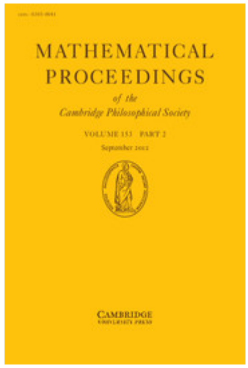

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Lagrangian subbundles of symplectic bundles over a curve

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Abstract

A symplectic bundle over an algebraic curve has a natural invariant s_{Lag} determined by the maximal degree of its Lagrangian subbundles. This can be viewed as a generalization of the classical Segre invariants of a vector bundle. We give a sharp upper bound on s_{Lag} which is analogous to the Hirschowitz bound on the classical Segre invariants. Furthermore, we study the stratifications induced by s_{Lag} on moduli spaces of symplectic bundles, and get a full picture for the case of rank four.

1. *Introduction*

Let X be a smooth projective curve of genus $g \geqslant 2$ over $\mathbb{C}$. Let G be a connected reductive algebraic group over $\mathbb{C}$, and $P \subset G$ a parabolic subgroup. Given a G-bundle V over X, we have an associated G/P-bundle $\pi : V/P \to X$. For a section σ of π, consider the normal bundle N_σ over $\sigma(X) \cong X$ in V/P. We define

$$s(V; P) := \min_{\sigma}\{\deg N_\sigma\},$$

where σ runs through all sections of V/P.

It is easy to show that $s(V; P) > -\infty$; see Holla–Narasimhan [**10**, lemma 2·1]. A section σ is called a *minimal section* of V if $\deg N_\sigma = s(V; P)$. According to Ramanathan [**20**, definition 2·13], V is a semistable G-bundle if and only if $s(V;\, P) \geqslant 0$ for every maximal parabolic subgroup P of G. In general, it can be said that the invariant $s(V;\, P)$ measures the degree of stability of V with respect to P. The invariant $s(-,\, P)$ is a lower semicontinuous function on any parameter space of G-bundles over X, so defines a stratification on the moduli space of semistable G-bundles over X.

The geometry of this stratification has been extensively studied in the case $G = \mathrm{GL}_n$. To review some of the results, first note that a topologically trivial GL_n-bundle V is nothing but a vector bundle of rank n and degree zero, which we also denote by V. For each $1 \leqslant r \leqslant n-1$, there is a maximal parabolic subgroup P_r of GL_n which is unique up to conjugation. A section σ of the associated Grassmannian bundle V/P_r gives a rank r subbundle E of V, and vice versa. Since $N_\sigma = \mathrm{Hom}\,(E,\, V/E)$, we get

$$s(V;\, P_r) = \min\{-n \deg E \;:\; E \subset V,\ \mathrm{rk}(E) = r\}.$$

This coincides with the well-known invariant $s_r(V)$ of a vector bundle V, sometimes called the *rth Segre invariant*. In the literature, a rank r subbundle E is called a *maximal subbundle* of V if $s_r(V) = -n \deg E$.

The first result on the invariant s_r is the upper bound

$$s_r(V) \leqslant g \cdot r(n-r).$$

This was obtained by Nagata [**17**] for $n = 2$ and by Mukai and Sakai [**16**] in general. Later, Hirschowitz obtained the following sharp bound [**6**, théorème 4·4]; see also [**4**]:

PROPOSITION 1·1. *For a bundle V as above, we have*

$$s_r(V) \leqslant r(n-r)(g-1) + \varepsilon,$$

where $0 \leqslant \varepsilon < n$ *and* $r(n-r)(g-1) + \varepsilon \equiv 0 \mod n$.

Next, let us recall the results on the stratification defined by $s_r(V)$. Let $\mathcal{U}(n)$ be the moduli space of semistable bundles over X of rank n and degree zero. For each integer s divisible by n, we define

$$\mathcal{U}(n;\, r, s) := \{V \in \mathcal{U}(n) \;:\; s_r(V) \leqslant s\}.$$

By Proposition 1·1, we have $\mathcal{U}(n;\, r, s) = \mathcal{U}(n)$ if $s \geqslant r(n-r)(g-1)$.

PROPOSITION 1·2. (Brambila–Paz–Lange [**3**], Russo–Teixidor i Bigas [**21**]). *For each integer k with $0 < k \leqslant k_0 := \lfloor r(n-r)(g-1)/n \rfloor$, the locus $\mathcal{U}(n;\, r, kn)$ is an irreducible closed subvariety of $\mathcal{U}(n)$. Also, for each $k \leqslant k_0$, we have*

$$\mathrm{codim}\, \mathcal{U}(n;\, r, kn) = r(n-r)(g-1) - kn.$$

Moreover, each variety $\mathcal{U}(n;\, r, kn)$ can be described by using the extension spaces of fixed type, see [**21**, theorem 0·1].

On the other hand, not much seems to have been studied on the properties of the invariant $s(V;\, P)$ of a G-bundle V in general, except the following universal upper bound on $s(V;\, P)$ obtained by Holla and Narasimhan [**10**, theorem 1·1]:

PROPOSITION 1·3. *Fix a parabolic subgroup P of G. For every G-bundle V, we have*

$$s(V;\, P) \leqslant g \cdot \dim(G/P). \tag{1·1}$$

Note that when $G = \mathrm{GL}_n$, this coincides with the Mukai–Sakai bound discussed above, which is not sharp. As in the vector bundle case, the invariant $s(V; P)$ induces a stratification on the moduli space of semistable principal G-bundles over X.

In this paper, we study the geometry of the stratification when $G = \mathrm{Sp}_{2n}$. We write $\mathcal{M}_{2n}$ for the moduli space of semistable principal Sp_{2n}-bundles over X (see Ramanathan [**20**]). A vector bundle W will be called *symplectic* if there exists a nondegenerate bilinear alternating form $\omega\colon W \otimes W \to \mathcal{O}_X$. Such a bundle always has even rank $2n$. It is easy to see that a vector bundle of rank $2n$ is symplectic if and only if the associated principal GL_{2n}-bundle admits a reduction of structure group to Sp_{2n}. There is a natural map from $\mathcal{M}_{2n}$ to the moduli space $\mathcal{SU}(2n, \mathcal{O}_X)$ of semistable bundles of rank $2n$ with trivial determinant, which is an embedding (Serman [**22**]). Henceforth, we identify $\mathcal{M}_{2n}$ with its image in $\mathcal{SU}(2n, \mathcal{O}_X)$.

A subbundle E of W is called *isotropic* if $\omega|_{E\otimes E} = 0$. By linear algebra, the rank of an isotropic subbundle is at most n; an isotropic subbundle of rank n is called a *Lagrangian* subbundle. A Lagrangian subbundle E of W corresponds to a section of the associated Sp_{2n}/P-bundle W/P, where P is the maximal parabolic subgroup of Sp_{2n} preserving a fixed Lagrangian subspace of $\mathbb{C}^{2n}$. Let us abbreviate $s(W; P)$ to $s_{\mathrm{Lag}}(W)$, where the subscript indicates "Lagrangian". Since Sp_{2n}/P is none other than the Lagrangian Grassmannian, the normal bundle of the section corresponding to E is given by $\mathrm{Sym}^2 E^*$. Recall that for any vector bundle V, we have

$$\deg(\mathrm{Sym}^2 V) = (\mathrm{rk}(V) + 1)\deg(V),$$

so

$$s_{\mathrm{Lag}}(W) = \min\{-(n+1)\deg E \;:\; E \text{ a Lagrangian subbundle of } W\}.$$

According to the bound (1·1), we get

$$s_{\mathrm{Lag}}(W) \leqslant \frac{n(n+1)}{2} g. \tag{1·2}$$

In other words, every symplectic bundle W admits a Lagrangian subbundle of degree at least $-(n/2)g$.

One may compare this with the Hirschowitz bound on the nth Segre invariant s_n, which says that as a vector bundle, W admits a subbundle of half rank with degree at least $-\lceil (n/2)(g-1)\rceil$. We prove that this slightly nicer bound is still valid for symplectic bundles.

THEOREM 1·4. *For every symplectic bundle W of rank $2n$, we have*

$$s_{\mathrm{Lag}}(W) \leqslant \frac{1}{2}\left(n(n+1)(g-1) + (n+1)\varepsilon\right),$$

where $\varepsilon \in \{0, 1\}$ is such that $n(g-1) + \varepsilon$ is even. This bound is sharp in the sense that the equality holds for a general bundle W in the moduli space $\mathcal{M}_{2n}$ of semistable symplectic bundles of rank $2n$ over X.

Next, consider the stratification on $\mathcal{M}_{2n}$ given by the invariant s_{Lag}. For each $k > 0$, let

$$\mathcal{M}_{2n}^k := \{W \in \mathcal{M}_{2n} \;:\; s_{\mathrm{Lag}}(W) \leqslant (n+1)k\}.$$

By semicontinuity, $\mathcal{M}_{2n}^k$ is a closed subvariety of $\mathcal{M}_{2n}$ and s_{Lag} induces a stratification on $\mathcal{M}_{2n}$. In particular when $n = 1$, since Sp_1 is isomorphic to SL_2, this reduces to the stratification already studied on the moduli space $\mathcal{SU}(2, \mathcal{O}_X)$ of semistable bundles of rank two with trivial determinant. We prove the following result on the stratification on the moduli space

$\mathcal{M}_4$ of semistable rank four symplectic bundles over X. Note that $\mathcal{M}_4^{g-1}$ is the whole space $\mathcal{M}_4$ by Theorem 1·4.

THEOREM 1·5. *For each e with* $1 \leqslant e \leqslant g-1$, *the locus* $\mathcal{M}_4^e$ *is an irreducible closed subvariety of* $\mathcal{M}_4$ *of dimension* $7(g-1)+3e$.

