Different representations of the Levi-Civita Bertotti Robinson solution

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Abstract The Levi-Civita Bertotti Robinson (LBR) spacetime is investigated in various coordinate systems. By means of a general formalism for constructing coordinates in conformally flat spacetimes, coordinate transformations between the different coordinate systems are deduced. We discuss the motion of the reference frames in which the different coordinate systems are comoving. Furthermore we characterize the motion of the different reference frames by their normalized timelike Killing vector fields, i.e. by the four velocity fields of the reference particles. We also deduce the formulae in the different coordinate systems for the embedding of the LBR spacetime in a flat 6-dimensional manifold. In particular we discuss a scenario with a spherical domain wall having LBR spacetime outside the wall and flat spacetime inside. We also discuss the internal flat spacetime using the same coordinate systems as in the external LBR spacetime with continuous metric at the wall. Among the different cases one represents a Milne-LBR universe model with a part of the Milne universe inside the wall and an infinitely extended LBR universe outside it. In an appendix we define combinations of trigonometric and hyperbolic functions that we call k-functions and present a new k-function calculus.

1. Introduction

Conformally flat spacetimes have vanishing Weyl tensor. The line element of such spacetimes can in general be given the form of a conformal factor times the Minkowski line element. The coordinates in which the line element takes this form are called conformally flat spacetime (CFS) coordinates.

The FRW universe models are conformally flat. We have recently given a systematic description of these universe models in CFS coordinates [1-3].

In the present article we shall give a similar treatment of the LBR spacetime which was found by T. Levi-Civita [4,5] already in 1917, and was rediscovered by B. Bertotti [6] and E. Robinson [7] in 1959. It was proved by N. Tariq and B. O. J. Tupper [8] and by N. Tariq and R. G. McLenaghan [9], and later emphasized by H. Stephani et al. [10] that the LBR spacetime is the only conformally flat solution of the Einstein-Maxwell equations which is homogeneous and has a non-null Maxwell field. The physical interpretation of the solution has been discussed by D. Lovelock [11,12], P. Doland [13] and the present authors [14].

Our article is organized as follows. In section 2 we present a new method for finding different coordinates of the LBR spacetime. We give a general formalism in section 3 for finding coordinate transformations between the canonical CFS system and an arbitrary coordinate system. Section 4 is the main part of the article. Here we find the different coordinate systems and give a thorough discussion of their properties and of the reference frames in which they are comoving. In section 5 we discuss a particularly interesting example, a Milne-LBR universe model where there is LBR spacetime outside a charged domain wall with a radius equal to the distance corresponding to its charge, and there is a part of the Milne universe inside the domain wall. The motions of the reference frames are further characterized in section 6, where we calculate the four-acceleration of the reference particles from the Killing vectors. In section 7 we present embedding parametrizations for the different coordinate representations of the LBR spacetime in a 6-dimensional, flat spacetime. Our results are summarized in section 8. We define k-functions, which are combinations of trigonometric and hyperbolic functions, in an appendix where we also present the k-calculus of these functions.

2. A new method for finding different representations of the LBR spacetime

By the *Levi-Civita Bertotti Robinson* (LBR) *spacetime* we shall mean a conformally flat and static spacetime which is a solution of the Einstein-Maxwell equations with an electromagnetic field having a constant energy momentum tensor. This solution has usually been called the Bertotti Robinson solution, but it was actually discovered by T. Levi-Civita already in 1917 [4,5]. Hence we shall call it the Levi-Civita Bertotti Robinson solution.

In a previous paper [14] we have given a new interpretation of the LBR solution. According to our interpretation this solution describes a static, spherically symmetric and conformally flat spacetime with a radial electrical field outside a charged domain wall. There is Minkowski spacetime inside the wall.

It is well known that the LBR solution can be represented by a spacetime which is the product of a 2-dimensional anti de Sitter space and a spherical surface [13]. Hence the line element may be written in a spherically symmetric form with an angular part which is $K^2 d\Omega^2$, where K is a constant. According to our interpretation [14] the constant K is equal to the radius R_Q of the domain wall. Also the radius of the domain wall is determined by its charge Q so that $R_Q = [G/(4\pi\epsilon_0 c^4)]^{1/2} Q$, i.e. R_Q is the length corresponding to the charge Q. The line element may then be given the form

$$ds^2 = -e^{2\alpha(\check{t},\check{r})} d\check{t}^2 + e^{2\beta(\check{t},\check{r})} d\check{r}^2 + R_Q^2 d\Omega^2 \ . \tag{1}$$

This line element is rather general, and only in the case where the Weyl tensor vanishes does it describe the LBR spacetime. M. Gürzes and Ö. Sarioğlu [15] have shown that a D-dimensional conformally flat LBR spacetime, which is a product of a 2-dimensional anti de Sitter spacetime and a $(D-2)$-dimensional spherical surface, permits a cosmological constant proportional to $1-(D-3)^2$. Hence in the 4-dimensional LBR spacetime the cosmological constant vanishes, which has earlier been noted by V. I. Khlebnikov and É. Shelkovenko [16] and by J. Podolský and M. Ortaggio [17].

Using the radial coordinate $\check{r}$ in the line element invites the interpretation of the spacetime as a spherically symmetric space in the spacetime $\mathbf{R}^4$. An alternative interpretation is also possible. Neglecting the time dimension in the 2-dimensional anti de Sitter space and a spatial dimension in the spherical surface, replacing it by a circle, the spacetime can be interpreted as a cylinder. Then the electrical field is directed along the axis of the cylinder. We here want to consider both physical interpretations. The spacetime with a domain wall will be called the WLBR spacetime, and the spacetime with a product of a 2-dimensional anti de Sitter space and a spherical surface will be called the PLBR spacetime. We use LBR in statements concerning both WPBL and PLBR. Note that in the PLBR interpretation the coordinate $\check{r}$ shall not be interpreted as a radial coordinate.

In the present case it follows from the geodesic equation that a free particle instantaneously at rest has an acceleration

$$\ddot{\check{r}} = -\Gamma^{\check{r}}{}_{\check{t}\check{t}}\,\dot{\check{t}}^2 = -e^{-2\beta}\alpha_{,\check{r}} \ , \tag{2}$$

where the dot denotes differentiation with respect to the proper time of the particle. Hence there is attractive gravity, i.e. the acceleration of gravity points in the negative $\mathbf{e}_{\check{r}}$-direction, if α is an increasing function of $\check{r}$ and repulsive gravitation if α is a decreasing function of $\check{r}$.

With the line element (1) the condition that the Weyl tensor vanishes takes the form

$$e^{-2\beta}(\alpha_{,\check{r}\check{r}} + \alpha_{,\check{r}}^2 - \alpha_{,\check{r}}\beta_{,\check{r}}) - e^{-2\alpha}(\beta_{,\check{t}\check{t}} + \beta_{,\check{t}}^2 - \alpha_{,\check{t}}\beta_{,\check{t}}) = \frac{1}{R_Q^2} \ . \tag{3}$$

Calculating the components of the Einstein tensor from the line element (1) and using Einstein's field equations it follows that when equation (3) is fullfilled, the mixed components of the energy momentum tensor reduce to

$$T^{\check{t}}{}_{\check{t}} = T^{\check{r}}{}_{\check{r}} = -T^{\theta}{}_{\theta} = -T^{\phi}{}_{\phi} = -\frac{1}{\kappa R_Q^2} \ , \tag{4}$$

which represents a constant radial electric field, as is the case in the LBR spacetime. This shows that the LBR spacetime does not allow a non-vanishing cosmological constant.

In the section 4 equation (3) will be solved under different coordinate conditions. The solutions found is the subsections 4.Ia and 4.Ib will turn out to be special cases of the line element

$$ds^2 = [R_Q/G(x^0,x^1)]^2[-(dx^0)^2 + (dx^1)^2] + R_Q^2 d\Omega^2 \ , \tag{5}$$

where x^0 is a time coordinate, x^1 is a radial coordinate and $d\Omega^2$ is a solid angle element.

In the next section we shall develop a formalism for finding transformations between the coordinates where the line element takes the form (5) and the CFS coordinates.

3. Conformally flat spacetime coordinates for the LBR spacetime

We want to write the line element (5) of a spacetime with spherically symmetric space in terms of conformally flat spacetime (CFS) coordinates (T,R). Then the line element takes the form of a conformal factor $C(T,R)^2$ times the Minkowski line element,

$$ds^2 = C(T,R)^2 ds_M^2 = C(T,R)^2(-dT^2 + dR^2 + R^2 d\Omega^2) \ . \tag{6}$$

In order to perform this we shall generalize the method developed in reference [1].

We then use transformations of the form

$$T = \frac{1}{2}\left[f(x^0 + x^1) + g(x^0 - x^1)\right] \quad , \quad R = \frac{1}{2}\left[f(x^0 + x^1) - g(x^0 - x^1)\right] \tag{7}$$

where f and g are functions that must satisfy an identity deduced below. A transformation of this form can be described as a composition of three simple transformations. The first transforms from the coordinates x^0 and x^1 in the line element (5) to light cone coordinates (null coordinates) associated with a Minkowski diagram referring to the (x^0, x^1) coordinate system

$$u = x^0 + x^1 \quad , \quad v = x^0 - x^1 \ . \tag{8}$$

In the Minkowski diagram this rotates the previous coordinate system by $-\pi/4$ and scales it by a factor $\sqrt{2}$. The scaling is performed for later convenience. The coordinate u is constant for light moving in the negative x^1-direction, and v in the positive x^1-direction. The second transforms u and v to the coordinates

$$U = f(u) \quad , \quad V = g(v) \ . \tag{9}$$

Finally, we scale and rotate with the inverse of the transformation (8),

$$T = \frac{U+V}{2} \quad , \quad R = \frac{U-V}{2} \ . \tag{10}$$

The inverse of the transformation (10) is

$$U = T + R \quad , \quad V = T - R \ , \tag{11}$$

showing that U and V are light cone coordinates associated with a Minkowski diagram referring to the CFS coordinate system. The coordinate U is constant for light moving in the negative R-direction and V in the positive R-direction. Note that

$$T^2 - R^2 = UV \ . \tag{12}$$

Taking the differentials of T and R we get

$$-dT^2 + dR^2 = -dU dV = -f'(u)g'(v)\, du\, dv = f'(u)g'(v)(-(dx^0)^2 + (dx^1)^2) \ . \tag{13}$$

Comparing the expressions (5) and (6) for the line element and using the previous formula, we find

$$C(T,R)^2 = \frac{R_Q^2}{f'(u)g'(v)G(x^0,x^1)^2} \tag{14}$$

where x^0, x^1, u and v are functions of T and R, and

$$C(T,R)^2 = \frac{R_Q^2}{R^2} \ . \tag{15}$$

From equations (14) and (15) it follows that

$$f'(u)g'(v)\, G(x^0,x^1)^2 = R^2 \ . \tag{16}$$

By (7) and (8) equation (16) may be written as

$$f'(u)g'(v)\,G\left(\tfrac{u+v}{2},\tfrac{u-v}{2}\right)^2=\tfrac{1}{4}\,[f(u)-g(v)]^2\ . \tag{17}$$

Substituting $v=u$ we get the condition

$$f'(u)g'(u)\,G(u,0)^2=\tfrac{1}{4}\,[f(u)-g(u)]^2\ . \tag{18}$$

As shown in reference [1] if $G(u,0)=0$, the line element (6) can be written in the form (5) with $G(x^0,x^1)=S_k(x^1)$, where the function S_k is defined in equation (A.1). Then equation (17) reduces to

$$f'(u)g'(v)\,S_k\left(\tfrac{u-v}{2}\right)^2=\tfrac{1}{4}\,[f(u)-g(v)]^2\ . \tag{19}$$

Substituting $v=u$ and utilizing that $S_k(0)=0$, this equation gives $g(u)=f(u)$. Hence equation (19) may be written

$$f'(u)f'(v)\,S_k\left(\tfrac{u-v}{2}\right)^2=\tfrac{1}{4}\,[f(u)-f(v)]^2\ , \tag{20}$$

$$T=\tfrac{1}{2}\,[f(x^0+x^1)+f(x^0-x^1)]\quad ,\quad R=\tfrac{1}{2}\,[f(x^0+x^1)-f(x^0-x^1)]\ . \tag{21}$$

With the function [1]

$$f(x)=c\left[b+I_k\left(\tfrac{x-a}{2}\right)\right]^{-1}+d\ , \tag{22}$$

where a, b, c, d are arbitrary constants and the function $I_k(x)$ is defined in equation (A.4), the transformation (21) leads from (5) with $G(x^0,x^1)=S_k(x^1)$ to (6) with $C(T,R)$ given by equation (15) in the case of the LBR spacetime.

It follows from equations (1), (6) and (15) that the line element of the Minkowski spacetime inside the domain wall in the different coordinate systems takes the form

$$ds_M^2=\left(\frac{R(\check{t},\check{r})}{R_Q}\right)^2(-e^{2\alpha(\check{t},\check{r})}d\check{t}^2+e^{2\beta(\check{t},\check{r})}d\check{r}^2+R_Q^2d\Omega^2)\ . \tag{23}$$

The equations (1) and (23) give the general connection between the form of the line element of the WLBR spacetime outside the domain wall in an arbitrary coordinate system and the form of the line element of the flat spacetime inside the domain wall in the same coordinate system.

4. The LBR spacetime in different coordinate systems

Equation (3) will now be solved under different coordinate conditions.

Ia. Static metric and coordinates (η,χ) with $\beta(\chi)=\alpha(\chi)$.

In this case equation (3) reduces to

$$R_Q^2\alpha''-e^{2\alpha}=0 \tag{24}$$

where the prime means differentiation with respect to the radial coordinate. This equation may be written

$$R_Q^2(\alpha'^2)' = (e^{2\alpha})' \ . \tag{25}$$

Integration gives

$$R_Q^2\,\alpha'^2 = e^{2\alpha} - kc^2R_Q^2 \ , \tag{26}$$

where $c > 0$ is an integration constant and k takes the values 1, 0 or -1. The general solution of (26) is given by

$$e^{2\alpha} = c^2R_Q^2/S_k(\chi_0 + c\chi)^2 \ , \tag{27}$$

where $S_k(x)$ is the function defined in equation (A.1) in Appendix A. Here χ_0 is an integration constant and $c = 1$ when $k = 0$. The value $k = 0$ is a very important special case. Then one can introduce CFS coordinates simply by putting $\chi_0 = 0$. The line element with (η, χ) replaced by (T, R) then takes the form

$$ds^2 = \frac{R_Q^2}{R^2}\left(-dT^2 + dR^2 + R^2d\Omega^2\right) \tag{28}$$

with $-\infty < T < \infty$, $R > R_Q$ for the WLBR spacetime, and with $-\infty < T < \infty$, $-\infty < R < \infty$, $R \neq 0$ for the PLBR spacetime. This form of the line element is in agreement with equations (6) and (15). Note that the metric is static. This means that the coordinate clocks go with the same rate at all positions. The line element has the Minkowski form at the domain wall at $R = R_Q$. At this surface $g_{TT} = -1$, meaning that the coordinate clocks of the CFS system show the same time as standard clocks at rest at the domain wall. The fact that there exists a coordinate system so that the metric is static means that the LBR spacetime is static, although we will show later that there exist coordinates so that the metric of this spacetime is time dependent. This time dependence is due to the motion of the reference frame in which the coordinates are comoving.

As has been noted by O. J. C. Dias and J. P. S. Lemos [18] there is an interesting connection between the WLBR spacetime and the Reissner-Nordström spacetime, which is usually described by the line element

$$ds^2 = -\left(1 - \frac{R_S}{r} + \frac{R_Q^2}{r^2}\right)dt^2 + \left(1 - \frac{R_S}{r} + \frac{R_Q^2}{r^2}\right)^{-1}dr^2 + r^2d\Omega^2 \ , \tag{29}$$

where $R_S = 2GM/c^2$ is the Schwarzschild radius, and R_Q is the length corresponding to the electric charge Q. The extremal Reissner-Nordström spacetime has $R_S = 2R_Q$, and then the line element takes the form

$$ds^2 = -\left(1 - \frac{R_Q}{r}\right)^2dt^2 + \left(1 - \frac{R_Q}{r}\right)^{-2}dr^2 + r^2d\Omega^2 \ . \tag{30}$$

A Taylor expansion of $f(r) = (1 - R_Q/r)^2$ about $r = R_Q$ gives to 2. order in r, $f(r) \approx (r - R_Q)^2/R_Q^2$. Hence, the near-horizon limit of the line element for the extremal Reissner-Nordström spacetime takes the form

$$ds^2 = -\frac{(r - R_Q)^2}{R_Q^2}dt^2 + \frac{R_Q^2}{(r - R_Q)^2}dr^2 + R_Q^2d\Omega^2 \ , \tag{31}$$

where the angular part is correct only to 0. order in r. Introducing coordinates

$$R = (r - R_Q)^{-1} \quad , \quad T = t/R_Q^2 \; , \tag{32}$$

leads to the form (28) of the line element. Hence the line element of the near-horizon limit of the Reissner-Nordström spacetime has the same form as the line element of the LBR spacetime. But the coordinates R and r in equation (32) increase in opposite directions. If this is forgotten, gravity seems to be repulsive in the near-horizon limit of the Reissner-Nordström spacetime as expressed in terms of the CFS coordinate R, since α is a decreasing function of R. However, gravity is attractive in the near-horizon limit of the Reissner-Nordström spacetime. This is a coordinate independent property of the spacetime. In the LBR spacetime the CFS coordinate R increases in the direction away from the symmetry center, and there is repulsive gravity. The LBR spacetime is therefore very different from the near-horizon limit of the Reissner-Nordström spacetime.

We shall define the acceleration of gravity in a coordinate system with an arbitrary radial coordinate $\check{r}$ as the acceleration of a free particle instantaneously at rest and measured with standard measuring rods and clocks. Hence it is the component along the unit radial basis vector of the second derivative of the radial coordinate with respect to the proper time of the particle,

$$a^{\hat{\check{r}}} = (g_{\check{r}\check{r}})^{1/2}\, \ddot{\check{r}} \; . \tag{33}$$

In the present case $\ddot{\check{r}} = \ddot{R}$ where $\ddot{R}$ is given by the geodesic equation

$$\ddot{R} = -\Gamma^{R}_{\ TT}\dot{T}^2 = \frac{\Gamma^{R}_{\ TT}}{g_{TT}} \; . \tag{34}$$

For the WLBR spacetime this gives

$$a^{\hat{R}} = \sqrt{g_{RR}}\, \ddot{R} = \frac{1}{R_Q} \; , \tag{35}$$

i.e. in the CFS system the acceleration of gravity is constant and directed away from the domain wall.

We will show that the solutions (27) with $k = 1$ and $k = -1$ represent the same spacetime as the solution with $k = 0$. This will be shown by demonstrating that there exists a coordinate transformation that transforms the line elements of the solutions (27) with $k = 1$ and $k = -1$ to the form (28). Putting $c = 1$ and $\chi_0 = 0$, the line element (1) with the solution (27) takes the form

$$ds^2 = \frac{R_Q^2}{S_k(\chi)^2}(-d\eta^2 + d\chi^2) + R_Q^2 d\Omega^2 \; . \tag{36}$$

In the case $k = 1$ the coordinate clocks showing η go at the same rate as a standard clock at $\chi = \pi/2$, scaled by the factor R_Q.

Note that the form (36) of the line element is valid for all values of k. In the case $k = 0$ the line element reduces to form (28) with (T, R) replaced by (η, χ).

In order to find a coordinate transformation between the (η, χ)-coordinates and the CFS coordinates we apply the formalism in section 3. By choosing $a = 0$, $b = 0$, $c = 1$ and $d = 0$ in equation (22) we obtain the generating function

$$f(x) = B\, T_k\big(x/2\big) \tag{37}$$

where $T_k(x)$ is defined in equation (A.3) and B is a positive constant satisfying

$$(1-|k|)B = 2(1-|k|) \ . \tag{38}$$

Hence B equals 2 when $k = 0$, and has an arbitrary positive value when $k = 1$ and $k = -1$. Using the generating function (37) as shown in Appendix B, the transformation (21) between the (η, χ)-system and the CFS system takes the form

$$T = \frac{BS_k(\eta)}{C_k(\eta)+C_k(\chi)} \quad , \quad R = \frac{BS_k(\chi)}{C_k(\eta)+C_k(\chi)} \ , \tag{39}$$

where $C_k(x)$ is defined in equation (A.2).

