \quad V_\lambda^N(x) = \sup_{\bar{\tau} \in \mathcal{S}_0^N(\hat{U}, \mathbb{F})} \mathbf{E}_x \left\{ \sum_{i=1}^N e^{-r\tau_i} g(X_{\tau_i}) \mathbf{1}_{\{\tau_i < \infty\}} \right\},$$

with payoff function g , we make more specific assumptions on g later.

For some preliminary analysis, define the operator L as follows

$$(6.4) \quad (Lf)(x) = \frac{\psi_r'(x)}{S'(x)}f(x) - \frac{f'(x)}{S'(x)}\psi_r(x).$$

Then we have the following lemma.

Lemma 6.1. *Assume that the function f is continuous and r -excessive for diffusion X . Then the function $x \mapsto \frac{\lambda(R_{r+\lambda}f)(x)}{\psi_r(x)}$ is decreasing.*

Proof. Since the function ψ_r is r -harmonic, a straightforward differentiation yields

$$\begin{aligned} (LR_{r+\lambda}f)'(x) &= -\psi_r(x)(\mathcal{A} - r)(R_{r+\lambda}f)(x)m'(x) \\ &= -\psi_r(x)(\lambda(R_{r+\lambda}f)(x) - f(x))m'(x) \geq 0, \end{aligned}$$

for all $x \in \mathbf{R}_+$. Here, the operator L is defined in (6.4) and the last inequality is given by Prop. II.2.3 in [8]. Thus

$$\begin{aligned} \frac{d}{dx} \left(\frac{\lambda(R_{r+\lambda}f)(x)}{\psi_r(x)} \right) &= -\frac{S'(x)}{\psi_r^2(x)} \lambda(L(R_{r+\lambda}f))(x) \\ &= \frac{S'(x)}{\psi_r^2(x)} \lambda \int_0^x \psi_r(y)(\lambda(R_{r+\lambda}f)(y) - f(y))m'(y)dy \leq 0, \end{aligned}$$

for all $x \in \mathbf{R}_+$. □

In what follows, we pin down sufficient conditions for the optimal stopping rules to be one-sided threshold rules.

Proposition 6.2. *Assume that*

- *the function $g \in C^2(\mathbf{R}_+ \setminus D) \cap C(\mathbf{R}_+)$ such that limits $\lim_{x \rightarrow y \pm} g'(x)$ and $\lim_{x \rightarrow y \pm} g''(x)$ are finite for all $y \in D$. Here, D is a countable subset of \mathbf{R}_+ which has no accumulation points,*

- the function $x \mapsto \frac{g(x)}{\psi_r(x)}$ has a unique finite global maximum at \hat{x} and is decreasing for all $x > \hat{x}$,
- the function $(A - r)g$ is decreasing for $x > \hat{x}_\infty$, where $\hat{x}_\infty := \inf S^\infty$, see Remark 5.1.

Then, for all $i = 1, \dots, N$

- the function $x \mapsto \frac{H^i(x)}{\psi_r(x)}$ has a finite global maximum at a point $x_i^* \leq \hat{x}$,
- $S_i = [x_i^*, \infty)$ and the optimal stopping times $\tau_1^*, \dots, \tau_N^*$ read as

$$\tau_i^* = \inf\{t \geq \tau_{i-1}^* + U_{i-1} : X_t \geq x_{N-i+1}^*\},$$

- the value functions read as

$$(6.5) \quad V_\lambda^i(x) = \begin{cases} H^i(x), & x \geq x_i^* \\ \frac{H^i(x_i^*)}{\psi_r(x_i^*)} \psi_r(x), & x \leq x_i^*, \end{cases}$$

where the function H^i is defined in (5.1).

Proof. Since $V_\lambda^0 = 0$, the claim follows immediately for $i = 1$ by [2, Theorem 3]. We proceed by induction: assume that the claim holds for index $i - 1$.

- (1) We know from the standard theory of optimal stopping, see, e.g., [35], that the value function V_λ^{i-1} is finite and r -excessive. By [8, Proposition II.2.3], this implies that $V_\lambda^{i-1}(x) \geq \lambda(R_{r+\lambda} V_\lambda^{i-1})(x)$ for all $x \in \mathbf{R}_+$. Thus

$$(6.6) \quad \frac{H^i(x)}{\psi_r(x)} \leq \frac{g(x) + V_\lambda^{i-1}(x)}{\psi_r(x)} \leq \frac{g(\hat{x})}{\psi_r(\hat{x})} + \frac{H^{i-1}(x_{i-1}^*)}{\psi_r(x_{i-1}^*)},$$