In the case of genus two, this was proven in [**9**]. The key ingredient of the proof there was a symplectic version of Lange and Narasimhan's description [**15**] of the Segre invariant using secant varieties (also see [**4**] for a higher rank version in the case of vector bundles). In this paper, we generalize the method and results of [**9**] to the case of arbitrary genus. We will see an interesting variant of Lange and Narasimhan's picture in the case of symplectic bundles: a relation between the invariant s_{Lag} and the higher secant varieties of certain fibre bundles over X.

This paper is organized as follows. In Section 2, we provide the basic setup for our discussion. In particular, a relation between the higher secant varieties and the Segre invariants will be established in Theorem 2·12. In Section 3, we prove Theorem 1·4 using this relation combined with the Terracini Lemma. In Section 4, we study symplectic bundles of rank four in more detail. We will see that the relation discussed in Section 2 can be improved to yield a nice picture in this case (Theorem 4·3). This enables us to prove Theorem 1·5. Finally a remark will be given on the comparison between the two stratifications defined by s_2 and s_{Lag}.

A variant of Hirschowitz's lemma is required in various places. To streamline the arguments, the proof of this lemma is postponed to the appendix.

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers.

2. *Symplectic extensions and lifting criteria*

In this section, we establish basic results on symplectic extensions.

2·1. *Symplectic extensions and symmetric principal parts*

Let $W \to X$ be a symplectic bundle and $E \subset W$ a subbundle. There is a natural short exact sequence

$$0 \longrightarrow E^{\perp} \longrightarrow W \longrightarrow E^* \longrightarrow 0,$$

where $E^{\perp}$ is the orthogonal complement of E with respect to the symplectic form on W. If E is a Lagrangian subbundle of W, then $E = E^{\perp}$.

Definition 2·1. An extension $0 \to E \to W \to E^* \to 0$ will be called *symplectic* if W admits a symplectic structure with respect to which the subbundle E is Lagrangian.

A symplectic extension defines a class

$$\delta(W) \in H^1(X, \mathrm{Hom}\,(E^*, E)) = H^1(X, E \otimes E).$$

Note that we have the decomposition

$$H^1(X, E \otimes E) \ \cong\ H^1(X, \mathrm{Sym}^2 E) \oplus H^1(X, \wedge^2 E).$$

LEMMA 2·2.

(i) *An extension* $0 \to E \to W \to E^* \to 0$ *is symplectic if and only if* W *is isomorphic as a vector bundle to an extension with class belonging to* $H^1(X, \mathrm{Sym}^2 E)$.

(ii) *If E is simple, then the extension W is symplectic if and only if $\delta(W)$ itself belongs to $H^1(X, \mathrm{Sym}^2 E)$.*

Proof. Due to S. Ramanan; see [**8**, Section 2] for a proof.

2·2. *Cohomological criterion for lifting*

We recall the notion of a *bundle-valued principal part* (see Kempf [**12**] for corresponding results on line bundles). For any bundle V over X, we have an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow V \longrightarrow \underline{\mathrm{Rat}}(V) \longrightarrow \underline{\mathrm{Prin}}(V) \longrightarrow 0 \tag{2·1}$$

where $\underline{\mathrm{Rat}}(V)$ is the sheaf of rational sections of V and $\underline{\mathrm{Prin}}(V)$ the sheaf of principal parts with values in V. We denote their groups of global sections by $\mathrm{Rat}(V)$ and $\mathrm{Prin}(V)$ respectively. The sheaves $\underline{\mathrm{Rat}}(V)$ and $\underline{\mathrm{Prin}}(V)$ are flasque, so we have the cohomology sequence

$$0 \longrightarrow H^0(X, V) \longrightarrow \mathrm{Rat}(V) \longrightarrow \mathrm{Prin}(V) \longrightarrow H^1(X, V) \longrightarrow 0. \tag{2·2}$$

We denote by $\overline{s}$ the principal part of $s \in \mathrm{Rat}(V)$, and we write $[p]$ for the class in $H^1(X, V)$ of $p \in \mathrm{Prin}(V)$. Any extension class in $H^1(X, V)$ is of the form $[p]$ for some $p \in \mathrm{Prin}(V)$, which is far from unique in general.

Now consider an extension of vector bundles

$$0 \longrightarrow E \longrightarrow W \longrightarrow E^* \longrightarrow 0 \tag{2·3}$$

and an elementary transformation F of E^* defined by the sequence

$$0 \longrightarrow F \xrightarrow{\mu} E^* \longrightarrow \tau \longrightarrow 0$$

for some torsion sheaf τ. We say that F *lifts to* W if there is a sheaf injection $F \to W$ such that the composition $F \to W \to E^*$ coincides with the elementary transformation μ. We quote two results from [**8**]:

LEMMA 2·3 ([**8**, corollary 3·5 and criterion 3·6]). *Suppose that $h^0(X, \mathrm{Hom}\,(E^*, E)) = 0$ and E is simple. Let W be an extension of type (2·3) with class $\delta(W) \in H^1(X, E \otimes E)$.*

(i) *There is a one-to-one correspondence between principal parts $p \in \mathrm{Prin}(E \otimes E)$ such that $[p] = \delta(W)$, and elementary transformations F of E^* lifting to W as a subbundle, given by $p \longleftrightarrow \mathrm{Ker}\,(p\colon E^* \to \underline{\mathrm{Prin}}(E))$.*

(ii) *Suppose $\delta(W) = [p]$ belongs to $H^1(X, \mathrm{Sym}^2 E)$, so W is a symplectic extension. Then the subbundle lifting from* $\mathrm{Ker}\,(p)$ *is isotropic if and only if ${}^t p = p$.*

It will be convenient to make the following definition:

Definition 2·4. The *degree* of a principal part $p \in \mathrm{Prin}(E \otimes E)$ is defined as the degree of the torsion sheaf $\mathrm{Im}(p : E^* \to \underline{\mathrm{Prin}}(E))$.

2·3. *Subvarieties of the extension spaces*

Let V be a vector bundle with $h^1(X, V) \neq 0$. We describe a rational map of the scroll $\mathbb{P}V$ into the projective space $\mathbb{P}H^1(X, V)$. Let $\pi : \mathbb{P}V \to X$ be the projection. We have the following sequence of identifications:

$$\begin{aligned} H^1(X, V) &\cong H^0(X, K_X \otimes V^*)^\vee \\ &\cong H^0(X, K_X \otimes \pi_* \mathcal{O}_{\mathbb{P}V}(1))^\vee \\ &\cong H^0(X, \pi_*\,(\pi^* K_X \otimes \mathcal{O}_{\mathbb{P}V}(1)))^\vee \\ &\cong H^0(\mathbb{P}V, \pi^* K_X \otimes \mathcal{O}_{\mathbb{P}V}(1))^\vee. \end{aligned}$$

Hence via the linear system $|\pi^* K_X \otimes \mathcal{O}_{\mathbb{P}V}(1)|$, we have a map $\varphi\colon \mathbb{P}V \dashrightarrow \mathbb{P}H^1(X, V)$. For $V = \mathrm{Hom}\,(E^*, E) = E \otimes E$, we get a map

$$\varphi\colon \mathbb{P}(E \otimes E) \dashrightarrow \mathbb{P}H^1(X, E \otimes E).$$

Definition 2·5. For each $x \in X$, we denote by $\Delta|_x$ the projectivization of the set of all rank one linear maps $E^*|_x \to E|_x$. The union

$$\Delta = \bigcup_{x \in X} (\Delta|_x)$$

is a fibre bundle inside the scroll $\mathbb{P}(E \otimes E)$, which we call the *decomposable locus*. Note that Δ has a fibre subbundle given as the image of the Veronese embedding $\mathbb{P}E \hookrightarrow \mathbb{P}(E \otimes E)$; clearly,

$$\mathbb{P}E \;=\; \Delta \,\cap\, \mathbb{P}\mathrm{Sym}^2 E.$$

If $v \in E$ is nonzero, we will abuse notation and denote also by v the point of $\mathbb{P}E$ corresponding to the vector v. By restricting φ to $\mathbb{P}E \subset \mathbb{P}\mathrm{Sym}^2 E$, we get

$$\psi : \mathbb{P}E \dashrightarrow \mathbb{P}H^1(X, \mathrm{Sym}^2 E).$$

Next, we give another way to define φ and ψ, which will be convenient in what follows. This idea was used by Kempf and Schreyer [**13**, Section 1] to define the canonical map $X \to |K_X|^\vee$, and generalized in [**9**, Section 3·3].

For each $x \in X$, we have an exact sequence of sheaves

$$0 \longrightarrow V \longrightarrow V(x) \longrightarrow \frac{V(x)}{V} \longrightarrow 0,$$

which is a subsequence of (2·1). Taking global sections, we obtain

$$0 \longrightarrow H^0(X, V) \longrightarrow H^0(X, V(x)) \longrightarrow V(x)|_x \longrightarrow H^1(X, V) \longrightarrow \cdots \qquad (2\cdot 4)$$

The restriction of the map $\varphi : \mathbb{P}V \dashrightarrow \mathbb{P}H^1(X, V)$ to the fibre $\mathbb{P}V|_x$ is identified with the projectivized coboundary map in (2·4). In view of (2·2), it can be identified with the map taking $v \in \mathbb{P}V$ to the cohomology class of a V-valued principal part supported at x with a simple pole in the direction of v. Similarly, the restriction $\psi : \mathbb{P}E \dashrightarrow \mathbb{P}H^1(X, \mathrm{Sym}^2 E)$ can be identified with the map taking $v \in E|_x$ to the cohomology class of a $\mathrm{Sym}^2 E$-valued principal part supported at x with a simple pole along $v \otimes v$.