We have shown in Apppendix B how the inverse transformation is obtained from the generating function

$$f(x) = 2T_k^{-1}(x/B) \ , \tag{40}$$

giving the result

$$I_k(\eta) = \frac{B^2-k(T^2-R^2)}{2BT} \quad , \quad I_k(\chi) = \frac{B^2+k(T^2-R^2)}{2BR} \tag{41}$$

when $T \neq 0$, where $I_k(x)$ is defined in equation (A.4). In the case $T = 0$ we have that $\eta = 0$. Note that the formulae (36) - (41) are valid for all values of k. A special case of the line element (36) with $k = -1$ has been used by A. C. Ottewill and P. Taylor [19] in connection with quantum field theory on the LBR spacetime.

The world lines of points on the domain wall are given by the second of equations (39) with $R = R_Q$, which leads to

$$C_k(\eta) = (B/R_Q)S_k(\chi) - C_k(\chi) \ . \tag{42}$$

Introducing the constant

$$\chi_Q = I_k^{-1}(B/R_Q) \ , \tag{43}$$

equation (42) takes the form

$$S_k(\chi_Q)\,C_k(\eta) = S_k(\chi - \chi_Q) \tag{44}$$

which can also be written as

$$\chi = \chi_Q + S_k^{-1}(S_k(\chi_Q)\,C_k(\eta)) \ . \tag{45}$$

The point of intersection $(0, \chi_0)$ with the χ-axis, where χ_0 is the coordinate radius of the domain wall in the (η, χ)-system at the point of time $\eta = 0$, is found by inserting $\eta = 0$ in equation (45). Using that $C_k(0) = 1$ for all values of k we then obtain a physical interpretation of the constant χ_Q,

$$\chi_Q = \chi_0/2 \ . \tag{46}$$

From equation (43) it then follows that

$$B = R_Q I_k(\chi_0/2) \ . \tag{47}$$

When $k = 1$ equation (42) takes the form

$$\cos\eta = (B/R_Q)\sin\chi - \cos\chi \tag{48}$$

which is plotted in Figure 1 as the left hand boundary of the hatched region. It follows that in the case $k = 1$ the WLBR spacetime is represented in the (η, χ)-plane by the hatched region in Figure 1, which is given by

$$\chi_Q + \arcsin(\sin\chi_Q \cos\eta) < \chi < \pi - |\eta| \quad , \quad -\pi < \eta < \pi \ . \tag{49}$$

We want to find the corresponding region in the (η, χ)-system representing the PLBR spacetime. From equation (21) we obtain

$$T + R = f(\eta + \chi) \quad , \quad T - R = f(\eta - \chi) \ . \tag{50}$$

Hence $\eta + \chi$ and $\eta - \chi$ must belong to the domain $(-\pi, \pi)$ of the generator function in equation (37) with $k = 1$. This gives the region

$$|\eta| + |\chi| < \pi \quad , \quad \chi \neq 0 \tag{51}$$

when $k = 1$, as illustrated in Figure 1.

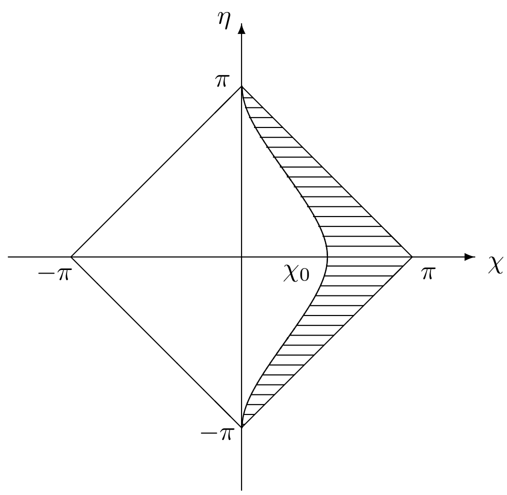

Figure 1. The square represents the PLBR spacetime for $k = 1$ in the (η, χ)-system given by (51). The hatched region represents the WLBR spacetime in the (η, χ)-system given by (49). The left hand curve represents the world line of a point on the domain wall as given by equation (48) where χ_0 is given by (46).

When $k = -1$ the region representing the PLBR spacetime is the whole (η, χ) coordinate space except the η-axis, but $T + R$ and $T - R$ must belong to the range $(-B, B)$ of the generator function in equation (37) with $k = -1$, which gives the region

$$|T| + |R| < B \quad , \quad R \neq 0 \ . \tag{52}$$

Hence in this case the (η,χ)-system does not cover the whole PLBR spacetime, but the constant B secures the possibility of choosing the region given in (52) to be arbitrarily large. The WLBR spacetime for $k=-1$ is given by

$$\chi > \chi_Q + \operatorname{arcsinh}(\sinh\chi_Q \cosh\eta) \quad , \quad -\infty < \eta < \infty \ . \tag{53}$$

The world lines of fixed particles $\chi=\chi_1$ in the (η,χ)-system as described in the CFS system is found from equations (41), which gives

$$(R-R_1)^2 - T^2 = R_1^2 + kB^2 \quad , \quad R_1 = -kBI_k(\chi_1) \ . \tag{54}$$

when $k=1$ and $k=-1$. For $k=0$ we get $R=\chi_1$. The corresponding simultaneity curves $\eta=\eta_1$ are given by

$$(T-T_1)^2 - R^2 = T_1^2 + kB^2 \quad , \quad T_1 = -kBI_k(\eta_1) \ . \tag{55}$$

when $k=1$ and $k=-1$. For $k=0$ we get $T=\eta_1$. Note that the (η,χ)-coordinates and the CFS coordinates are comoving in the same reference frame when $k=0$. Using the transformation (41) the line element (36) is given the form (28). This shows that the solution (27) represents the LBR spacetime for all values of k.

With the line element (36) the coordinate acceleration of a free particle instantaneously at rest is

$$a^\chi = \ddot\chi = -\Gamma^\chi_{\ \eta\eta}\,\dot\eta^2 = -|g_{\eta\eta}|^{-1}\,\Gamma^\chi_{\ \eta\eta} \ , \tag{56}$$

since $\dot\eta = |g_{\eta\eta}|^{-1/2}$ for such a particle. Calculating the Christoffel symbol $\Gamma^\chi_{\ \eta\eta}$ from the line element (36) we obtain

$$\Gamma^\chi_{\ \eta\eta} = -I_k(\chi) \ . \tag{57}$$

The acceleration of gravity in the (η,χ)-system is defined as the component of $a^\chi \mathbf{e}_\chi$ along the unit basis vector $\mathbf{e}_{\hat\chi}$, giving

$$a^{\hat\chi} = \sqrt{g_{\chi\chi}}\,a^\chi = C_k(\chi)/R_Q \ . \tag{58}$$

Hence for $k=1$ the acceleration of gravity in the (η,χ)-system is $a^{\hat\chi} = (1/R_Q)\cos\chi$ so that $a^{\hat\chi} > 0$ for $0<\chi<\pi/2$ and $a^{\hat\chi} < 0$ for $\pi/2 < \chi < \pi$. This is different from the situation in the CFS system, where the acceleration of gravity is directed away from the domain wall everywhere according to equation (35). However, in the (η,χ)-system there is a region $\pi/2<\chi<\pi$ where the acceleration of gravity is directed towards the domain wall. This is due to the motion of the reference frame in which (η,χ) are comoving coordinates, as will be explained below. The charged domain wall is at rest in the (T,R)-system. Hence the CFS coordinates are those of a static, but not inertial, reference frame. In this case the world lines are given by equation (54) with $k=1$ which represents the hyperbolae shown in Figure 2.

From this figure it seems that the (η,χ)-system covers only a part of the WLBR spacetime. The worldlines of fixed particles in (η,χ)-system are hyperbolae which never enter the future region above the asymptotes. This is however not the case because R_1 can have different values depending on χ_1. If R_1 is moved to the left towards $R=0$

the hyperbolae are straightened out. Hence the (η, χ)-system covers all of the WLBR spacetime.

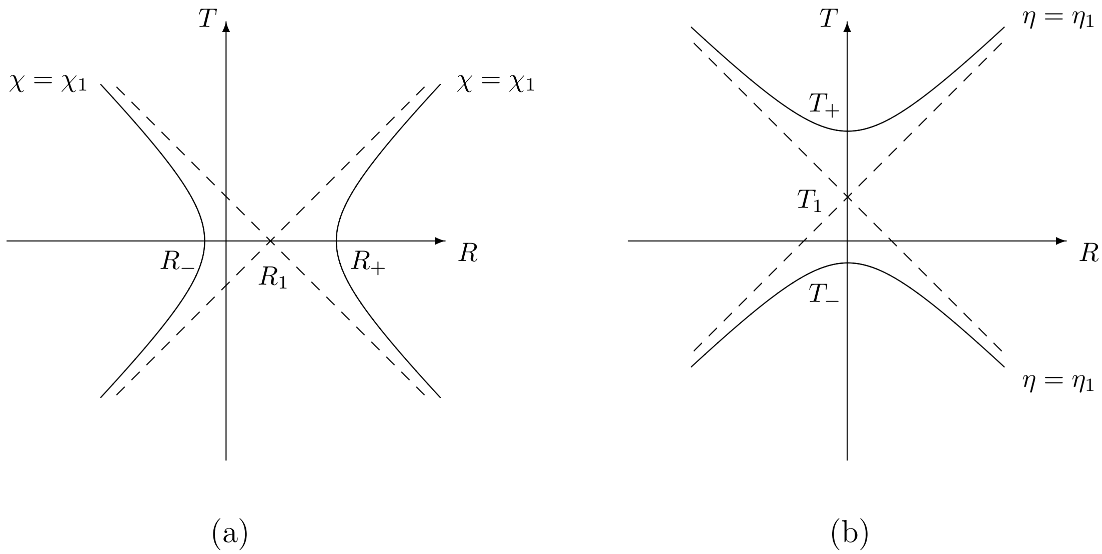

Figure 2. (a) The world lines of points with $\chi = \chi_1$ as given by equation (54) with $k = 1$. Here $R_- = R_1 - \sqrt{R_1^2 + B^2}$ and $R_+ = R_1 + \sqrt{R_1^2 + B^2}$. Note that $R_+ > 0$. A particle in the WLBR spacetime can only follow a hyperbola in the region to the right of $R = R_Q$. However, this limitation does not exist in the PLBR spacetime. (b) The simultaneity curves with $\eta = \eta_1$. Here $T_- = T_1 - \sqrt{T_1^2 + B^2}$ and $T_+ = T_1 + \sqrt{T_1^2 + B^2}$.

It is known from the description of the WLBR spacetime with CFS coordinates, where the metric is static and the domain wall is at rest, that there is repulsive gravitation outside the domain wall. Nevertheless in the region $R > \sqrt{B^2 + T^2}$ in Figure 3 the acceleration of gravity is directed towards the wall in the (η, χ) coordinate system. This apparent contradiction will be explained by comparing the acceleration of fixed points in the (η, χ)-system with the acceleration of free particles relative to the CFS coordinate system.

The world line of a particle at rest in the (η, χ)-system is given by equation (54). From this it follows that the velocity and the acceleration of the particle in the CFS system are

$$\left(\frac{dR}{dT}\right)_{\chi=\chi_1} = \frac{T_2}{R_2 - R_1} \quad , \quad \left(\frac{d^2R}{dT^2}\right)_{\chi=\chi_1} = \frac{R_1^2 + B^2}{(R_2 - R_1)^3} \tag{59}$$

at an arbitrary point (T_2, R_2).

We now consider a free particle with Lagrangian function

$$L = \frac{R_Q^2}{2R^2}(-\dot{T}^2 + \dot{R}^2) \ , \tag{60}$$

where the dot denotes differentiation with respect to the proper time of the particle. Since the metric is static,

$$p_T = \frac{\partial L}{\partial \dot{T}} = -\frac{R_Q^2}{R^2}\frac{dT}{d\tau} \tag{61}$$

is a constant of motion. Together with the four-velocity identity

$$\frac{R_Q^2}{R^2}(-\dot{T}^2 + \dot{R}^2) = -1 \tag{62}$$

this leads to (for a particle moving outwards)

$$\left(\frac{dR}{dT}\right)_F = \frac{\dot{R}}{\dot{T}} = \frac{1}{R}\sqrt{R^2 - \left(\frac{R_Q}{p_T}\right)^2} \ . \tag{63}$$

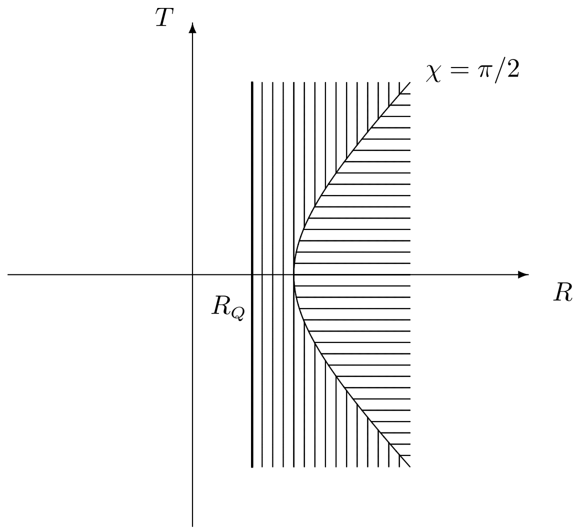

Figure 3. The WLBR spacetime in the (T, R)-system. The hyperbola $R^2 - T^2 = B^2$ corresponding to $\chi = \pi/2$ separates the WLBR spacetime in two regions. To the right of this hyperbola an observer at rest in the (η, χ)-system experiences an acceleration of gravity directed towards the domain wall, and to the left away from the domain wall. The reason for this is explained in the text.

We now demand that the free particle passes through the point (T_2, R_2) with the same velocity as the particle with $\chi = \chi_1$. Using the equations (63) and (54) we obtain

$$p_T = \frac{R_Q(R_2 - R_1)}{R_2\sqrt{R_1^2 + B^2}} \ . \tag{64}$$

Integrating equation (63) we find the equation for the world line of the particle,

$$R^2 - (T - T_0)^2 = R_0^2 \ , \tag{65}$$

where

$$R_0 = \frac{R_Q}{p_T} = \frac{R_2\sqrt{R_1^2 + B^2}}{R_2 - R_1} \ , \tag{66}$$

and T_0 is a constant of integration. With the boundary condition $R(T_2) = R_2$ it follows that

$$T_0 = -\frac{R_1 T_2}{R_2 - R_1} \ . \tag{67}$$

Note that $(dR/dT)_F = 0$ for $R = R_0$. Hence the particle falls from rest at $R = R_0$ at the point of time $T = T_0$. The fact that T_0 depends upon R_1, i.e. on χ_1, means that different reference particles in the (η, χ)-system are instantaneously at rest relative to the CFS system at different points of time. Differentiating we find that the acceleration of the free particle at $R = R_2$ is

$$\left(\frac{d^2R}{dT^2}\right)_F = \frac{R_1^2 + B^2}{R_2(R_2 - R_1)^2} \ . \tag{68}$$

From equations (59) and (68) we have that

$$N = \frac{(d^2R/dT^2)_{\chi=\chi_1}}{(d^2R/dT^2)_F} = \frac{R_2}{R_2 - R_1} \ . \tag{69}$$

From the definition of R_1 in equation (54) and the transformation (41) for $k = 1$ it follows that

$$R_1 = \frac{R_2^2 - T_2^2 - B^2}{2R_2} \ . \tag{70}$$

This implies that

$$N = \frac{2R_2^2}{R_2^2 + T_2^2 + B^2} \ . \tag{71}$$

Hence $N > 1$ for $R_2^2 - T_2^2 > B^2$, i.e. to the right of the hyperbola in Figure 3, which means that the reference particles of the (η, χ)-system have a greater outwards acceleration than a free particle. This is the reason why an observer at rest in the (η, χ)-system experiences that the acceleration of gravity is directed in the negative χ-direction. The wall has a decreasing radius in the (η, χ)-system. This is, however, a coordinate effect. In reality the wall is static and the (η, χ) coordinate system is comoving in an expanding reference frame.

The acceleration of gravity vanishes in the (η, χ)-system on the hyperbola $\chi = \pi/2$ in Figure 3. This leads to the following physical interpretation of the constant B. As seen from equation (54) the point $(0, B)$ in the CFS system corresponds to the point $(0, \pi/2)$ in the (η, χ)-system where the acceleration of gravity vanishes. From equation (58) it follows that a particle with $\chi = \pi/2$ moves freely. Equation (54) gives $R_1 = 0$ for this particle. The coordinate acceleration of this particle at the point of time $T = 0$ as given by equation (68) is $1/B$. Hence B is the inverse of the coordinate acceleration of a free particle at $(0, B)$.

We shall now consider the case $k = -1$. The world lines of reference particles with $\chi = \chi_1$ are given by equation (54) and are shown in Figure 4. We see that the reference points in the WLBR spacetime accelerate in the negative R direction. Hence in this reference frame the acceleration of gravity is directed outwards and is larger than in the static CFS system. This is verified by the expression (58) which implies that in this case $g = (1/R_Q)\cosh\chi \geq 1/R_Q$.

We now consider the flat spacetime inside the shell. The line element of the Minkowski spacetime in this region has the following form in the CFS coordinate system,

$$ds_M^2 = -dT^2 + dR^2 + R^2 d\Omega^2 \ . \tag{72}$$

Inserting $e^\alpha = R_Q/S_k(\chi)$ from the line element (36) and the expression (39) for R into the line element (23) we find the form of the line element (72) in the (η, χ)-system.

$$ds_M^2 = \frac{B^2}{[C_k(\eta) + C_k(\chi)]^2}[-d\eta^2 + d\chi^2 + S_k(\chi)^2 d\Omega^2] \ . \tag{73}$$

Comparing with the line element (36) and using equation (42) we see that the metric is continuous at the domain wall. Note that the line element (73) reduces to (72) for $k = 0$ replacing (η, χ) with (T, R). In this case a spherical surface with radius R has area $4\pi R^2$ in the region inside the domain wall. Outside the domain wall, on the other hand, a spherical surface with radius R has area $4\pi R_Q^2$ which is independent of R. The reason for this strange result is that the space $T =$ constant is curved outside the domain wall.

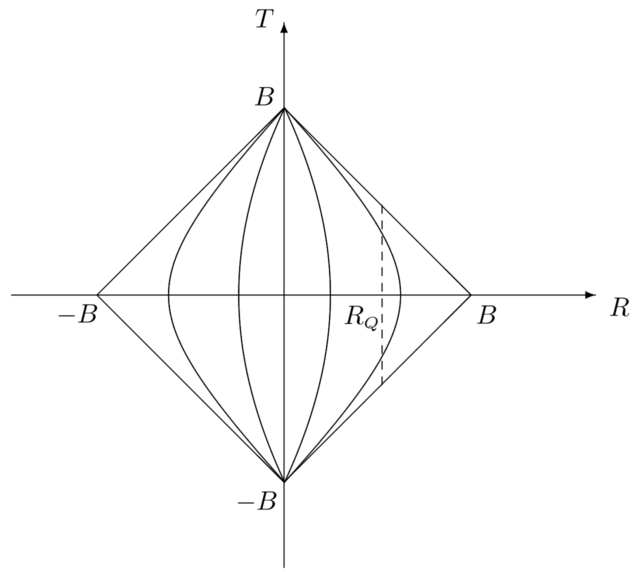

Figure 4. The square represents that part of the PLBR spacetime which is described by the (η, χ)-system with $k = -1$. The world lines of points with $\chi = \chi_1$ as given by equation (54). The region to the right of the vertical line $R = R_Q$ represents a part of the WLBR spacetime when $B > R_Q$ in accordance with equation (47).

Calculating the acceleration of gravity inside the shell as experienced by an observer at rest in the (η, χ)-system in the same way as in equation (58), we find

$$a_M^{\hat{\chi}} = -k S_k(\chi)/B \ . \tag{74}$$

In order to find the discontinuity of the acceleration of gravity in the (η, χ)-system at the domain wall, it is sufficient to consider the point of time $\eta = 0$. Then the domain wall has the position $\chi = 2\chi_Q$, where χ_Q is given in equation (43). Inserting this into equation (74) and using equations (A.18) and (A.34) we find the acceleration of gravity just inside the domain wall,

$$a_M^{\hat{\chi}} = -\frac{2kR_Q^2}{B^2 + kR_Q^2}\,\frac{1}{R_Q} \ . \tag{75}$$

We see that the acceleration of gravity depends on the value of k. There is no acceleration if $k = 0$ because in this case the (η, χ)-system is comoving in a static reference frame in flat spacetime. When $k = 1$ there is an acceleration of gravity towards the point $\chi = 0$. In this case the (η, χ)-system is comoving in a reference frame accelerating in the outwards direction. In the case $k = -1$ we must have $B > R_Q$ in order that the WLBR spacetime shall exist outside the domain wall as seen in Figure 4. Then the acceleration of gravity points outwards, meaning that the reference frame of the (η, χ)-system is accelerating inwards.