for all $x \in \mathbf{R}_+$, that is, the function $x \mapsto \frac{H^i(x)}{\psi_r(x)}$ is bounded. In addition, we know from Lemma 6.1 that the function $x \mapsto \frac{\lambda(R_{r+\lambda} V_\lambda^{i-1})(x)}{\psi_r(x)}$ is decreasing. Thus the function $x \mapsto \frac{H^i(x)}{\psi_r(x)}$ is decreasing for all $x > \hat{x}$. Since the interval $(0, \hat{x}_\infty)$ is a subset of the continuation region for all $i = 1, \dots, N$ (see Remark 5.1), we can assume, without loss of generality, that $g(x) = 0$ for all $x \in (0, \hat{x}_\infty)$. Thus we observe using (6.6) that $\frac{H^i(x)}{\psi_r(x)} \leq \frac{H^{i-1}(x_{i-1}^*)}{\psi_r(x_{i-1}^*)} \leq \frac{H^i(x_{i-1}^*)}{\psi_r(x_{i-1}^*)}$ for all $x \in (0, \hat{x}_\infty)$. By continuity, we conclude that the function $x \mapsto \frac{H^i(x)}{\psi_r(x)}$ has at least one maximum point on $(\hat{x}_\infty, \hat{x})$. Let x_i^* denote the smallest maximum point.

- (2) By [13, Lemma 1.1], we know that $x_i^* \in S^i$. By [5, Theorem 2] we furthermore know that $\inf\{t \geq 0 : X_t = x_i^*\}$ is an optimal stopping time for all starting points in $(0, x_i^*)$ and by

the minimality of x_i^* we obtain that $S^i \subseteq [x_i^*, \infty)$. On the other hand, by Remark 5.1 it holds that $[x_{i-1}^*, \infty) = S^{i-1} \subseteq S^i$.

- (3) We now show that $x \mapsto \frac{H^i(x)}{\psi_r(x)}$ is non-increasing on $[x_i^*, x_{i-1}^*]$. To this end, consider the function

$$I^i(x) = \frac{\psi_r'(x)}{S'(x)} H^i(x) - \frac{H^{i'}(x)}{S'(x)} \psi_r(x).$$

Since ψ_r is r -harmonic, straightforward differentiation yields

$$I^{i'}(x) = -\psi_r(x)(\mathcal{A} - r)H^i(x)m'(x),$$

for all $x \in \mathbf{R}_+$. Since the boundaries are natural, we find that $I^i(0) = 0$ and, consequently, that

$$I^i(x) = - \int_0^x \psi_r(y)(\mathcal{A} - r)H^i(y)m'(y)dy.$$

On the other hand, since

$$I^i(x) = -\frac{\psi_r^2(x)}{S'(x)} \frac{d}{dx} \left(\frac{H^i(x)}{\psi_r(x)} \right),$$

we conclude

$$(6.7) \quad \frac{d}{dx} \left(\frac{H^i(x)}{\psi_r(x)} \right) = \frac{S'(x)}{\psi_r^2(x)} \int_0^x \psi_r(y)(\mathcal{A} - r)H^i(y)m'(y)dy.$$

We know that the resolvent $(R_{r+\lambda}V_\lambda^{i-1})$ satisfies the relation $(\mathcal{A} - (r + \lambda))(R_{r+\lambda}V_\lambda^{i-1}) = -V_\lambda^{i-1}$. Thus

$$(6.8) \quad (\mathcal{A} - r)H^i(x) = (\mathcal{A} - r)g(x) + \lambda(\lambda(R_{r+\lambda}V_\lambda^{i-1})(x) - V_\lambda^{i-1}(x)).$$

Since the boundary ∞ is natural, it follows from [28, Lemma 2.1] that

$$V^{i-1}(x) = \frac{H^{i-1}(x_{i-1}^*)}{\psi_r(x_{i-1}^*)} \psi_r(x) = \lambda \left(R_{r+\lambda} \frac{H^{i-1}(x_{i-1}^*)}{\psi_r(x_{i-1}^*)} \psi_r \right) (x)$$

for all $x \leq x_{i-1}^*$. By invoking the representation (6.2), we find that

$$\begin{aligned} & \lambda(R_{r+\lambda}V_\lambda^{i-1})'(x) - V_\lambda^{i-1}'(x) \\ &= \frac{\lambda}{B_{r+\lambda}} \left(\varphi'_{r+\lambda}(x) \int_0^x \psi_{r+\lambda}(y)V_\lambda^{i-1}(y)m'(y)dy + \psi'_{r+\lambda}(x) \int_x^\infty \varphi_{r+\lambda}(y)V_\lambda^{i-1}(y)m'(y)dy \right) \\ & - \frac{\lambda}{B_{r+\lambda}} \frac{H^{i-1}(x_{i-1}^*)}{\psi_r(x_{i-1}^*)} \left(\varphi'_{r+\lambda}(x) \int_0^x \psi_{r+\lambda}(y)\psi_r(y)m'(y)dy + \psi'_{r+\lambda}(x) \int_x^\infty \varphi_{r+\lambda}(y)\psi_r(y)m'(y)dy \right) \\ & \leq 0. \end{aligned}$$

Thus, we find using (6.8) that the integrand in (6.7) is non-increasing on $(\hat{x}_\infty, x_{i-1}^*)$.