LEMMA 2·6. *Suppose $g \geqslant 2$.*

(i) *If E is general in $\mathcal{U}(n, d)$ with $\mu(E) < -1$, then $\varphi\colon \mathbb{P}(E \otimes E) \dashrightarrow \mathbb{P}H^1(X, E \otimes E)$ is an embedding.*

(ii) *If E is general in $\mathcal{U}(n, d)$ with $\mu(E) \leqslant -1/2$, then φ is base-point free.*

(iii) *If $g \geqslant 3$ and E is general in $\mathcal{U}(2, d)$, $d < 0$, then $\psi : \mathbb{P}E \dashrightarrow \mathbb{P}H^1(X, \mathrm{Sym}^2 E)$ is an embedding.*

Proof. (i) For any divisor D on X, write $V(D)$ for the bundle $V \otimes \mathcal{O}_X(D)$. One can easily see that the map $\varphi : \mathbb{P}V \dashrightarrow \mathbb{P}H^1(X, V)$ is base point free if $H^0(X, V(x)) = 0$ for all $x \in X$, and an embedding if $H^0(X, V(D)) = 0$ for all effective divisors D of degree 2.

Since E is stable, $E \otimes E(D)$ is semistable. If $\mu(E) < -1$, then $E \otimes E(D)$ has negative degree for every $D \in \mathrm{Sym}^2 X$, and so $H^0(X, E \otimes E(D)) = 0$.

(ii) An argument similar to that in (i) shows that if $\mu(E) \leqslant -1/2$ and E is general, then $H^0(X, E \otimes E(x)) = 0$ for all $x \in X$.

(iii) By (i), φ is an embedding for $d \leqslant -2$. Now suppose $d = -1$. By (ii), we know that φ is base-point free, and hence so is ψ.

To show that ψ is an embedding it suffices to show that, for any $x, y \in X$:

(i) no two principal parts of the form

$$\frac{e \otimes e}{z} \quad \text{and} \quad \frac{f \otimes f}{w}$$

are identified in $H^1(X, \mathrm{Sym}^2 E)$ for any nonzero $e \in E|_x$ and $f \in E|_y$, where z and w are local coordinates at x and y respectively, and

(ii) the principal part

$$\frac{e \otimes e}{z^2}$$

is not cohomologically zero for any local section e nonzero at x.

The first statement is equivalent to saying that there does not exist a symmetric map

$$\alpha \colon E^* \longrightarrow E(x + y)$$

with principal part equal to

$$\frac{e \otimes e}{z} - \frac{f \otimes f}{w}.$$

Suppose there exists such an α. First, assume that α is generically injective. Note that any such α has rank one at x and y. Hence the non-zero section $\det(\alpha)$ of the line bundle

$$\mathrm{Hom}\,(\det E^*, \det E(2x + 2y)) \cong (\det E)^2(2x + 2y)$$

vanishes at x and y. Thus $(\det E)^2(x + y)$ is trivial, so $(\det E)^{-2}$ is effective. But if $g \geqslant 3$, a general $E \in \mathcal{U}(2, -1)$ does not have this property.

Next, assume that α is of generic rank 1. Then α factorizes as

$$E^* \longrightarrow M \longrightarrow E(x + y)$$

for some rank 1 sheaf M of degree m. The surjection $E^* \to M$ and the injection $M \to E(x + y)$ imply that E admits a line subbundle of degree at least

$$\max\{\deg M^*, \deg M(-x - y)\} \;=\; \max\{-m, m - 2\} \;\geqslant -1.$$

This shows that E is not general in $\mathcal{U}(2, -1)$ for $g \geqslant 3$ by a known property of the Segre invariant of E, which is analogous to Proposition 1·1: for a rank 2 bundle E of degree d, define

$$s_1(E) := \min\{d - 2 \deg L\}$$

where L runs through all the line subbundles of E. Then by Lange–Narasimhan [**15**, proposition 3·1], we have

$$g - 1 \leqslant s_1(E) \leqslant g \tag{2·5}$$

for a general $E \in \mathcal{U}(2, d)$. This shows in particular that for $g \geqslant 3$, a general $E \in \mathcal{U}(2, -1)$ does not admit a line subbundle of degree $\geqslant -1$.

Hence we conclude that there is no such symmetric map α. This proves the first statement. The second statement can be proven by a similar argument with $x = y$.

2·4. *Irreducibility of symmetric principal parts*

In this subsection, we prove a technical result on spaces of symmetric principal parts. This will be used later when we discuss relations between principal parts, symplectic extensions and secant varieties.

LEMMA 2·7. *For a fixed vector bundle E over X and for each $k > 0$, the space of symmetric principal parts in* $\mathrm{Prin}(E \otimes E)$ *of degree k is irreducible.*

Proof. A principal part of degree k is called *general* if it is supported at k distinct points of X. It is clear that the general symmetric principal parts in $\mathrm{Prin}(E \otimes E)$ of degree k are parameterized by a (quasi-projective) irreducible variety. Thus the irreducibility of all symmetric principal parts will follow once we show the following:

Every symmetric principal part $p \in$ $\mathrm{Prin}(E \otimes E)$ *of degree k is a limit (in the analytic topology) of a sequence of general symmetric principal parts of degree k.*

To show this, it suffices to consider the case where p is supported at a single point $x \in X$. In this case, we define the *order* of p at x to be the smallest integer m such that $\mathrm{Im}(p)$ is contained in $E(mx)/E$. Let z be a local coordinate at x and let $e_1, e_2, \ldots, e_n$ be a frame for E near x. Then p is locally expressed as

$$p = \frac{1}{z^m}\left(\psi_m + z \cdot \psi_{m-1} + \cdots + z^{m-1}\psi_1\right),$$

where $\psi_1, \psi_2, \ldots, \psi_m$ are symmetric tensors in $e_1, e_2, \ldots, e_n$.

We claim that after a suitable change of the local frame, every symmetric principal part p of degree k and order m can be expressed as

$$p = \sum_{i=1}^{r} p_i \quad \text{with} \quad p_i = \frac{e_i \otimes e_i}{z^{m_i}},$$

where $m = m_1 \geqslant m_2 \geqslant \cdots \geqslant m_r \geqslant 0$ and $\sum_{i=1}^{r} m_i = k$.

Once we have this expression, it is easy to see that p is a limit of a sequence of general symmetric principal parts of degree k: for each i, choose m_i distinct complex numbers $c_1, c_2, \ldots, c_{m_i}$ and consider the one-parameter family of principal parts

$$p_i(t) = \frac{e_i \otimes e_i}{(z - c_1 t)(z - c_2 t)\cdots(z - c_{m_i} t)}.$$

For $t \neq 0$, the sum $\sum_{i=1}^{r} p_i(t)$ is a general principal part of degree $\sum_{i=1}^{r} m_i = k$, and $\sum_{i=1}^{r} p_i(0) = p$. Thus p is the limit of a sequence of general symmetric principal parts of degree k.

Now we prove the claim by invoking a diagonalization process of matrices. In local coordinates, p appears as

$$p = \frac{1}{z^m} S$$

for an $n \times n$ symmetric matrix S over $\mathbb{C}[z]$. However, since we are concerned with the principal parts only, the entries of S can be regarded as the elements of the ring $R = \mathbb{C}[z]/(z^m)$. In this context, the claim will be proven if we show that S is congruent in $M_n(R)$ to a diagonal matrix. More precisely, it suffices to show that there exists a matrix $P \in M_n(R)$ which is nonsingular in the sense that $\det P$ is a unit in R, such that

$${}^tPSP = \mathrm{diag}(z^{d_1}, z^{d_2}, \ldots, z^{d_r}), \quad \text{where} \quad 0 = d_1 \leqslant d_2 \leqslant \cdots \leqslant d_r.$$

This can be shown by the Gram–Schmidt process. The standard process provides a way to diagonalize symmetric matrices with entries in $\mathbb{C}$, but the same process can be applied to our situation because

(i) an element in R given by $a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}$ has an inverse in R if and only if $a_0 \neq 0$, and

(ii) every unit has a square root in R.

(For related discussion, we refer the reader to [**1**, corollary (2·44)].) Thus after applying the Gram–Schmidt process to S, we get a diagonal matrix whose diagonal entries are powers of z.

2·5. *Geometric criterion for lifting*

Firstly we adapt some ideas from [**4**, Section 3] to the present situation.

Definition 2·8. Let $V \to X$ be a vector bundle. Then an elementary transformation

$$0 \longrightarrow \tilde{V} \longrightarrow V \longrightarrow \tau \longrightarrow 0$$

is called *general* if $\tau \cong \bigoplus_{i=1}^{k} \mathbb{C}_{x_i}$ for distinct points $x_1, \ldots, x_k$ of X.

The word "general" is justified as follows. As in [**4**, Section 3], one can consider a parameter space $\mathcal{Q}_k(V)$ of elementary transformations of V of degree $\deg(V) - k$ and observe that the locus of general elementary transformations is open and dense in $\mathcal{Q}_k(V)$.

Definition 2·9. Let Y be a closed subvariety of a projective space $\mathbb{P}^N$. The *kth secant variety* of Y, denoted by $\mathrm{Sec}^k Y$, is the Zariski closure of the union of all the k-secants to Y, that is, all the projective subspaces of $\mathbb{P}^N$ spanned by k distinct points of Y.