The acceleration of gravity just outside the domain wall as given by equation (58) is found in a similar way using equations (A.19) and (A.34) with the result

$$a^{\hat{\chi}} = \frac{B^2 - kR_Q^2}{B^2 + kR_Q^2} \frac{1}{R_Q} . \tag{76}$$

When $k = 0$ the (η, χ)-system is comoving in the same reference frame as the CFS coordinates, which is at rest relative to the domain wall. In this case the acceleration of gravity in the (η, χ)-system just outside the domain wall is equal to $1/R_Q$ just as in the CFS system. When $k = 1$ and $B > R_Q$, the acceleration of gravity is directed away from the domain wall. But when $B < R_Q$ it is directed towards the domain wall. If $B = R_Q$ the acceleration of gravity vanishes. This behaviour can be understood by considering Figure 3. In the case $B = R_Q$ the hyperbola $\chi = \pi/2$ touches the domain wall at $R = R_Q$ when $T = 0$, corresponding to $\eta = 0$. For $B > R_Q$ the hyperbola moves to the right, and for $B < R_Q$ to the left.

For all values of k and B the discontinuity of the acceleration of gravity at the domain wall is

$$a^{\hat{\chi}} - a_M^{\hat{\chi}} = \frac{1}{R_Q} . \tag{77}$$

This shows that the domain wall produces repulsive gravity.

Ib. Time dependent metric and coordinates (τ, ρ) with $\beta(\tau) = \alpha(\tau)$.

In spite of the fact that the LBR spacetime is static, it may be described in terms of coordinates comoving with a reference frame expanding in such a way that the line element takes a time dependent form.

Assuming that the metric functions are independent of the radial coordinate, equation (3) reduces to

$$R_Q^2 \ddot{\alpha} + e^{2\alpha} = 0 \tag{78}$$

which may be written

$$R_Q^2 (\dot{\alpha}^2)^{\dot{}} = -(e^{2\alpha})^{\dot{}} \tag{79}$$

with general solution

$$R_Q^2 \dot{\alpha}^2 = -e^{2\alpha} + a^2 R_Q^2 , \tag{80}$$

where $a > 0$ is an integration constant. The general solution of (80) is given by

$$e^{\alpha} = aR_Q / \cosh(a(\tau - \tau_0)) , \tag{81}$$

Here τ_0 is an integration constant. Choosing $a = 1$ and $\tau_0 = 0$ the line element (1) takes the form

$$ds^2 = \frac{R_Q^2}{\cosh^2 \tau} (-d\tau^2 + d\rho^2) + R_Q^2 d\Omega^2 \tag{82}$$

where $-\infty < \tau < \infty$ and $-\infty < \rho < \infty$. The form of this line element when the proper time of the reference particles is used as a time coordinate is given in equation (189).

We want to investigate whether particles with constant ρ are free, and hence whether their world lines fullfill the geodesic equation. The radial component of this equation then reduces to

$$\ddot{\rho} = -\Gamma^{\rho}{}_{\tau\tau}\dot{\tau}^2 \ . \tag{83}$$

Calculating the Christoffel symbol from the line element (82) we find that $\Gamma^{\rho}{}_{\tau\tau} = 0$. Hence a particle with constant ρ has vanishing acceleration. It is a free particle. Accordingly the (τ, ρ)-system is comoving with free particles.

From equation (82) it follows that the coordinate clocks of the (τ, ρ)-system go at a rate

$$\dot{\tau} = \frac{d\tau}{ds} = \frac{\cosh\tau}{R_Q} \ , \tag{84}$$

which is increasing relative to the rate of standard clocks at rest in the reference frame where (τ, ρ) are comoving coordinates. Note that the coordinate time τ is not equal to the proper time t of the reference particles with constant ρ. The relationship between τ and t will be treated in section IIIb where the proper time will be used as coordinate time.

In this reference frame the physical distances in the radial direction are extremely small when $\tau \to -\infty$. However the space expands in the radial direction and the radial scale factor has a maximal value equal to R_Q when $\tau = 0$. Then space contracts in the radial direction towards vanishingly small distances in the infinitely far future.

We shall find the transformation relating the line elements (28) and (82). In this case $G(x^0, x^1) = \cosh(x^0)$ in the line element (5) so that $G(x^0, 0) \neq 0$. Hence we need two generating functions. We introduce the generating functions

$$f(x) = -B\coth(x/2) \quad , \quad g(x) = B\tanh(x/2) \ , \tag{85}$$

using the same constant B as in equation (37) when $k = 1$ and $k = -1$ in order to simplify the transformations. By means of equations (7), using the procedure shown in Appendix B, we find the following transformation from the (τ, ρ)-coordinates to the CFS cordinates,

$$T = -\frac{B\cosh\rho}{\sinh\tau + \sinh\rho} \quad , \quad R = -\frac{B\cosh\tau}{\sinh\tau + \sinh\rho} \ . \tag{86}$$

This transforms the region $\tau + \rho < 0$ in the (τ, ρ)-system to the region $T + R > B$, $|T - R| < B$ in the CFS system, and the region $\tau + \rho > 0$ in the (τ, ρ)-system to the region $T + R < -B$, $|T - R| < B$ in the CFS system. As shown in Appendix B the inverse transformation is found from the generating functions

$$f(x) = -2\,\mathrm{arccoth}(x/B) \quad , \quad g(x) = 2\,\mathrm{arctanh}(x/B) \ , \tag{87}$$

which gives

$$\tanh\tau = \frac{(T^2 - R^2) - B^2}{2BR} \quad , \quad \tanh\rho = \frac{(R^2 - T^2) - B^2}{2BT} \ . \tag{88}$$

From the second of equations (86) with $R = R_Q$ it follows that the charged domain wall which represents the inner boundary of the WLBR spacetime, moves according to

$$\sinh\rho = -(B/R_Q)\cosh\tau - \sinh\tau \tag{89}$$

in the (τ,ρ)-system. It follows that the WLBR spacetime is represented in the (τ,ρ)-plane by the hatched region in Figure 5, which is given by

$$-\text{arcsinh}\left((B/R_Q)\cosh\tau+\sinh\tau\right)<\rho<-\tau\ . \tag{90}$$

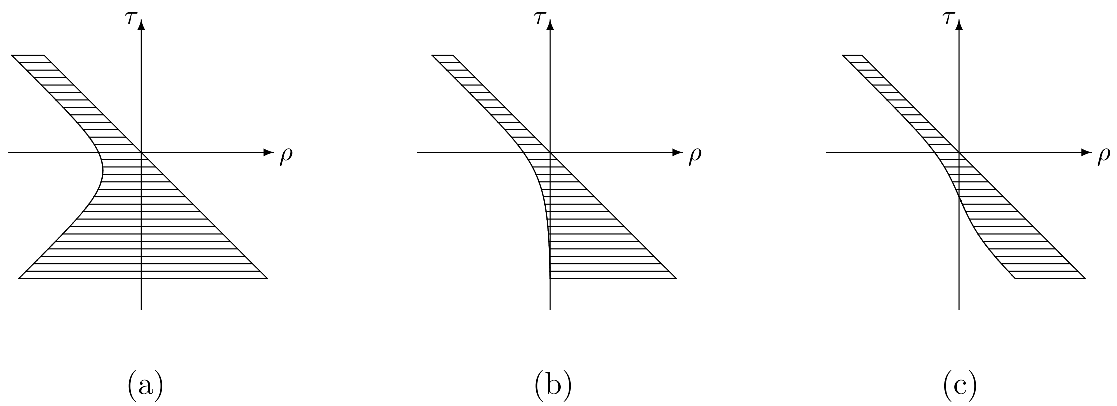

Figure 5. The hatched region represents the WLBR spacetime in the (τ,ρ) coordinate system. The left boundaries are the world lines of a fixed point on the domain wall $R=R_Q$ as given by equation (89). The cases $B>R_Q$, $B=R_Q$ and $B<R_Q$ are shown in (a), (b) and (c) respectively.

The coordinate velocity of the domain wall is

$$\frac{d\rho}{d\tau}=\frac{R_Q+B\tanh\tau}{\sqrt{(R_Q/\cosh\tau)^2+(B+R_Q\tanh\tau)^2}}\ . \tag{91}$$

Using equation (91) and looking at Figure 5(a) we see that in the case $B>R_Q$ the domain wall initially has velocity in the positive ρ-direction with decelerating motion. It stops at an event given by

$$\tau_1=-\text{arccoth}\,(B/R_Q)\quad,\quad\rho_1=-\text{arcsinh}\sqrt{\frac{B-R_Q}{B+R_Q}}\ , \tag{92}$$

and moves in the negative ρ-direction. This means that the (τ,ρ)-system accelerates in the positive R-direction. From equation (91) it follows that

$$\lim_{\tau\to-\infty}\frac{d\rho}{d\tau}=\frac{B-R_Q}{|B-R_Q|}\quad,\quad\lim_{\tau\to\infty}\frac{d\rho}{d\tau}=-1 \tag{93}$$

when $B\neq R_Q$. Hence we see that when $B>R_Q$ the domain wall has initially a velocity close to the velocity of light in the positive ρ-direction, and finally the same velocity in the negative ρ-direction.

Next we consider the case $B=R_Q$. Then the expression for the velocity of the domain wall can be written

$$\frac{d\rho}{d\tau}=\frac{e^\tau}{\sqrt{1+e^{2\tau}}}\ . \tag{94}$$

From this equation it follows that

$$\lim_{\tau\to-\infty}\frac{d\rho}{d\tau}=0\quad,\quad\lim_{\tau\to\infty}\frac{d\rho}{d\tau}=-1 \tag{95}$$

In this case the domain wall has initially a vanishing coordinate velocity, but it accelerates slowly in the negative ρ-direction and ends up approaching the velocity of light.

We finally consider the case $0 < B < R_Q$. In this case the motion of the domain wall is more complicated. In the limit that $\tau \to -\infty$ the domain wall moves nearly with the velocity of light in the negative ρ-direction. Then it decellerates and obtains a minimal velocity $-\sqrt{1-(B/R_Q)^2}$ when it passes $\rho = 0$. Afterwards it accelerates again and approaches the velocity of light in the infinite future.

Since the domain wall is at rest in the CFS system, all of this reflects the motion of the reference frame in which the (τ, ρ)-coordinates are comoving.

We shall find the transformation relating the line elements (36) and (82). Combining the generating functions (85) with the inverse of the generating function (37) with $k = 1$, we obtain the generating functions

$$f(x) = -2\arctan(\coth\tfrac{x}{2}) \quad , \quad g(x) = 2\arctan(\tanh\tfrac{x}{2}) \tag{96}$$

which give the transformation

$$\cot\eta = -\frac{\sinh\tau}{\cosh\rho} \quad , \quad \cot\chi = -\frac{\sinh\rho}{\cosh\tau} \ , \tag{97}$$

as shown in Appendix B. This transforms the region $\tau + \rho < 0$ in the (τ, ρ)-system to the region $\pi/2 < \eta + \chi < \pi$, $|\eta - \chi| < \pi/2$ in the (η, χ)-system, and the region $\tau + \rho > 0$ in the (τ, ρ)-system to the region $-\pi < \eta + \chi < -\pi/2\,,\ |\eta - \chi| < \pi/2$ in the (η, χ)-system.

The inverse transformation is found in a similar way using the generating functions

$$f^{-1}(x) = -2\,\mathrm{arctanh}(\cot\tfrac{x}{2}) \quad , \quad g^{-1}(x) = 2\,\mathrm{arctanh}(\tan\tfrac{x}{2}) \tag{98}$$

which give the transformation

$$\tanh\tau = -\frac{\cos\eta}{\sin\chi} \quad , \quad \tanh\rho = -\frac{\cos\chi}{\sin\eta} \ . \tag{99}$$

In the present case the transformation (97) and its invers can also be found from the equations (39), (41), (86) and (88). Combining the first equation in (88) and (41) for $k = 1$ and substituting for R/T from (86) we get

$$\cot\eta = -\frac{R}{T}\tanh\tau = -\frac{\sinh\tau}{\cosh\rho} \ . \tag{100}$$

Note that the hyperbola $\chi = \pi/2$ in Figure 3 corresponds to $\rho = 0$.

As shown above a free particle has constant ρ, say $\rho = \rho_1$. Hence it follows from the second of the transformation equations (99) that the world line of a free particle as described in the (η, χ)-system is given by

$$\cos\chi = k_1 \sin\eta \quad , \quad k_1 = -\tanh\rho_1 \ . \tag{101}$$

We will now show that this is a solution of the Lagrangian equation for a free particle moving radially. With the line element (36) and $k = 1$ the Lagrangian is

$$L = \frac{R_Q^2}{2\sin^2\chi}\left(-\dot{\eta}^2 + \dot{\chi}^2\right) \ . \tag{102}$$

The conserved momentum conjugate to the time coordinate is

$$p_\eta = \frac{\partial L}{\partial \dot\eta} = -\frac{R_Q^2}{\sin^2\chi}\,\dot\eta\ , \tag{103}$$

giving

$$\dot\eta = -(p_\eta/R_Q^2)\sin^2\chi\ . \tag{104}$$

The 4-velocity identity then takes the form

$$\dot\eta^2 = (1/R_Q^2)\sin^2\chi + \dot\chi^2\ . \tag{105}$$

The last two equations lead to

$$\dot\chi - (1/R_Q)\sqrt{(p_\eta/R_Q)^2\sin^2\chi - 1}\,\sin\chi = 0\ . \tag{106}$$

We now transform from differentiation with respect to the proper time of the particle to differentiation with respect to the coordinate time η by means of equation (104), and find that the solution of this differential equation with the initial condition $\chi(0) = \pi/2$ is

$$\cos\chi = -\sqrt{1 - (R_Q/p_\eta)^2}\,\sin\eta\ . \tag{107}$$

This is in accordance with equation (101) if the conserved energy of the particle is

$$p_\eta = -R_Q\cosh\rho_1\ . \tag{108}$$

Inserting the expression (81) for e^α with $a = 1$ and $\tau_0 = 0$, and (86) for R into the line element (23), we obtain the form of the line element for the Minkowski spacetime inside the domain wall in the (τ, ρ) coordinates,

$$ds_M^2 = \frac{B^2}{(\sinh\tau + \sinh\rho)^2}\left(-d\tau^2 + d\rho^2 + \cosh^2\tau\, d\Omega^2\right)\ . \tag{109}$$

It follows from equation (82), the last of equations (86) with $R = R_Q$, and the line element (109) that the metric is continuous at the shell.

In the (τ, ρ)-system the acceleration of gravity is

$$a^{\hat\rho} = \frac{1}{B}\cosh\rho \tag{110}$$

which is positive. Hence an observer at rest in this coordinate system experiences an acceleration of gravity in the outwards direction in the flat spacetime inside the wall, which means that the reference frame of these coordinates is accelerating in the inwards direction.

IIa. Static metric and coordinates $(\tilde t, \tilde r)$ with $\beta(\tilde r) = -\alpha(\tilde r)$.

In this case equation (3) reduces to

$$\alpha'' + 2\alpha'^2 = \frac{e^{-2\alpha}}{R_Q^2}\ , \tag{111}$$

which may be written

$$(e^{2\alpha})'' = \frac{2}{R_Q^2} \ . \tag{112}$$

The general solution of this equation can be written as

$$e^{2\alpha} = D + \left(\frac{\tilde{r} - \tilde{r}_0}{R_Q}\right)^2 \ , \tag{113}$$

where D and $\tilde{r}_0$ are constants. The line element (1) then takes the form

$$ds^2 = -\left[D + \left(\frac{\tilde{r} - \tilde{r}_0}{R_Q}\right)^2\right] d\tilde{t}^2 + \left[D + \left(\frac{\tilde{r} - \tilde{r}_0}{R_Q}\right)^2\right]^{-1} d\tilde{r}^2 + R_Q^2 d\Omega^2 \ , \tag{114}$$

This form of the line element with $R_Q = 1$, $\tilde{r}_0 = 0$ and $D = \pm 1$ has been used by Ottewill and Taylor [19] and by V. Cardoso, O. J. C. Dias and J. P. S. Lemos [20]. It may further be noted that A. S. Lapedes [21] has studied particle creation in the LBR spacetime. He then considered three forms of the line element (114) with $\tilde{r}_0 = 0$ and $D = -1, 1, 0$ respectively, with a rescaling of $\tilde{t}$ and $\tilde{r}$ by R_Q, and constructed a Penrose diagram for the LBR spacetime. The coordinate clocks showing $\tilde{t}$ go at the same rate as a standard clock at $\tilde{r} = \tilde{r}_0$ scaled by the factor $\sqrt{D}$ when $D > 0$. For $D = 1$ and $\tilde{r}_0 = 0$ the components $g_{\tilde{t}\tilde{t}}$ and $g_{\tilde{r}\tilde{r}}$ have the same form as those of the anti De Sitter metric in static coordinates, while the angular part of the line element represents a spherical surface.

Writing $D = kA^2$ where $k = \mathrm{sgn}(D)$ and

$$A = \begin{cases} \sqrt{|D|} & \text{when} \quad k \neq 0 \\ R_Q & \text{when} \quad k = 0 \end{cases} \ , \tag{115}$$

the transformation between the $(\tilde{t}, \tilde{r})$- and the (η, χ)-system used in the line element (36) is given by

$$I_k(\chi) = \frac{\tilde{r}_0 - \tilde{r}}{R_Q A} \quad , \quad \eta = \frac{A}{R_Q}\, \tilde{t} \ . \tag{116}$$

The transformation has been chosen so that χ and $\tilde{r}$ increases in the same direction. The transformation from χ to $\tilde{r}$ is

$$\tilde{r} = \tilde{r}_0 - R_Q A\, I_k(\chi) \ . \tag{117}$$

It follows that that the $(\tilde{t}, \tilde{r})$-system is comoving with the reference particles of the same reference frame as the (η, χ)-system. Inserting equation (117) into equation (113) and using the relation (A.13) we obtain

$$e^{2\alpha} = A^2\, S_k(\chi)^{-2} \ . \tag{118}$$

Note that this expression in consistent with the line element (36) due to the relation between η and $\tilde{t}$ in the transformation (116). Differentiating equation (117) we get

$$d\tilde{r} = R_Q\, A\, S_k(\chi)^{-2} d\chi \ . \tag{119}$$

Using (118) and (119) we see that the line element (114) takes the form (36).

It follows from equations (49) and (117) for $k = 1$ that in this case the WLBR spacetime is represented by the hatched region in Figure 6 given by

$$-\pi R_Q / A < \tilde{t} < \pi R_Q / A \tag{120}$$

and

$$\tilde{r}_0 - R_Q A \cot\left(\chi_Q + \arcsin\left(\sin\chi_Q \cos\eta\right)\right) < \tilde{r} < \tilde{r}_0 + R_Q A \cot|\eta| \ , \tag{121}$$

where $\eta = (A/R_Q)\,\tilde{t}$ and $\chi_Q = \text{arccot}(B/R_Q)$.

In the PLBR spacetime all the values $k = -1, 0, 1$ are allowed. For $k = 1$ the PLBR spacetime is represented by the region between the horizontal lines in Figure 6 given by

$$-\pi R_Q/A < \tilde{t} < \pi R_Q/A \quad , \quad -\infty < \tilde{r} < \infty \ . \tag{122}$$

The part to the left of the hatched region corresponds to $0 < R < R_Q$ in the CFS system, while the part to the right corresponds to $R < 0$.

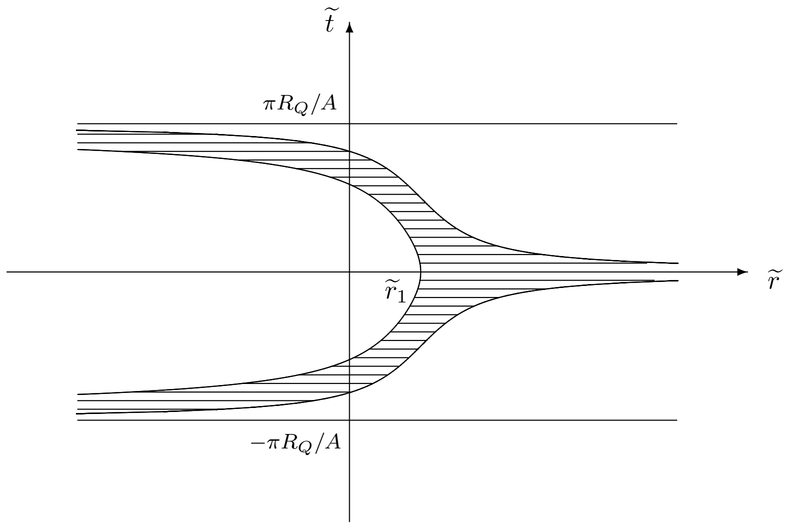

Figure 6. The hatched region represents the WLBR spacetime in the $(\tilde{t}, \tilde{r})$-system for $k = 1$. The boundary of the this region follows from the inequalities (121), and $\tilde{r}_1 = \tilde{r}_0 - R_Q A \cot(2\chi_Q)$ which follows from the left inequality in (121) with $\eta = 0$.