Therefore the function is $x \mapsto \frac{H^i(x)}{\psi_r(x)}$ is non-increasing on $[x_i^*, x_{i-1}^*]$.

- (4) To prove that $S^i = [x_i^*, \infty)$ it remains to be shown that $[x_i^*, x_{i-1}^*] \subseteq S^i$. To this end, assume that there exist $y_1 < y_2 \in [x_i^*, x_{i-1}^*]$ such that $(y_1, y_2) \notin S^i$. By [5, Theorem 3] (or also [13]), there exists $\lambda \in (0, 1)$ such that y_1, y_2 are maximum points of $\frac{H^i}{\lambda\varphi_r + (1-\lambda)\psi_r}$. Without loss of generality, we standardize the functions ψ_r and φ_r such that $\varphi_r(y_1) = \psi_r(y_1)$. Then – using that H^i/ψ_r is non-increasing on $[x_{i-1}^*, x_i^*]$ and $\varphi_r(y_2) < \psi_r(y_2)$ – we obtain

$$\frac{H^i}{\lambda\varphi_r + (1-\lambda)\psi_r}(y_1) = \frac{H^i}{\varphi_r}(y_1) \geq \frac{H^i}{\varphi_r}(y_2) > \frac{H^i}{\lambda\varphi_r + (1-\lambda)\psi_r}(y_2),$$

which is a contradiction.

Now, all other claims hold by the general theory of optimal stopping. \square

Proposition 6.2 is formulated for increasing payoff satisfying additional regularity conditions. We point out that one can formulate an analogous result also for decreasing payoffs in terms of the ratio function $x \mapsto \frac{g(x)}{\varphi_r(x)}$ and obtain another wide class of solvable optimal multiple stopping problems with one-sided optimal threshold rules – for analogous results in optimal single stopping, see, e.g., [27] and [28].

6.2. Lévy processes. For Lévy processes X , we consider the resolvent kernel G_r given by

$$G_r(x, y)dy = \int_0^\infty e^{-rt} \mathbf{P}_x(X_t \in dy)dt.$$

Assuming that M and I are independent random variables with distributions of $\sup_{t \leq T} X_t$ and $\inf_{t \leq T} X_t$, respectively, where T is an independent exponential time with parameter r , the Wiener-Hopf-factorization (see [24, Theorem 6.16]) states that

$$X_T \stackrel{d}{=} M + I.$$

Therefore,

$$rG_r(x, y) = \begin{cases} \int_{-\infty}^{y-x} f_I(t)f_M(y-x-t)dt, & y < x \\ \int_{y-x}^\infty f_M(t)f_I(y-x-t)dt, & y > x, \end{cases}$$

where f_I, f_M denote the densities of I and M , that we assume to exist.

Unfortunately, these densities are often not easy to find explicitly. Interesting classes of processes with two-sided jumps, where the Wiener-Hopf factors can be inverted analytically, are Lévy

processes with phase-type jumps and meromorphic Lévy processes, see [3] and [23]. In the following, we consider the particularly interesting class of spectrally negative Lévy processes, since our results can be presented without technical difficulties in this case. Here, the resolvent kernel G_r can be given semi-explicitly in terms of the scale-function $W^{(r)}$ of the process and the right inverse Φ of the Laplace exponent as

$$(6.9) \quad G_r(x, y) = \Phi'(r)e^{-\Phi(r)(y-x)} - W^{(r)}(x - y)$$

and x^* , σ are given as above, see [24, Corollary 8.9]. The resolvent is then given by

$$(R_r f)(x) = \int_{\mathbb{R}} f(y)G_r(x, y)dy.$$

This representation can be used as a key for solving multiple stopping problems for underlying Lévy processes in our setting. But also in this case, the scale function is not known in closed form in general, but a fast evaluation of these functions is essential to apply our approach. A way out is to approximate a general jump distribution by phase-type jumps. We refer to [21] for a discussion of the numerical approximation of general scale functions by scale functions of phase-type spectrally one-sided Lévy processes, which can be written in terms of (complex) exponentials.