Now we will consider a bundle $E \to X$ of rank n which satisfies either condition (i) or condition (iii) in Lemma 2·6 so that either

$$\varphi \colon \mathbb{P}(E \otimes E) \to \mathbb{P}H^1(X, E \otimes E) \quad \text{or} \quad \psi : \mathbb{P}E \longrightarrow \mathbb{P}H^1(X, \mathrm{Sym}^2 E)$$

is an embedding. Consider the following diagram:

$$\begin{tikzcd}
\Delta \arrow[r] & \mathbb{P}(E \otimes E) \arrow[r, "\varphi"'] & \mathbb{P}H^1(X, E \otimes E) \\
\mathbb{P}E \arrow[r] \arrow[u] & \mathbb{P}(\mathrm{Sym}^2 E) \arrow[r] \arrow[u] & \mathbb{P}H^1(X, \mathrm{Sym}^2 E), \arrow[u]
\end{tikzcd}$$

where the composition of the two maps on the bottom row is ψ. If ψ is an embedding, these two maps are inclusions, whereas if φ is an embedding, all the arrows are inclusions.

LEMMA 2·10. *Consider a bundle W fitting into a nontrivial symplectic extension of E^* by E with class $\delta(W) \in \mathbb{P}H^1(X, \mathrm{Sym}^2 E)$.*

(i) *If the class $\delta(W)$ corresponds to a general point of* $\mathrm{Sec}^k \mathbb{P}E$, *then W admits an isotropic lifting of a general elementary transformation F of E^* with* $\deg(E^*/F) \leqslant k$.

(ii) *If W admits an isotropic lifting of an elementary transformation F of E^* with* $\deg(E^*/F) \leqslant k$, *then $\delta(W)$ belongs to* $\mathrm{Sec}^k \mathbb{P}E$.

Remark 2·11. An analogous lifting condition for the map φ was considered in [**4**, theorem 4·4], where a criterion on the lifting of an elementary transformation of E^* without the isotropy condition, was given in terms of the higher secant variety of the scroll $\Delta \subset \mathbb{P}H^1(X, E \otimes E)$ instead of $\mathbb{P}E \subset \mathbb{P}H^1(X, \mathrm{Sym}^2 E)$.

Proof. Basically we follow the same idea underlying the proof of [**4**, theorem 4·4], except that we use the language of principal parts here.

To show (i), suppose $\delta(W)$ lies on a secant plane spanned by k general points of $\mathbb{P}E$. This means that $\delta(W)$ can be defined by a linear combination $p = \sum \lambda_i p_i$ of at most k symmetric principal parts p_i supported at distinct points, where $p_i \in \mathbb{P}E|_{x_i}$. By Lemma 2·3 (ii), the kernel of $p\colon E^* \to \underline{\mathrm{Prin}}(E)$ lifts to W isotropically. Also, $\mathrm{Ker}\,(p)$ is a general elementary transformation of E^* whose degree is $\geqslant \deg E^* - k$.

As for (ii): By Lemma 2·3 (ii), an elementary transformation F of E^* lifts to W isotropically if and only if $\delta(W) = [p]$ for some symmetric principal part p such that

$$F \subseteq \mathrm{Ker}\,\big(p\colon E^* \longrightarrow \underline{\mathrm{Prin}}(E)\big).$$

First assume that F is a general elementary transformation so that p is a linear combination of k symmetric principal parts $p_1, \ldots, p_k$ supported at distinct points $x_1, \ldots, x_k$, with

$$p_i \in \Delta|_{x_i} \subseteq (E \otimes E)|_{x_i} \cong \frac{(E \otimes E)\,(x_i)}{(E \otimes E)} \tag{2·6}$$

for each i. Since p is symmetric, each p_i belongs to $\Delta|_{x_i} \cap \mathbb{P}\mathrm{Sym}^2 E|_{x_i} = \mathbb{P}E|_{x_i}$. By our alternative definition of $\varphi\colon \Delta \to \mathbb{P}H^1(X, E \otimes E)$ immediately before Lemma 2·6, the point $\delta(W) = [p]$ lies on the secant plane spanned by k distinct points of $\mathbb{P}E$. In particular, $\delta(W) \in \mathrm{Sec}^k(\mathbb{P}E)$.

The proof of (ii) is completed by passing to the limit, using Lemma 2·7.

From the above discussion, we obtain:

THEOREM 2·12. *Let E and W be as above. If $\delta(W) \in \mathrm{Sec}^k\mathbb{P}E$, then we have $s_{\mathrm{Lag}}(W) \leqslant (n+1)(k + \deg E)$.*

Proof. By Lemma 2·10 (i), if $\delta(W)$ is a general point of $\mathrm{Sec}^k\mathbb{P}E$, then there is some elementary transformation $F \subset E^*$ lifting to W isotropically such that $\deg(F) \geqslant \deg E^* - k$. Hence by definition, $s_{\mathrm{Lag}}(W) \leqslant (n+1)(k + \deg E)$. By the semicontinuity of the invariant s_{Lag}, this inequality still holds for any W with $\delta(W) \in \mathrm{Sec}^k\mathbb{P}E$.

Remark 2·13. (1) In [**4**, theorem 4·4], it was proven that if $\delta(W) \in \mathrm{Sec}^k\Delta$ then we have $s_n(W) \leqslant 2n(k + \deg E)$, that is, W contains a rank n subsheaf of degree $\deg(E^*) - k$, which is not necessarily isotropic. We will return to this phenomenon in Section 4·3.

(2) One can ask if the converse holds in Theorem 2·12. This is a subtle question. Certainly the condition $s_{\mathrm{Lag}}(W) \leqslant (n+1)(k + \deg E)$ implies that W admits an isotropic subbundle F of rank n and degree $\geqslant \deg E^* - k$. But in general F need not come from an elementary transformation of E^*, due to the possible existence of a diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & W & \longrightarrow & E^* & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & G & \longrightarrow & F & \longrightarrow & H & \longrightarrow & 0 \end{array} \tag{2·7}$$

where $\mathrm{rk}(G) \geqslant 1$. In other words, for $n > 1$, it can happen that a maximal Lagrangian subbundle of W may intersect E in a nonzero subsheaf. But in Section 4·1, we will show that this kind of diagram appears only in a restricted way for $n = 2$.

3. *Upper bound on* s_{Lag}

One can easily see that the invariant s_{Lag} has no lower bound by considering, for instance, a decomposable bundle $E \oplus E^*$ where E is a bundle of arbitrarily large degree. In this section, we prove Theorem 1·4 which gives us the sharp upper bound on s_{Lag}.

In [**4**], the Hirschowitz bound in Proposition 1·1 was reproved using the relation between the invariant s_r and the geometry of the higher secant varieties of the ruled varieties in the extension spaces. We would like to adapt the same idea to the case of symplectic bundles by applying the results in the previous section. We begin with the following observations.

PROPOSITION 3·1. *A general symplectic bundle is stable as a vector bundle.*

Proof. By essentially the argument of Ramanan [**19**, Section 4] (see [**7**] for details), a stable principal Sp_{2n}-bundle W corresponds to an orthogonal direct sum of subbundles W_i which are mutually nonisomorphic, with each W_i a symplectic bundle which is stable as a vector bundle. Hence the dimension of the sublocus in $\mathcal{M}_{2n}$ of symplectic bundles which are strictly semistable vector bundles, is bounded by $\dim \mathcal{M}_{2n-2} + \dim \mathcal{M}_2$. Near $W \in \mathcal{M}_{2n}$, the moduli space looks like the quotient of $H^1(X, \mathrm{Sym}^2 W)$ by the finite group of symplectic automorphisms of W. Since $h^0(X, \mathrm{Sym}^2 W) = 0$,

$$\dim \mathcal{M}_{2n} = -\chi(X, \mathrm{Sym}^2 W) = n(2n+1)(g-1).$$

This is greater than

$$\dim \mathcal{M}_{2n-2} + \dim \mathcal{M}_2 = (n-1)(2n-1)(g-1) + 3(g-1) = (2n^2 - 3n + 4)(g-1)$$

if $n \geqslant 2$. Thus a general bundle in $\mathcal{M}_{2n}$ is stable as a vector bundle.

LEMMA 3·2. *A general symplectic bundle* $W \in \mathcal{M}_{2n}$ *has a Lagrangian subbundle* E *of degree* $\leqslant -n(g-1)/2$ *which is general as a vector bundle.*

Proof. First we show that every symplectic bundle W has a Lagrangian subbundle. Let $U \subset X$ be an open set over which W is trivial. Any Lagrangian subspace of $\mathbb{C}^{2n}$ yields a Lagrangian subbundle of $W|_U$. This corresponds to a rational reduction of structure group of the associated Sp_{2n}-bundle of W to the maximal parabolic subgroup preserving a fixed Lagrangian subspace. Since X has dimension one, this can be extended to a reduction of structure group over the whole of X, giving a Lagrangian subbundle of W (cf. Hartshorne [**5**, proposition I·6·8]).

Now let W_0 be a general symplectic bundle. Since W_0 admits a Lagrangian subbundle, say E_0, we may present W_0 as an extension

$$0 \longrightarrow E_0 \longrightarrow W_0 \longrightarrow E_0^* \longrightarrow 0$$

with class $\delta(W_0) \in H^1(X, \mathrm{Sym}^2 E_0)$. Now it is well known that there is a versal deformation $\mathcal{E} \to S \times X$ of E_0 parameterized by an irreducible variety S, whose general member is a general stable bundle. Consider the direct image $R^1 q_*(\mathrm{Sym}^2 \mathcal{E})$, where $q : S \times X \to S$ is the projection. This sheaf is locally free over the open subset S° of S consisting of the bundles E with $h^0(X, \mathrm{Sym}^2 E) = 0$. We claim that E_0 belongs to S°: Indeed, a nonzero map $E_0^* \to E_0$ would induce an endomorphism of W_0 given by composition $W_0 \to E_0^* \to E_0 \to W_0$. This is clearly nonzero and nilpotent. But this contradicts the stability of W_0.