For $k = -1$ the WLBR spacetime is represented in the $(\tilde{t}, \tilde{r})$-system by a region given by

$$-\infty < \tilde{t} < \infty \ , \ \tilde{r}_0 - R_Q A \coth\left(\chi_Q + \text{arcsinh}\left(\sinh\chi_Q \cosh\eta\right)\right) < \tilde{r} < \tilde{r}_0 - R_Q A \ , \tag{123}$$

where $\eta = (A/R_Q)\,\tilde{t}$ and $\chi_Q = \text{arccoth}(B/R_Q)$. For $k = 0$ it is represented by the region

$$-\infty < \tilde{t} < \infty \ , \ \tilde{r}_0 - R_Q < \tilde{r} < \tilde{r}_0 \ . \tag{124}$$

We shall now give a physical interpretation of the constants $\tilde{r}_0$ and D valid for all values of k by considering the motion of a free particle. The acceleration of a free particle instantaneously at rest is here given by

$$a^{\tilde{r}} = \ddot{\tilde{r}} = -\Gamma^{\tilde{r}}_{\tilde{t}\tilde{t}}\, \dot{\tilde{t}}^{\,2} \ . \tag{125}$$

Using the line element (114) we get

$$\dot{\tilde{t}} = \frac{d\tilde{t}}{d\tau} = |g_{\tilde{t}\tilde{t}}|^{-1/2} = \left[D + \left(\frac{\tilde{r} - \tilde{r}_0}{R_Q}\right)^2\right]^{-1/2} \tag{126}$$

where τ is the proper time of the particle. Furthermore

$$\Gamma^{\tilde{r}}{}_{\tilde{t}\tilde{t}} = \frac{\tilde{r}-\tilde{r}_0}{R_Q^2}\left[D + \left(\frac{\tilde{r}-\tilde{r}_0}{R_Q}\right)^2\right] . \tag{127}$$

This gives for the acceleration of gravity in the $(\tilde{t}, \tilde{r})$-system

$$a^{\hat{\tilde{r}}} = \sqrt{g_{\tilde{r}\tilde{r}}}\, a^{\tilde{r}} = \frac{\tilde{r}_0-\tilde{r}}{R_Q^2}\left[D + \left(\frac{\tilde{r}-\tilde{r}_0}{R_Q}\right)^2\right]^{-1/2} . \tag{128}$$

This means that $\tilde{r} = \tilde{r}_0$ is the position where the acceleration of gravity vanishes in the $(\tilde{t}, \tilde{r})$-system. In the WLBR spacetime a free particle is falling towards the domain wall in the region $\tilde{r} > \tilde{r}_0$ and away from the domain wall in the region $\tilde{r} < \tilde{r}_0$. In section 5 we shall show that this is due to the motion of the reference frame in which $(\tilde{t}, \tilde{r})$ are co-moving coordinates. In the case $k = 1$ the constant $\tilde{r}_1$ in Figure 6 represents the position of the shell in the $(\tilde{t}, \tilde{r})$-system at the point of time $T = 0$. The constant $A = \sqrt{D}$ has the following physical interpretation. The coordinate clocks $\tilde{t}$ go at a constant rate equal that of the standard clocks at the domain wall at the point of time $T = 0$ scaled by the factor A. The line element (114) can now be written as

$$ds^2 = -\left[D + R_Q^2 a(\tilde{r})^2\right] d\tilde{t}^{\,2} + \left[D + R_Q^2 a(\tilde{r})^2\right]^{-1} d\tilde{r}^{\,2} + R_Q^2 d\Omega^2 . \tag{129}$$

In the previous cases with $k = 1$ the PLBR spacetime corresponds to only a part of the coordinate region in the $(\tilde{t}, \tilde{r})$-system. However, when $k = -1$ the $(\tilde{t}, \tilde{r})$-system covers only a part of the PLBR spacetime as shown in Figure 4.

Combining the transformations (116) and (39), and using the identities (A.34), we obtain the coordinate transformation from $(\tilde{t}, \tilde{r})$ to (T, R) in the following form

$$T = \frac{B\sqrt{R_Q^2 D + (\tilde{r}-\tilde{r}_0)^2}\, S_k(A\tilde{t}/R_Q)}{\sqrt{R_Q^2 D + (\tilde{r}-\tilde{r}_0)^2}\, C_k(A\tilde{t}/R_Q) \pm (\tilde{r}_0 - \tilde{r})} , \tag{130}$$

$$R = \frac{ABR_Q}{(\tilde{r}_0 - \tilde{r}) \pm \sqrt{R_Q^2 D + (\tilde{r}-\tilde{r}_0)^2}\, C_k(A\tilde{t}/R_Q)} . \tag{131}$$

In the cases $k = 0$ and $k = -1$ we use plus when $\tilde{r} < \tilde{r}_0$ and minus when $\tilde{r} > \tilde{r}_0$. In the case $k = 1$ we use plus for all values of $\tilde{r}$. This generalizes and modifies the corresponding transformation for $k = 1$, $\tilde{r}_0 = 0$ and $A = B = D = 1$ given by Griffiths and Podolosky [22].

When $k = 1$ this transformation maps the region $-\pi R_Q/A < \tilde{t} < \pi R_Q/A$, $\tilde{r} < \tilde{r}_0 + R_Q A \cot(A|\tilde{t}|/R_Q)$ in the PLBR spacetime shown in Figure 6 onto the right half plane in the CFS system, and the region $-\pi R_Q/A < \tilde{t} < \pi R_Q/A$, $\tilde{r} > \tilde{r}_0 + R_Q A \cot(A|\tilde{t}|/R_Q)$ in the PLBR spacetime onto the left half plane in the CFS system. When $k = -1$ the transformation maps the region $-\infty < \tilde{t} < \infty$, $\tilde{r} < \tilde{r}_0 - R_Q A$ in the PLBR spacetime onto the triangle $0 < R < B$, $|T| < B - R$ in the CFS system, and the region $-\infty < \tilde{t} < \infty$, $\tilde{r} > \tilde{r}_0 + R_Q A$ onto the triangle $-B < R < 0$, $|T| < B + R$ in the CFS system. The WLBR spacetime described by the inequalities (123) is mapped onto the triangle $R_Q < R < B$,

$|T| < B - R$ in the CFS system. Using the relationship (A.11) we find that the inverse transformation can be written

$$I_k\left(\frac{A\tilde{t}}{R_Q}\right) = \frac{B^2-k(T^2-R^2)}{2BT} \quad , \quad \frac{\tilde{r}_0-\tilde{r}}{R_Q A} = \frac{B^2+k(T^2-R^2)}{2BR} \ . \tag{132}$$

when $T \neq 0$. In the case $T = 0$, we have $t = 0$. From the last one of the transformation equations (132) it follows that for $k = 1$ the hyperbola of Figure 3 where the acceleration of gravity vanishes is given by $\tilde{r} = \tilde{r}_0$, in agreement with equation (128).

In the case $k = 0$ the transformation (130), (131) reduces to

$$T = \tilde{t} \quad , \quad R = \frac{R_Q^2}{\tilde{r}_0 - \tilde{r}} \tag{133}$$

which has been chosen so that $\mathbf{e}_{\tilde{r}}$ points in the same direction as $\mathbf{e}_R$. This transformation maps the region $-\infty < \tilde{t} < \infty$, $\tilde{r} < \tilde{r}_0$ onto the right half plane in the CFS system, and the region $-\infty < \tilde{t} < \infty$, $\tilde{r} > \tilde{r}_0$ onto the left half plane in the CFS system. The inverse transformation is

$$\tilde{t} = T \quad , \quad \tilde{r} = \tilde{r}_0 - \frac{R_Q^2}{R} \ . \tag{134}$$

In this case the line element (114) takes the form

$$ds^2 = -\left(\frac{\tilde{r}-\tilde{r}_0}{R_Q}\right)^2 d\tilde{t}^{\,2} + \left(\frac{R_Q}{\tilde{r}-\tilde{r}_0}\right)^2 d\tilde{r}^{\,2} + R_Q^2 d\Omega^2 \ , \tag{135}$$

which was considered in reference [19] with $\tilde{r}_0 = 0$.

In these coordinates we shall write down the form of the line element in the flat spacetime inside the domain wall only for the case $k = 0$ when the external metric is given by the equation (135). Inserting the expression (133) for R in the line element (23) then leads to

$$ds_M^2 = -d\tilde{t}^{\,2} + \left(\frac{R_Q}{\tilde{r}-\tilde{r}_0}\right)^2 \left[\left(\frac{R_Q}{\tilde{r}-\tilde{r}_0}\right)^2 d\tilde{r}^{\,2} + R_Q^2 \, d\Omega^2\right] \ . \tag{136}$$

It follows from the transformation (134) that $(\tilde{r} - \tilde{r}_0)/R_Q = 1$ when $R = R_Q$, showing that the metric is continuous at the domain wall.

Calculating the Christoffel symbols from the line element (136) shows that there is vanishing acceleration of gravity inside the domain wall in this coordinate system. The reason is that for $k = 0$ the $(\tilde{t}, \tilde{r})$ coordinates are comoving in a static reference frame in this region.

IIb. Time dependent metric and coordinates $(\bar{t}, \bar{r})$ with $\beta(\bar{t}) = -\alpha(\bar{t})$.

In this case equation (3) reduces to

$$\ddot{\beta} + 2\,\dot{\beta}^{\,2} = -\frac{e^{-2\beta}}{R_Q^2} \ , \tag{137}$$

which may be written

$$\left(e^{2\beta}\right)^{\cdot\cdot} = -\frac{2}{R_Q^2} \ . \tag{138}$$

The general solution of this equation can be written as

$$e^{2\beta} = D - \left(\frac{\bar{t}-\bar{t}_0}{R_Q}\right)^2 \ , \tag{139}$$

where D and $\bar{t}_0$ are constants. A special case of this solution with $\bar{t}_0 = 0$ and $D = 1$ has earlier been found by N. Dadhich [23]. The line element (1) then takes the form

$$ds^2 = -\left[D - \left(\frac{\bar{t}-\bar{t}_0}{R_Q}\right)^2\right]^{-1} d\bar{t}^2 + \left[D - \left(\frac{\bar{t}-\bar{t}_0}{R_Q}\right)^2\right] d\bar{r}^2 + R_Q^2 d\Omega^2 \ , \tag{140}$$

From equation (139) we see that the constant D must be positive, and we introduce the constant $A = \sqrt{D}$ as in section IIa. Here the standard measuring rods have a time dependent length, and the coordinate rods have a constant length equal to the length of the standard rods at the point of time $\bar{t} = \bar{t}_0$ scaled by the factor A. The allowed range of the time $\bar{t}$ is

$$\bar{t}_0 - R_Q A < \bar{t} < \bar{t}_0 + R_Q A \ . \tag{141}$$

The transformation between the $(\bar{t}, \bar{r})$- and the (τ, ρ)-system used in the line element (82) is given by

$$\tanh\tau = \frac{\bar{t}-\bar{t}_0}{R_Q A} \quad , \quad \rho = \frac{A}{R_Q}\,\bar{r} \ . \tag{142}$$

The transformation has been chosen so that τ and $\bar{t}$ increases in the same direction. The second of these equations shows that the $(\bar{t}, \bar{r})$-system is comoving with the same reference frame as the (τ, ρ)-system. Hence particles with $\bar{r} =$ constant are moving freely. The transformation from τ to $\bar{t}$ is

$$\bar{t} = \bar{t}_0 + R_Q A \tanh\tau \ . \tag{143}$$

Inserting equation (143) into equation (139) we obtain

$$e^{2\beta} = D/\cosh^2\tau \ . \tag{144}$$

Differentiating equation (143) we get

$$d\bar{t} = (R_Q A/\cosh^2\tau)\, d\tau \ . \tag{145}$$

Using (144) and (145) we see that the line element (140) takes the form (82).

Combining the transformations (86) and (142) we obtain the coordinate transformation from $(\tilde{t}, \tilde{r})$ to (T, R) in the following form

$$T = \frac{B\sqrt{R_Q^2 D - (\bar{t}-\bar{t}_0)^2}\ \cosh(A\bar{r}/R_Q)}{(\bar{t}_0 - \bar{t}) - \sqrt{R_Q^2 D - (\bar{t}-\bar{t}_0)^2}\ \sinh(A\bar{r}/R_Q)} \ , \tag{146}$$

$$R = \frac{ABR_Q}{(\bar{t}_0 - \bar{t}) - \sqrt{R_Q^2 D - (\bar{t}-\bar{t}_0)^2}\ \sinh(A\bar{r}/R_Q)} \ . \tag{147}$$

The inverse transformation is given by

$$\frac{\bar{t}-\bar{t}_0}{R_Q A} = \frac{(T^2 - R^2) - B^2}{2BR} \quad , \quad \tanh\left(\frac{A\bar{r}}{R_Q}\right) = \frac{(R^2 - T^2) - B^2}{2BT} \tag{148}$$

IIIa. Static metric and coordinates $(\hat{t},\hat{r})$ with $\alpha=\alpha(\hat{r})$ and $\beta=0$.

In this case the radial coordinate $\hat{r}$ is equal to physical distance in the radial direction. Equation (3) then reduces to

$$\alpha''+\alpha'^{\,2}=\frac{1}{R_Q^2}\ , \tag{149}$$

which may be written

$$R_Q^2(e^{\alpha})''-e^{\alpha}=0\ . \tag{150}$$

The general solution of this equation is

$$e^{\alpha}=c_1e^{\hat{r}/R_Q}+c_2e^{-\hat{r}/R_Q} \tag{151}$$

or alternatively

$$e^{\alpha}=c_3\cosh(\hat{r}/R_Q)+c_4\sinh(\hat{r}/R_Q)\ , \tag{152}$$

where c_i, $i=1,2,3,4$ are constants. Here the coordinate clocks go with a position independent rate equal to that of the standard clocks at $\hat{r}=0$ scaled by the factor c_3. This solution (151) was found already in 1917 by T. Levi-Civita [4,5], and was later mentioned in [23] and in [24] with $c_3=0$.

We are now going to find the coordinate transformation between the physical coordinates $(\hat{t},\hat{r})$ and the CFS coordinates (T,R). In this connection we will also deduce the transformation between χ and $\hat{r}$ and between $\tilde{r}$ and $\hat{r}$. Since $\hat{r}$ represents the physical radial distance we have from equation (36) that

$$d\hat{r}=\frac{R_Q}{|S_k(\chi)|}\,d\chi\ . \tag{153}$$

Integration using the identities (A.10), (A.18) and (A.33) gives

$$\hat{r}=\hat{r}_0-\mathrm{sgn}S_k(\chi)\,R_Q\ln\left|I_k\Big(\frac{\chi}{1+|k|}\Big)\right|\ , \tag{154}$$

where $\hat{r}_0$ is a constant. With a suitable scaling of the time coordinate the transformation between the $(\hat{t},\hat{r})$-system and the (η,χ)-system is given by

$$\left|I_k\left(\frac{\chi}{1+|k|}\right)\right|=e^{\pm\frac{\hat{r}_0-\hat{r}}{R_Q}}\quad ,\quad \eta=\frac{A}{R_Q}\,\hat{t}\ , \tag{155}$$

where A is defined in equation (115). Comparing with equation (116) we see that $\hat{t}=\tilde{t}$. In the case $k=-1$ we use the upper sign when $\hat{r}<\hat{r}_0$ and the lower sign when $\hat{r}>\hat{r}_0$. In the cases $k=0$ and $k=1$ we use the upper sign for all $\hat{r}$. These rules mean that $\hat{r}$ increases in the same direction as χ. Using the identity (A.31) with $x=\chi/2$ combined with equations (116) and (155) we obtain

$$\pm a_{-k}\left(\frac{\hat{r}_0-\hat{r}}{R_Q}\right)=I_k(\chi)=\frac{\tilde{r}_0-\tilde{r}}{R_QA} \tag{156}$$

with the same rule for choosing the signs as above, meaning that $\hat{r}$ and $\tilde{r}$ increases in the same direction. This implies that

$$\hat{r}=\hat{r}_0\mp R_Q\,a_{-k}^{-1}\left(\frac{\tilde{r}_0-\tilde{r}}{R_QA}\right)\ . \tag{157}$$

The inverse transformation is given by

$$\tilde{r} = \tilde{r}_0 \mp R_Q A\, a_{-k}\left(\tfrac{\hat{r}_0 - \hat{r}}{R_Q}\right) \ . \tag{158}$$

Since the relationship between $\tilde{r}$ and $\hat{r}$ is time-independent, the $\hat{r}$-coordinate is comoving in the same reference frame as the $\tilde{r}$-coordinate.

It follows from equations (49) and (156) for $k = 1$ that in this case the WLBR spacetime is given by

$$-\pi R_Q/A < \hat{t} < \pi R_Q/A \tag{159}$$

and

$$\hat{r}_0 - R_Q \operatorname{arcsinh}(\cot(\chi_Q + \arcsin(\sin\chi_Q \cos\eta))) < \hat{r} < \hat{r}_0 + R_Q \operatorname{arcsinh}(\cot|\eta|) \ , \tag{160}$$

where $\eta = (A/R_Q)\,\hat{t}$ and χ_Q is given in equation (46).

In the PLBR spacetime all the values $k = -1, 0, 1$ are allowed. For $k = 1$ the PLBR spacetime is represented by

$$-\pi R_Q/A < \hat{t} < \pi R_Q/A \quad , \quad -\infty < \hat{r} < \infty \ . \tag{161}$$

For $k = -1$ the WLBR spacetime is represented in the $(\hat{t}, \hat{r})$-system by a region given by

$$-\infty < \hat{t} < \infty \ , \ \hat{r}_0 - R_Q \operatorname{arccosh}(\coth(\chi_Q + \operatorname{arcsinh}(\sinh\chi_Q \cosh\eta))) < \hat{r} < \hat{r}_0 \ , \tag{162}$$

where $\eta = (A/R_Q)\,\hat{t}$. For $k = 0$ it is represented by the region

$$-\infty < \hat{t} < \infty \ , \ \hat{r} > \hat{r}_0 + R_Q \ln(R_Q) \ . \tag{163}$$

From equations (156), (A.10), (A.13) and (A.35) we obtain

$$S_k(\chi) = 1 \Big/ a_k\left(\tfrac{\hat{r}_0 - \hat{r}}{R_Q}\right) \quad , \quad C_k(\chi) = \pm\, b_k\left(\tfrac{\hat{r}_0 - \hat{r}}{R_Q}\right) \ . \tag{164}$$

Using the formulae (164) and (155) it follows that the line element (36) takes the form

$$ds^2 = -A^2\, a_k\left(\tfrac{\hat{r}_0 - \hat{r}}{R_Q}\right)^2 d\hat{t}^2 + d\hat{r}^2 + R_Q^2 d\Omega^2 \ . \tag{165}$$

The connection between the general solution (151) and the form (165) of the line element is given by

$$c_1 = \tfrac{kA}{1+|k|}\, e^{-\hat{r}_0/R_Q} \quad , \quad c_2 = \tfrac{A}{1+|k|}\, e^{\hat{r}_0/R_Q} \ . \tag{166}$$

Inserting the relations (164) into equations (39) we obtain the transformation between the physical coordinates $(\hat{t}, \hat{r})$ and the CFS coordinates,

$$T = \frac{B\, a_k\left(\frac{\hat{r}_0 - \hat{r}}{R_Q}\right) S_k\left(\frac{A\hat{t}}{R_Q}\right)}{a_{-k}\left(\frac{\hat{r}_0 - \hat{r}}{R_Q}\right) + a_k\left(\frac{\hat{r}_0 - \hat{r}}{R_Q}\right) C_k\left(\frac{A\hat{t}}{R_Q}\right)} \ , \tag{167}$$

$$R = \frac{B}{a_{-k}\left(\frac{\hat{r}_0 - \hat{r}}{R_Q}\right) + a_k\left(\frac{\hat{r}_0 - \hat{r}}{R_Q}\right) C_k\left(\frac{A\hat{t}}{R_Q}\right)} \ . \tag{168}$$

Using equation (156) and the identity (A.35) we see that this transformation is consistent with equations (130) and (131). The inverse transformation is given by

$$I_k\left(\frac{A\hat{t}}{R_Q}\right) = \frac{B^2-k(T^2-R^2)}{2BT} \quad , \quad \pm\, a_{-k}\left(\frac{\hat{r}_0-\hat{r}}{R_Q}\right) = \frac{B^2+k(T^2-R^2)}{2BR} \ . \tag{169}$$

In the case $k=0$ the transformation (167), (168) reduces to

$$T=\hat{t} \quad , \quad R=e^{\frac{\hat{r}-\hat{r}_0}{R_Q}} \tag{170}$$

which has been chosen so that $\mathbf{e}_{\hat{r}}$ points in the same direction as $\mathbf{e}_R$. The inverse transformation is

$$\hat{t}=T \quad , \quad \hat{r}=\hat{r}_0+R_Q \ln R \ . \tag{171}$$

Then the line element (165) takes the form

$$ds^2 = -R_Q^2 e^{-2(\hat{r}-\hat{r}_0)/R_Q}\, d\hat{t}^2 + d\hat{r}^{\,2} + R_Q^2 d\Omega^2 \ . \tag{172}$$

The line elements (28) and (172) are related by the transformation (170) with $\hat{r}_0 = R_Q$. In this case the $\hat{r}$-coordinate and the R-coordinate are comoving in the same reference frame. Although different choices of c_i, $i=1,2,3,4$ all represent conformally flat solutions of the field equations with the same energy momentum tensor representing a constant, radial electrical field, the physical properties of the solutions are different.