To give a basis for obtaining explicit examples, we again concentrate on examples that lead to optimal stopping times of threshold-type. As a main tool, we use the theory developed in [15].

Proposition 6.3. *Let X be a spectrally positive Lévy process and let g be such that there exists a continuous function \tilde{f} such that*

- $g(x) = (R_r \tilde{f})(x)$ for all x ,¹
- the function \tilde{f} is non-decreasing.

Then for each $N \in \mathbb{N}$, there exists $x_N^ \geq \hat{x}_\infty$ such that $S^N = [x_N^*, \infty)$ and the value function has the form $V^N = R_r \sigma^N$ for some continuous non-decreasing function σ^N , which fulfills the recursive equation*

$$\sigma^N = \tilde{f} + \lambda R_{r+\lambda} \sigma^{N-1} \text{ on } [x_N^*, \infty) \text{ and } \sigma^N = 0 \text{ on } (-\infty, x_N^*].$$

Proof. We proceed by induction on N and assume that

$$S^{N-1} = [x_{N-1}^*, \infty), \quad V^{N-1} = R_r \sigma^{N-1},$$

¹under appropriate smoothness assumptions on g , this means that $\tilde{f} = (r - \mathcal{A})g$

where σ^{N-1} is continuous and non-decreasing. Note that $\sigma^{N-1}(x) = 0$ for $x \leq x_{N-1}^*$. By Theorem 5.1, we are faced with the optimal stopping problem with reward function

$$g + \lambda R_{r+\lambda} V^{N-1} = R_r \tilde{f} + \lambda R_{r+\lambda} R_r \sigma^{N-1} = R_r \tilde{f}^N,$$

where $\tilde{f}^N = \tilde{f} + \lambda R_{r+\lambda} \sigma^{N-1}$. \tilde{f}^N is also continuous and non-decreasing. Writing

$$\hat{f}(z) = \frac{1}{r} \int_{-\infty}^0 \tilde{f}^N(z+y) \mathbf{P}_0(I_T \in dy),$$

we immediately see that \hat{f} fulfills the assumptions of [15, Theorem 2.5], i.e. the set S^N is of the form $[x_N^*, \infty)$ for some x_N^* , and [15, Proposition 2.16] yields the desired resolvent-representation for V^N . \square

Remark 6.1.

- *In the setting of Proposition 6.3, the Lévy process X may jump into the stopping set, i.e. overshoot may occur. The assumption of the process having only positive jumps guarantees that the processes does not exit the stopping set by a jump, when started in the stopping set. This is the essential assumption for [15, Proposition 2.16] to be applicable.*
- *Note that for each spectrally positive Lévy process X , the process $-X$ is a spectrally negative Lévy process, so that Proposition 6.3 can immediately be extended to spectrally negative Lévy processes when the stopping sets are of the form $(-\infty, x_*]$.*
- *The arguments in the previous proof were not directly based on the special structure of the process X . The arguments can be generalized to more general real-valued Hunt processes with only positive jumps under appropriate additional assumptions following the results in [15] carefully. Because a more detailed discussion on Hunt processes is needed to state the corresponding results, we do not give the details here. Nonetheless, this class of processes is rich and interesting. It covers spectrally positive Lévy processes perturbed by running infimum ([25]), refracted Lévy processes ([26]), Lévy processes reflected at their infima ([36]), and impulse controlled spectrally positive Lévy processes ([42]). In all these examples, the resolvent can be obtained analytically from the scale functions of the corresponding Lévy process, which make these processes accessible for our approach.*

To illustrate the previous result, we come back to the problem discussed in Subsections 4.1 and 4.2 for $N = \infty$. For general underlying geometric spectrally positive Lévy processes, it is clear

from the discussion in Subsection 4.2 that

$$e^x - K = R_r \tilde{f}(x),$$

with $\tilde{f}(x) = c_1 e^x - c_2$, where c_1, c_2 are given in (4.2). From this explicit representation, it can immediately be seen that the assumptions of Proposition 6.3 are fulfilled and that the optimal stopping times are one-sided.

7. AN ILLUSTRATION OF THE FINITE STOPPING PROBLEM

To obtain more explicit results, we now consider the problem discussed in Subsections 4.1 and 4.2 for $N = \infty$ in the finite stopping case where the underlying process X follows a geometric Brownian motion and apply the results described in Subsection 6.1. The scale density S' reads as $S'(x) = x^{-\frac{2\mu}{\sigma^2}}$ and the speed density m' reads as $m'(x) = \frac{2}{(\sigma x)^2} x^{\frac{2\mu}{\sigma^2}}$. It is well known that the differential operator $\mathcal{A} = \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx}$. For the sake of finiteness, we assume that $\mu < r$ and $\mu - \frac{1}{2}\sigma^2 > 0$. This guarantees that the optimal exercise thresholds are finite and are attained almost surely in finite time. The minimal excessive functions ψ and φ can be written as

$$\begin{cases} \psi_r(x) = x^b, & \psi_{r+\lambda}(x) = x^\beta, \\ \varphi_r(x) = x^a, & \varphi_{r+\lambda}(x) = x^\alpha, \end{cases}$$

where the constants

$$\begin{cases} b = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1, & \beta = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1, \\ a = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0, & \alpha = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} < 0. \end{cases}$$

It is a simple computation to show that the Wronskian $B_{r+\lambda} = 2\sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}}$.