The associated projective bundle $\mathbb{P}(R^1 q_*(\mathrm{Sym}^2 \mathcal{E})|_{S^\circ})$ over S° will be denoted by $\pi : P \to S^\circ$. By Lange [**14**, corollary 4·5], there is an exact sequence of bundles over $P \times X$:

$$0 \longrightarrow (\pi \times \mathrm{Id}_X)^* \mathcal{E} \otimes p^* \mathcal{O}_P(1) \longrightarrow \mathcal{W} \longrightarrow (\pi \times \mathrm{Id}_X)^* \mathcal{E}^* \longrightarrow 0,$$

where $p\colon P \times X \to P$ is the projection, with the property that for each δ in the fibre $\pi^{-1}([E])$ in P, the restriction $\mathcal{W}|_{\{\delta\}\times X}$ is isomorphic to the symplectic extension of E^* by E defined by δ. From the construction of S°, a general point of P corresponds to a symplectic bundle admitting a Lagrangian subbundle which is general as a vector bundle.

Finally we check that $\deg(E) =: -e \leqslant -n(g-1)/2$. From the above argument, the dimension of the space of symplectic bundles admitting a Lagrangian subbundle of degree $-e$ is bounded above by

$$\dim \mathcal{U}(n, -e) + h^1(X, \mathrm{Sym}^2 E) - 1 \;=\; n^2(g-1) + 1 + (n+1)e + \frac{1}{2}n(n+1)(g-1) - 1.$$

Since this should be at least $\dim \mathcal{M}_{2n} = n(2n+1)(g-1)$, we obtain $e \geqslant n(g-1)/2$.

To prove Theorem 1·4, it suffices to show that a general $W \in \mathcal{M}_{2n}$ has a Lagrangian subbundle of degree at least $\lceil n(g-1)/2 \rceil$. By Lemma 3·2, we know that W has a Lagrangian subbundle E which is general in $\mathcal{U}(n, -e)$ for some e.

If $e \leqslant \lceil n(g-1)/2 \rceil$ then we are done. Otherwise, $\mu(E) < -(g-1)/2$, and by Lemma 2·6 (i), we may assume the map $\varphi\colon \mathbb{P}(E \otimes E) \to \mathbb{P}H^1(X, E \otimes E)$ is an embedding. Consider its restriction $\psi\colon \mathbb{P}E \to \mathbb{P} := \mathbb{P}H^1(X, \mathrm{Sym}^2 E)$. By Theorem 2·12, if the class $\delta(W) \in \mathbb{P}$ belongs to the subvariety $\mathrm{Sec}^k \mathbb{P}E$, then W has a Lagrangian subbundle of degree at least $e - k$. For the moment, suppose that $\mathbb{P}E$ has no secant defect in $\mathbb{P}$, so that for each $k \geqslant 1$, we have

$$\begin{aligned} \dim \mathrm{Sec}^k \mathbb{P}E &= \min\{k \dim \mathbb{P}E + k - 1,\ \dim \mathbb{P}\} \\ &= \min\left\{ k(n+1),\ (n+1)e + \frac{1}{2}n(n+1)(g-1) \right\} - 1. \end{aligned}$$

By straightforward computation, $\mathrm{Sec}^k \mathbb{P}E = \mathbb{P}$ if $k \geqslant k_0 := e + \lceil n(g-1)/2 \rceil$. This implies that W has a Lagrangian subbundle of degree at least if $k \geqslant k_0 := e + \lceil n(g-1)/2 \rceil$, as was desired.

Hence the proof will be completed once we show the following.

PROPOSITION 3·3. *For a general bundle $E \in \mathcal{U}(n, -e)$, $e \gg 0$, the subvariety $\mathbb{P}E$ of $\mathbb{P} := \mathbb{P}H^1(X, \mathrm{Sym}^2 E)$ has no secant defect.*

The rest of this section is devoted to the proof of Proposition 3·3. First we recall the Terracini Lemma [**23**]:

LEMMA 3·4. *Let $Z \subset \mathbb{P}^N$ be a projective variety and let $z_1, \ldots, z_k$ be general points of Z. Then* $\dim \mathrm{Sec}^k Z = \dim\langle \mathbb{T}_{z_1} Z, \ldots, \mathbb{T}_{z_k} Z\rangle$*, where $\mathbb{T}_{z_i} Z$ is the embedded tangent space to Z in $\mathbb{P}^N$ at z_i.*

To apply the Terracini Lemma, let us find a description of the embedded tangent spaces of $\mathbb{P}E \subset \mathbb{P}$. Let v be a nonzero vector of $E|_x$, and consider the elementary transformation $0 \to E \to \hat{E} \to \mathbb{C}_x \to 0$ such that the kernel of $E|_x \to \hat{E}|_x$ is spanned by v. Consider the induced elementary transformation

$$0 \longrightarrow \mathrm{Sym}^2 E \longrightarrow \mathrm{Sym}^2 \hat{E} \longrightarrow \tau \longrightarrow 0$$

where τ is a torsion sheaf of degree $(n+1)$.

LEMMA 3·5. *For a general bundle $E \in \mathcal{U}(n, -e)$, with $e \gg 0$, the embedded tangent space $\mathbb{T}_v(\mathbb{P}E)$ to $\mathbb{P}E$ at v in $\mathbb{P}$ is given by*

$$\mathbb{T}_v(\mathbb{P}E) \;=\; \mathbb{P}\,\mathrm{Ker}(H^1(X, \mathrm{Sym}^2 E) \longrightarrow H^1(X, \mathrm{Sym}^2 \hat{E})).$$

Proof. Recall that Δ is the subvariety of $\mathbb{P}(E \otimes E) \subset \mathbb{P}H^1(X, E \otimes E)$ consisting of decomposable vectors. We have the commutative diagram

$$\begin{array}{ccccc} H^1(X, E \otimes E) & \xrightarrow{f} & H^1(X, \hat{E} \otimes \hat{E}) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ H^1(X, \mathrm{Sym}^2 E) & \xrightarrow{g} & H^1(X, \mathrm{Sym}^2 \hat{E}) & \longrightarrow & 0, \end{array} \tag{3·1}$$

where the vertical arrows are inclusions. By [**4**, lemma 5·3], the embedded tangent space of Δ at $v \otimes v$ in $\mathbb{P}H^1(X, E \otimes E)$ is given by $\mathbb{T}_{v\otimes v}\Delta = \mathbb{P}(\mathrm{Ker}\ f)$. From the inclusion $\mathbb{P}E \subset \Delta$ and the above diagram (3·1), we get

$$\mathbb{T}_v \mathbb{P}E \subseteq \mathbb{T}_{v\otimes v}\Delta \ \cap\ \mathbb{P}H^1(X, \mathrm{Sym}^2 E) = \mathbb{P}\,\mathrm{Ker}\,(f) \ \cap\ \mathbb{P}H^1(X, \mathrm{Sym}^2 E) = \mathbb{P}\,\mathrm{Ker}\,(g).$$

Thus it suffices to show that $\mathbb{P}\,\mathrm{Ker}\,(g)$ has dimension $n = \dim \mathbb{P}E$. Since E is general, we may assume $\hat{E}$ is stable. Then $\mathrm{Sym}^2 E$ and $\mathrm{Sym}^2 \hat{E}$ are semistable of negative degree, so they have no nonzero sections. Thus $\mathrm{Ker}\,(g)$ has dimension

$$h^1(X, \mathrm{Sym}^2 E) - h^1(X, \mathrm{Sym}^2 \hat{E}) \ = \ \deg(\mathrm{Sym}^2 \hat{E}) - \deg(\mathrm{Sym}^2 E) \ = \ n + 1,$$

as desired.

Proof of Proposition 3·3. Let F be the general elementary transformation of E associated to the points $v_1, \ldots, v_k \in \mathbb{P}E$. The bundle F fits into an exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow \bigoplus_{i=1}^{k} \mathbb{C}_{x_i} \longrightarrow 0, \tag{3·2}$$

for k distinct points $x_1, \ldots, x_k$ such that v_i lies over x_i. For each i, let E_i be the intermediate sheaf ($E \subset E_i \subset F$) defined by the elementary transformation associated to the v_i:

$$0 \longrightarrow E \longrightarrow E_i \longrightarrow \mathbb{C}_{x_i} \longrightarrow 0.$$

By the Terracini Lemma and Lemma 3·5, the dimension of $\mathrm{Sec}^k \mathbb{P}E \subset \mathbb{P}$ is equal to the dimension of the linear span of the union of

$$\mathbb{P}\,\mathrm{Ker}\left(H^1(X, \mathrm{Sym}^2 E) \longrightarrow H^1(X, \mathrm{Sym}^2 E_i)\right) \tag{3·3}$$

for $1 \leqslant i \leqslant k$. Since F^* is precisely the intersection of all the E_i^* inside E^*, the linear span of the spaces (3·3) coincides with

$$\mathbb{P}\,\mathrm{Ker}\left(H^1(X, \mathrm{Sym}^2 E) \longrightarrow H^1(X, \mathrm{Sym}^2 F)\right).$$

Thus now it remains to show

$$\begin{aligned} &\dim \mathrm{Ker}\left(H^1(X, \mathrm{Sym}^2 E) \longrightarrow H^1(X, \mathrm{Sym}^2 F)\right) \\ &\quad = \min\left\{k(n+1), (n+1)e + \frac{1}{2}n(n+1)(g-1)\right\}. \end{aligned}$$

From the vanishing of $H^0(X, \mathrm{Sym}^2 E)$, one checks that the left-hand side equals $k(n+1) - h^0(X, \mathrm{Sym}^2 F)$. Since F is obtained from a general elementary transformation of a general bundle E, it is a general element of $\mathcal{U}(n, -e+k)$.