This may be most clearly seen by utilizing the geodesic equation. Inserting the solution (151) in the line element (1) we find that a free particle instantaneously at rest has a coordinate acceleration

$$a^{\hat{r}} = \ddot{\hat{r}} = -\Gamma^{\hat{r}}_{\ \hat{t}\hat{t}}\ \dot{\hat{t}}^{\,2} \ . \tag{173}$$

The acceleration of gravity in the $(\hat{t},\hat{r})$-system is the component of $a^{\hat{r}}\mathbf{e}_{\hat{r}}$ along the unit basis vector $\mathbf{e}_{\hat{\hat{r}}}$. Since $g_{\hat{r}\hat{r}}=1$, we have that $a^{\hat{\hat{r}}} = a^{\hat{r}}$. A reference particle with given values of $\hat{r},\theta,\phi$ is at rest relative a reference particle with given values of χ,θ,ϕ. Hence the transformation from the (η,χ)-system to the $\hat{t},\hat{r}$-system is a so called internal transformation, i.e. a coordinate transformation inside a single reference frame. In addition, the unit radial vector in the (η,χ)-system is identical to the unit radial vector in the $\hat{t},\hat{r}$-system, $\mathbf{e}_{\hat{r}}=\mathbf{e}_{\chi}$. These two conditions mean that $a^{\hat{\hat{r}}} = a^{\hat{\chi}}$. Using equation (164) we obtain

$$a^{\hat{\hat{r}}} = \pm\frac{1}{R_Q}\, b_k\left(\frac{\hat{r}_0-\hat{r}}{R_Q}\right) \ . \tag{174}$$

This expression for the acceleration of gravity can be positive or negative. We shall here discuss these possibilities for the WLBR spacetime.

The reason for these differences is found in the different motions of the $(\hat{t},\hat{r})$-systems relative to the CFS system for different values of k and $\hat{r}$. The world line of a reference point $\hat{r}=\hat{r}_1$ as described with reference to the (T,R)-system is given by equation (54). By means of equation (156) the constant R_1 in equation (54) can be expressed in terms of the coordinate $\hat{r}_1$ as $R_1 = \mp k B a_{-k}((\hat{r}_0-\hat{r}_1)/R_Q)$. The world line is shown for $k=1$ in Figure 1.

The form (172) of the line element for the WLBR spacetime in a uniform electric field outside a charged domain wall shows that the time does not proceed infinitely far from the domain wall. This coordinate velocity of light moving radially outwards is

$$\frac{d\hat{r}}{d\hat{t}} = e^{-(\hat{r}-R_Q)/R_Q} \ . \tag{175}$$

Hence $\lim_{\hat{r}\to\infty} d\hat{r}/d\hat{t} = 0$. There is, however, no horizon at a finite distance from the wall.

Again we shall write down the form of the line element in the flat spacetime inside the domain wall only for the case $k = 0$. Inserting e^{α} from the external line element (172) and the expression (170) for R we obtain the internal line element in the $(\hat{t}, \hat{r})$ coordinates,

$$ds_M^2 = -d\hat{t}^2 + (1/R_Q^2)\, e^{2(\hat{r}-\hat{r}_0)/R_Q}\, (d\hat{r}^{\,2} + R_Q^2 d\Omega^2) \ . \tag{176}$$

It follows from equation (171) that the $\hat{r}$ coordinate of the shell is

$$\hat{r}_Q = \hat{r}_0 + R_Q \ln R_Q \ , \tag{177}$$

showing that $e^{2(\hat{r}-\hat{r}_0)/R_Q} = R_Q^2$ at $\hat{r} = \hat{r}_Q$. Inserting this in the line elements (172) and (176) shows that metric is continuous at the shell.

As in the $(\tilde{t}, \tilde{r})$ coordinates there is vanishing acceleration of gravity inside the domain wall in the $(\hat{t}, \hat{r})$ coordinate system for the case that $k = 0$ because then it is comoving in a static reference frame in this region.

IIIb. Time dependent metric and coordinates (t, r) with $\alpha = 0$, $\beta = \beta(t)$.

With $e^{\beta(t)} = a(t)$ the line element then takes the form

$$ds^2 = -dt^2 + a(t)^2 dr^2 + R_Q^2 d\Omega^2 \ . \tag{178}$$

Here t corresponds to the cosmic time of the FRW universe models, i.e. it is the proper time of clocks with fixed spatial coordinates, and $a(t)$ is a scale factor describing the expansion of space in the radial direction. There is no expansion in the directions orthogonal to the radius vector.

Calculating the Christoffel symbols from this line element we find that

$$\Gamma^r_{\ tt} = \Gamma^\theta_{\ tt} = \Gamma^\phi_{\ tt} = 0 \ . \tag{179}$$

From the geodesic equation it follows that a free particle instantaneously at rest will remain at rest in this coordinate system. Hence the coordinates r, θ, ϕ are comoving with free particles. Therefore (t, r, θ, ϕ) are the coordinates of an inertial reference frame. They may be called inertial coordinates in the PLBR spacetime. These coordinates are analogous to the standard coordinates used in the FRW universe models.

The Einstein-Maxwell equations then take the form

$$R_Q^2 \ddot{a} + a = 0 \ , \tag{180}$$

where the dots denote differentiation with respect to the proper time of the reference particles. With the line element (178) this is also the condition that the Weyl tensor vanishes. The general solution of equation (180) is

$$a(t) = d_1 \cos(t/R_Q) + d_2 \sin(t/R_Q) \tag{181}$$

where d_1 and d_2 are constants.

We shall find the transformation relating this line element to the line element (28) of the LBR spactime in CFS coordinates. In this connection we will also deduce the transformation between τ and t and between $\bar{t}$ and t. Since t represents the proper time of clocks with fixed spatial coordinates it follows from the line element (82) that

$$dt = \frac{R_Q}{\cosh\tau}\, d\tau = \frac{R_Q \cosh\tau}{1+\sinh^2\tau}\, d\tau \ . \tag{182}$$

Integration gives

$$t = t_0 + R_Q \arctan(\sinh\tau) \ , \tag{183}$$

where t_0 is a constant. With a suitable scaling of the radial coordinate the transformation from the (t, r)-system to the (τ, ρ)-system is given by

$$\sinh\tau = \tan\left(\frac{t-t_0}{R_Q}\right) \quad , \quad \rho = \frac{A}{R_Q}\, r \ , \tag{184}$$

transforming the region $t_0 - R_Q\pi/2 < t < t_0 + R_Q\pi/2\,,\ -\infty < r < \infty$ in the inertial system to the region $-\infty < \tau < \infty$, $-\infty < \rho < \infty$ in the (τ, ρ)-system. It also follows that

$$\cosh\tau = 1/\cos\left(\frac{t-t_0}{R_Q}\right) \ . \tag{185}$$

Combining this with equation (142) we obtain

$$\sin\left(\frac{t-t_0}{R_Q}\right) = \tanh\tau = \frac{\bar{t}-\bar{t}_0}{R_Q A} \ , \tag{186}$$

which implies that

$$t = t_0 + R_Q\ \arcsin\left(\frac{\bar{t}-\bar{t}_0}{R_Q A}\right) \ . \tag{187}$$

The inverse transformation is given by

$$\bar{t} = \bar{t}_0 + R_Q A\ \sin\left(\frac{t-t_0}{R_Q}\right) \ . \tag{188}$$

Using the formulae (182), (185) and (184) it follows that the line element (82) takes the form

$$ds^2 = -dt^2 + A^2\cos^2\left(\frac{t-t_0}{R_Q}\right) dr^2 + R_Q^2 d\Omega^2 \ . \tag{189}$$

In these coordinates the line element has a form similar to those of the Friedmann Robertson Walker universe models with radial scale factor $a(t) = A\cos((t-t_0)/R_Q)$. The coordinate time t corresponds to the cosmic time as measured by clocks comoving with free particles. There is initially an expansion in the radial direction, turning to contraction at the point of time $t = t_0$. Hence t_0 is the point of time with maximal physical distances.

The connection between the general solution (181) and the form (189) of the line element is given by

$$d_1 = A\ \cos\left(\frac{t_0}{R_Q}\right) \quad , \quad d_2 = A\ \sin\left(\frac{t_0}{R_Q}\right) \ . \tag{190}$$

Inserting the relations (184) and (185) into equations (86) we obtain the transformation between the inertial coordinates (t, r) and the CFS coordinates,

$$T = \frac{B \cos\left(\frac{t-t_0}{R_Q}\right) \cosh\left(\frac{A\,r}{R_Q}\right)}{\sin\left(\frac{t_0-t}{R_Q}\right) - \cos\left(\frac{t-t_0}{R_Q}\right) \sinh\left(\frac{A\,r}{R_Q}\right)} \,, \tag{191}$$

$$R = \frac{B}{\sin\left(\frac{t_0-t}{R_Q}\right) - \cos\left(\frac{t-t_0}{R_Q}\right) \sinh\left(\frac{A\,r}{R_Q}\right)} \,, \tag{192}$$

transforming the region $t_0 - R_Q\pi/2 < t < t_0 + R_Q\pi/2\,,\ -\infty < r < \infty$ in the inertial system to the region $|T+R| > B$, $|T-R| < B$ in the CFS system (see Figure 7). Using equation (186) we see that this transformation is consistent with equations (146) and (147). The inverse transformation is given by

$$\sin\left(\frac{t-t_0}{R_Q}\right) = \frac{(T^2-R^2)-B^2}{2BR} \quad , \quad \tanh\left(\frac{A\,r}{R_Q}\right) = \frac{(R^2-T^2)-B^2}{2BT} \,. \tag{193}$$

Note that the denominators cannot vanish in the regions specified above.

The world lines of fixed points $r = r_1$ in the inertial frame with reference to the CFS system are given by

$$R^2 - (T-T_1)^2 = B^2 - T_1^2 \quad , \quad T_1 = -B \tanh\left(\frac{A\,r_1}{R_Q}\right) \,, \tag{194}$$

which represents hyperbolae as shown in Figure 7.

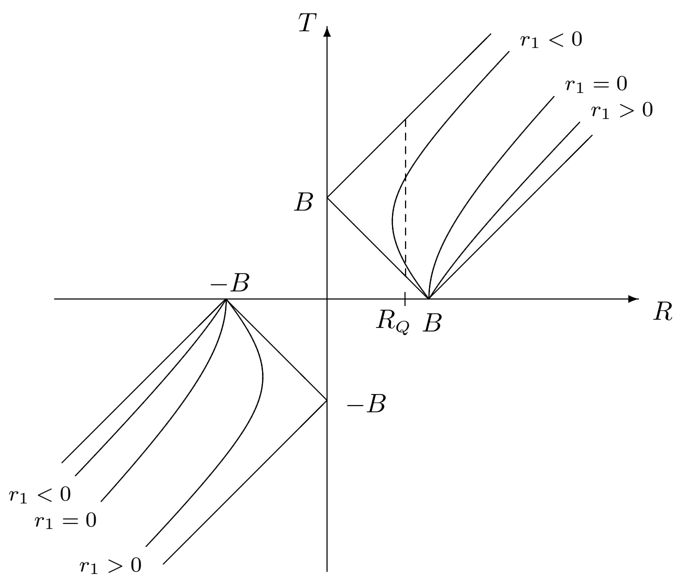

Figure 7. The world lines of freely falling particles with $r = r_1$ as shown in the CFS system for different values of r_1. The comoving coordinates (t, r) cover a part of the PLBR spacetime given by $|T-R| < B$, $|T+R| > B$. The region to the right of the vertical line $R = R_Q$ represents a part of the WLBR spacetime.

Differentiating equation (194) we find the coordinate velocity of a particle with $r = r_1$ in the CFS system. The initial velocity of the particle at $T = 0$, $R = B$ is

$$\left(\frac{dR}{dT}\right)_{T=0} = -\frac{T_1}{B} = \tanh\left(\frac{A\,r_1}{R_Q}\right) . \tag{195}$$

We want to find the region in the (t, r)-system corresponding to WLBR spacetime. This region is given by $R > R_Q$. From equation (192) it follows that this corresponds to

$$0 < \sin\left(\frac{t_0 - t}{R_Q}\right) - \cos\left(\frac{t - t_0}{R_Q}\right) \sinh\left(\frac{A\,r}{R_Q}\right) < \frac{B}{R_Q} , \tag{196}$$

which gives

$$t_0 - R_Q\pi/2 < t < t_0 + R_Q\pi/2 \tag{197}$$

and

$$\frac{R_Q \sin\left(\frac{t - t_0}{R_Q}\right) - B}{R_Q \cos\left(\frac{t - t_0}{R_Q}\right)} < \sinh\left(\frac{A\,r}{R_Q}\right) < \tan\left(\frac{t - t_0}{R_Q}\right) . \tag{198}$$

The WLBR spacetime is shown as the hatched region in Figure 8. The part to the left of the hatched region corresponds to $0 < R < R_Q$ in the CFS system, while the part to the right corresponds to $R < 0$.

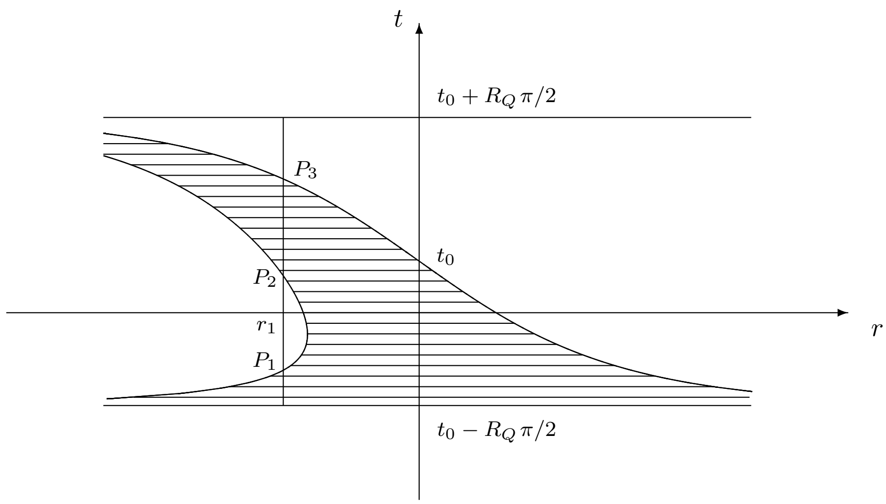

Figure 8. The region between the horizontal lines in this figure represents that part of the PLBR spacetime covered by the (t, r) coordinate system. The hatched region in this figure represents the WLBR spacetime in the (t, r)-system, with $0 < R < R_Q$ to the left of this region and $R < 0$ to the right. Consider the vertical line $r = r_1$ where $r_1 < 0$. This is the world line of a free particle with $r_1 < 0$ in Figure 7. The initial point with $t = t_0 - R_Q\pi/2$ corresponds to an event with coordinates $(0, B)$ in the CFS system. According to equation (195) the initial velocity of a particle with $r_1 < 0$ is directed inwards. The world line between the events P_1 and P_2 is only possible in the PLBR spacetime which has no domain wall. At the event P_2 the particle enters the WLBR spacetime, and at P_3 it leaves the WLBR spacetime as R and T approaches infinity. Then it appears in the PLBR spacetime as R and T comes from minus infinity. Finally it arrives at $(0, -B)$ in the CFS system when $t = t_0 + R_Q\pi/2$.

Let us consider a particle falling freely with outwards directed initial velocity from $R = R_Q$ at the event P_2. This particle follows the world line $r = r_1 < 0$ as shown in Figure 7. As observed in the CFS system it accelerates away from the wall as seen from equation (35). Hence there is repulsive gravitation. However, it follows from the line element (189) that as observed by freely falling observers, the 3-space $t =$ constant first expands and then contracts in the radial direction. This strange behaviour can be understood by considering the equation of geodesic deviation.

In comoving coordinates with tangent vector $\mathbf{u} = (1, 0, 0, 0)$ for the geodesic curves the equation takes the form

$$\frac{d^2 s^i}{dt^2} + R^i{}_{0j0} s^j = 0 \ . \tag{199}$$

With the line element (189) this equation reduces to

$$\frac{d^2 s^r}{dt^2} + \frac{1}{R_Q^2} s^r = 0 \ , \tag{200}$$

having the solution

$$s^r = A \cos\left(\frac{t - t_0}{R_Q}\right) \ , \tag{201}$$

which is equal to the scale factor in the line element (189). This then provides an explanation for the surprising contraction of the space between the events (t_0, r_1) and P_3 in Figure 8. The transition from expansion to contractions happens at $t = t_0$, corresponding to the simultaneity curve $T^2 - R^2 = B^2$ as seen from the first of the equations (193). The world line of an observer with $r = r_1$ will intersect this simultaneity curve only when $r_1 < 0$. Hence only these observers will experience contraction.

A simple special case of the line element (189) is obtained by choosing $t_0 = R_Q \pi/2$ and $A = 1$, giving

$$ds^2 = -dt^2 + \sin^2(t/R_Q)\, dr^2 + R_Q^2 d\Omega^2 \ . \tag{202}$$

A deeper understanding of the t coordinate may be obtained by giving a parametric description of a free particle in the PLBR spacetime with the proper time t of the particle as parameter. We consider a particle with $r = r_1$ in the inertial coordinate system, with world line given in equation (194). The particle is instantaneously at rest at the point P with CFS coordinates (T_1, R_1) where $R_1 = \sqrt{B^2 - T_1^2}$. We shall now apply Lagrangian dynamics in the CFS system to this particle. Putting the velocity (63) equal to zero at the point P shows that the constant of motion p_T for this particle is

$$p_T = -\frac{R_Q}{R_1} \ , \tag{203}$$

where the minus sign has been chosen in order that

$$\dot{T} = -\frac{R^2}{R_Q^2} p_T = \frac{R^2}{R_1 R_Q} > 0 \ . \tag{204}$$

The first equality follows from equation (61). Inserting this into the four-velocity identity (62) and integrating leads to

$$R = R_1 / \sin\left(\frac{t_1 - t}{R_Q}\right) \ , \tag{205}$$

where t_1 is a constant of integration. Inserting the expression (205) into (204) and integrating gives

$$T = T_2 + R_1 \cot\left(\frac{t_1 - t}{R_Q}\right) \ , \tag{206}$$

where T_2 is a new constant of integration. Demanding that equations (205) and (206) is a parametric representation of the hyperbola (194) gives $T_2 = T_1$. From equations (205) and (206) we then have

$$\cos\left(\frac{t_1 - t}{R_Q}\right) = \frac{T - T_1}{R} \ . \tag{207}$$

The constant t_1 is now determined by eliminating t from equations (207) and the first of the transformation formulae (193) at the point P. This gives

$$t_1 - t_0 = R_Q\left(\frac{\pi}{2} - \arcsin\frac{R_1}{B}\right) = R_Q \arcsin\frac{T_1}{B} \ . \tag{208}$$

The equations (205) to (208) give a parametric representation of the world lines of free particles with $r = r_1$ as shown in Figure 7.