Assume that the exercise payoff reads as $g(x) = (x - K)^+$, where K is a fixed strike price. Furthermore, fix the parameter $\lambda > 0$. Then the optimal multiple stopping problem reads as

$$V_\lambda^N(x) = \sup_{\bar{\tau}} \mathbf{E}_x \left\{ \sum_{i=1}^N e^{-r\tau_i} (X_{\tau_i} - K)^+ \right\},$$

for all $i = 1, \dots, N$. It is known from the literature that for $N = 1$, the optimal stopping threshold and the value read as

$$x_1^* = \frac{b}{b-1}K, \quad V_\lambda^1(x) = \begin{cases} x - K, & x \geq x_1^* \\ \frac{x_1^* - K}{x_1^{*b}} x^b, & x \leq x_1^*. \end{cases}$$

For $N > 1$, it is easily checked in this example that the assumptions of Proposition 6.2 are fulfilled, which implies that the stopping sets are one-sided. To characterize the corresponding thresholds for $i = 2, \dots, N$, the objective is to find the state

$$x_i^* = \operatorname{argmax} \left(x \mapsto \frac{H^i(x)}{x^b} \right).$$

This yields the necessary condition

$$(7.1) \quad \left[-\frac{d}{dx} \left(\frac{\lambda(R_{r+\lambda} V_\lambda^{i-1})(x)}{x^b} \right) \right]_{x=x_i^*} = \left[\frac{d}{dx} \left(\frac{(x-K)^+}{x^b} \right) \right]_{x=x_i^*}.$$

Inductively, we assume that

$$x_{i-1}^* = \operatorname{argmax} \left(x \mapsto \frac{H^{i-1}(x)}{x^b} \right), \quad V_\lambda^{i-1}(x) = \begin{cases} H^{i-1}(x), & x \geq x_{i-1}^* \\ \frac{H^{i-1}(x_{i-1}^*)}{x_{i-1}^{*b}} x^b, & x \leq x_{i-1}^*. \end{cases}$$

For brevity, denote $c_{i-1}^* = \frac{H^{i-1}(x_{i-1}^*)}{x_{i-1}^{*b}}$. Let $x \leq x_{i-1}^*$. Then we find that

$$\begin{aligned} \lambda(R_{r+\lambda} V_\lambda^{i-1})(x) &= \frac{2\lambda}{\sigma^2 B_{r+\lambda}} \left[x^\alpha \int_0^x y^\beta c_{i-1}^* y^b y^{\frac{2\mu}{\sigma^2}-2} dy + x^\beta \int_x^{x_{i-1}^*} y^\alpha c_{i-1}^* y^b y^{\frac{2\mu}{\sigma^2}-2} dy \right. \\ &\quad \left. + x^\beta \int_{x_{i-1}^*}^\infty y^\alpha H^{i-1}(y) y^{\frac{2\mu}{\sigma^2}-2} dy \right]. \end{aligned}$$

Integration by parts yields

$$\begin{aligned} \lambda(R_{r+\lambda} V_\lambda^{i-1})(x) &= \frac{2\lambda}{\sigma^2 B_{r+\lambda}} \left[c_{i-1}^* \frac{\kappa + \gamma}{\kappa \gamma} x^b + x^\beta \frac{1}{\kappa} \int_{x_{i-1}^*}^\infty y^{-\kappa} \frac{d}{dy} \left(\frac{H^{i-1}(y)}{y^b} \right) dy \right] \\ &= V_\lambda^{i-1}(x) + x^\beta \frac{\gamma}{\gamma + \kappa} \int_{x_{i-1}^*}^\infty y^{-\kappa} \frac{d}{dy} \left(\frac{H^{i-1}(y)}{y^b} \right) dy, \end{aligned}$$

where

$$\begin{cases} \kappa = \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}, \\ \gamma = \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \end{cases}$$