Hence the required equality follows from the result on $\dim H^0(X, \mathrm{Sym}^2 F)$ which will be discussed in Lemma A·1 and Corollary A·3 in the appendix.

Remark 3·6 (Hirschowitz bound for Gp_{2n}-bundles).

The statement of Theorem 1·4 is easily generalized to principal Gp_{2n}-bundles. Recall that the group Gp_{2n} of *conformally symplectic transformations* is the image of the multiplication map $\mathrm{Sp}_{2n} \times \mathbb{C}^* \to \mathrm{GL}_{2n}$. A principal Gp_{2n}-bundle corresponds to a vector bundle of rank $2n$ carrying a symplectic form with values in a line bundle L which may be different from $\mathcal{O}_X$; equivalently, admitting an antisymmetric isomorphism $W \xrightarrow{\sim} W^* \otimes L$. Such W has determinant L^n and hence $\deg W = n \deg(L)$. (See Biswas–Gomez [**2**] for more details on Gp_{2n}-bundles.)

If $E \subset W$ is a Lagrangian subbundle then, by [**8**, criterion 2·1], we get an extension

$$0 \longrightarrow E \longrightarrow W \longrightarrow \mathrm{Hom}\,(E, L) \longrightarrow 0$$

with class $\delta(W) \in H^1(X, L^{-1} \otimes \mathrm{Sym}^2 E)$, and conversely. Arguing as above, one can show that W admits a Lagrangian subbundle of degree at least $-\lceil n(g-1-\deg(L))/2 \rceil$.

4. *Geometry of the strata in rank four*

In this section, we prove Theorem 1·5 on the geometry of the strata on $\mathcal{M}_4$. For the case of genus 2, it has already been proven in [**9**]. *Throughout this section, we assume $g \geqslant 3$.*

4·1. *Symplectic extensions of rank four*

First we study more details on symplectic extensions of rank four. The goal is to prove the converse of the statement in Theorem 2·12 in the case of rank four. As was mentioned in Remark 2·13(2), we need to know how often diagrams of the form (2·7) appear.

For $e \in \{1, 2, \ldots, g-1\}$, let $E \to X$ be a vector bundle of rank two and degree $-e$. Assume that E is general, so $g - 1 \leqslant s_1(E) \leqslant g$ as was remarked in (2·5). Equivalently, assume that any line subbundle of E has degree at most $(-e-g+1)/2$.

LEMMA 4·1. *Let E be as above and consider an extension* $0 \to E \to W \to E^* \to 0$ *(not necessarily symplectic). Let F be a rank two subbundle of W such that the intersection of E and F is generically of rank one. Then:*

(i) $\deg F \leqslant -(g-1)$*;*

(ii) *If* $\deg E = \deg F = -(g-1)$*, then the intersection of E and F is a line subbundle of W of degree* $-(g-1)$.

Proof. (i) Let L be the line subbundle of W associated to the intersection of E and F. By the assumptions, we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & W & \longrightarrow & E^* & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & L & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where M is a subsheaf of E^*. Since E is general,

$$\deg(L) \leqslant \frac{1}{2}(-e-g+1) \quad \text{and} \quad \deg(M) \leqslant \frac{1}{2}(e-g+1).$$

Hence $\deg(F) = \deg(L) + \deg(M) \leqslant -(g-1)$.

(ii) By the above inequalities, the condition $\deg E = \deg F = -(g-1)$ implies that $\deg L = -(g-1)$ and $\deg M = 0$. If the quotient sheaf E^*/M had nonzero torsion, then E^* would admit a quotient line bundle of degree $< g-1$, contradicting the generality of

E. Hence M must be a line subbundle of E^*, which shows that the intersection of E and F coincides with L.

The following is an immediate consequence of Lemma 4·1 (i).

COROLLARY 4·2. *Let E be a general rank two bundle of degree $-e$ with $g-1 \leqslant s_1(E) \leqslant g$. Consider a symplectic extension $0 \to E \to W \to E^* \to 0$ corresponding to a general point $\delta(W)$ of $\mathbb{P}H^1(X, \mathrm{Sym}^2 E)$.*

- (i) *If $1 \leqslant e \leqslant g-2$, then any Lagrangian subbundle of W other than E itself, of degree $\geqslant -e$, comes from an elementary transformation of E^*.*
- (ii) *If $e = g-1$, then any Lagrangian subbundle of W of degree $> -e$ comes from an elementary transformation of E^*.*

From this, we get a nice geometric criterion on lifting which improves Theorem 2·12 in the case of rank 4. Recall the embedding criteria on ψ of Lemma 2·6 (iii), confirming that for $g \geqslant 3$ and for a general bundle $E \in \mathcal{U}(2, -e)$ with $e > 0$, the map $\psi\colon \mathbb{P}E \to \mathbb{P}H^1(X, \mathrm{Sym}^2 E)$ is an embedding. Hence in this case, we get a surface $\psi(\mathbb{P}E) \cong \mathbb{P}E$ inside $\mathbb{P}H^1(X, \mathrm{Sym}^2 E)$.

THEOREM 4·3. *Assume $g \geqslant 3$ and $1 \leqslant e \leqslant g-1$. Let E be a general bundle in $\mathcal{U}(2, -e)$, and consider a nontrivial symplectic extension W of E^* by E. For each k with $1 \leqslant k \leqslant 2e-1$, the following conditions are equivalent*:

- (i) *W admits an isotropic lifting of an elementary transformation F of E^* with* $\deg F \geqslant e-k$*;*
- (ii) *$\delta(W) \in \mathrm{Sec}^k \mathbb{P}E$ in $\mathbb{P}H^1(X, \mathrm{Sym}^2 E)$;*
- (iii) *$s_{\mathrm{Lag}}(W) \leqslant (n+1)(k-e)$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) were already shown in Lemma 2·10 (ii) and Theorem 2·12 respectively. The implication (iii) $\Rightarrow$ (i) can be readily seen as follows.

The condition $s_{\mathrm{Lag}}(W) \leqslant (n+1)(k-e)$ implies that W admits a Lagrangian subbundle F of degree at least $e-k$. By Corollary 4·2, if $\deg E = -e < e-k \leqslant \deg F$, then F comes from an elementary transformation of E^*.

4·2. *Stratification on $\mathcal{M}_4$*

For any positive integer e, let $\mathcal{U}(2, -e)^s$ be the moduli space of stable bundles over X of rank 2 and degree $-e$. According to Narasimhan and Ramanan [**18**, proposition 2·4], there exist a finite étale cover $\pi_e\colon \tilde{U}_e \to \mathcal{U}(2, -e)^s$ and a bundle $\mathcal{E}_e \to \tilde{U}_e \times X$ with the property that $\mathcal{E}_e|_{\{E\}\times X} \cong \pi_e(E)$ for all $E \in \tilde{U}_e$ (for odd e, we can take π_e to be the identity map since $\mathcal{U}(2, -e)$ is a fine moduli space).

Now by Riemann-Roch and semistability, for each $E \in \mathcal{U}(2, -e)^s$, we have

$$\dim H^1(X, \mathrm{Sym}^2 E) = 3e + 3(g-1).$$

Therefore, the sheaf $R^1 p_* \mathrm{Sym}^2(\mathcal{E}_e)$ is locally free of rank $3(e+g-1)$ on $\tilde{U}_e$. Consider its projectivization $\mu\colon \mathbb{P}_e \to \tilde{U}_e$.

Now we have a diagram

$$
\begin{tikzcd}
\mathbb{P}_e \arrow[dd, "\mu"'] & \mathbb{P}_e \times X \arrow[l, "r"'] \arrow[d, "\mu\times \mathrm{Id}_X"] & \\
& \tilde{U}_e \times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\tilde{U}_e & & X.
\end{tikzcd}
\tag{4·1}
$$

We write $r : \mathbb{P}_e \times X \to \mathbb{P}_e$ for the projection. By Lange [**14**, corollary 4·5], there is an exact sequence of vector bundles

$$0 \longrightarrow (\mu \times \mathrm{Id}_X)^*\mathcal{E}_e \otimes r^* O_{\mathbb{P}_e}(1) \longrightarrow \mathcal{W}_e \longrightarrow (\mu \times \mathrm{Id}_X)^*\mathcal{E}_e^* \longrightarrow 0$$

over $\mathbb{P}_e \times X$, with the property that for $\delta \in \mathbb{P}_e$ with $\mu(\delta) = E$, the restriction of $\mathcal{W}_e$ to $\{\delta\} \times X$ is isomorphic to the symplectic extension of E^* by E defined by $\delta \in \mathbb{P}H^1(X, \mathrm{Sym}^2 E)$.

There arises a basic question regarding these extension spaces: Is a bundle W corresponding to a general point $\delta \in \mathbb{P}H^1(X, \mathrm{Sym}^2 E)$ stable, if E is taken to be general? (The same question for the vector bundles was answered affirmatively by Brambila–Paz and Lange [**3**], and Russo and Teixidor i Bigas [**21**].) The machinery obtained in the previous subsection enables us to answer this question.

LEMMA 4·4. *Let $E \in \mathcal{U}(2, -e)$, $1 \leqslant e \leqslant g-1$, be a general bundle such that $g - 1 \leqslant s_1(E) \leqslant g$. Then a general point of $\mathbb{P}H^1(X, \mathrm{Sym}^2 E)$ corresponds to a stable symplectic bundle.*

Proof. By Theorem 1·4, a general symplectic bundle W in $\mathcal{M}_4$ satisfies $s_{\mathrm{Lag}}(W) = 3(g-1)$. This shows the above statement for $e = g - 1$.