We shall now show that this parametric description of the path of a free particle with the proper time of the particle as parameter is in agreement with the transformation (193) from the CFS coordinates to the comoving coordinates of the particle. We have that

$$\sin\left(\frac{t - t_0}{R_Q}\right) = \sin\left(\arcsin\frac{T_1}{B} - \frac{t_1 - t}{R_Q}\right) = \frac{T_1}{B}\cos\left(\frac{t_1 - t}{R_Q}\right) - \frac{R_1}{B}\sin\left(\frac{t_1 - t}{R_Q}\right) \ . \tag{209}$$

Inserting equations (205) and (207) gives

$$\sin\left(\frac{t - t_0}{R_Q}\right) = \frac{T_1(T - T_1) - R_1^2}{BR} = \frac{T_1 T - B^2}{BR} \ . \tag{210}$$

From equation (194) and the second of the equations (193) it follows that

$$T_1 = \frac{(T^2 - R^2) + B^2}{2T} \ . \tag{211}$$

Inserting this into equation (210) we finally obtain the first of the transformation equations (193).

In Figure 7 we have drawn the world lines of particles with different values of r_1. All of the particles come from the point $(B, 0)$ and move so that T and R approaches infinity when t increases towards t_0 as seen from equations (205) and (206). When t passes t_1 the values of T and R switch to minus infinity, and all particles approach the point $(-B, 0)$ when t increases towards $t_0 + \pi R_Q/2$. This highly surprising behaviour may be understood by noting that according to the line element (28) the physical distances in the PLBR spacetime approach zero when $|R|$ approaches infinity.

The reference particles of the (t, r)-system are freely falling. Their world lines are hyperbolae corresponding to particles with constant proper acceleration. This is in accordance with the fact that the acceleration of gravity as given in equation (35) is constant in the LBR spacetime.

In the final part of this section we shall present the transformations between the previous coordinate systems and the inertial one. Combining equation (186) with the transformation (142) we obtain

$$\bar{t} = \bar{t}_0 + R_Q A \sin\left(\frac{t-t_0}{R_Q}\right) \quad , \quad \bar{r} = r \ . \tag{212}$$

The inverse transformation is

$$\sin\left(\frac{t-t_0}{R_Q}\right) = \frac{\bar{t}-\bar{t}_0}{R_Q A} \quad , \quad r = \bar{r} \ . \tag{213}$$

Hence

$$\frac{d\bar{t}}{dt} = A \cos\left(\frac{t-t_0}{R_Q}\right) = A/\cosh\tau \ , \tag{214}$$

showing that the coordinate clocks showing $\bar{t}$ go at an increasingly slower rate than the standard clocks showing t.

We shall find the transformation relating the line element (1) of the LBR spactime in inertial and physical coordinates respectively. Here α in the line element (1) is given by equation (151) and $\beta = 0$. Combining the transformation (97) with the equations (184) and (185) we obtain the transformation

$$\cot\eta = \tan\left(\frac{t_0-t}{R_Q}\right)\Big/\cosh\left(\frac{A\,r}{R_Q}\right) \quad , \quad \cot\chi = -\cos\left(\frac{t-t_0}{R_Q}\right)\sinh\left(\frac{A\,r}{R_Q}\right) \ . \tag{215}$$

Now using $\eta = (A/R_Q)\,\hat{t}$ and (156) we obtain the transformation

$$\cot\left(\frac{A\,\hat{t}}{R_Q}\right) = \tan\left(\frac{t_0-t}{R_Q}\right)\Big/\cosh\left(\frac{A\,r}{R_Q}\right) \ , \ \sinh\left(\frac{\hat{r}-\hat{r}_0}{R_Q}\right) = \cos\left(\frac{t-t_0}{R_Q}\right)\sinh\left(\frac{A\,r}{R_Q}\right) \tag{216}$$

The inverse transformation is found in a similar way by combining the transformation (99) with equation (186) which gives

$$\sin\left(\frac{t_0-t}{R_Q}\right) = \frac{\cos\eta}{\sin\chi} \quad , \quad \tanh\left(\frac{A\,r}{R_Q}\right) = -\frac{\cos\chi}{\sin\eta} \ . \tag{217}$$

Introducing $\hat{t}$ and using the equations (164) we obtain the transformation

$$\sin\left(\frac{t_0-t}{R_Q}\right) = \cosh\left(\frac{\hat{r}-\hat{r}_0}{R_Q}\right)\cos\left(\frac{A\,\hat{t}}{R_Q}\right) \ , \ \tanh\left(\frac{A\,r}{R_Q}\right) = \tanh\left(\frac{\hat{r}-\hat{r}_0}{R_Q}\right)\Big/\sin\left(\frac{A\,\hat{t}}{R_Q}\right) \ . \tag{218}$$

Hence the world line of a free particle with $r = r_1$ as described with reference to the $(\hat{t}, \hat{r})$-system is given by

$$\tanh\left(\frac{\hat{r}-\hat{r}_0}{R_Q}\right) = a_1 \sin\left(\frac{A\,\hat{t}}{R_Q}\right) \quad , \quad a_1 = \tanh\left(\frac{A\,r_1}{R_Q}\right) \ . \tag{219}$$

The coordinate transformations from the CFS system to the inertial system and the $(\hat{t}, \hat{r})$-system with $k = -1$ are defined on the disjoint domains $|T + R| > B$, $|T - R| < B$ and $|T + R| < B$, $|T - R| < B$, $R \neq 0$ respectively. There is therefore no coordinate transformation from the inertial system to the $(\hat{t}, \hat{r})$-system in this case. For $k = 0$ we have

$$\hat{t} = \frac{B\cos\left(\frac{t-t_0}{R_Q}\right)\cosh\left(\frac{A\,r}{R_Q}\right)}{\sin\left(\frac{t_0-t}{R_Q}\right) - \cos\left(\frac{t-t_0}{R_Q}\right)\sinh\left(\frac{A\,r}{R_Q}\right)} \ , \tag{220}$$

$$B\,e^{(\hat{r}_0 - \hat{r})/R_Q} = \sin\left(\frac{t_0-t}{R_Q}\right) - \cos\left(\frac{t-t_0}{R_Q}\right)\sinh\left(\frac{A\,r}{R_Q}\right) . \tag{221}$$

We can also find the transformation between the $(\tilde{t}, \tilde{r})$-system and the inertial system. Combining the transformations (215) and (116) we obtain

$$\cot\left(\frac{A\,\tilde{t}}{R_Q}\right) = \tan\left(\frac{t_0-t}{R_Q}\right)\Big/\cosh\left(\frac{A\,r}{R_Q}\right) \quad , \quad \tilde{r} = \tilde{r}_0 + R_Q A\cos\left(\frac{t-t_0}{R_Q}\right)\sinh\left(\frac{A\,r}{R_Q}\right) \tag{222}$$

The inverse transformation is given by

$$\sin\left(\frac{t_0-t}{R_Q}\right) = \frac{1}{R_Q}\cos\left(\frac{A\,\tilde{t}}{R_Q}\right)\sqrt{R_Q^2 D + (\tilde{r}-\tilde{r}_0)^2} \ , \tag{223}$$

$$\tanh\left(\frac{A\,r}{R_Q}\right) = \frac{\tilde{r}-\tilde{r}_0}{\sin\left(\frac{A\,\tilde{t}}{R_Q}\right)\sqrt{R_Q^2 D+(\tilde{r}-\tilde{r}_0)^2}} \ . \tag{224}$$

Equation (128) gives the following equation of motion for a free particle in the $(\tilde{t}, \tilde{r})$-system,

$$R_Q^2\,\frac{d^2\tilde{r}}{dt^2} + \tilde{r} - \tilde{r}_0 = 0 \ , \tag{225}$$

where t is the proper time of the particle. This is the equation of harmonic motion about the position $\tilde{r} = \tilde{r}_0$ as noted by Dadhich [23]. His interpretation is that a free particle would execute harmonic oscillation about $\tilde{r} = \tilde{r}_0$. He has given an explanation of this motion in terms of electrostatic energy filling the LBR spacetime.

In our opinion, however, there is another explanation for this motion. Equation (225) has the general solution

$$\tilde{r} = \tilde{r}_0 + A_1\cos\left(\frac{t-t_0}{R_Q}\right) , \tag{226}$$

where A_1 and t_0 are integration constants. According to equation (222) a fixed point $r = r_1$ in a freely moving reference frame has a radial coordinate given by the above equation with $A_1 = R_Q A\sinh(Ar_1/R_Q)$. Hence $(\tilde{t}, \tilde{r})$ are comoving coordinates in a reference frame that performs harmonic relatively to a freely falling reference frame. This is the reason for the oscillating motion of a free particle in the $(\tilde{t}, \tilde{r})$-system which was noted by Dadhich, assuming that $-\infty < t < \infty$.

In the context of the LBR as interpreted in the present article, the situation is different. The coordinate region in (t, r)-system of the LBR spacetime is given by the inequalities (197) and $-\infty < r < \infty$. This restriction of the time interval means that the oscillating character of the motion of a free particle in the $(\tilde{t}, \tilde{r})$-system as given by equation (226) vanishes.

Choosing $A = 1$, $B = R_Q$, $t_0 = 0$ and introducing the coordinates $\tilde{\tau} = t$, $x = r$,

$y = R_Q\phi$ and $z = R_Q(\theta - \pi/2)$ in equation (189), the PLBR line element takes the form [19]

$$ds^2 = -d\tilde{\tau}^2 + \cos^2(\tilde{\tau}/R_Q)\,dx^2 + \cos^2(z/R_Q)\,dy^2 + dz^2 \ . \tag{227}$$

Using the formulae (191) and (192) we see that this form of the line element is obtained from (28) by the transformation

$$T = \frac{R_Q \cos(\tilde{\tau}/R_Q)\cosh(x/R_Q)}{\sin(\tilde{\tau}/R_Q) - \cos(\tilde{\tau}/R_Q)\sinh(x/R_Q)} \ , \tag{228}$$

$$R = \frac{R_Q}{\sin(\tilde{\tau}/R_Q) - \cos(\tilde{\tau}/R_Q)\sinh(x/R_Q)} \ , \tag{229}$$

$$\theta = \frac{z}{R_Q} + \frac{\pi}{2} \quad , \quad \phi = \frac{y}{R_Q} \tag{230}$$

Note that x, y and z are not to be interpreted as Cartesian coordinates.

IV. A new type of coordinate systems for the case $k = -1$.

When $k = -1$ in equation (36) there exist different types of coordinates for the LBR spacetime obeying the coordinate conditions $\beta = \alpha$, $\beta = -\alpha$ and $\beta = 0$. Here we will introduce coordinates (η', χ') with $\beta = \alpha$, $(\tilde{t}', \tilde{r}')$ with $\beta = -\alpha$ and $(\hat{t}', \hat{r}')$ with $\beta = 0$ different from the coordinates (η, χ), $(\tilde{t}, \tilde{r})$ and $(\hat{t}, \hat{r})$ respectively.

In order to find the transformation between the line element (28) and the line element (36) with marked coordinates,

$$ds^2 = \frac{R_Q^2}{\sinh^2 \chi'}(-d\eta'^2 + d\chi'^2) + R_Q^2 d\Omega^2 \ , \tag{231}$$

we replace the generating function (37) by

$$f(x) = Be^x \ . \tag{232}$$

Like the function (37) it satisfies the condition (20). It is obtained from equation (22) with $a = 0$, $b = -1$, $c = 2B$ and $d = B$. This leads to the transformation

$$T = Be^{\eta'} \cosh \chi' \quad , \quad R = Be^{\eta'} \sinh \chi' \ . \tag{233}$$

The inverse transformation is

$$Be^{\eta'} = \sqrt{T^2 - R^2} \quad , \quad \tanh \chi' = \frac{R}{T} \ . \tag{234}$$

In the (T, R)-system, each reference particle with $\chi' =$ constant in the coordinate system has a constant velocity

$$V = \frac{R}{T} = \tanh \chi' \tag{235}$$

which is less than 1. According to this equation χ' is the rapidity of a reference particle with radial coordinate χ'.

Figure 9 shows the (η', χ')-system in a Minkowski diagram referring to the CFS system of the observer at $\chi' = 0$. It follows from equations (234) that the world lines of the reference particles with $\chi' =$ constant are straight lines, and the curves of the space $\eta' =$ constant are hyperbolae with centre at the origin as shown in the diagram in Figure 9.

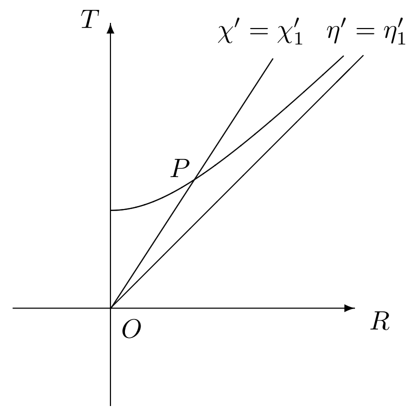

Figure 9. Minkowski diagram for the LBR spacetime with reference to the CFS coordinates (T, R). Here the line OP is the world line of a reference particle with $\chi' = \chi'_1$. The hyperbola represents a simultaneity curve $\eta' = \eta'_1$.

One rather sublime point should be noted. Since the LBR line element has similar coordinate expressions in (36) and (231), one might think that for instance the kinematics of free particles are identical in these coordinate systems. Calculating the acceleration of a free particle from the geodesic equation, one finds identical coordinate expressions for the Christoffel symbols. Hence it seems that the coordinate acceleration of a free particle at a given event is the same in the marked and unmarked coordinate systems. This is, however, not the case since the transformations (41) and (234) imply that the χ- and χ'-coordinates of an event with coordinates (T_1, R_1) are different.

Inserting the expression $e^{\alpha} = R_Q / \sinh \chi'$ from the line element (231) and the expression (233) for R, we find the line element of the flat spacetime inside the domain wall in the (η', χ') coordinates,

$$ds_M^2 = B^2 e^{2\eta'} (-d\eta'^2 + d\chi'^2 + \sinh^2 \chi' d\Omega^2) \ . \tag{236}$$

From equations (231), (233) with $R = R_Q$ and (236) it is seen that the metric is continuous at the domain wall.

Combining the transformations (233) and (116) we obtain the transformation

$$T = e^{A\tilde{t}/R_Q} \frac{B(\tilde{r}'_0 - \tilde{r}')}{\sqrt{(\tilde{r}' - \tilde{r}'_0)^2 - R_Q^2 A^2}} \ , \tag{237}$$

$$R = e^{A\tilde{t}'/R_Q} \frac{ABR_Q}{\sqrt{(\tilde{r}' - \tilde{r}'_0)^2 - R_Q^2 A^2}} \ , \tag{238}$$

with inverse transformation given by

$$B e^{A\tilde{t}'/R_Q} = \sqrt{T^2 - R^2} \quad , \quad \tilde{r}' = \tilde{r}'_0 - R_Q AT/R \ . \tag{239}$$

Combining the transformations (233) and (164) we obtain the transformation

$$T = B\,e^{A\hat{t}'/R_Q}\coth\left(\frac{\hat{r}'_0 - \hat{r}'}{R_Q}\right) \quad , \quad R = B\,e^{A\hat{t}'/R_Q}/\sinh\left(\frac{\hat{r}'_0 - \hat{r}'}{R_Q}\right) \ . \tag{240}$$

with inverse transformation given by

$$B\,e^{A\hat{t}'/R_Q} = \sqrt{T^2 - R^2} \quad , \quad \cosh\left(\frac{\hat{r}'_0 - \hat{r}'}{R_Q}\right) = \frac{T}{R} \ . \tag{241}$$

This transformation has earlier be considered by Zaslavskii [30].

V. Cylindrical coordinates.

We shall now consider an axially symmetric space using cylindrical coordinates ρ, θ, z, assuming that the line element has the form

$$ds^2 = R_Q^2\,[-f\,dT^2 + \tfrac{1}{f}(d\rho^2 + dz^2 + \rho^2 d\theta^2)\,] \tag{242}$$

where $f = f(\rho, z)$. Demanding that the Weyl tensor vanishes, we find that

$$f(\rho, z) = a(\rho^2 + z^2) + bz + c \tag{243}$$

where a, b and c are constants.

In general the energy momentum tensor has the following physical interpretation. Since the tensor is symmetrical, the eigenvectors of the tensor can be chosen to be orthonormal with one timelike and three spacelike vectors. These vectors will then represent an orthonormal basis that may be associated with an observer with four velocity equal to the timelike eigenvector $\mathbf{u} = \mathbf{e}_0$. The eigenvalue λ_0 is interpreted as the energy density measured by this observer, and the eigenvalues λ_i are interpreted as the stresses he measures. For $a = 1$, $b = c = 0$ the vectors of the observer's orthonormal basis are

$$\mathbf{e}_0 = \frac{1}{R_Q\sqrt{\rho^2+z^2}}\,\mathbf{e}_t \ , \ \mathbf{e}_1 = \frac{\sqrt{\rho^2+z^2}}{R_Q\,\rho}\,\mathbf{e}_\theta \ , \ \mathbf{e}_2 = \tfrac{1}{R_Q}(z\,\mathbf{e}_z + \rho\,\mathbf{e}_\rho) \ , \ \mathbf{e}_3 = \tfrac{1}{R_Q}(\rho\,\mathbf{e}_z - z\,\mathbf{e}_\rho) \ . \tag{244}$$

The corresponding eigenvalues of the energy momentum tensor are

$$\lambda_0 = \lambda_2 = -\frac{1}{\kappa R_Q^2} \quad , \quad \lambda_1 = \lambda_3 = \frac{1}{\kappa R_Q^2} \ . \tag{245}$$

These eigenvalues are recognized as those of the energy momentum tensor of an electrical field. The line element then takes the form

$$ds^2 = R_Q^2\,[-(\rho^2 + z^2)\,dT^2 + \tfrac{1}{\rho^2+z^2}(d\rho^2 + dz^2 + \rho^2 d\theta^2)\,] \ . \tag{246}$$

As shown by D. Garfinkle and E. N. Glass [25] this may also be found by transforming the line element (28) to cylindrical coordinates by means of

$$\rho = \frac{\sin\theta}{R} \quad , \quad z = \frac{\cos\theta}{R} \ , \tag{247}$$

or inversely

$$R = \frac{1}{\sqrt{\rho^2+z^2}} \quad , \quad \tan\theta = \frac{\rho}{z} \ , \tag{248}$$

Note that the charged domain wall defining the inner boundary of the WLBR spacetime according to our interpretation is now given by $\rho^2 + z^2 = R_Q^{-2}$.

From equations (23), (246) and (248) we find that in the cylinder coordinates the line element of the Minkowski spacetime inside the domain wall takes the form

$$ds_M^2 = -dT^2 + \frac{1}{(\rho^2+z^2)^2}\left(d\rho^2 + dz^2 + \rho^2 d\theta^2\right) \ . \tag{249}$$

It follows from equations (246), (248) with $R = R_Q$ and (249) that the metric is continuous at the domain wall.