On the other hand, let $x \geq x_{i-1}^*$. Then

$$\begin{aligned} \lambda(R_{r+\lambda} V_\lambda^{i-1})(x) &= \frac{2\lambda}{\sigma^2 B_{r+\lambda}} \left[x^\alpha \int_0^{x_{i-1}^*} y^\beta c_{i-1}^* y^b y^{\frac{2\mu}{\sigma^2}-2} dy + x^\alpha \int_{x_{i-1}^*}^x y^\beta H^{i-1}(y) y^{\frac{2\mu}{\sigma^2}-2} dy \right. \\ &\quad \left. + x^\beta \int_x^\infty y^\alpha H^{i-1}(y) y^{\frac{2\mu}{\sigma^2}-2} dy \right]. \end{aligned}$$

Again, integration by parts yields

$$\begin{aligned}
\lambda(R_{r+\lambda}V_\lambda^{i-1})(x) &= \frac{2\lambda}{\sigma^2 B_{r+\lambda}} \left[x^\alpha c_{i-1}^* \frac{1}{\gamma} x_{i-1}^{*\gamma} + x^\alpha \int_{x_{i-1}^*}^x y^{\gamma-1} \frac{H^{i-1}(y)}{y^b} dy \right. \\
&\quad \left. + x^\beta \int_x^\infty y^{-\kappa-1} \frac{H^{i-1}(y)}{y^b} dy \right] \\
&= \frac{2\lambda}{\sigma^2 B_{r+\lambda}} \left[x^\alpha c_{i-1}^* \frac{1}{\gamma} x_{i-1}^{*\gamma} + \frac{1}{\gamma} x^\alpha \left(x^\gamma \frac{H^{i-1}(x)}{x^b} - x_{i-1}^* \gamma \frac{H^{i-1}(x_{i-1}^*)}{x_{i-1}^{*\frac{b}{\gamma}}} \right) \right. \\
&\quad - \frac{1}{\gamma} x^\alpha \int_{x_{i-1}^*}^x y^\gamma \frac{d}{dy} \left(\frac{H^{i-1}(y)}{y^b} \right) dy \\
&\quad \left. + \frac{1}{\kappa} x^\beta \left(x^{-\kappa} \frac{H^{i-1}(x)}{x^b} + \int_x^\infty y^{-\kappa} \frac{d}{dy} \left(\frac{H^{i-1}(y)}{y^b} \right) dy \right) \right] \\
&= H^{i-1}(x) + \frac{2\lambda}{\sigma^2 B_{r+\lambda}} \left[\frac{1}{\kappa} x^\beta \int_x^\infty y^{-\kappa} \frac{d}{dy} \left(\frac{H^{N-1}(y)}{y^b} \right) dy \right. \\
&\quad \left. - \frac{1}{\gamma} x^\alpha \int_{x_{i-1}^*}^x y^\gamma \frac{d}{dy} \left(\frac{H^{N-1}(y)}{y^b} \right) dy \right].
\end{aligned}$$

Summarizing, a round of differentiation yields

$$\begin{aligned}
(7.2) \quad \frac{d}{dx} \left(\frac{\lambda(R_{r+\lambda}V_\lambda^{i-1})(x)}{x^b} \right) &= \frac{\kappa\gamma}{\kappa + \gamma} \left(x^{\kappa-1} \int_{\max\{x, x_{i-1}^*\}}^\infty y^{-\kappa} \frac{d}{dy} \left(\frac{H^{i-1}(y)}{y^b} \right) dy \right. \\
&\quad \left. + x^{-\gamma-1} \int_{x_{i-1}^*}^{\max\{x, x_{i-1}^*\}} y^\gamma \frac{d}{dy} \left(\frac{H^{i-1}(y)}{y^b} \right) dy \right),
\end{aligned}$$

for all $x \in \mathbf{R}_+$. For brevity, denote

$$\Delta_{i-1} = \frac{\kappa\gamma}{\kappa + \gamma} \int_{x_{i-1}^*}^\infty y^{-\kappa} \frac{d}{dy} \left(\frac{H^{i-1}(y)}{y^b} \right) dy.$$

Then the necessary condition (7.1) can be expressed as

$$x_i^* - b(x_i^* - K) = -\Delta_{i-1} x_i^{*\beta}.$$

By further simplification we obtain

$$1 - \frac{x_i^*}{x_1^*} = -\frac{\Delta_{i-1} x_1^{*\beta-1}}{b-1} \left(\frac{x_i^*}{x_1^*} \right)^\beta.$$

Since $\beta > 1$ and $\Delta_i < 0$, we observe that this necessary condition has a unique solution $x_i^* < x_1^*$. In particular, this condition implies that there is a unique coefficient $y_i^* \in (0, 1)$ such that $x_i^* = y_i^* x_1^*$.