Now assume $1 \leqslant e < g - 1$. Let W be a symplectic bundle corresponding to a general point of $\mathbb{P}H^1(X, \mathrm{Sym}^2 E)$. Consider any line subbundle L of W. If L is contained in the subbundle E, then $\deg(L) < 0$ by the stability of E. Otherwise L would yield an invertible subsheaf of E^* via the composition $L \to W \to E^*$. From the condition on $s_1(E) = s_1(E^*)$, we have

$$\deg L \leqslant \frac{e-g+1}{2} < 0$$

by generality of E. Therefore, any line subbundle of W has negative degree. Since symplectic bundles are self-dual, the same holds for subbundles of rank three.

Finally we consider the Lagrangian subbundles of W. By Theorem 4·3, we have $s_{\mathrm{Lag}}(W) \leqslant 0$ if and only if $\delta(W) \in \mathrm{Sec}^e \mathbb{P}E$. But

$$\dim \mathrm{Sec}^e \mathbb{P}E \ \leqslant\ 2e + (e-1) = 3e - 1,$$

while $\dim \mathbb{P}H^1(X, \mathrm{Sym}^2 E) = 3e + 3g - 4$. Therefore, the bundles with $s_{\mathrm{Lag}}(W) \leqslant 0$ form a proper subset in $\mathbb{P}H^1(X, \mathrm{Sym}^2 E)$.

Now we go back to the diagram (4·1). There is a classifying map $\gamma_e : \mathbb{P}_e \dashrightarrow \mathcal{M}_4$ induced by the bundle $\mathcal{W}_e \to \mathbb{P}_e \times X$. By Lemma 4·4, this map γ_e is defined over a nonempty dense subset of $\mathbb{P}_e$.

Recall the definition of the stratification on $\mathcal{M}_4$ given by the invariant s_{Lag}: for each $e > 0$, consider the subvarieties of $\mathcal{M}_4$ defined by

$$\mathcal{M}_4^e := \{W \in \mathcal{M}_4 \ : \ s_{\mathrm{Lag}}(W) \leqslant 3e\}.$$

Now we can show the following result which implies Theorem 1·5.

THEOREM 4·5.

(i) *For each e with* $1 \leqslant e \leqslant g-1$, *the map* γ_e *is generically finite and its image is dense in* $\mathcal{M}_4^e$.

(ii) *For each e with* $1 \leqslant e \leqslant g-1$, *the locus* $\mathcal{M}_4^e$ *is irreducible of dimension* $7(g-1)+3e$.

(iii) *For* $e \leqslant g-2$, *a general point of* $\mathcal{M}_4^e$ *corresponds to a symplectic bundle which has a unique maximal Lagrangian subbundle. In particular for odd* $e \leqslant g-2$, *the locus* $\mathcal{M}_4^e$ *is birationally equivalent to the fibration* $\mathbb{P}_e$ *over* $\mathcal{U}(2, -e)^s$.

Proof. First consider the case $e = g-1$. From the bound on s_{Lag} given in Theorem 1·4, we have $\mathcal{M}_4 = \mathcal{M}_4^{g-1}$. In particular, $\mathcal{M}_4^{g-1}$ is irreducible of dimension $10(g-1)$. Moreover, the fact that a general symplectic bundle W satisfies $s_{\mathrm{Lag}}(W) = 3(g-1)$ implies that $\gamma_{g-1} : \mathbb{P}_{g-1} \dashrightarrow \mathcal{M}_4$ is dominant. But $\dim \mathbb{P}_{g-1} = 10(g-1)$, so γ_{g-1} must be generically finite.

In general, it is clear that the image of γ_e lands on the stratum $\mathcal{M}_4^e$. Since the source $\mathbb{P}_e$ is irreducible, so is the image of γ_e. Now we show that it is dense in $\mathcal{M}_4^e$. Any W in $\mathcal{M}_4^e$ is fitted into a symplectic extension of E^* by E for some E of degree $-e$ which might be unstable. But every such E is contained in an irreducible family of bundles whose general member is a stable bundle in $\mathcal{U}(2, -e)$. This shows that $\mathcal{M}_4^e$ is the closure of the image of γ_e. In particular, $\mathcal{M}_4^e$ is irreducible for each e.

Now assume $1 \leqslant e \leqslant g-2$ and consider a general point of $\mathbb{P}_e$; precisely, a symplectic extension

$$0 \longrightarrow E \longrightarrow W \longrightarrow E^* \longrightarrow 0$$

where $E \in \mathcal{U}(2, -e)^s$ is general, so $s_1(E) \geqslant g-1$. Suppose that W admits a Lagrangian subbundle F of degree $\geqslant -e$ other than E. Then by Corollary 4·2 (i), such an F is an elementary transformation of E^* which lifts to W. By Lemma 2·10 (ii) we have $\delta(W) \in \mathrm{Sec}^{2e}(\mathbb{P}E)$. But

$$\dim \mathrm{Sec}^{2e}(\mathbb{P}E) \leqslant 6e - 1 \ < \ 3e + 3g - 4 = \dim \mathbb{P}H^1(X, \mathrm{Sym}^2 E).$$

Thus an extension represented by a general point of $\mathbb{P}_e|_E$ contains no Lagrangian subbundle of degree $\geqslant -e$ apart from E itself. This implies that a general symplectic bundle in $\mathcal{M}_4^e$ has a unique maximal Lagrangian subbundle. In other words, W is represented only in the fibres of $\mathbb{P}_e \to \tilde{U}_e$ over the finite subset $\pi_e^{-1}(E)$ for the étale cover $\pi_e : \tilde{U}_e \to \mathcal{U}(2, -e)^s$. Moreover, there is only one extension class in $\mathbb{P}H^1(X, \mathrm{Sym}^2 E)$ whose associated bundle is isomorphic to W.

This shows that γ_e is generically finite onto its image, of degree equal to that of π_e. In particular, for odd $e \leqslant g-2$, γ_e is generically injective since π_e is the identity map, and so $\mathcal{M}_4^e$ is birationally equivalent to the fibration $\mathbb{P}_e$ over $\mathcal{U}(2, -e)^s$. Also, for all e we obtain $\dim \mathcal{M}_4^e \ = \ \dim \mathbb{P}_e \ = \ 7(g-1) + 3e$.

4·3. *Nonisotropic maximal subbundles*

By Serman [**22**], for $n > 1$ the forgetful map taking a symplectic bundle to the equivalence class of the underlying vector bundle gives an embedding of $\mathcal{M}_{2n}$ in the moduli space

$\mathcal{SU}(2n, \mathcal{O}_X)$ of semistable bundles of rank $2n$ with trivial determinant. Thus it is interesting to compare the two stratifications on $\mathcal{M}_{2n}$ given by s_n and s_{Lag}. We will focus on the rank four case.

Suppose X has genus $g \geqslant 4$. Let F_1 and F_2 be a pair of mutually nonisomorphic stable bundles of rank two and trivial determinant. The direct sum $W := F_1 \oplus F_2$ is a symplectic bundle with $s_2(W) = 0$ and $s_{\mathrm{Lag}}(W) > 0$. We will now use Lemma 2·10 and Remark 2·11 to illuminate this phenomenon.

Let E be a general bundle of rank two and determinant $\mathcal{O}_X(-x)$ for some point $x \in X$. The constant function 1 defines a global rational section α of $\det(E)$ which has a simple pole at x and is elsewhere regular. Since E has rank two, for *any* linearly independent $v, w \in E|_x$ we have, up to a constant,

$$\overline{\alpha} = \frac{v \wedge w}{z} = \frac{v \otimes w}{z} - \frac{w \otimes v}{z}. \tag{4·2}$$

Thus the cohomology class

$$\left[\frac{v \otimes w}{z}\right] = \left[\frac{w \otimes v}{z}\right]$$

in $H^1(X, E \otimes E)$ defines a symplectic extension of E^* by E.

Write $p = v \otimes w/z$. By Lemma 2·6 (ii), the map $\varphi\colon \Delta \to \mathbb{P}H^1(X, E \otimes E)$ is base point free. Since $[p] = \varphi(v \otimes w)$, it is a nontrivial cohomology class.

By Lemma 2·3, the kernel F of $p\colon E^* \to \underline{\mathrm{Prin}}(E)$ lifts to a *nonisotropic* subbundle of W. This subbundle is an elementary transformation $0 \to F \to E^* \to \mathbb{C}_x \to 0$, so has trivial determinant (as it must, since $h^0(X, \wedge^2 F^*) > 0$).

This behaviour can be explained geometrically as follows. Since

$$\left[\frac{v \otimes w}{z}\right] = \left[\frac{w \otimes v}{z}\right] \tag{4·3}$$

for *any* linearly independent v and w, the map $\varphi\colon \Delta|_x \to \mathbb{P}H^1(X, E \otimes E)$ is not an embedding. In fact it is a double cover

$$\Delta|_x \cong \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}\mathrm{Sym}^2 E|_x \cong \mathbb{P}^2$$

ramified over the plane conic $\mathbb{P}E|_x$. (In particular, it factorizes via $\mathbb{P}H^1(X, \mathrm{Sym}^2 E)$.) By (4·3), any extension class lying in this $\mathbb{P}^2$ lies on (a 1-secant to) the quadric bundle Δ, but a general such class does not lie on (a 1-secant to) the conic bundle $\mathbb{P}E$. Therefore, by Lemma 2·10, an elementary transformation of degree $\deg(E^*) - 1 = 0$ lifts to a nonisotropic subbundle of W, but W has no Lagrangian subbundle of degree zero.