There is no acceleration of gravity in the reference frame in which these coordinates are comoving. The unusual form of the spatial part of the line element is a coordinate effect. The space is defined by T = constant just as in the CFS coordinate system. Hence it is a Euclidean space described by using non-standard coordinate measuring rods that are related to the standard rods by the transformation (247). In these coordinates the space looks like a curved, but conformally flat space. In Cartesian and spherical coordinates, respectively, this line element takes the form

$$ds^2 = -dT^2 + \frac{dx^2+dy^2+dz^2}{(x^2+y^2+z^2)^2} = -dT^2 + \frac{1}{r^4}\left(dr^2 + r^2 d\Omega^2\right) \ . \tag{250}$$

VI. Light-like coordinates.

In spherical coordinates the line element on a 2-sphere with radius R_Q has the form

$$ds_2^2 = R_Q^2 d\Omega^2 = R_Q^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{251}$$

One can project the spherical surface onto a plane by means of stereographic coordinates given by

$$\zeta = \sqrt{2}\, R_Q \tan\frac{\theta}{2}\ e^{i\phi} \quad , \quad \overline{\zeta} = \sqrt{2}\, R_Q \tan\frac{\theta}{2}\ e^{-i\phi} \tag{252}$$

with inverse transformation

$$\tan\frac{\theta}{2} = \frac{1}{R_Q}\sqrt{\frac{\zeta\overline{\zeta}}{2}} \quad , \quad \cos 2\phi = \mathrm{Re}\left(\frac{\zeta}{\overline{\zeta}}\right) \ . \tag{253}$$

Taking the differentials and inserting into equation (251) we find the line element of the 2-sphere parametrized with the stereographic coordinate ζ representing two real coordinates, i.e. the real and imaginary part of ζ,

$$ds_2^2 = \frac{8R_Q^4\, d\zeta\, d\overline{\zeta}}{(2R_Q^2 + \zeta\overline{\zeta})^2} \ , \tag{254}$$

where $\overline{\zeta}$ is the complex conjugate of ζ. In terms of these coordinates and the light cone coordinates U and V the line element of the LBR spacetime takes the form

$$ds^2 = -\frac{8R_Q^4\, dU\, dV}{(2R_Q^2 + UV)^2} + \frac{8R_Q^4\, d\zeta\, d\overline{\zeta}}{(2R_Q^2 + \zeta\overline{\zeta})^2} \ . \tag{255}$$

This form of the line element has earlier been considered by M. Ortaggio [26] and later mentioned by Ortaggio and Podolský [27] and Griffiths and Podolský [22]. The coordinate transformation between the light cone coordinates (U, V) and the CFS coordinates is

$$T = \frac{B}{2}\left(\frac{\sqrt{2}\,R_Q + V}{\sqrt{2}\,R_Q - V} - \frac{\sqrt{2}\,R_Q - U}{\sqrt{2}\,R_Q + U}\right) \ , \tag{256}$$

$$R = \frac{B}{2}\left(\frac{\sqrt{2}\,R_Q + V}{\sqrt{2}\,R_Q - V} + \frac{\sqrt{2}\,R_Q - U}{\sqrt{2}\,R_Q + U}\right) \ . \tag{257}$$

The inverse transformation is

$$U = -\sqrt{2}\,R_Q\,\frac{T - R + B}{T - R - B} \ , \tag{258}$$

$$V = \sqrt{2}\,R_Q\,\frac{T + R - B}{T + R + B} \ . \tag{259}$$

Introducing coordinates

$$\tilde{u} = -\frac{R_Q^2}{U} \quad , \quad \tilde{v} = -\frac{4R_Q^2 U}{2R_Q^2 + UV} \ , \tag{260}$$

the line element (255) takes the form

$$ds^2 = -\frac{\tilde{v}^2}{R_Q^2}\,d\tilde{u}^2 - 2\,d\tilde{u}\,d\tilde{v} + \frac{8R_Q^4\,d\zeta\,d\bar{\zeta}}{(2R_Q^2 + \zeta\bar{\zeta})^2} \tag{261}$$

which has earlier been studied by Podolský and Ortaggio [17].

The inverse of the transformation (260) is

$$U = -\frac{R_Q^2}{\tilde{u}} \quad , \quad V = 2\left(\tilde{u} - \frac{2R_Q^2}{\tilde{v}}\right) \ . \tag{262}$$

Inserting these expressions into the equations (256) and (257), we obtain the transformation between the $(\tilde{u}, \tilde{v})$ coordinates and the CFS coordinates,

$$T = \frac{B[(\sqrt{2}\,\tilde{u} + R_Q)(\sqrt{2}\,\tilde{u} - R_Q)\tilde{v} - 4R_Q^2\tilde{u}]}{(\sqrt{2}\,\tilde{u} - R_Q)[2\sqrt{2}\,R_Q^2 - (\sqrt{2}\,\tilde{u} - R_Q)\tilde{v}]} \ , \tag{263}$$

$$R = \frac{2\sqrt{2}\,BR_Q^3}{(\sqrt{2}\,\tilde{u} - R_Q)[2\sqrt{2}\,R_Q^2 - (\sqrt{2}\,\tilde{u} - R_Q)\tilde{v}]} \ . \tag{264}$$

Inserting $T = -Bt/R_Q$, $R = -Br/R_Q$, $\tilde{u} = -u/2$ and $\tilde{v} = -2v$ in equations (263) and (264) gives the equations immediately preceding (A1) in the appendix of reference [17]. The inverse transformation is found by inserting the exressions (258) and (259) for U and V into equation (260), giving

$$\tilde{u} = \frac{R_Q(T - R - B)}{\sqrt{2}\,(T - R + B)} \ , \tag{265}$$

$$\tilde{v} = \frac{R_Q[R^2 - (T + B)^2]}{\sqrt{2}\,BR} \ . \tag{266}$$

5. A Milne-LBR universe model

We consider the flat spacetime inside the domain wall. In the (η', χ')-system the line element is given by (236). Introducing the proper time τ' of the reference particles in the (η', χ')-system as a time coordinat we have

$$d\tau' = Be^{\eta'} d\eta' \ . \tag{267}$$

Integrating with the initial condition $\tau'(0) = 0$, we obtain

$$\tau' = Be^{\eta'} \ , \tag{268}$$

and the line element of the flat spacetime inside the domain wall takes the form

$$ds_M^2 = -d\tau'^2 + a(\tau')^2 (d\chi'^2 + \sinh^2 \chi' d\Omega^2) \tag{269}$$

where the scale factor is

$$a(\tau') = \tau' \ . \tag{270}$$

This line element represents the Milne spacetime, which is simply the flat Minkowski spacetime as described from a uniformly expanding reference frame. The coordinates (τ', χ') will here be call the Milne coordinates. The transformation between the CFS coordinates and the Milne coordinates is obtained from equation (233) with the substitution $Be^{\eta'} = \tau'$, giving

$$T = \tau' \cosh \chi' \quad , \quad R = \tau' \sinh \chi' \ . \tag{271}$$

The inverse transformation is

$$\tau' = \sqrt{T^2 - R^2} \quad , \quad \tanh \chi' = \frac{R}{T} \ . \tag{272}$$

In these coordinates the line element of the WLBR spacetime outside the domain wall takes the form

$$ds^2 = \frac{R_Q^2}{\sinh^2 \chi'} \left(-\frac{d\tau'^2}{\tau'^2} + d\chi'^2 \right) + R_Q^2 d\Omega^2 \ , \tag{273}$$

It follows from the transformation (272) that the world lines of particles with $\chi' =$ constant are straight lines both inside and outside the domain wall as illustrated in Figure 9. Imagine observers with constant value of χ'. The coordinate time τ' both inside and outside the domain wall is equal to the proper time of these observers. Note from the form of the line element (273) that τ' is not equal to the proper time of standard clocks with $\chi' =$ constant outside the domain wall. The rate of the proper time of these clocks is given by

$$d\tau'_W = \frac{R_Q}{\tau' \sinh \chi'} d\tau' \ . \tag{274}$$

This formula shows that a standard clock outside the domain wall with $\chi' =$ constant goes at an increasingly slower rate than a standard clock inside the domain wall. The reason is that the clocks with $\chi' =$ constant move in the outwards direction in the static CFS system. This does not change the rate of the clocks inside the domain wall because they have constant velocity and there is no gravitational field in this region. However, outside the domain wall there is an outwards directed gravitational field. Hence a clock with constant χ' comes lower in this field, and therefore its rate decreases.

We will now consider clocks in the WLBR region with a fixed physical distance from the domain wall, $R = R_1$. It follows from the transformation (271) that the χ' coordinate of this clock is given by $\tau' \sinh \chi' = R_1$. Hence equation (274) shows that these clocks go at a constant rate.

Observers comoving with the reference particles inside the domain wall, $\chi' =$ constant, will observe that the domain wall collapses towards them. The physical distance from an observer at the origin to an object with coordinate χ' is

$$l = a(\tau')\chi' \ . \tag{275}$$

The physical velocity of the object relative to the observer is

$$\dot{l} = \dot{a}\chi' + a\dot{\chi}' = Hl + a\dot{\chi}' \ , \tag{276}$$

where $H = \dot{a}/a$ is the Hubble parameter. The first term is the velocity of the Hubble flow as given by Hubble's law, i.e. in the present case the velocity of "the river of space" [28] in the Milne universe, and the second term represents the so-called peculiar velocity of the object, i.e. its velocity through space.

The physical velocity of the domain wall is found by inserting $R = R_Q$ in the second of the transformation equations (271), which gives

$$\sinh \chi' = R_Q/\tau' \ . \tag{277}$$

Hence the coordinate velocity of the domain wall is

$$\dot{\chi}' = -\frac{R_Q/\tau'}{\sqrt{R_Q^2 + \tau'^2}} \ , \tag{278}$$

and its physical velocity is

$$\dot{l}_Q = \text{arcsinh}(R_Q/\tau') - \frac{R_Q}{\sqrt{R_Q^2 + \tau'^2}} \ . \tag{279}$$

Surprisingly the domain wall has a non-vanishing physical velocity in the Milne universe inside the wall which is even infinitely great initially, and then decreases to zero in an infinitely far future. Hence the Hubble flow dominates over the peculiar motion all the time. Integrating with the initial condition $l(0) = 0$ we find the physical distance from the observer at the center to the domain wall,

$$l = \tau' \text{arcsinh}(R_Q/\tau') = \tau'\chi' \ . \tag{280}$$

The chosen initial condition is necessary in order to obtain a result in accordance with the expression for l in equation (275). Taking the limit when $\tau' \to \infty$ we find that the final distance of the domain wall from the observer at the center is $l = R_Q$.

The physical velocity of the domain wall in the CFS system inside the wall vanishes. Hence, as described by an observer at the center of these coordinates, which coincides with the center of the Milne coordinates, the domain wall is at rest. Since we talk about physical velocity and physical distance one might think that these quantities should be coordinate invariant. The reason that this is not so, is that the spaces of the CFS system and the Milne universe are different simultaneity spaces.

6. The Killing vector field defining the motion of the reference frames

The LBR spacetime has a timelike Killing vector field which is most easily seen in the coordinate systems in which the metric is static. Then the timelike coordinate basis vector is a timelike Killing vector [29].

From the line element (36) it follows that $\mathbf{K} = \mathbf{e}_\eta = \partial/\partial\eta$ is a Killing vector. In order to make it explicit that there are three different cases we define unit vectors in the direction of $\mathbf{K}$ by

$$\mathbf{V} = \frac{\mathbf{K}}{\sqrt{K_\mu K^\mu}} = \frac{\mathbf{K}}{\sqrt{-g_{\eta\eta}}} = \frac{S_k(\chi)}{R_Q}\,\mathbf{e}_\eta \ . \tag{281}$$

These timelike unit vectors $\mathbf{V}$ can be interpreted as the 4-velocity of reference particles following trajectories of the Killing vector field $\mathbf{K}$, i.e. it is the 4-velocity of the reference particles defining the reference frame in which the (η, χ)-coordinates are comoving. The vectors $\mathbf{V}$ given in equation (281) for $k = -1, 0, 1$ may be distinguished by the magnitude of the 4-accelerations of the particles. The 4-acceleration of a particle with a world line having $\mathbf{V}$ as a unit tangent vector is

$$\mathbf{A} = A^\chi \mathbf{e}_\chi = V^\chi_{;\nu} V^\nu \mathbf{e}_\chi = (V^\chi_{,\nu} V^\nu + \Gamma^\chi_{\ \alpha\beta}\, V^\alpha V^\beta)\,\mathbf{e}_\chi \ . \tag{282}$$

Since the only non-vanishing component of $\mathbf{V}$ is V^η, this expression reduces to

$$\mathbf{A} = \Gamma^\chi_{\ \eta\eta}\,(V^\eta)^2 \mathbf{e}_\chi \ . \tag{283}$$

Using the expression (57) for the Christoffel symbol we obtain

$$\mathbf{A} = -I_k(\chi)\,\frac{S_k(\chi)^2}{R_Q^2}\,\mathbf{e}_\chi = -\frac{S_k(2\chi)}{2R_Q^2}\,\mathbf{e}_\chi \ . \tag{284}$$

The square of the acceleration scalar of this reference particle is

$$A^2 = A_\mu A^\mu = \frac{C_k(\chi)^2}{R_Q^2} \ . \tag{285}$$

The physical meaning of the acceleration scalar of an arbitrary particle is that it represents the ordinary acceleration of the particle as measured with standard clocks and measuring rods relative to a local inertial frame in which the particle is instantaneously at rest. In other words it represents the acceleration of the particle relative to a free particle. This is called the *proper acceleration* of the particle and will here be denoted by A_k for reasons that will be apparent below. It follows that the acceleration of gravity as defined in equation (33) is equal to minus the proper acceleration of the reference particles defining the motion of a reference frame.

The acceleration of a free particle instantaneously at rest in the (η, χ)-system is given by equation (58). This acceleration is due to the non-inertial character of the reference frame. Hence the proper acceleration of a reference particle in the (η, χ)-system is given by

$$A_k = -\frac{C_k(\chi)}{R_Q} \tag{286}$$

for $k = 1, 0, -1$.

In the $(\tilde{t}, \tilde{r})$-system the Killing vector is $\mathbf{K} = (R_Q/A)\,\mathbf{e}_{\tilde{t}} = (R_Q/A)\,\partial/\partial\tilde{t}$. Using the transformation (116) and the formula (A.13) we find that the proper acceleration of the reference particle is given by

$$A_k = \frac{\tilde{r} - \tilde{r}_0}{R_Q\sqrt{(\tilde{r} - \tilde{r}_0)^2 + kA^2R_Q^2}} \ . \tag{287}$$

This expression has earlier been deduced by Lapedes [21] with $\tilde{r}_0 = 0$ and $A = 1/R_Q$. We shall here use equation (287) to discuss the motion of the reference frame in which the (η, χ), $(\tilde{t}, \tilde{r})$ and $(\hat{t}, \hat{r})$ coordinate systems are comoving in the WLBR spacetime. The equation shows how the reference particles move in the radial direction. We first consider the case $k = 0$. Equation (134) implies that $\tilde{r} < \tilde{r}_0 - R_Q$ in the WLBR spacetime. Hence $\tilde{r} < \tilde{r}_0$ in this region. Equation (287) shows that in this case the reference particles have a constant acceleration $A_0 = -1/R_Q$ which is directed towards the domain wall relative to a free particle, with just the magnitude that keeps it at rest relative to the domain wall.

We then consider the case $k = 1$. At $\tilde{r} = \tilde{r}_0$ the reference particles have vanishing proper acceleration, i.e. they are freely falling. When $\tilde{r} < \tilde{r}_0$ in the region given by the inequalities (121) we then have $-1/R_Q \leq A_1 < 0$. This means that a reference particle in this region accelerates away from the domain wall, but with a smaller acceleration than that of a free particle. Hence in this region the reference frame accelerates inwards relative to a local inertial frame. When $\tilde{r} > \tilde{r}_0$ in the region given by the inequalities (121) we have that $0 < A_1 \leq 1/R_Q$, and the reference frame accelerates outwards relative to a local inertial reference frame.

Finally we consider the case $k = -1$. In this case the proper acceleration of the reference particles is directed towards the domain wall and has a magnitude greater than $1/R_Q$, i.e. greater than that of a free particle. Thus the reference frame accelerates towards the domain wall.

The proper acceleration of the reference particles depends upon k in the following way

$$|A_1| \leq 1/R_Q \quad , \quad |A_0| = 1/R_Q \quad \text{and} \quad |A_{-1}| \geq 1/R_Q \ . \tag{288}$$

This gives a physical meaning of the constant k. It tells whether the magnitude of the proper acceleration of the reference particles is smaller than, equal to, or greater than that of a free particle. If $k = 1$ the reference frame accelerates away from the domain wall, if $k = 0$ it is at rest relative to the domain wall, and if $k = -1$ it accelerates towards the domain wall.

Finally, in the $(\hat{t}, \hat{r})$-system the Killing vector is $\mathbf{K} = (R_Q/A)\,\mathbf{e}_{\hat{t}} = (R_Q/A)\,\partial/\partial\hat{t}$. From equation (286) and the transformation (156) we get

$$A_k = \mp\frac{1}{R_Q}\, b_k\!\left(\frac{\hat{r}_0 - \hat{r}}{R_Q}\right) \ , \tag{289}$$

which is consistent with the expression (174) for the acceleration of gravity in the $(\hat{t}, \hat{r})$-system.

7. Embedding of the LBR spacetime in a flat six-dimensional manifold

In order to exhibit the topological structure of the LBR spacetime Dias and Lemos [18] considered the embedding of the LBR spacetime in a flat six-dimensional manifold $M^{2,4}$.

We shall here show how LBR spacetime is parametrized in $M^{2,4}$ in the six main coordinate systems that we have considered in this paper. The coordinates in $M^{2,4}$ are denoted by $(z_0, z_1, z_2, z_3, z_4, z_5)$, and for $k = \pm 1$ the line element of $M^{2,4}$ has the form

$$ds^2 = -dz_0^2 + k\,dz_1^2 - k\,dz_2^2 + dz_3^2 + dz_4^2 + dz_5^2 \ . \tag{290}$$

Note that z_1 and z_2 are exchanged when k changes sign. The LBR 4-submanifold is determined by the two constraints

$$z_0^2 - k\,z_1^2 + k\,z_2^2 = R_Q^2 \ , \tag{291}$$

$$z_3^2 + z_4^2 + z_5^2 = R_Q^2 \ . \tag{292}$$

The first of these constraints defines the AdS_2 hyperboloid, and the second defines the 2-sphere of radius R_Q.

From equation (1) it follows that the spherical part of the LBR submanifold is invariant. Hence the parametrization of the 2-sphere takes the same form in all the coordinate systems,

$$z_3 = R_Q \sin\theta\cos\phi \quad , \quad z_4 = R_Q \sin\theta\sin\phi \quad , \quad z_5 = R_Q\cos\theta \ . \tag{293}$$

This satisfies the constraint (292) and gives the last three terms of equation (290).

We shall now consider the different parametrizations of the AdS hyperboloid satisfying the constraint (291) and giving the first three terms at the right hand side of equation (290) using the coordinate systems mentioned above. In CFS coordinates the parametrization takes the form

$$z_0 = R_Q\frac{T}{R} \quad , \quad z_1 = R_Q\frac{B^2 + k(T^2 - R^2)}{2BR} \quad , \quad z_2 = R_Q\frac{B^2 - k(T^2 - R^2)}{2BR} \ , \tag{294}$$

giving

$$-dz_0^2 + k\,dz_1^2 - k\,dz_2^2 = \frac{R_Q^2}{R^2}\left(-dT^2 + dR^2\right) \ . \tag{295}$$

Note that the cases $k = 1$ and $k = -1$ give the same parametrization, but with the coordinates z_1 and z_2 exchanged. A special case of this parametrization has earlier been considered by O. B. Zaslavskii [30].

From the transformation (41) it follows that with the (η, χ)-coordinates the parametrization of the AdS hyperboloid in $M^{2,4}$ takes the form

$$z_0 = R_Q\frac{S_k(\eta)}{S_k(\chi)} \quad , \quad z_1 = R_Q I_k(\chi) \quad , \quad z_2 = R_Q\frac{C_k(\eta)}{S_k(\chi)} \ . \tag{296}$$

Using equations (A.11), (A.13), (A.32) and (A.33) one may show that this parametrization fullfills equation (290) and the constraint (291).