To close the section, we fix $N = 5$ and compute numerically the optimal stopping thresholds x_i^* for $i = 1, \dots, 5$. The parameter configuration reads as $r = 0.05$, $\mu = 0.008$, $\sigma = 0.125$, $\lambda = 0.1$ and $K = 2$. In this case, the threshold $\hat{x}_\infty = \frac{\beta K}{\beta-1} \approx 2.593508$.

x_1^*	x_2^*	x_3^*	x_4^*	x_5^*
3.317653	3.079880	2.971528	2.738782	2.643230

Table 1. The optimal exercise threshold x_i^* , $i = 1, \dots, 5$ under the parameter configuration $r = 0.05$, $\mu = 0.008$, $\sigma = 0.125$, $\lambda = 0.1$ and $K = 2$.

We observe from Table 1 that the thresholds x_i^* , $i = 1, \dots, 5$, form a decreasing sequence as a function of the number of stopping times left. This is in line with our general theory. Furthermore, Table 1 indicates that the thresholds x_i^* converge to the threshold \hat{x}_∞ as the number of stopping times left increases. This observation is also in line with our general theory.

8. DISCUSSION

We study the multiple stopping problem under the assumption that the refraction times are exponentially distributed. It turns out that this assumption makes the problem more tractable than the ordinary multiple stopping problem in the case that the resolvent measure for the underlying Markov process is known. More precisely, the solution of infinite stopping problem can be obtained whenever a corresponding ordinary optimal stopping problem can be solved using the resolvent technique, which is the case in many classes of problems. In the finite stopping case, it is necessary to evaluate iterated integrals involving the resolvent measure. To this end, one needs a representation of the resolvent that can be evaluated fast to obtain feasible expressions.

Overall, the resolvent technique turns out to be a useful tool for multiple exercise problems whenever the resolvent measure is available.

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REFERENCES

- [1] Alexandrov, N. and Hambly, B. M. (2010). A dual approach to multiple exercise option problems under constraints, *Mathematical Methods of Operations Research*, 71/3, 503 – 533
- [2] Alvarez, L. H. R. (2001). Reward functionals, salvage values and optimal stopping, *Math. Methods Oper. Res.*, 54/2, 315 – 337
- [3] Asmussen, S., Avram, F., and Pistorius, M. (2004). Russian and American put options under exponential phase-type Lévy models, *Stochastic Process. Appl.*, 109/1, 79–111.

- [4] Beibel, M. and Lerche, H.R. (1997). A new look at optimal stopping problems related to mathematical finance, *Statist. Sinica*, 7/1, 93–108.
- [5] Beibel, M. and Lerche, H.R. (2000). A note on optimal stopping of regular diffusions under random discounting, *Teor. Veroyatnost. i Primenen.*, 45/4, 657–669.
- [6] Bender, C. (2011). Primal and dual pricing of multiple exercise options in continuous time, *SIAM Journal on Financial Mathematics*, 2/1, 562–586
- [7] Bender, C. (2011). Dual pricing of multi-exercise options under volume constraints, *Finance and Stochastics*, 15/1, 1 – 26
- [8] Blumenthal, R. M. and Gettoor, R. K. (1968). *Markov processes and potential theory*, Academic press, New York and London
- [9] Borodin, A. and Salminen, P. (2002). *Handbook on Brownian Motion – Facts and Formulæ*, Birkhäuser, Basel
- [10] Carmona, R. and Dayanik, S. (2008). Optimal multiple stopping of linear diffusions, *Mathematics for Operations Research*, 33/2, 446 – 460
- [11] Carmona, R. and Touzi, N. (2008). Optimal multiple stopping and valuation of swing options, *Mathematical Finance*, 18/2, 239 – 268
- [12] Christensen, S. (2013). On the Solution of General Impulse Control Problems Using Superharmonic Functions, to appear in *Stochastic Processes and their Applications*
- [13] Christensen, S. and Irle, A. (2011). A harmonic function technique for the optimal stopping of diffusions, *Stochastics*, 83/4–6, 347 – 363
- [14] Christensen, S., Irle, A., and Jürgens, S. (2013). Optimal multiple stopping with random waiting times, *Sequential Analysis*, 32/3, 297–318.
- [15] Christensen, S., Salminen, P., and Ta, B. (2013). Optimal stopping of strong Markov processes, *Stochastic Processes and their Applications*, 123/3, 1138 – 1159.
- [16] Christensen, S. and Salminen, P. (2013). Riesz representation and optimal stopping with two case studies, *Preprint*.
- [17] Chung, K.L. and Walsh, J.B. (2005) *Markov processes, Brownian motion, and time symmetry*, 2nd ed., Springer Verlag, Berlin – Heidelberg – New York – Tokyo
- [18] Croce, F. and Mordecki, E. (2013). Explicit solutions in one-sided optimal stopping problems for one-dimensional diffusions, *preprint: ArXiv e-prints 1302.0712*.
- [19] Doob, J. L. (1984). *Classical Potential Theory and Its Probabilistic Counterpart*, Springer Verlag, Berlin – Heidelberg – New York – Tokyo
- [20] McDonald, R. and Siegel, D. (1986). The value of waiting to invest, *Quarterly J. Econ.*, 101, 707–727.
- [21] Egami, M. and Yamazaki, K. (2014). Phase-type Fitting of Scale Functions for Spectrally Negative Levy Processes, forthcoming in *Journal of Computational and Applied Mathematics*.
- [22] Hu, Y. and Øksendal, B. (1998). Optimal time to invest when the price processes are geometric Brownian motions, *Finance Stoch.*, 2/3, 295–310.
- [23] Kuznetsov, A., Kyprianou, A. E. and Pardo, J. C. (2012). Meromorphic Lévy processes and their fluctuation identities, *Ann. Appl. Probab.*, 22/3, 1101–1135