Remark 4·6. In fact W admits a pair of nonisotropic subbundles of trivial determinant given by $\mathrm{Ker}\,(p)$ and $\mathrm{Ker}\,({}^t p)$. It is not hard to see that these are of the form F and $F^{\perp}$. Furthermore, the bundle W splits as the direct sum $F \oplus F^{\perp}$, by [**7**, theorem 2·3].

Building upon this idea, one can show the following:

Suppose $g \geqslant 4$. For any e and f with $0 \leqslant f < e$ and $f + 2e - 1 \leqslant 2(g-2)/3$, there exists a stable rank four symplectic bundle W with $s_2(W) = 4f$ and $s_{\mathrm{Lag}}(W) = 3e$. In other words, W has a maximal subbundle of degree $-f$ which is nonisotropic, and a maximal Lagrangian *subbundle of degree $-e < -f$.*

The principle is as above: for certain E of degree $-e$ and special determinant, one can construct a class δ in $H^1(X, \mathrm{Sym}^2 E)$ lying on $\mathrm{Sec}^{e+f}\Delta \setminus \mathrm{Sec}^{2e-1}\mathbb{P}E$. By Lemma 2·10, the extension $0 \to E \to W \to E^* \to 0$ defined by δ will have a nonisotropic subbundle of rank two and degree $-f$, but no Lagrangian subbundle of degree greater than $-e$. The details are tedious, so we omit the calculation. We mention only that the condition $f + 2e - 1 \leqslant 2(g-2)/3$ comes from Hwang–Ramanan [**11**, proposition 3·2], which guarantees the vanishing of $H^0(X, ad_E(D))$ for certain effective divisors D.

Appendix A. A variant of Hirschowitz' lemma

Hirschowitz' lemma [**6**, 4·6] assures us that the tensor product of two general bundles is nonspecial (see also Russo–Teixidor i Bigas [**21**, theorem 1·2]). This is useful in many situations; for instance the proof [**4**, p. 12] of Hirschowitz' bound on the Segre invariants. Here we will prove a variant of Hirschowitz' lemma.

LEMMA A·1. *Let F be a general stable bundle of rank n and degree d. If $d \leqslant n(g-1)/2$, then $h^0(X, F \otimes F) = 0$.*

Proof. The proof will be completed in three steps.

Step 1. Let $\mathbb{F}$ be a deformation of F given by a nonzero class $\delta \in H^1(X, \mathrm{End}(F))$. For a given nonzero symmetric map $\alpha : F^* \to F$, we want to know when there exists an extension $\tilde{\alpha} : \mathbb{F}^* \to \mathbb{F}$ inducing the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & \mathbb{F} & \longrightarrow & F & \longrightarrow & 0 \\ & & \Big\uparrow{\scriptstyle \alpha} & & \Big\uparrow{\scriptstyle \tilde{\alpha}} & & \Big\uparrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & F^* & \longrightarrow & \mathbb{F}^* & \longrightarrow & F^* & \longrightarrow & 0. \end{array} \tag{A 1}$$

We have induced maps

$$\alpha_\flat : H^1(X, \mathrm{End}(F^*)) \longrightarrow H^1(X, \mathrm{Hom}\,(F^*, F))$$
$$\text{and} \quad \alpha^\sharp : H^1(X, \mathrm{End}(F)) \longrightarrow H^1(X, \mathrm{Hom}\,(F^*, F)).$$

Note that the class of the dual deformation $\mathbb{F}^*$ is given by $-{}^t\delta \in H^1(X, \mathrm{End}(F^*))$. By straightforward computation, we can check that the maps α on the outer terms in the diagram (A 1) extend to a map $\tilde{\alpha} : \mathbb{F}^* \to \mathbb{F}$ if and only if $-\alpha_\flat {}^t\delta = \alpha^\sharp \delta$ in $H^1(X, \mathrm{Hom}\,(F^*, F))$.

Now since α is symmetric, we have

$$-\alpha_\flat {}^t\delta = -({}^t\alpha)_\flat {}^t\delta = -{}^t(\alpha^\sharp \delta).$$

Thus we obtain:

LEMMA A·2. *A nonzero symmetric map $\alpha : F^* \to F$ extends to $\tilde{\alpha}$ if and only if $\alpha^\sharp\delta$ belongs to the subspace $H^1(X, \wedge^2 F) \subseteq H^1(X, \mathrm{Hom}\,(F^*, F))$.*

Step 2. Rank 2 case: It suffices to consider the boundary case when $\deg F = g - 1$. Let F be a general bundle in $\mathcal{U}(2, g-1)$. Since $\wedge^2 F = \det F$ is general in $\mathrm{Pic}^{g-1}(X)$, we have $H^0(X, \wedge^2 F) = 0$. Thus it suffices to show the vanishing of $H^0(X, \mathrm{Sym}^2 F)$.

First we show that there is no nonzero map $F^* \to F$ whose image is of rank 1: since $s_1(F) = g - 1$, every line subbundle of F has degree $\leqslant 0$. Furthermore, by [**15**, corollary 3·2], F has only finitely many maximal line subbundles (of degree 0). Thus if there were a nonzero map $\alpha\colon F^* \to F$ with image M of rank 1, then $\deg M = 0$ and F should have both M and M^* as its maximal subbundles. This implies either $M \cong M^*$ or there is a sequence

$$0 \longrightarrow M \oplus M^* \longrightarrow F \longrightarrow \tau \longrightarrow 0$$

for some torsion sheaf τ of degree $g-1$. By dimension counting, one can check that neither of these conditions is satisfied by a general $F \in \mathcal{U}(2, g-1)$.

Next we show that if $\mathrm{Sym}^2 F$ has a nonzero section α, then it does not extend to every deformation of F. We have seen that $\alpha : F^* \to F$ must be generically surjective. Hence the induced cohomology map

$$\alpha^\sharp\colon H^1(X, \mathrm{End}(F)) \longrightarrow H^1(X, \mathrm{Hom}\,(F^*, F))$$

is surjective. By the assumptions $h^0(X, \mathrm{Sym}^2 F) > 0$ and $\chi(\mathrm{Sym}^2 F) = 0$, we see that $h^1(X, \mathrm{Sym}^2 F) > 0$. This implies that the image of $\alpha^\sharp$ is not contained in $H^1(X, \wedge^2 F)$. By Lemma A·2, α does not extend to $\tilde{\alpha}$ for some deformation $\mathbb{F}$ of F. Therefore, $H^0(X, F \otimes F)$ vanishes for a general $F \in \mathcal{U}(2, g-1)$.

Step 3. Induction for higher rank cases: Now we consider bundles of rank $n \geqslant 3$. Firstly, suppose n is even. By semicontinuity, it suffices to find a bundle $F_0 \in \mathcal{U}(n, n(g-1)/2)$ such that $H^0(X, F_0 \otimes F_0) = 0$. We let $F_0 = F_1 \oplus F_2$ where F_1 is general in $\mathcal{U}(2, g-1)$ and F_2 is general in $\mathcal{U}(n-2, (n-2)(g-1)/2)$, so that F_0 is a polystable bundle of rank n and degree $n(g-1)/2$. From the generality condition and the induction hypothesis, $F_1 \otimes F_1$ and $F_2 \otimes F_2$ have no nonzero sections. Also $h^0(X, F_1 \otimes F_2) = 0$ by Hirschowitz' lemma [**6**, 4·6], [**21**, theorem 1·2]. Therefore, $F_0 \otimes F_0$ has no nonzero sections.

Next suppose both n and g are odd. Let $F_0 = F_1 \oplus F_2$ for a general line bundle F_1 of degree $(g-1)/2$ and a general $F_2 \subset \mathcal{U}(n-1, (n-1)(g-1)/2)$. By the same argument as above, we see that $h^0(X, F_0 \otimes F_0) = 0$.

Finally suppose that n is odd and g is even. In this case, it suffices to find a bundle $F_0 \in \mathcal{U}(n, (n(g-1)-1)/2)$ such that $H^0(X, F_0 \otimes F_0) = 0$. Let F_0 be a general extension

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_2 \longrightarrow 0$$

for a general line bundle F_1 of degree $(g-2)/2$ and a general $F_2 \in \mathcal{U}(n-1, (n-1)(g-1)/2)$. From the stability of F_1 and F_2, it is easy to check that F_0 is also stable. Again by the above argument, we see that $h^0(X, F_0 \otimes F_0) = 0$.

COROLLARY A·3. *If F is a general bundle of rank n and degree $d \geqslant n(g-1)/2$, then*

$$h^0(X, \mathrm{Sym}^2 F) = (n+1)d - \frac{1}{2}n(n+1)(g-1).$$

Proof. For a theta characteristic κ of X, we have

$$\begin{aligned} h^1(X, \mathrm{Sym}^2 F) &\cong h^0(X, K_X \otimes \mathrm{Sym}^2 F^*) \\ &= h^0(X, \mathrm{Sym}^2(\kappa \otimes F^*)). \end{aligned}$$

Since $\deg(\kappa \otimes F^*) = n(g-1) - d \leqslant n(g-1)/2$, we can apply Lemma A·1 to get the vanishing of $H^1(X, \mathrm{Sym}^2 F)$. By Riemann-Roch, we get the wanted result on $h^0(X, \mathrm{Sym}^2 F)$.

Remark A·4. In the case when either n is even or g is odd, the assignment $F \mapsto F \otimes F$ defines a morphism

$$\mathcal{U}\left(n, \frac{1}{2}n(g-1)\right) \longrightarrow \mathcal{U}(n^2, n^2(g-1)),$$

since a tensor product of semistable bundles is semistable. The target moduli space has a generalized theta divisor, whose support consists of semistable bundles with nonzero sections. Lemma A·1 implies that the image of this morphism is not contained in the generalized theta divisor.

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