Using the transformation (88) we find that the parametrization that transforms between the line element (82) with (τ, ρ)-coordinates and the first three terms of (290) with $k = -1$ is

$$z_0 = R_Q\frac{\cosh\rho}{\cosh\tau} \quad , \quad z_1 = -R_Q\tanh\tau \quad , \quad z_2 = -R_Q\frac{\sinh\rho}{\cosh\tau} \ . \tag{297}$$

With the $(\tilde{t}, \tilde{r})$-coordinates equation (132) gives

$$z_0 = \left[kR_Q^2 + \left(\frac{\tilde{r} - \tilde{r}_0}{A} \right)^2 \right]^{1/2} S_k \left(\frac{A\,\tilde{t}}{R_Q} \right) \ , \tag{298}$$

$$z_1 = \frac{\tilde{r}_0 - \tilde{r}}{A} \ , \tag{299}$$

$$z_2 = \left[kR_Q^2 + \left(\frac{\tilde{r} - \tilde{r}_0}{A} \right)^2 \right]^{1/2} C_k \left(\frac{A\,\tilde{t}}{R_Q} \right) \ . \tag{300}$$

From equation (148) it follows that the parametrization with the $(\bar{t}, \bar{r})$-coordinates has the form

$$z_0 = \left[R_Q^2 - \left(\frac{\bar{t} - \bar{t}_0}{A} \right)^2 \right]^{1/2} \cosh \left(\frac{A\,\bar{r}}{R_Q} \right) \ , \tag{301}$$

$$z_1 = \frac{\bar{t}_0 - \bar{t}}{A} \ , \tag{302}$$

$$z_2 = -\left[R_Q^2 - \left(\frac{\bar{t} - \bar{t}_0}{A} \right)^2 \right]^{1/2} \sinh \left(\frac{A\,\bar{r}}{R_Q} \right) \ , \tag{303}$$

corresponding to $k = -1$ in equations (290) and (291). With $(\hat{t}, \hat{r})$-coordinates equation (169) leads to the parametrization

$$z_0 = R_Q\, a_k\!\left(\frac{\hat{r}_0 - \hat{r}}{R_Q} \right) S_k \left(\frac{A\,\hat{t}}{R_Q} \right) \ , \tag{304}$$

$$z_1 = R_Q\, a_{-k}\!\left(\frac{\hat{r}_0 - \hat{r}}{R_Q} \right) \ , \tag{305}$$

$$z_2 = R_Q\, a_k\!\left(\frac{\hat{r}_0 - \hat{r}}{R_Q} \right) C_k \left(\frac{A\,\hat{t}}{R_Q} \right) \ . \tag{306}$$

In order to show that this parametrization fullfills the constraint (291), one has to use equation (A.35). Equation (193) leads to the following parametrization in (t, r)-coordinates,

$$z_0 = R_Q \cos \left(\frac{t - t_0}{R_Q} \right) \cosh \left(\frac{A\,r}{R_Q} \right) \ , \tag{307}$$

$$z_1 = -R_Q \sin \left(\frac{t - t_0}{R_Q} \right) \ , \tag{308}$$

$$z_2 = -R_Q \cos \left(\frac{t - t_0}{R_Q} \right) \sinh \left(\frac{A\,r}{R_Q} \right) \ , \tag{309}$$

again corresponding to $k = -1$ in equations (290) and (291).

The parametrization of the AdS hyperboloid in (η', χ')-coordinates takes the form

$$z_0 = R_Q \coth \chi' \quad , \quad z_1 = kR_Q \frac{a_k(\eta')}{\sinh \chi'} \quad , \quad z_2 = kR_Q \frac{a_{-k}(\eta')}{\sinh \chi'} \ . \tag{310}$$

With $(\tilde{t}', \tilde{r}')$-coordinates the parametrization of the AdS hyperboloid is

$$z_0 = \frac{\tilde{r}'_0 - \tilde{r}'}{A} \ , \tag{311}$$

$$z_1 = k\left[\left(\frac{\tilde{r}' - \tilde{r}'_0}{A}\right)^2 - R_Q^2\right]^{1/2} a_k\left(\frac{A\,\tilde{t}'}{R_Q}\right) \ , \tag{312}$$

$$z_2 = k\left[\left(\frac{\tilde{r}' - \tilde{r}'_0}{A}\right)^2 - R_Q^2\right]^{1/2} a_{-k}\left(\frac{A\,\tilde{t}'}{R_Q}\right) \ . \tag{313}$$

With $(\hat{t}', \hat{r}')$-coordinates the parametrization is

$$z_0 = R_Q \cosh\left(\frac{\hat{r}'_0 - \hat{r}'}{R_Q}\right) \ , \tag{314}$$

$$z_1 = kR_Q \sinh\left(\frac{\hat{r}'_0 - \hat{r}'}{R_Q}\right) a_k\left(\frac{A\,\hat{t}'}{R_Q}\right) \ , \tag{315}$$

$$z_2 = kR_Q \sinh\left(\frac{\hat{r}'_0 - \hat{r}'}{R_Q}\right) a_{-k}\left(\frac{A\,\hat{t}'}{R_Q}\right) \ . \tag{316}$$

We shall now consider the case $k = 0$. Then the line element of $M^{2,4}$ has the form

$$ds^2 = -dz_0^2 - dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 + dz_5^2 \ . \tag{317}$$

The LBR 4-submanifold is determined by the constraint (292) and

$$z_0^2 + z_1^2 - z_2^2 = R_Q^2 \ . \tag{318}$$

The (η, χ)-system with $k = 0$ coincides with the CFS system, and the line element takes the form (28). In this case the parametrization of the AdS hyperboloid is given by (294) with $T = \eta$, $R = \chi$ and $k = -1$,

$$z_0 = R_Q \frac{\eta}{\chi} \quad , \quad z_1 = R_Q \frac{B^2 - (\eta^2 - \chi^2)}{2B\chi} \quad , \quad z_2 = R_Q \frac{B^2 + (\eta^2 - \chi^2)}{2B\chi} \ . \tag{319}$$

The reason why we have inserted $k = -1$ instead of $k = 0$ is that the case $k = 0$ concerns the type of coordinate system which we consider, while the $k = -1$ value in equation (294) concerns the parametrization.

These parametrizations of the AdS hyperboloid in the $M^{2,4}$ manifold makes it clear that the line elements (28), (36), (82), (114), (140), (165) and (189) describe the same LBR spacetime.

Ortaggio and Podolský [27] have used the following embedding parametrization of the LBR spacetime in the coordinates introduced in section 4.VI,

$$z_0 = \sqrt{2}R_Q^2 \frac{U + V}{2R_Q^2 + UV} \quad , \quad z_1 = \sqrt{2}R_Q^2 \frac{U - V}{2R_Q^2 + UV} \quad , \quad z_2 = R_Q \frac{2R_Q^2 - UV}{2R_Q^2 + UV} \ , \tag{320}$$

$$z_3 = \sqrt{2}\,R_Q^2 \frac{\zeta + \bar{\zeta}}{2R_Q^2 + \zeta\bar{\zeta}} \quad , \quad z_4 = -i\sqrt{2}\,R_Q^2 \frac{\zeta - \bar{\zeta}}{2R_Q^2 + \zeta\bar{\zeta}} \quad , \quad z_5 = R_Q \frac{2R_Q^2 - \zeta\bar{\zeta}}{2R_Q^2 + \zeta\bar{\zeta}} \ , \tag{321}$$

corresponding to $k = -1$ in equations (290) and (291). Note that the coordinates (U, V) in equations (320) and (321) are related to the coordinates (u, v) used in equations (1) -

(5) in reference [27] by $U = u$ and $V = -v$.

8. Conclusion

The LBR solution of Einstein's field equations was found more than 90 years ago by T. Levi-Civita [4,5] and rediscovered in 1959 by B. Bertotti [6] and I. Robinson [7]. The solution was interpreted physically as a spacetime with an electric or a magnetic field with constant field strength. However the source of the electrical field remained rather obscure. We recently used Israel's formalism [31] for describing singular shells in general relativity to investigate the physical properties of a shell with LBR spacetime outside the shell and flat spacetime inside it, and found [14] that the source then had to be a charged domain wall with a radius equal to the distance corresponding to its charge. From equation (44) in reference [14] we see that the radius of the shell is one half of its Schwarzschild radius.

We have found different coordinate representations of the LBR spacetime by taking a general form (1) of a spherically symmetric line element as our point of departure, permitting the metric functions to depend upon the radial and the time coordinate. The differential equation (3) obtained from the requirement that the spacetime is conformally flat, i.e. that the Weyl curvature tensor vanishes, was then solved under different coordinate conditions. Remarkably, with the general form (1) of the line element and the requirement that the Weyl tensor vanishes, Einstein's field equations restrict the energy-momentum tensor to be of a form (4) representing a constant electric or magnetic field. In the present article we have only discussed the case of an electric field.

Next we have given a general prescription for finding coordinate transformations between the "canonical" CFS coordinate system in which the line element of the LBR spacetime is equal to a conformal factor times the Minkowski line element, and the coordinate representations obtained by the method based on solving equation (1). In sections 4 and 6 of this article we have given a detailed discussion of the kinematical properties of the reference frames both outside and inside the domain wall, in which the coordinate systems are comoving.

We have found that in several coordinate systems there are three cases which we have parameterized by the constant k having the values 1, 0 or -1. The corresponding reference frames have different motions. In the case $k = 0$ the (η, χ) coordinate system is comoving in the same referenc frame as that of the CFS coordinates. The domain wall at $R = R_Q$ of the WLBR spacetime is static in this reference frame, and the acceleration of gravity is constant and equal to $1/R_Q$. In the case $k = 1$ the (η, χ) coordinate system is comoving with a reference frame that accelerates away from the domain wall in the WLBR spacetime. Then the acceleration of gravity is smaller that that in the static case $(k = 0)$, and even directed towards the domain wall for $R > \sqrt{B^2 + T^2}$. In the case $k = -1$ the (η, χ) coordinate system is comoving with a reference frame that accelerates towards the domain wall in the WLBR spacetime. Hence observers in this reference frame will experience an acceleration of gravity directed away from the domain wall larger than $1/R_Q$.

In section 5 we have presented a Milne-LBR universe model with a part of the Milne universe inside the domain wall and an infinitely extended LBR spacetime outside it.

Finally we have considered embedding of the LBR spacetime in a flat, 6-dimensional manifold, $M^{2,4}$, and deduced the parameterizations of this embedding for the six main coordinate systems considered in the present article.

Appendix A. Calculus of k-functions

In this appendix we shall define functions which we call k-functions and deduce their main properties. Motivated by the angular part of the Robertson-Walker line element in standard coordinates it is natural to introduce the function

$$S_k(x) = \begin{cases} \sin x & \text{for} \quad k=1 \\ x & \text{for} \quad k=0 \\ \sinh x & \text{for} \quad k=-1 \end{cases} . \tag{A.1}$$

In the present paper we shall need several functions of similar type defined by

$$C_k(x) = \begin{cases} \cos x & \text{for} \quad k=1 \\ 1 & \text{for} \quad k=0 \\ \cosh x & \text{for} \quad k=-1 \end{cases} , \tag{A.2}$$

$$T_k(x) = \begin{cases} \tan x & \text{for} \quad k=1 \\ x & \text{for} \quad k=0 \\ \tanh x & \text{for} \quad k=-1 \end{cases} \tag{A.3}$$

and

$$I_k(x) = \begin{cases} \cot x & \text{for} \quad k=1 \\ 1/x & \text{for} \quad k=0 \\ \coth x & \text{for} \quad k=-1 \end{cases} . \tag{A.4}$$

Motivated by the scale factor of the DeSitter line element we also introduce

$$a_k(x) = \begin{cases} \cosh x & \text{for} \quad k=1 \\ e^x & \text{for} \quad k=0 \\ \sinh x & \text{for} \quad k=-1 \end{cases} \tag{A.5}$$

and

$$b_k(x) = \begin{cases} \tanh x & \text{for} \quad k=1 \\ 1 & \text{for} \quad k=0 \\ \coth x & \text{for} \quad k=-1 \end{cases} . \tag{A.6}$$

Note that

$$b_k(x) = \frac{a_{-k}(x)}{a_k(x)} . \tag{A.7}$$

The series expansions for the function $S_k(x)$ and $C_k(x)$ are

$$S_k(x) = x + \sum_{k=1}^{\infty} \frac{(-k)^n}{(2n+1)!} x^{2n+1} \tag{A.8}$$

and

$$C_k(x) = 1 + \sum_{k=1}^{\infty} \frac{(-k)^n}{(2n)!} x^{2n} \ . \tag{A.9}$$

Furthermore

$$T_k(x) = \frac{S_k(x)}{C_k(x)} \quad , \quad I_k(x) = \frac{C_k(x)}{S_k(x)} \tag{A.10}$$

and

$$C_k(x)^2 + kS_k(x)^2 = 1 \ , \tag{A.11}$$

which implies that

$$1 + kT_k(x)^2 = C_k(x)^{-2} \tag{A.12}$$

and

$$I_k(x)^2 + k = S_k(x)^{-2} \quad , \quad C_k(x)^2 = \frac{I_k(x)^2}{I_k(x)^2 + k} \ . \tag{A.13}$$

We have the following addition formulae

$$S_k(x+y) = S_k(x)\, C_k(y) + C_k(x) S_k(y) \ , \tag{A.14}$$

$$C_k(x+y) = C_k(x)\, C_k(y) - kS_k(x) S_k(y) \ , \tag{A.15}$$

$$T_k(x+y) = \frac{T_k(x) + T_k(y)}{1 - kT_k(x)T_k(y)} \tag{A.16}$$

and

$$I_k(x+y) = \frac{I_k(x)I_k(y) - k}{I_k(x) + I_k(y)} \ . \tag{A.17}$$

With $y = x$ this gives

$$S_k(2x) = 2S_k(x)\, C_k(x) \ , \tag{A.18}$$

$$C_k(2x) = C_k(x)^2 - kS_k(x)^2 \ , \tag{A.19}$$

$$T_k(2x) = \frac{2T_k(x)}{1 - kT_k(x)^2} \tag{A.20}$$

and

$$I_k(2x) = \frac{I_k(x)^2 - k}{2I_k(x)} \ . \tag{A.21}$$

From equations (A.16) and (A.17) we also obtain

$$T_k^{-1}(x) + T_k^{-1}(y) = T_k^{-1}\left(\frac{x+y}{1-kxy}\right) \tag{A.22}$$

and

$$I_k^{-1}(x) + I_k^{-1}(y) = I_k^{-1}\left(\frac{xy-k}{x+y}\right) \ . \tag{A.23}$$

Furthermore

$$S_k(-x) = -S_k(x) \quad , \quad C_k(-x) = C_k(x) \tag{A.24}$$

and

$$T_k(-x) = -T_k(x) \quad , \quad I_k(-x) = -I_k(x) \ . \tag{A.25}$$

Combining equations (A.14) - (A.17) we also have that

$$S_k(x) + S_k(y) = 2S_k(\tfrac{x+y}{2})\,C_k(\tfrac{x-y}{2}) \ , \tag{A.26}$$

$$S_k(x) - S_k(y) = 2S_k(\tfrac{x-y}{2})\,C_k(\tfrac{x+y}{2}) \ , \tag{A.27}$$

$$C_k(x) + C_k(y) = 2\,C_k(\tfrac{x+y}{2})\,C_k(\tfrac{x-y}{2}) \ , \tag{A.28}$$

$$C_k(x) - C_k(y) = -2kS_k(\tfrac{x+y}{2})\,S_k(\tfrac{x-y}{2}) \tag{A.29}$$

and

$$T_k(x/2) = \frac{S_k(x)}{1+C_k(x)} \quad , \quad I_k(x/2) = \frac{1+C_k(x)}{S_k(x)} \ . \tag{A.30}$$

Using (A.10), (A.18) and (A.19) we obtain

$$I_k(2x) = \frac{C_k(2x)}{S_k(2x)} = \frac{C_k(x)^2 - kS_k(x)^2}{2S_k(x)\,C_k(x)} = \frac{1}{2}\left[\,I_k(x) - k\,I_k(x)^{-1}\right] \ . \tag{A.31}$$

The derivatives of the k-functions are

$$S'_k(x) = C_k(x) \quad , \quad C'_k(x) = -kS_k(x) \tag{A.32}$$

and

$$T'_k(x) = C_k(x)^{-2} \quad , \quad I'_k(x) = -S_k(x)^{-2} \ . \tag{A.33}$$

The following identities will also be needed

$$|S_k(I_k^{-1}(x))| = \frac{1}{\sqrt{x^2+k}} \quad , \quad C_k(I_k^{-1}(x)) = xS_k(I_k^{-1}(x)) \ . \tag{A.34}$$

From the definition (A.5) it follows that

$$a_k(x)^2 - a_{-k}(x)^2 = k \tag{A.35}$$

and

$$a'_k(x) = a_{-k}(x) \ . \tag{A.36}$$

Appendix B. From generating functions to transformations

We shall here show how the transformation (39) is deduced from the generating function (37). From equation (7) with $g = f$, $x^0 = \eta$ and $x^1 = \chi$ and using equations (A.10), (A.26) and (A.28) it follows that

$$T + R = f(\eta+\chi) = B\,T_k\Big(\frac{\eta+\chi}{2}\Big) = B\,\frac{2\,S_k\Big(\frac{\eta+\chi}{2}\Big)\,C_k\Big(\frac{\eta-\chi}{2}\Big)}{2\,C_k\Big(\frac{\eta+\chi}{2}\Big)\,C_k\Big(\frac{\eta-\chi}{2}\Big)} = B\,\frac{S_k(\eta)+S_k(\chi)}{C_k(\eta)+C_k(\chi)} \ . \tag{B.1}$$

In the same way we find

$$T - R = B\,\frac{S_k(\eta)-S_k(\chi)}{C_k(\eta)+C_k(\chi)} \ , \tag{B.2}$$

which gives the transformation (39).

Next we show how the transformation (41) is deduced from the generating function (40). Using this generating function and equation (7), replacing T by η, x^0 by T and x^1 by R, we obtain

$$\eta = \frac{1}{2}\left[f(T+R) + f(T-R)\right] = T_k^{-1}\left(\tfrac{T+R}{B}\right) + T_k^{-1}\left(\tfrac{T-R}{B}\right) \ . \tag{B.3}$$

From equation (A.22) it then follows that

$$\eta = T_k^{-1}\left(\frac{2BT}{B^2 - k(T^2 - R^2)}\right) \ . \tag{B.4}$$

Hence we obtain the first of equations (41). The second is found in the same way.

We shall now deduce the transformation (86) between the (τ,ρ)-koordinates and the CFS coordinates. From equation (7) with $x^0 = \tau$ and $x^1 = \rho$ and using the generating functions (85), it follows that

$$T + R = -B\,\coth\left(\tfrac{\tau+\rho}{2}\right) = -B\,\frac{2\,\cosh\left(\frac{\tau+\rho}{2}\right)\cosh\left(\frac{\tau-\rho}{2}\right)}{2\,\sinh\left(\frac{\tau+\rho}{2}\right)\cosh\left(\frac{\tau-\rho}{2}\right)} = -B\,\frac{\cosh\tau + \cosh\rho}{\sinh\tau + \sinh\rho} \ . \tag{B.5}$$

In the same way we find

$$T - R = B\,\frac{\cosh\tau - \cosh\rho}{\sinh\tau + \sinh\rho} \ , \tag{B.6}$$

which gives the transformation (86).

Next we show how the transformation (88) is deduced from the generating functions (87). Using these generating functions and equation (7), replacing T by τ, x^0 by T and x^1 by R, we obtain

$$\tau = \frac{1}{2}\left[f(T+R) + g(T-R)\right] = -\mathrm{arctanh}\left(\tfrac{B}{T+R}\right) + \mathrm{arctanh}\left(\tfrac{T-R}{B}\right) \ . \tag{B.7}$$

Hence we find

$$\tau = \mathrm{arctanh}\left(\frac{\frac{T-R}{B} - \frac{B}{T+R}}{1 - \left(\frac{T-R}{B}\right)\left(\frac{B}{T+R}\right)} \cdot \frac{B\,(T+R)}{B\,(T+R)}\right) = \mathrm{arctanh}\left(\frac{(T^2 - R^2) - B^2}{2BR}\right) \ . \tag{B.8}$$

Hence we obtain the first of equations (88). The second equation is found in the same way.

We shall now deduce the transformation (97) between the (τ,ρ)- and the (η,χ)-coordinates. From equation (7) with $x^0 = \tau$ and $x^1 = \rho$ and using the generating functions (96), it follows that

$$\eta = \frac{1}{2}\left[f(\tau+\rho) + g(\tau-\rho)\right] = -\arctan\left(\coth\tfrac{\tau+\rho}{2}\right) + \arctan\left(\coth\tfrac{\tau-\rho}{2}\right) \ , \tag{B.9}$$

which may be rewritten as

$$\eta = -\arctan\left(\frac{\coth\frac{\tau+\rho}{2} - \tanh\frac{\tau-\rho}{2}}{1 + \coth\frac{\tau+\rho}{2}\tanh\frac{\tau-\rho}{2}} \cdot \frac{\sinh\frac{\tau+\rho}{2}\cosh\frac{\tau-\rho}{2}}{\sinh\frac{\tau+\rho}{2}\cosh\frac{\tau-\rho}{2}}\right) \ . \tag{B.10}$$

Multiplication and using the addition formulae for hyperbolic functions give

$$\eta = -\arctan\left(\frac{\cosh\frac{\tau+\rho}{2}\cosh\frac{\tau-\rho}{2} - \sinh\frac{\tau+\rho}{2}\sinh\frac{\tau-\rho}{2}}{\sinh\frac{\tau+\rho}{2}\cosh\frac{\tau-\rho}{2} + \cosh\frac{\tau+\rho}{2}\sinh\frac{\tau-\rho}{2}}\right) = -\arctan\left(\frac{\cosh\rho}{\sinh\tau}\right) \ . \tag{B.11}$$

Hence we obtain the first of equations (97). The second equation is found in a similar way.

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