- [24] Kyprianou, A. E. (2006). *Introductory lectures on fluctuations of Lévy processes with applications*, Springer Verlag, Berlin
- [25] Kyprianou, A. E. and Zhou, X. (2009). General tax structures and the Lévy insurance risk model, *J. Appl. Probab.*, 46/4, 1146–1156.
- [26] Kyprianou, A. E. and Loeffen, R. L. (2012). Refracted Lévy processes, *Ann. Inst. Henri Poincaré Probab. Stat.*, 46/1, 24–44
- [27] Lempa, J. (2012). Optimal stopping with random exercise lag, *Mathematical Methods of Operations Research*, 75/3, 273 – 286
- [28] Lempa, J. (2012). Optimal stopping with information constraint, *Applied mathematics and optimization*, 66/2, 147 – 173
- [29] Majd, S. and Pindyck, R. S. (2004). *Time to build, option value and investment decisions*, in *Real Options And Investment Under Uncertainty* (eds. Schwartz, E.S. and Trigeorgis, L.), MIT press
- [30] Meinshausen, N. and Hambly, B. M. (2004). Monte-Carlo methods for the valuation of multiple-exercise options, *Mathematical Finance*, 14/4, 557 – 583
- [31] Mordecki, E. (2002). Optimal stopping and perpetual options for Lévy processes, *Finance Stoch.*, 6(4):473 – 493
- [32] Mordecki, E. and Salminen, P. (2007). Optimal stopping of Hunt and Lévy processes. *Stochastics*, 79(3-4):233–251, 2007.
- [33] Nishide, K. and Rogers, L. C. G. (2011). Optimal time to exchange two baskets, *J. Appl. Probab.*, 48/1, 21–30.
- [34] Olsen, T. E. and Stensland, G. (1992). On optimal timing of investment when cost components are additive and follows geometric diffusions, *Journal of Economic Dynamics and Control*, 16, 39–51.
- [35] Peskir, G. and Shiryaev, A. (2006). *Optimal stopping and free boundary problems*, Birkhäuser, Basel
- [36] Pistorius, M. R. (2004). On exit and ergodicity of the spectrally one-sided Lévy process reflected at its infimum. *J. Theoret. Probab.*, 17/1, 183–220.
- [37] Rogers, L. C. G. and Williams, D. (2001). *Diffusions, Markov processes and Martingales: volume 1*, Cambridge university press
- [38] Salminen, P. (1985). Optimal stopping of one-dimensional diffusions. *Math. Nachr.*, 124:85–101.
- [39] Schoenmakers, J. (2012). A pure martingale dual for multiple stopping, *Finance and Stochastics*, 16, 319 – 334
- [40] Shiryaev, A. (2008). *Optimal stopping rules: volume 8*, Springer-Verlag, Berlin
- [41] Stettner, L. (1983). On impulsive control with long run average cost criterion, *Studia Math.*, 76(3):279–298.
- [42] Yamazaki, K. (2014). Inventory Control for Spectrally Positive Levy Demand Processes. arXiv:1303.5163.
- [43] Zeghal, A. B. and Mnif, M. (2006). Optimal multiple stopping and valuation of swing options in Lévy models, *International Journal of Theoretical and Applied Finance*, 9/8, 1267 – 1297

(Jukka Lempa) SCHOOL OF BUSINESS, FACULTY OF SOCIAL SCIENCES, OSLO AND AKERSHUS UNIVERSITY COLLEGE,
P.O. BOX 4 ST. OLAVS PLASS, NO – 0130 OSLO, E-MAIL: JUKKA.LEMPA@HIOA.NO